ON REGULAR HÉNON-LIKE RENORMALIZATION

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ABSTRACT. We develop a renormalization theory of non-perturbative dissipative Hénon-like maps with combinatorics of bounded type. A key novelty of our approach is the incorporation of Pesin theoretic ideas to the renormalization method, which enables us to control the small-scale geometry of dynamics in the higher-dimensional setting. We show that, under certain regularity conditions on the return maps coming from a measure-independent quantitative formulation of Pesin theory, renormalizations of Hénon-like maps have *a priori* bounds. Then using this estimate, we obtain the following results. First, we show that infinite regular Hénon-like renormalizability is a finite-time checkable condition. Second, we prove that Hénon-like maps converge under renormalization to the same renormalization attractor as for 1D unimodal maps. Lastly, we show that every infinitely renormalizable Hénon-like map is *regularly unicritical*: there exists a unique orbit of tangencies between strong-stable and center manifolds, and outside a slow-exponentially shrinking neighborhood of this orbit, the dynamics behaves as a uniformly partially hyperbolic system.

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1. Introduction

The set of real quadratic polynomials, after normalization, can be represented as the following one-parameter family:

$$\mathfrak{Q} := \{f_a(x) := x^2 + a \mid a \in \mathbb{R}\}. \tag{1.1}$$

We refer to $\mathfrak{Q}$ as the *(real) quadratic family*. Despite its elementary form, the dynamics in $\mathfrak{Q}$ turns out to be incredibly rich and fascinatingly complicated. In fact, the study of this family has been a focal point in the field of one-dimensional dynamics for nearly three decades (see e.g. [L2]).

At the heart of this topic lies the renormalization theory of unimodal maps, which analyzes the appearance of small-scale self-similarity in these systems. It was first introduced to the subject independently by Feigenbaum [Fe] and Coullet–Tresser [CoTr] in the mid 1970's. They observed that under successive renormalizations (or "zoom-ins"), the small-scale dynamics of a unimodal map asymptotically approach a *universal* limit sequence that only depends on the combinatorial type of the original system. As a conjectural explanation of this phenomenon, they proposed that renormalization can be viewed as an operator acting on the space of unimodal maps, and that the set of renormalization limits form a hyperbolic attractor $\mathfrak{A}$ for this operator. A rigorous mathematical proof of this conjecture was completed in 1999, through the combined efforts of Sullivan [Su], McMullen [Mc] and Lyubich [L1].

In dimension two, the role of the quadratic family is assumed by the *Hénon family*:

$$\mathfrak{H} := \{F_{a,b}(x, y) := (x^2 + a - by, x) \mid a, b \in \mathbb{R}\}. \tag{1.2}$$

The elements in $\mathfrak{H}$ are referred to as *Hénon maps*. We identify $\mathfrak{Q}$ with the line $b = 0$ consisting of degenerate Hénon maps. This way, $\mathfrak{H}$ can be viewed as a two-dimensional extension of $\mathfrak{Q}$. A Hénon map $F_{a,b}$ is said to be *perturbative* if $|b| \ll 1$, as it can be obtained by making a small 2D perturbation to the 1D system $F_{a,0} \sim f_a$.

The Hénon maps were introduced by Hénon in 1969 as simplified models of the Poincaré sections of the Lorenz model [He]. Since then, these maps have been some of the most widely studied examples in two-dimensional dynamics. For some particularly notable results about Hénon maps, see [BeCa], [] and []. However, despite these remarkable developments, the dynamics in $\mathfrak{H}$ still remains a wide open area of research. Given the enormous success of renormalization theory in dimension one, it is natural to try and extend this technique to the two-dimensional setting.

Consider an infinitely renormalizable quadratic polynomial f_{a_*}. Numerical evidence suggests that (at least for some combinatorial types) there exists a real analytic curve

$$\gamma(b) = (a(b), b) \quad \text{for} \quad b \in [0, 1)$$

extending from $\gamma(0) := (a_*, 0)$ such that the Hénon map $F_{\gamma(b)}$ is infinitely renormalizable, and has the same asymptotics as f_{a_*} under renormalization. See [GaTr].

In the perturbative regime, this conjecture can be verified for bounded type combinatorics using the following argument. Since the 1D renormalization attractor $\mathfrak{A}$ is hyperbolic, infinitely renormalizable quadratic polynomials converge to $\mathfrak{A}$ under renormalization in a robust way. Using this fact, one can show that the 1D renormalization convergence in the quadratic family extends to nearby 2D infinitely renormalizable Hénon maps. This argument has been applied to the period-doubling case by Coullet-Eckmann-Koch [CoEcKo], Gambaudo–Treser–van Strien [GavSTr] and De Carvalho–Lyubich–Martens [DCLMa]; and to the stationary case by Hazard [Ha].

The goal of this paper is to extend the 1D renormalization theory of unimodal maps to a *non-perturbative* 2D setting. A natural 2D analogue of a unimodal map is given by a *Hénon-like map*: a diffeomorphism $F : D \to F(D) \Subset D$ of the form $F(x, y) = (f(x, y), x)$ defined on a rectangle $0 \in D \subset \mathbb{R}^2$, such that for any fixed y, the 1D map $f(\cdot, y)$ is unimodal. One may visualize the action of F as bending D into a U-shape, and then turning it on its side. See Figure 1. We refer to $\Pi_{1\mathrm{D}}(F)(x) := f(x, 0)$ as the *1D profile of F*.

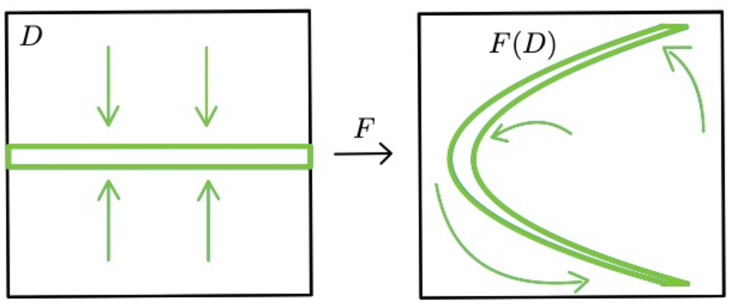

Figure 1. Hénon-like mapping.

The Hénon-like map F is *(Hénon-like) renormalizable* if for some integer $R \geq 2$, there is an R-periodic subdomain $\mathcal{B}^1 \Subset D$, and the return map $F^R|_{\mathcal{B}^1}$ is again Hénon-like after a smooth change-of-coordinates $\Phi : \mathcal{B}^1 \to D^1$. In this case, the map Φ and the pair (F^R, Φ) are referred to as a *straightening chart* and a *Hénon-like return* respectively. We define the *(Hénon-like) renormalization* $\mathcal{R}(F)$ of F as the Hénon-like map obtained via a suitable affine rescaling of $\Phi \circ F^R \circ \Phi^{-1}$ that normalizes the width of the domain D^1. See Figure 2. Lastly, a *centered straightening chart* $\Psi : \mathcal{B}^1 \to B^1$ is a chart that induces the same vertical and horizontal foliations over $\mathcal{B}^1$, and preserves the arclengths along the vertical and horizontal lines through the origin in B^1.

A key novelty of our approach is the incorporation of Pesin theoretic ideas to the renormalization method. This involves keeping track of the *regularity* of points, which can then be used to control the geometry of dynamics in the higher-dimensional setting (see Appendix A). We give loose definitions of these notions below. For the precise definitions, see Subsection 2.5.

Let p be a point in D, and let E_p be a tangent direction at p. For $M \in \mathbb{N} \cup \{\infty\}$, we say that p is *M-times forward regular along E_p* if under DF^m for $1 \leq m \leq M$, there is sufficiently dominant exponential contraction along E_p. Similarly, p is *M-times backward regular along E_p* if under DF^m for $1 \leq m \leq M$, there is sufficiently dominant exponential expansion along E_p.

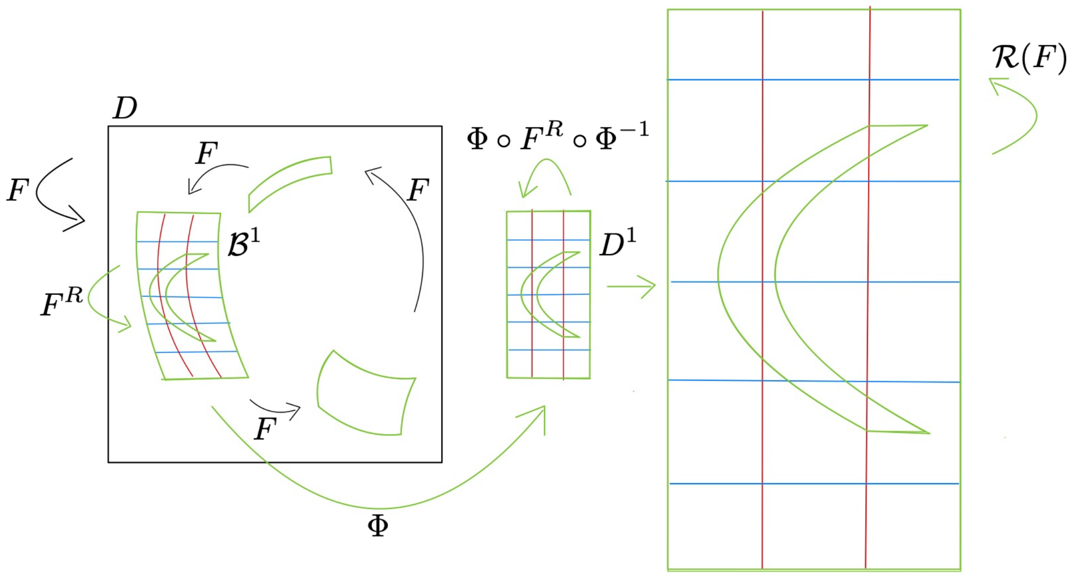

Figure 2. Hénon-like renormalization.

Consider the Hénon-like return (F^R, Φ). For $p \in \mathcal{B}^1$, let E^v_p and E^h_p be the tangent directions at p that are mapped by $D\Phi$ to the genuine vertical and horizontal directions respectively. We say that (F^R, Φ) is *regular*, and that F is *regularly Hénon-like renormalizable* if

i) every $p \in \mathcal{B}^1$ is R-times forward regular along E^v_p;
ii) every $q \in F^R(\mathcal{B}^1)$ is R-times backward regular along E^h_q; and
iii) at every $p \in \mathcal{B}^1$, the tangent directions E^v_p and E^h_p are uniformly transversal.

Under this regularity assumption, we establish the following uniform control on the small-scale geometry of the dynamics of a Hénon-like map that holds at all renormalization depths. An estimate of this kind is commonly referred to as *a priori* bounds, and is typically the key ingredient needed to develop a functioning renormalization theory. The precise version is stated as Theorem A in Section 3.

***A Priori* Bounds.** *Suppose for some $N \in \mathbb{N} \cup \{\infty\}$, a Hénon-like map F is N-times regularly Hénon-like renormalizable with bounded type combinatorics. Then for all $1 \leq n \leq N$, the distortion along the horizontal direction of the nth return map is uniformly bounded.*

A priori bounds has far-reaching consequences for renormalization of Hénon-like maps, which we summarize below as three main results. They are given informally here in order to better convey their conceptual meaning to the readers. For their precise statements, see Section 3.

The first main result describes the asymptotics of Hénon-like maps under regular Hénon-like renormalization. The precise version is stated as Theorem E in Section 3.

Renormalization Convergence. *Suppose a Hénon-like map F is infinitely regularly Hénon-like renormalizable with bounded type combinatorics. Then the following statements hold.*

i) The centered straightening charts for the renormalizations of F converge super-exponentially fast.
ii) The renormalizations of F converge to the space of 1D systems (i.e. their dependence on the second coordinate goes to zero) super-exponentially fast.
iii) The 1D profiles of the renormalizations of F converge to the 1D renormalization attractor for unimodal maps exponentially fast.

The second main result addresses the problem of guaranteeing the existence of infinitely regularly Hénon-like renormalizable maps. It is actually a combination of two theorems: Theorem D and Theorem E in Section 3. See also Remark 3.1.

Finite-time Checkability. *For bounded type combinatorics, infinite regular Hénon-like renormalizability is a finite-time checkable condition.*

Applying the previous two main results to the Hénon family $\mathfrak{H}$, it becomes theoretically possible to verify numerically if the curves of infinitely renormalizable Hénon maps extend arbitrarily close to $b = 1$ (although the computations involved would become infinitely complex as the Jacobian gets closer and closer to 1). See Examples 3.2 and 3.3 for more details.

Renormalization convergence gives us extremely precise information about what the dynamics of a Hénon map looks like when it is "zoomed-in" at a certain point (which we later identify as the *critical value* of the map). The last main result concerns the global geometry of the dynamics over the entire renormalization limit set. A more detailed version of this result is stated in Theorem F in Section 3.

Regular Unicriticality. *An infinitely regularly Hénon-like renormalizable map with bounded type combinatorics is* regularly unicritical *on its renormalization limit set.*

The notion of regular unicriticality is introduced and studied in [CLPY1]. It is a new type of axiomatic dynamics defined on uniquely ergodic sets. Loosely speaking, it means that the system features a unique *critical orbit*: an orbit of tangencies between strong-stable and center manifolds. Moreover, outside a slow-exponentially shrinking neighborhood of this orbit, every point is uniformly regular. See Definition 2.4. In [CLPY1], it is shown that, despite the presence of the critical orbit adding highly non-linear "bends" in the picture, the global geometry of the dynamics for a regularly unicritical system can be understood almost as explicitly as for a uniformly partially hyperbolic system.

1.1. **Notations and conventions.** Unless otherwise specified, we adopt the following notations and conventions.

Any diffeomorphism on a domain in $\mathbb{R}^2$ is assumed to be orientation-preserving. The projective tangent space at a point $p \in \mathbb{R}^2$ is denoted by $\mathbb{P}^2_p$.

Given a number $\eta > 0$, we use $\bar{\eta}$ to denote any number that satisfy

$$\eta < \bar{\eta} < C\eta^D$$

for some uniform constants $C > 1$ and $D > 1$ (if $\eta > 1$) or $D \in (0,1)$ (if $\eta < 1$) that are independent of the map being considered. Additionally, we allow $\bar{\eta}$ to absorb any uniformly bounded coefficient or power. So for example, if $\bar{\eta} > 1$, then we may write

$$\text{“ } 10\bar{\eta}^5 = \bar{\eta} \text{ ”}.$$

Similarly, we use $\underline{\eta}$ to denote any number that satisfy

$$c\eta^d < \underline{\eta} < \eta$$

for some uniform constants $c \in (0,1)$ and $d \in (0,1)$ (if $\eta > 1$) or $d > 1$ (if $\eta < 1$) that are independent of the map being considered. As before, we allow $\underline{\eta}$ to absorb any uniformly bounded coefficient or power. So for example, if $\underline{\eta} > 1$, then we may write

$$\text{“ } \tfrac{1}{3}\underline{\eta}^{1/4} = \underline{\eta} \text{ ”}.$$

These notations apply to any positive real number: e.g. $\bar{\varepsilon} > \varepsilon$, $\underline{\delta} < \delta$, $\bar{L} > L$, etc.

Note that $\bar{\eta}$ can be much larger than η (similarly, $\underline{\eta}$ can be much smaller than η). Sometimes, we may avoid the β or $\underline{\eta}$ notation when indicating numbers that are somewhat or very close to the original value of η. For example, if $\eta \in (0,1)$ is a small number, then we may denote $\eta' := (1-\bar{\eta})\eta$. Then $\underline{\eta} \ll \eta' < \eta$.

We use n, m, i, j to denote integers (and less frequently l, k). The letter i is never the imaginary number. Typically (but not always), $n \in \mathbb{N}$ and $m \in \mathbb{Z}$. We typically use N, M to indicate fixed integers (often related to variables n, m).

We typically denote constants used for estimate bounds by $C, K \geq 1$ (less frequently $c > 0$).

We use calligraphic font $\mathcal{U}, \mathcal{T}, \mathcal{I}$, etc, for objects in the phase space and regular fonts U, T, I, etc, for corresponding objects in the linearized/uniformized coordinates. A notable exception are for the invariant manifolds W^{ss}, W^c.

We use p, q to indicate points in the phase space, and z, w for points in linearized/uniformized coordinates.

For any set $X_m \subset \Omega$ with a numerical index $m \in \mathbb{Z}$, we denote

$$X_{m+l} := F^l(X_m)$$

for all $l \in \mathbb{Z}$ for which the right-hand side is well-defined. Similarly, for any direction $E_{p_m} \in \mathbb{P}^2_{p_m}$ at a point $p_m \in \Omega$, we denote

$$E_{p_{m+l}} := DF^l(E_{p_m}).$$

Define

$$\pi_h(x,y) := x, \quad \pi_v(x,y) := y, \quad \Pi_h(x,y) := (x,0) \quad \text{and} \quad \Pi_v(x,y) := (0,y).$$

2. Preliminaries

2.1. **Renormalization of unimodal maps.** Let $I \subset \mathbb{R}$ be an interval. A C^2-map $f : I \to I$ is *unimodal* if it has a unique critical point $c \in I$, which of quadratic type: i.e. $f'(c) = 0$ and $f''(c) \neq 0$. Denote the critical value of f by $v := f(c)$. We say that f is *normalized* if $c = 0$ and $f''(c) = 2$. Let $\gamma \in \{r, \omega\}$, where $r \geq 2$ is an integer. The space of normalized C^γ-unimodal maps is denoted $\mathfrak{U}^\gamma$.

A unimodal map $f : I \to I$ is *topologically renormalizable* if there exists an *R-periodic* subinterval $I^1 \subset I$:

$$f^i(I^1) \cap I^1 = \varnothing \quad \text{for} \quad 1 \leq i < R \quad \text{and} \quad f^R(I^1) \Subset I^1.$$

We say that f is *(valuably) renormalizable* if $f^R(I^1)$ contains the critical value v.

If f is valuably renormalizable, then the return map $f^R|_{I^1}$ is also unimodal. We define the renormalization of f to be

$$\mathcal{R}_{1\mathrm{D}}(f) := S \circ f^R|_{I^1} \circ S^{-1},$$

where $S : \mathbb{R} \to \mathbb{R}$ is the unique affine map such that $\mathcal{R}_{1\mathrm{D}}(f) \in \mathfrak{U}^\gamma$.

2.2. **Hénon-like maps.** Let $D := I \times J \subset \mathbb{R}^2$ be a rectangle, where $0 \in I \Subset J \subset \mathbb{R}$ are intervals. A C^2-diffemorphism $F : D \to F(D) \Subset D$ is *Hénon-like* if F is of the form

$$F(x, y) = (f(x, y), x) \quad \text{for} \quad (x, y) \in D, \tag{2.1}$$

such that for any $y \in J$, the map $f(\cdot, y) : I \to I$ is a unimodal map. We say that F is *normalized* if $f(\cdot, 0)$ is normalized. The set of normalized C^γ-Hénon-like maps is denoted $\mathfrak{H}\mathfrak{L}^\gamma$.

For $\beta \in (0, 1]$, we say that F is *β-thin (in C^γ)* if

$$\|\partial_y f\|_{C^{\gamma-1}} < \beta.$$

The space of β-thin Hénon-like maps in $\mathfrak{H}\mathfrak{L}^\gamma$ is denoted $\mathfrak{H}\mathfrak{L}^\gamma_\beta$.

For any 1D map $g : I \to I$, define its *2D embedding* $\iota(g) : I \times \mathbb{R} \to I \times \mathbb{R}$ by

$$\iota(g)(x, y) := (g(x), x). \tag{2.2}$$

For any 2D map $G : D \to D$, define its *1D profile* $\Pi_{1\mathrm{D}}(G) : I \to I$ by

$$\Pi_{1\mathrm{D}}(G)(x) := \pi_h \circ G(x, 0). \tag{2.3}$$

Note that we have $\Pi_{1\mathrm{D}} \circ \iota(g) = g$.

The space of degenerate C^γ-Hénon-like maps is given by $\mathfrak{H}\mathfrak{L}^\gamma_0 := \iota(\mathfrak{U}^\gamma)$. A map $F \in \mathfrak{H}\mathfrak{L}^\gamma_\beta$ is said to be *perturbative* if $\beta \ll 1$, as it can be obtained by making a small 2D perturbation to a 1D system in $\mathfrak{H}\mathfrak{L}^\gamma_0$.

2.3. **Charts.** For $z \in \mathbb{R}^2$, let $E^{gv}_z, E^{gh}_z \in \mathbb{P}^2_z$ denote the genuine vertical and horizontal directions at z respectively.

A C^r*-chart* is a C^r-diffeomorphism $\Phi : \mathcal{B} \to B$ from a quadrilateral $\mathcal{B} \subset \mathbb{R}^2$ to a rectangle $B = I \times J \subset \mathbb{R}^2$, where $I, J \subset \mathbb{R}$ are intervals. The *vertical/horizontal direction* $E^{v/h}_p \in \mathbb{P}^2_p$ *at* $p \in \mathcal{B}$ *(associated to* Φ*)* are given by

$$E^{v/h}_p := D\Phi^{-1}\left(E^{gv/gh}_{\Phi(p)}\right).$$

The chart Φ is said to be *genuinely vertical/horizontal* if $E^{v/h}_p = E^{gv/gh}_p$ for all $p \in \mathcal{B}$.

A *vertical leaf in* $\mathcal{B}$ is a curve l^v such that

$$l^v \subseteq \Phi^{-1}(\{a\} \times \pi_v(B)) \quad \text{for some} \quad a \in \pi_h(B).$$

If the above containment is an equality, then l^v is said to be *full*. A *(full) horizontal leaf* l^h *in* $\mathcal{B}$ is defined analogously.

Let $p \in \mathcal{B}$ and $E_p \in \mathbb{P}^2_p$. Denote $z := \Phi(p)$ and $E_z := D\Phi(E_p)$. For $t > 0$, the direction E_p is said to be t*-vertical in* $\mathcal{B}$ if

$$\frac{\measuredangle(E_z, E^{gv}_z)}{\measuredangle(E_z, E^{gh}_z)} < t.$$

A t*-horizontal direction in* $\mathcal{B}$ is analogously defined.

A C^0-curve $\Gamma^v \subset \mathcal{B}$ is said to be *vertical in* $\mathcal{B}$ if $\Phi(\Gamma^v)$ is a vertical graph in B in the usual sense. That is, there exists an interval $I^v \subseteq \pi_v(B)$ and a map $g_v : I^v \to \pi_h(B)$ such that

$$\Phi(\Gamma^v) = \mathcal{G}^v(g_v) := \{(g_v(y), y) \mid y \in I^v\}.$$

If $I^v = \pi_v(B)$, then Γ^v is said to be *vertically proper in* $\mathcal{B}$. A *horizontal* or a *horizontally proper curve* Γ^h *in* $\mathcal{B}$ is analogously defined. If Γ^v is C^r, and $\|g_v'\|_{C^{r-1}} \leq t$ for some $t \geq 0$, then we say that Γ^v is t*-vertical (in* C^r*) in* $\mathcal{B}$. Note that Γ^v is a (vertically proper) 0-vertical curve if and only if it is a (full) vertical leaf.

If Γ^v is C^2, and g_v has a unique critical point $c \in I^v$ of quadratic type: $g_v'(c) = 0$ and

$$\kappa_\Phi(\Gamma^v) := g_v''(c) \neq 0, \tag{2.4}$$

then Γ^v is a *vertical quadratic curve in* $\mathcal{B}$. We refer to $\kappa_\Phi(\Gamma^v)$ as the *valuable curvature of* Γ^v *in* $\mathcal{B}$.

Let $\mathcal{E}^v : \mathcal{B} \to T^1(\mathcal{B})$ be the C^{r-1}-unit vector field given by

$$\mathcal{E}^v(p) := D\Phi^{-1}(E^{gv}_{\Phi(p)}).$$

A C^{r-1}-unit vector field $\tilde{\mathcal{E}}^v : \mathcal{U} \to T^1(\mathcal{U})$ defined on a domain $\mathcal{U} \subset \mathcal{B}$ is said to be t*-vertical in* C^{r-1} *in* $\mathcal{B}$ for some $t \geq 0$ if $\|\tilde{\mathcal{E}}^v - \mathcal{E}^v\|_{C^{r-1}} \leq t$.

Let $\tilde{\Phi} : \tilde{\mathcal{B}} \to \tilde{B}$ be another chart with $\tilde{\mathcal{B}} \subset \mathcal{B}$. We define the following relations between Φ and $\tilde{\Phi}$. Let $\tilde{\Phi} : \tilde{\mathcal{B}} \to \tilde{B}$ be another chart with $\tilde{\mathcal{B}} \subset \mathcal{B}$. We define the following relations between Φ and $\tilde{\Phi}$.

- We say that $\tilde{\mathcal{B}}$ is *vertically proper in* $\mathcal{B}$ if every full vertical leaf in $\tilde{\mathcal{B}}$ is vertically proper in $\mathcal{B}$.

- We say that Φ and $\tilde{\Phi}$ are *horizontally equivalent on* $\tilde{\mathcal{B}}$ if every horizontal leaf in $\tilde{\mathcal{B}}$ is a horizontal leaf in $\mathcal{B}$.
- For $t \geq 0$, we say that $\tilde{\mathcal{B}}$ *is t-vertical in* $\mathcal{B}$ if Φ and $\tilde{\Phi}$ are horizontally equivalent, and the unit vector field given by
$$\tilde{\mathcal{E}}^v(p) := D\tilde{\Phi}^{-1}(E^{gv}_{\tilde{\Phi}(p)}) \quad \text{for} \quad p \in \tilde{\mathcal{B}}$$
is t-vertical in C^{r-1} in $\mathcal{B}$.
- We say that Φ and $\tilde{\Phi}$ are *equivalent on* $\tilde{\mathcal{B}}$ if $\tilde{\mathcal{B}}$ is 0-vertical in $\mathcal{B}$.

Let $\Psi : \mathcal{B} \to B$ be a chart satisfying the following properties.

- There exists $q \in \mathcal{B}$ such that $\Psi(q) = 0 \in B$.
- Let
$$\mathcal{I}^h(t) := \Psi^{-1}(t, 0) \quad \text{for} \quad t \in \pi_h(B),$$
and
$$\mathcal{I}^v(s) := \Psi^{-1}(0, s) \quad \text{for} \quad s \in \pi_v(B).$$
Then $\|(\mathcal{I}^{h/v})'\| \equiv 1$.

In this case, we say that Ψ is *centered (at q)*. Clearly, for any chart $\Phi : \mathcal{B} \to D$ and any point $q \in \mathcal{B}$, there exists a unique chart $\Psi : \mathcal{B} \to B$ equivalent to Φ that is centered at q.

Suppose that $\Psi : (\mathcal{B}, q) \to (B, 0)$ is centered at some point $q \in \mathcal{B}$. Let $\Gamma^h \subset \mathcal{B}$ be a horizontal C^r-curve, so that $\Psi(\Gamma^h)$ is the horizontal graph in B of a C^r-map $g_h : I^h \to \pi_v(B)$ defined on an interval $I^h \subset \pi_h(B)$. We say that Γ^h is *t-horizontal in C^r in $\mathcal{B}$* if $\|g_h\|_{C^r} \leq t$. In particular, Γ^h is 0-horizontal in $\mathcal{B}$ if and only if Γ^h is a subarc of the full horizontal leaf containing q.

2.4. **Hénon-like renormalization.** Consider a C^{r+1}-Hénon-like map $F : D \to D$. We say that F is *topologically renormalizable* if there exists an R-periodic Jordan domain $\mathcal{B}^1 \Subset D$ for some integer $R \geq 2$:
$$F^i(\mathcal{B}^1) \cap \mathcal{B}^1 = \varnothing \quad \text{for} \quad 1 \leq i < R \quad \text{and} \quad F^R(\mathcal{B}^1) \Subset \mathcal{B}^1.$$
If, additionally, $\mathcal{B}^1$ contains $(v, 0)$, where v is the critical value of the unimodal map $\Pi_{1\mathrm{D}}(F)$, and there exists a genuinely horizontal C^r-chart Φ from $\mathcal{B}^1$ to a rectangle $D^1 \subset \mathbb{R}^2$ such that the map $\tilde{F}_1 := \Phi \circ F \circ \Phi^{-1}$ is again Hénon-like, then F is said to be *(Hénon-like) renormalizable*. In this case, any chart $\Psi : \mathcal{B}^1 \to B^1$ equivalent to Φ is referred to as a *straightening chart*, and the pair (F^R, Ψ) is referred to as a *Hénon-like return*.

Denote $\tilde{f}_1 := \Pi_{1\mathrm{D}}(\tilde{F}_1)$, and let $S : \mathbb{R} \to \mathbb{R}$ be the unique affine map such that $S \circ \tilde{f}_1 \circ S^{-1} \in \mathfrak{U}^r$. Define $\mathcal{S}$ as the affine map on $\mathbb{R}^2$ given by $\mathcal{S}(x, y) := (S(x), S(y))$. The *(Hénon-like) renormalization of F* is
$$\mathcal{R}(F) := \mathcal{S} \circ \Phi \circ F^R \circ (\mathcal{S} \circ \Phi)^{-1}.$$
Observe that $\mathcal{R}(F) \in \mathfrak{H}\mathfrak{L}^r$.

Remark 2.1. Note the loss of one degree of smoothness from F (which is C^{r+1}) to Φ and $\mathcal{R}(F)$ (which are C^r). This is to account for the loss of smoothness in the construction of regular charts given in Theorem A.2. This is not a critical issue, since it forces the loss of only one degree of smoothness no matter how many times F is renormalized.

2.5. **Definition of regularity.** Consider a C^r-diffeomorphism $F : D \to F(D) \Subset D$ defined on a domain $D \subset \mathbb{R}^2$. Let $L \geq 1$; $\varepsilon, \lambda \in (0,1)$ and $M \in \mathbb{N} \cup \{\infty\}$. A point $p \in D$ is *M-times forward $(L, \varepsilon, \lambda)$-regular along $E^+_p \in \mathbb{P}^2_p$* if for $s \in \{0,1\}$, we have

$$L^{-1}\lambda^{(1+\varepsilon)m} \leq \frac{\|DF^m|_{E^+_p}\|^{s+1}}{(\operatorname{Jac}_p F^m)^s} \leq L\lambda^{(1-\varepsilon)m} \quad \text{for all} \quad 1 \leq m \leq M. \tag{2.5}$$

Similarly, p is *M-times backward $(L, \varepsilon, \lambda)$-regular along $E^-_p \in \mathbb{P}^2_p$* if for $s \in \{0,1\}$, we have

$$L^{-1}\lambda^{(1+\varepsilon)m} \leq \frac{(\operatorname{Jac}_p F^{-m})^s}{\|DF^{-m}|_{E^-_p}\|^{s+1}} \leq L\lambda^{(1-\varepsilon)m} \quad \text{for all} \quad 1 \leq m \leq M. \tag{2.6}$$

The constants L, ε and λ are referred to as an *irregularity factor*, a *marginal exponent* and a *contraction base* respectively.

There exists a uniform constant $\varepsilon_1 \in (0,1)$ independent of F such that if (2.5) (or (2.6) resp.) holds with $\varepsilon \leq \varepsilon_1$, then the local dynamics of F near the forward (or backward resp.) orbit of p can be linearized up to the Mth iterate (see Theorem A.2). If $M = \infty$, this implies in particular that p has a well-defined C^r-smooth strong-stable manifold $W^{ss}(p)$ (or center manifold $W^c(p)$ resp.). It should be noted that the center manifold at an infinitely backward regular point p is not uniquely defined. However, its C^r-jet at p is unique (see Theorem A.16). Henceforth, any marginal exponent will be assumed to be less than ε_1.

2.6. **Regular Hénon-like returns.** A Hénon-like return $(F^R, \Psi : \mathcal{B}^1 \to B^1)$ is said to be *$(L, \varepsilon, \lambda)$-regular* if the following conditions hold. Let

$$E^{v/h}_p := D\Psi^{-1}\left(E^{gv/gh}_{\Psi(p)}\right).$$

- Every $p \in \mathcal{B}^1$ is R-times forward $(L, \varepsilon, \lambda)$-regular along E^v_p.
- Every $q \in F^R(\mathcal{B}^1) \Subset \mathcal{B}^1$ is R-times backward $(L, \varepsilon, \lambda)$-regular along E^h_q.
- For any $p \in \mathcal{B}^1$, we have $\measuredangle(E^v_p, E^h_p) > 1/L$.

In this case, we say that F is *$(L, \varepsilon, \lambda)$-regularly Hénon-like renormalizable.*

Example 2.2. Let $f : I \to I$ be a C^{r+1}-unimodal map. Suppose f is valuably renormalizable: there exists an R-periodic subinterval $I^1 \subset I$ such that $f^R(I^1)$ contains the critical value v of f. Then for $\varepsilon > 0$, there exists $\beta = \beta(f, R, \varepsilon) > 0$ such that the following holds. Let $F : D \to D$ be a β-thin C^{r+1}-Hénon-like map defined on a rectangle $D := I \times J$ with $\Pi_{1\mathrm{D}}(F) = f$. Then there exists an R-periodic quadrilateral $\mathcal{B}^1 \subset D$ containing $(v, 0)$ that is $\beta^{1-\varepsilon}$-close to $I^1 \times J$ in the Hausdorff topology, and a C^r-chart

$\Psi : \mathcal{B}^1 \to B^1$ centered at $(v, 0)$ that is $\beta^{1-\varepsilon}$-close to the identity in the C^r-topology such that (F^R, Ψ) is a $(1, \varepsilon, \beta)$-regular Hénon-like return. See Proposition 11.2.

2.7. **Nested Hénon-like returns.** A C^{r+1}-Hénon-like map $F : D \to D$ is N-times topologically renormalizable for some $N \in \mathbb{N} \cup \{\infty\}$ if there exist sequences

$$D =: \mathcal{B}^0 \Supset \mathcal{B}^1 \Supset \dots \quad \text{and} \quad 1 =: R_0 < R_1 < \dots$$

such that for $1 \leq n \leq N$, the set $\mathcal{B}^n$ is an R_n-periodic Jordan domain. If there exists $\mathbf{b} \geq 2$ such that

$$r_{n-1} := R_n / R_{n-1} \leq \mathbf{b} \quad \text{for all} \quad 1 \leq n \leq N,$$

then we say that the combinatorics of renormalization for F is *of (*$\mathbf{b}$*-)bounded type.* If $N = \infty$, then the *renormalization limit set* for F is

$$\Lambda_F := \bigcap_{n=1}^{\infty} \bigcup_{i=0}^{R_n - 1} F^{R_n + i}(\mathcal{B}^n). \tag{2.7}$$

Suppose for $1 \leq n \leq N$, there exist a C^r-straightening chart $\Psi^n : \mathcal{B}^n \to B^n$ such that (F^{R_n}, Ψ^n) is a Hénon-like return. Then the sequence

$$\{(F^{R_n}, \Psi^n : \mathcal{B}^n \to B^n)\}_{n=1}^{N} \tag{2.8}$$

said to be *nested*. Without loss of generality, we may assume that Ψ^n is a *centered straightening chart*: that is, Ψ^n is centered at some common point

$$v_0 \in \bigcap_{n=1}^{N} F^{R_n}(\mathcal{B}^n).$$

Let $\Phi^n : \mathcal{B}^n \to D^n$ be a chart equivalent to Ψ^n such that $\tilde{F}_n := \Phi^n \circ F^{R_n} \circ (\Phi^n)^{-1}$ is Hénon-like. Denote $\tilde{f}_n := \Pi_{1\mathrm{D}}(\tilde{F}_n)$, and let $S^n : \mathbb{R} \to \mathbb{R}$ be the unique affine map such that $S^n \circ \tilde{f}_n \circ (S^n)^{-1} \in \mathfrak{U}^r$. Define $\mathcal{S}^n$ as the affine map on $\mathbb{R}^2$ given by $\mathcal{S}^n(x, y) := (S^n(x), S^n(y))$. The nth (Hénon-like) renormalization of F is given by

$$F_n = \mathcal{R}^n(F) := \mathcal{S}^n \circ \Phi^n \circ F^{R_n} \circ (\mathcal{S}^n \circ \Phi^n)^{-1}. \tag{2.9}$$

Note that $\mathcal{R}^n(F) \in \mathfrak{H}\mathfrak{L}^r$. Lastly, we say that F is N-times $(L, \varepsilon, \lambda)$-regularly Hénon-like renormalizable for some $L \geq 1$ and $\varepsilon, \lambda \in (0, 1)$ if (F^{R_n}, Ψ^n) is $(L, \varepsilon, \lambda)$-regular for all $1 \leq n \leq N$.

Suppose that the combinatorics of renormalization for F are of $\mathbf{b}$-bounded type for some $\mathbf{b} \geq 2$. For many of our results, the specific values of the constants of regularity are not important, as long as ε is sufficiently small to compensate for the size of $\mathbf{b}$. That is, we have $\mathbf{b}\varepsilon^d < 1$ for some uniform constant $d \in (0, 1)$ independent of F. In this case, we will sometimes say that F is "N-times regularly Hénon-like renormalizable," without specifying the constants of regularity.

Definition 2.3. For $1 \leq n \leq N$, denote

$$I_0^n := \pi_h(B_0^n) \quad \text{and} \quad \mathcal{I}_m^n := F^m \circ (\Psi^n)^{-1}(I_0^n \times \{0\}) \quad \text{for} \quad m \in \mathbb{Z}.$$

The *nth valuable curvature* of the Hénon-like returns given in (2.8) is defined as

$$\kappa_n := \kappa_{\Psi^n}(\mathcal{I}^n_{R_n}) \tag{2.10}$$

(see (2.4)).

2.8. **Definition of regular unicriticality.** Consider a C^2-Hénon-like map $F : D \to D$. Suppose that F is infinitely renormalizable, and the renormalization limit set Λ_F supports a unique invariant probability measure μ. Then with respect μ, the Lyapunov exponents of F are 0 and $\log \lambda_\mu < 0$ for some $\lambda_\mu \in (0, 1)$ (see Proposition 14.1). By Oseledets theorem, μ-a.e. point $p \in \Lambda_F$ has strong-stable and center directions $E^{ss}_p, E^c_p \in \mathbb{P}^2_p$ such that

$$\lim_{n\to+\infty} \frac{1}{n} \log \|DF^n|_{E^{ss}_p}\| = \log \lambda_\mu \tag{2.11}$$

and

$$\lim_{n\to+\infty} \frac{1}{n} \log \|DF^{-n}|_{E^c_p}\| = 0. \tag{2.12}$$

Let $\varepsilon > 0$. Since $F|_{\Lambda_F}$ is uniquely ergodic, (2.11) ((2.12) resp.) implies that p is infinitely forward (backward resp.) $(L, \varepsilon, \lambda_\mu)$-regular for some $L = L(p, \varepsilon) \geq 1$ (see [CLPY1, Proposition 4.7]).

If $p \in \Lambda_F$ satisfies (2.11) and (2.12) with

$$E^*_p := E^{ss}_p = E^c_p,$$

then $\{F^m(p)\}_{m\in\mathbb{Z}}$ is referred to as a *regular critical orbit*. Note that in this case, the local strong-stable manifold $W^{ss}_{\mathrm{loc}}(p)$ and the center manifold $W^c(p)$ form a tangency at p. If this tangency is quadratic, then $\{F^m(p)\}_{m\in\mathbb{Z}}$ is referred to as a *regular quadratic critical orbit*.

For $t > 0$ and $p \in \mathbb{R}^2$, denote

$$\mathbb{D}_p(t) := \{q \in \mathbb{R}^2 \mid \operatorname{dist}(q, p) < t\}.$$

Definition 2.4. For $0 < \varepsilon < \delta < 1$, we say that F is (δ, ε)*-regularly unicritical on* Λ_F if the following conditions hold.

i) There is a regular quadratic critical orbit point $v_0 \in \Lambda_F$ (referred to as the *critical value* of F).
ii) For all $t > 0$, there exists $L(t) \geq 1$ such that for any $N \in \mathbb{N}$, if

$$p \in \Lambda_F \setminus \bigcup_{n=0}^{N-1} \mathbb{D}_{v_{-n}}(t\lambda_\mu^{\varepsilon n}), \tag{2.13}$$

then p is N-times forward $(L(t), \delta, \lambda_\mu)$-regular.

When δ and ε are implicit, we simply say that F is *regularly unicritical on* Λ_F.

3. Statements of the Main Theorems

Let $r \geq 2$ be an integer, and consider a C^{r+1}-Hénon-like map $F \in \mathfrak{HL}^{r+1}$ of the form (2.1). A quick computation shows that

$$\operatorname{Jac} F(x,y) = -\partial_y f(x,y).$$

If $\|\operatorname{Jac} F\| = 0$, then F does not depend on the second coordinate y. This means that F has the same dynamics as the unimodal map $\Pi_{1\mathrm{D}}(F)(\cdot) := f(\cdot, 0) \in \mathfrak{U}^{r+1}$. Hence, one may view the size of $\|\operatorname{Jac} F\|$ as a measure of how far F is from being a 1D system.

In this paper, we focus on the case when F is *dissipative*: $\|\operatorname{Jac} F\| \leq \lambda < 1$ for some $\lambda \in (0,1)$. Our goal is to understand what happens when such a map is renormalized many times. The following heuristics imply that the renormalizations of F rapidly become more and more one-dimensional.

Suppose that F is $N \in \mathbb{N} \cup \{\infty\}$ times renormalizable. For $1 \leq n \leq N$, the nth renormalization $\mathcal{R}^n(F)$ of F is given by (2.9). Assuming that the nonlinear component Φ^n of the nth rescaling map and its inverse remain uniformly bounded, we have

$$\|\operatorname{Jac} \mathcal{R}^n(F)\| = \|\operatorname{Jac} \Phi^n\| \cdot \|\operatorname{Jac} F^{R_n}\| \cdot \|\operatorname{Jac}(\Phi^n)^{-1}\| \asymp \|\operatorname{Jac} F^{R_n}\| \leq \lambda^{R_n}.$$

Since the return times R_n grow exponentially with n, we see that the renormalizations $\mathcal{R}^n(F)$ of F converge to the space of 1D systems super-exponentially fast. Thus, to understand the behavior of the 2D renormalization sequence $\{\mathcal{R}^n(F)\}_{n=1}^N \subset \mathfrak{HL}^r$, it suffices to study the following sequence of 1D profiles:

$$\{f_n := \Pi_{1\mathrm{D}} \circ \mathcal{R}^n(F)\}_{n=1}^N \subset \mathfrak{U}^r. \tag{3.1}$$

3.1. *A priori* bounds.

Our most difficult task is to establish the pre-compactness of the sequence given in (3.1). In other words, we must show that the 1D profiles f_n do not diverge (at least in the C^1-norm) as n increases. If $N = \infty$, this would imply that $\{f_n\}_{n=1}^\infty$ has meaningful limits. As is typical in the study of 1D systems, the key is to find a way of controlling the *distortion* of the return maps F^{R_n}.

Consider a C^1-diffeomorphism $G : U \to G(U)$ defined on a domain $U \subset \mathbb{R}^2$. For a C^1-curve $\gamma \subset U$, let $\phi_\gamma : [0, |\gamma|] \to \gamma$ be the arc-length parameterization of γ. Denote $G_\gamma := \phi_{G(\gamma)} \circ G \circ \phi_\gamma^{-1}$. The *distortion of G along γ* is defined as

$$\operatorname{Dis}(G, \gamma) := \sup_{s,t \in (0,|\gamma|)} \frac{|G'_\gamma(s)|}{|G'_\gamma(t)|}.$$

Theorem A. *Consider a C^6-Hénon-like map $F : D \to D$. Suppose for $N \in \mathbb{N} \cup \{\infty\}$, the map F has N nested regular Hénon-like returns given by (2.8) with combinatorics of bounded type. For $1 \leq n \leq N$, let γ^n be a genuine horizontal arc contained in $\mathcal{B}^n$. Then $\operatorname{Dis}(F^{R_n}, \gamma^n)$ is uniformly bounded.*

The proof of Theorem A is the content of Section 8.

The pre-compactness of the sequence (3.1) implies that there is definite scaling when we pass from one renormalization depth to the next. This fact has the following important geometric consequence for infinitely renormalizable Hénon-like maps.

Theorem B. *Consider a C^6-Hénon-like map $F : D \to D$. Suppose that F is infinitely regularly Hénon-like renormalizable with combinatorics of bounded type. Then the Hausdorff dimension of the renormalization limit set Λ_F given in* (2.7) *is less than* 1*. Consequently, Λ_F is totally disconnected, minimal, and supports a unique invariant probability measure μ.*

The proof of Theorem B is the content of Section 13.

3.2. **Renormalization convergence.** The following theorem summarizes the asymptotic behavior of an infinite regular Hénon-like renormalization sequence with combinatorics of bounded type.

Theorem C. *Let $r \geq 2$ be an integer, and consider a C^{r+4}-Hénon-like map $F : D \to D$. Suppose that F has infinite nested regular Hénon-like returns given by* (2.8) *with combinatorics of bounded type. Let $\varepsilon, \lambda \in (0,1)$ be the marginal exponent and the contraction base of regularity respectively. Then there exists a C^{r+3}-chart $\Phi : \mathcal{U} \to U$ centered at v_0 such that for all $n \in \mathbb{N}$ sufficiently large, the following properties hold.*

i) We have $\mathcal{B}^n \Subset \mathcal{U}$, and

$$\|\Phi \circ (\Psi^n)^{-1} - \mathrm{Id}\,\|_{C^{r+3}} < \lambda^{(1-\bar{\varepsilon})R_n}.$$

ii) The renormalization domain $\mathcal{B}^n$ can be extended vertically so that $|\pi_v \circ \Phi(\mathcal{B}^n)|$ is uniformly bounded below. Moreover, there exist a uniform constants $0 < \sigma_1 < \sigma_2 < 1$ such that

$$\sigma_1^n < |\pi_h \circ \Phi(\mathcal{B}^n)| < \sigma_2^n \quad \text{and} \quad |\pi_v \circ \Phi(F^{R_n}(\mathcal{B}^n))|^2 \asymp |\pi_h \circ \Phi(\mathcal{B}^n)|.$$

iii) The Hénon-like map $\mathcal{R}^n(F)$ is $\lambda^{(1-\bar{\varepsilon})R_n}$-thin in C^{r+3}.

iv) We have $\|\mathcal{R}^n(F)\|_{C^r} = O(1)$.

v) If $r \geq 4$, then there exists a universal constant $\rho \in (0,1)$ such that for any unimodal map $f_* \in \mathfrak{U}^r$ with the same asymptotic combinatorial type as F, we have

$$\|\Pi_{1\mathrm{D}} \circ \mathcal{R}^n(F) - \mathcal{R}^n_{1\mathrm{D}}(f_*)\|_{C^{r-1}} < \rho^n.$$

Let us briefly comment on how part v) in Theorem C relates to the existing renormalization theory of unimodal maps in literature. For $\gamma \in \{r, \omega\}$, the 1D renormalization $\mathcal{R}_{1\mathrm{D}}$ defined in Subsection 2.1 can be viewed as an operator acting on the Banach space $\mathfrak{U}^\gamma$ of unimodal maps. In [L1], Lyubich showed that $\mathcal{R}_{1\mathrm{D}}$ restricted to $\mathfrak{U}^\omega$ is an analytic operator that has a hyperbolic attractor $\mathfrak{A} \subset \mathfrak{U}^\omega$ with exactly one unstable dimension. This attractor is referred to as the *full renormalization horseshoe*.

Given an integer $\mathbf{b} \geq 2$, let $\mathfrak{A}_\mathbf{b}$ be the compact invariant subset of $\mathfrak{A}$ that consists of maps with combinatorics of $\mathbf{b}$-bounded type. In [dFdMPi], de Faria-de Melo-Pinto showed that for the action of the renormalization operator $\mathcal{R}_{1\mathrm{D}}$ on the much larger space $\mathfrak{U}^3 \supset \mathfrak{U}^\omega$, the set $\mathfrak{A}_\mathbf{b}$ is still a hyperbolic attractor with one unstable dimension.

Once we establish parts iii) and iv) in Theorem C, it follows by a general lemma (Lemma D.1) that the sequence of 1D profiles of the renormalizations of F is shadowed by actual orbits of 1D renormalization. Asymptotic convergence then follows from the hyperbolicity of 1D renormalization discussed above.

3.3. **Finite-time checkability.** If a quadratic polynomial $f : I \to I$ has a periodic subinterval $I^1 \subset I$, then the cycle of I^1 must go through the critical point of f exactly once. This ensures that the first return map on I^1 under f is again a unimodal map. In contrast, if a Hénon map $F : D \to D$ has a periodic subdomain $\mathcal{B}^1 \subset B$, it is not necessarily true that the first return map on $\mathcal{B}^1$ under F must also be Hénon-like. Intuitively, this contrast is due to the following important conceptual difference between the 1D case and the 2D case: for the dynamics of 2D diffeomorphisms, there is no definite single pinpoint location at which the action of the critical point takes place. Nevertheless, the following result states that if F is regular Hénon-like renormalizable up to a sufficiently deep depth, then all further topological renormalizations of F are necessarily also regular Hénon-like.

Theorem D. *Suppose we are given the following data:*

i) a Hénon-like map $F \in \mathfrak{H}\mathfrak{L}^6$*;*
ii) constants $\mathbf{b} \geq 2$*;* $L \geq 1$ *and* $\lambda \in (0,1)$*; and*
iii) an increasing sequence $\{R_n\}_{n=0}^{\infty}$ *such that* $R_0 = 1$ *and* $R_n/R_{n-1} \leq \mathbf{b}$ *for all* $n \in \mathbb{N}$*.*

Let $\varepsilon_0 \in (0,1)$ *and* $n_0 \in \mathbb{N} \cup \{0\}$ *be constants satisfying*

$$\mathbf{b}\varepsilon_0^d < 1, \tag{3.2}$$

and

$$C\lambda^{\varepsilon_0 R_{n_0}} < 1 \tag{3.3}$$

for some uniform constants $d \in (0,1)$ *(independent of* F*) and* $C \geq 1$ *(depending only on* L*,* λ*,* $\lambda^{1-\varepsilon_0}\|DF^{-1}\|$*,* $\|F\|_{C^3}$ *and* $\mathbf{b}$*). Suppose* F *has* n_1 *nested* $(L, \varepsilon_0, \lambda)$*-regular Hénon-like returns given by* (2.8) *for some* $n_1 \geq n_0$*. Suppose that*

$$K\lambda^{\varepsilon_0 R_{n_1}} < 1, \tag{3.4}$$

for some uniform constant $K = K(C, \|F^{R_{n_0}}|_{\mathcal{B}^{n_0}}\|_{C^6}, \kappa_{n_0}) \geq 1$ *(where* κ_{n_0} *is the* n_0*th valuable curvature given in* (2.10)*). Then there exists uniform constants* $\mathbf{L} \geq L$ *and* $\delta \in (\varepsilon_0, 1)$ *such that the following holds. Suppose that* F *is* N*-times topologically renormalizable for some* $n_1 \leq N \leq \infty$ *with return times* $\{R_n\}_{n=1}^{N}$*. Then all* N *renormalizations of* F *are* $(\mathbf{L}, \delta, \lambda)$*-regular Hénon-like (except possibly the last two if* $N < \infty$*).*

Quantitative Pesin theory combined with some standard arguments in one-dimensional dynamics is used to show that any topological return map within a sufficiently deep renormalization depth must be Hénon-like (see Section 6). A priori bounds is then needed to guarantee that regularity is preserved when we pass into deeper renormalization depths (see Section 10).

The significance of the previous theorem is that it turns infinite regular Hénon-like renormalizability of F into a finite condition, provided that F is known to be infinitely topologically renormalizable. The next result gives a criterion for guaranteeing the latter property.

Theorem E. *Suppose we are given the following data:*

i) a one-parameter family $\{F_a\}_{a\in\mathfrak{I}} \subset \mathfrak{H}\mathfrak{L}^6$ depending C^1-smoothly on a;
ii) constants $\mathbf{b} \geq 2$; $L \geq 1$ and $\lambda \in (0,1)$; and
iii) an increasing sequence $\{R_n\}_{n=0}^{\infty}$ such that $R_0 = 1$ and $R_n/R_{n-1} \leq \mathbf{b}$ for all $n \in \mathbb{N}$.

Let $\varepsilon_0 \in (0,1)$ and $n_1 \in \mathbb{N}$ be the constants given in (3.2) and (3.4). Suppose that for all $a \in \mathfrak{I}$, the map F_a is n_1-times $(L, \varepsilon_0, \lambda)$-regularly Hénon-like renormalizable, and that $\{\Pi_{1\mathrm{D}} \circ \mathcal{R}^{n_1}(F_a)\}_{a\in\mathfrak{I}}$ forms a full one-parameter family of 1D unimodal maps. Then for any $\mathbf{b}$-bounded renormalization type, there exists a parameter $a_* \in \mathfrak{I}$ such that $\mathcal{R}^{n_1}(F_{a_*})$ realizes this type.

The *renormalization type* of an infinitely regular Hénon-like renormalizable map referred to in Theorem E is defined in Section 6 (see (6.8)). It can be identified with the combinatorial type of some infinitely renormalizable unimodal map. The proof of Theorem E is the content of Section 11.

Remark 3.1. While it is not done in this paper, it is possible to obtain explicit estimates of the constants d, C and K in (3.2), (3.3) and (3.4). This means that for a given specific family of 2D maps (say, the Hénon family $\mathfrak{H}$), Theorems D and E turn infinite regular Hénon-like renormalizability in this family into an explicit finite-time checkable condition. This is illustrated in Examples 3.2 and 3.3.

Example 3.2. Consider a Hénon map $F_{a,b} \in \mathfrak{H}$ (see (1.2)) restricted to a suitable bounded subset $U \subset \mathbb{R}^2$. Then $F_{a,b}$ has uniformly bounded C^6-norm. Moreover, there exists a uniform constant $c = c(U) \geq 1$ such that $\|DF_{a,b}|_U^{-1}\| < c/|b|$. Lastly, the 0th valuable curvature of $F_{a,b}$ is exactly equal to 2.

For $\lambda \in (0,1)$, consider the one parameter family of Hénon maps $\mathfrak{H}_\lambda := \{F_{a,\lambda}\}_{a\in\mathbb{R}}$. Given $\mathbf{b} \geq 2$, a number $\varepsilon_0 \in (0,1)$ can be chosen so that (3.2) holds. Set $L = 1$. By the above observations, we see that for $\mathfrak{H}_\lambda$, the value of the uniform constant K given in (3.4) depends only on λ. Then let $\lambda_0 \in (0,1)$ be the largest number such that (3.4) is satisfied when we set $\lambda = \lambda_0$ and $n_1 = 0$ (i.e. $C\lambda_0^{\varepsilon_0} < 1$).

Fix $\lambda \in (0, \lambda_0)$. Since $\{\Pi_{1\mathrm{D}}(F_{a,\lambda})\}_{a\in\mathbb{R}} \equiv \mathfrak{Q}$ is a full family, it follows from the realization theorem that for any $\mathbf{b}$-bounded renormalization type, there exists a parameter value $a_*(\lambda) \in \mathbb{R}$ such that $F_{a_*(\lambda),\lambda}$ is infinitely regularly Hénon-like renormalizable with this combinatorics.

Note that the above argument is non-perturbative, and does not rely on the robustness of the 1D renormalization convergence. In particular, any numerical estimates on the quantities D and K would immediately yield a definite lower bound on the value of the Jacobian λ_0.

Example 3.3. Allowing for non-zero values of n_1 in Example 3.2 enables us to potentially find infinitely regularly Hénon-like renormalizable maps in the Hénon family with Jacobians arbitrarily close to 1 as follows.

Fix $\lambda \in (0,1)$, and consider the one-parameter family $\mathfrak{H}_\lambda$ of Hénon maps with Jacobian λ. Suppose we can find an interval $\mathfrak{I} \subset \mathbb{R}$ of parameters such that for each map $F_a := F_{a,\lambda}$ with $a \in \mathfrak{I}$, there exists a sequence of $N \in \mathbb{N}$ nested Hénon-like returns $\{(F_a^{R_n}, \Psi_a^n)\}_{n=1}^N$ with combinatorics of $\mathbf{b}$-bounded type. Additionally, suppose we can verify the following conditions.

i) The family $\{\Pi_{1\mathrm{D}} \circ \mathcal{R}^N(F_a)\}_{a \in \mathfrak{I}}$ depends smoothly on the parameter a, and is full.
ii) There exists $L \geq 1$ such that for all $a \in \mathfrak{I}$, the returns $\{(F_a^{R_n}, \Psi_a^n)\}_{n=1}^N$ are $(L, \varepsilon_0, \lambda)$-regular.
iii) We have $N \geq n_1$, where $n_1 \in \mathbb{N}$ is the smallest number such that (3.4) holds.

Then as before, we are guaranteed the existence of an infinitely regularly Hénon-like renormalizable map $F_{a_*(\lambda),\lambda}$ with $a_*(\lambda) \in \mathfrak{I}$ that realizes any given $\mathbf{b}$-bounded renormalization type.

We expect that for reasonably small values of λ (which would result in small values of n_1), it should be feasible to check these conditions numerically using a computer.

3.4. **Regular unicriticality.** Let $X \Subset D$ be a compact totally invariant set for a Hénon-like map F. We say that F is *uniformly partially hyperbolic on X* if every point $p \in X$ is infinitely forward and backward regular along some tangent direction E_p^{ss} at p, and the constants of regularity are uniform in p. The geometry of a 2D dynamical system is very well understood on uniformly partially hyperbolic sets. In particular, it is known that the leaves in the strong-stable and center laminations vary continuously, have uniformly bounded curvature, and are uniformly transverse to each other.

Suppose that F has infinite nested regular Hénon-like returns given by (2.8) with combinatorics of bounded type. It is shown in Theorem 7.1 that

$$\bigcap_{n=1}^{\infty} F^{R_n}(\mathcal{B}^n) = \{v_0\}. \tag{3.5}$$

We refer to v_0 as the *critical value of F*. Note that v_0 is both infinitely forward and backward regular. Thus, v_0 has well-defined strong-stable manifold $W^{ss}(v_0)$ and center manifold $W^c(v_0)$. The Hénon-likeness of the return maps under F forces $W^{ss}(v_0)$ and $W^c(v_0)$ to form a quadratic tangency at v_0. See Figure 3. Thus, the orbit $\mathcal{O}_{\mathrm{crit}}$ of v_0 is a regular quadratic critical orbit of F (as defined in Subsection 2.8).

The existence of $\mathcal{O}_{\mathrm{crit}}$ immediately implies that F is not uniformly partially hyperbolic on Λ_F. However, our last main theorem states that this is the only obstruction, and that uniform regularity still holds outside a slow-exponentially shrinking neighborhood of $\mathcal{O}_{\mathrm{crit}}$.

Theorem F. *Consider a C^6-Hénon-like map $F : D \to D$. Suppose that F has infinite nested regular Hénon-like returns given by* (2.8) *with combinatorics of bounded type.*

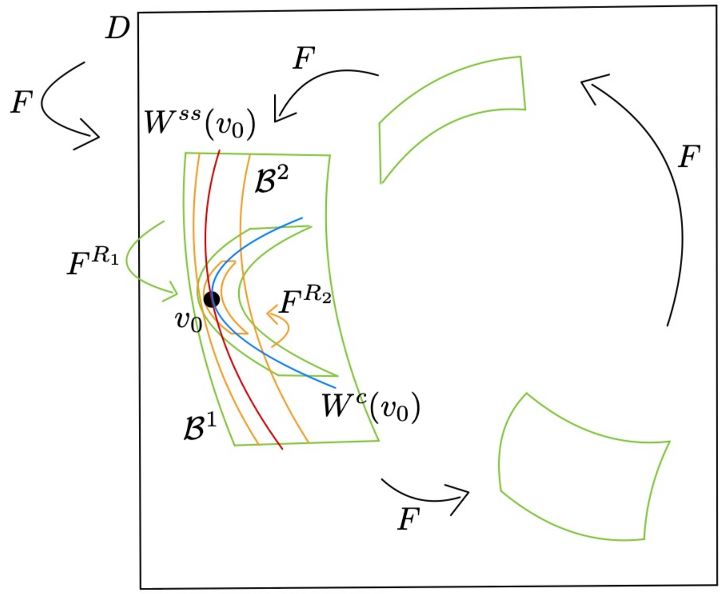

Figure 3. The critical value v_0 of an infinitely regularly Hénon-like renormalizable map F.

Then for any $\varepsilon > 0$, there exists $L_\varepsilon \geq 1$ such that for all $n \in \mathbb{N}$, the Hénon-like return (F^{R_n}, Ψ^n) is $(L_\varepsilon, \varepsilon, \lambda_\mu)$-regular. Moreover, F is regularly unicritical on the renormalization limit set Λ_F with the critical value v_0 given by (3.5)*.*

The study of 2D dynamics on a uniformly partially hyperbolic set X is greatly facilitated by the fact that X has a *local product structure*. This means that X can be covered by finitely many charts, called *regular Pesin boxes*, which endows the set with locally defined vertical (strong-stable) and horizontal (center) directions that are invariant under the dynamics.

In our setting, any covering of Λ_F by regular Pesin boxes must leave out points that are too close to the critical orbit $\mathcal{O}_{\text{crit}}$. In [CLPY1], we introduce new covering domains called *critical tunnels* and *valuable crescents* that uniformize the dynamics of F near $\mathcal{O}_{\text{crit}}$ (which is fundamentally non-linear in nature). See Figure 4. These new domains, together with regular Pesin boxes, completely cover Λ_F, resulting in a new type of dynamical structure that we call a *regular unicritical structure*.

Regular unicritical structures for regularly unicritical systems can fulfill a similar function as local product structures for uniformly partially hyperbolic systems. In [CLPY1], we use this new structure to characterize the local geometry of every strong-stable and center manifold in terms of its proximity to the critical orbit in an explicit way. Additionally, we prove the following converse of the unicriticality theorem.

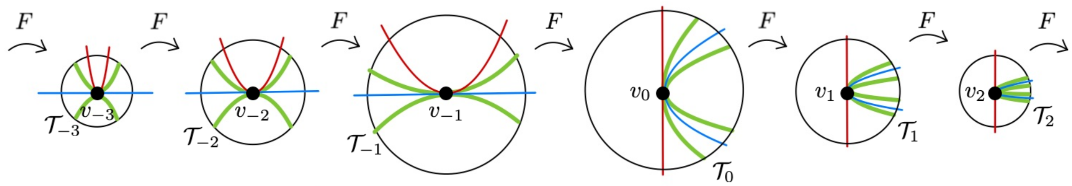

Figure 4. Regular quadratic critical orbit $\mathcal{O}_{\text{crit}} = \{v_m\}_{m\in\mathbb{Z}}$ contained in critical tunnels $\{\mathcal{T}_{-n}\}_{n=1}^{\infty}$ and valuable crescents $\{\mathcal{T}_n\}_{n=0}^{\infty}$. For $m \in \mathbb{Z}$, the strong-stable and center manifolds of v_m are indicated as red and blue curves respectively. The tunnel/crescent $\mathcal{T}_m$ is the pinched region bounded between two green curves that contains $W^c(v_m)$. The diameter of $\mathcal{T}_m$ shrinks slow-exponentially as $|m| \to \infty$.

Theorem 3.4 ([CLPY1]). *Let $F : \mathcal{D} \to F(\mathcal{D}) \Subset \mathcal{D}$ be a dissipative C^3-diffeomorphism defined on a Jordan domain $\mathcal{D} \subset \mathbb{R}^2$. Suppose that F is infinitely topologically renormalizable, and assume that F is regularly unicritical on the renormalization limit set. Then the renormalizations of F are eventually regular Hénon-like.*

4. Convergence of the Straightening Charts

Let $r \geq 2$ be an integer, and consider a C^{r+1}-Hénon-like map $F : D \to D$. For some $N \in \mathbb{N} \cup \{\infty\}$; $L \geq 1$ and $\varepsilon, \lambda \in (0, 1)$, suppose that F has N nested $(L, \varepsilon, \lambda)$-regular Hénon-like returns given by (2.8). Furthermore, assume that N is sufficiently large, so that for some smallest number $0 \leq n_0 \leq N$, we have

$$\overline{K_0}\lambda^{\varepsilon R_{n_0}} < \eta, \tag{4.1}$$

where $\eta \in (0, 1)$ is independent of F, and

$$K_0 = K_0(L, \lambda, \varepsilon, \lambda^{1-\varepsilon}\|DF^{-1}\|, \|DF\|_{C^r}, r) \geq 1 \tag{4.2}$$

is a uniform constant.

For $n_0 \leq n \leq N$ and $m \in \mathbb{Z}$, denote $\mathcal{B}^n_m := F^m(\mathcal{B}^n)$. Observe that $\mathcal{B}^{n+1}_{R_{n+1}} \Subset \mathcal{B}^n_{R_n}$. Let

$$v_0 \in \mathcal{Z}_0 := \bigcap_{n=1}^{N} \mathcal{B}^n_{R_n},$$

be a point to be specified later (as the *critical value of* F). Without loss of generality, assume that Ψ^n is centered at v_0.

In this section, we describe the asymptotic behavior of the centered straightening charts $\{\Psi^n\}_{n=1}^N$ for the renormalizations of F.

Remark 4.1. In Sections 4 and 5, we do not assume that the combinatorics of the renormalizations of F is necessarily of bounded type.

Define

$$I_0^n := \pi_h(B_0^n) \quad \text{and} \quad \mathcal{I}_0^n := (\Psi^n)^{-1}(I_0^n \times \{0\}).$$

Then it follows that $I_0^n \Subset I_0^1$ and $\Psi^n|_{\mathcal{I}_0^n} = \Psi^1|_{\mathcal{I}_0^n}$. Denote $\mathcal{I}_m^n := F^m(\mathcal{I}_0^n)$ for $m \in \mathbb{Z}$.

For $p_0 \in \mathcal{B}_0^n$, write $z_0 := \Psi^n(p_0)$, and let

$$E_{p_0}^h := D(\Psi^n)^{-1}(E_{z_0}^{gh}) \quad \text{and} \quad E_{p_0}^{v,n} := D(\Psi^n)^{-1}(E_{z_0}^{gv}).$$

Additionally, let

$$E_{p_{R_n-1}}^{h,n} := DF^{R_n-1}(E_{p_0}^h) \quad \text{and} \quad E_{p_{R_n-1}}^v := DF^{-1}(E_{p_{R_n}}^h) = DF^{R_n-1}(E_{p_0}^{v,n}).$$

By increasing L by a uniform amount if necessary (see Proposition A.1), we may assume that every $q \in \mathcal{B}_{R_n-1}^n$ is (R_n-1)-times backward $(L, \varepsilon, \lambda)$-regular along E_q^v.

Proposition 4.2 (Vertical extension of charts)**.** *For $n_0 \leq n \leq N$, the nth centered straightening chart can be extended to $\Psi^n : \hat{\mathcal{B}}_0^n \to \hat{B}_0^n$ such that the following properties hold.*

i) The quadrilateral $\hat{\mathcal{B}}_0^n$ is vertically proper and η-vertical in $\mathcal{B}_0^{n_0}$.

ii) We have $\|(\Psi^n)^{\pm 1}\|_{C^r} < K_0$, and

$$\|\Psi^n \circ (\Psi^{n+1})^{-1} - \mathrm{Id}\,\|_{C^r} < \lambda^{(1-\bar{\varepsilon})R_n}. \tag{4.3}$$

iii) Every point $q_0 \in \hat{\mathcal{B}}_0^n$ is R_n-times forward $(K_0, \varepsilon, \lambda)$-regular along $E_{q_0}^{v,n}$.

Proof. For $p_0 \in \mathcal{B}_0^n$, let

$$\{\Phi_{p_m} : \mathcal{U}_{p_m} \to U_{p_m}\}_{m=0}^{R_n}$$

be a linearization of F along the R_n forward orbit of p_0 with vertical direction $E_{p_0}^{v,n}$. Let $\mathcal{E}_{p_m}^{v,n} : \mathcal{U}_{p_m} \to T^1(\mathcal{U}_{p_m})$ be the C^r-unit vector field given by $\mathcal{E}_{p_m}^{v,n}(q) \in E_q^{v,n}$ for $q \in \mathcal{U}_{p_m}$.

Let $l_{p_0}^{v,n_0}$ be the full vertical leaf in $\mathcal{B}_0^{n_0}$ containing p_0. For $q_0 \in l_{p_0}^{v,n_0}$, let

$$\{\Phi_{q_m} : \mathcal{U}_{q_m} \to U_{q_m}\}_{m=0}^{R_{n_0}}$$

be a linearization of F along the R_{n_0} forward orbit of q_0 with vertical direction $E_{q_0}^{v,n_0}$ given by Theorem A.2.

Let M be a nearest integer to $R_{n_0}/2$. By Lemma A.3, we see that $\mathcal{U}_{p_M} \supset \mathbb{D}_{p_M}\left(\lambda^{\bar{\varepsilon}M}\right)$. Lemma A.11 implies that Corollary A.8 applies to all points in the M-times truncated regular neighborhood $\mathcal{U}_{q_0}^M$ at q_0. The R_{n_0}-times forward regularity at all points in $\mathcal{B}_0^{n_0}$ together with (4.1) implies that

$$\check{\mathcal{U}}_{q_M} := F^M(\mathcal{U}_{q_0}^M) \subset \mathcal{U}_{p_M}.$$

By Proposition A.1, q_M and p_M are M-times forward $(\lambda^{-\bar{\varepsilon}M}, \varepsilon, \lambda)_v$-regular along $E_{q_M}^{v,n_0}$ and $E_{p_M}^{v,n}$ respectively. Hence, Proposition A.9 implies that $\mathcal{E}_{p_M}^{v,n}|_{\check{\mathcal{U}}_{q_M}}$ is $\lambda^{(1-\bar{\varepsilon})M}$-vertical in C^0 in $\mathcal{U}_{q_M}$. Moreover, the bounds on $\|\Phi_{p_M}\|_{C^r}$ and $\|\Phi_{q_M}\|_{C^r}$ given by Theorem A.2 imply that

$$\|D\Phi_{q_M}(\mathcal{E}_{p_M}^{v,n})\|_{C^{r-1}} < \lambda^{-\bar{\varepsilon}M}.$$

Extend $\mathcal{E}^{v,n}_{p_0}$ to $\mathcal{U}^M_{q_0}$ as

$$\mathcal{E}^{v,n}_{p_0}|_{\mathcal{U}^M_{q_0}} := DF^{-M}(\mathcal{E}^{v,n}_{p_M}|_{\check{\mathcal{U}}_{q_M}}).$$

Then by Proposition A.14, we have

$$\|\mathcal{E}^{v,n}_{p_0} - \mathcal{E}^{v,n_0}_{q_0}\|_{C^r} \leq \lambda^{(1-\bar{\varepsilon})M}(1 + \|D\Phi^{-1}_{q_0}\|_{C^{r-1}})(1 + \|D\Phi_{q_M}(\mathcal{E}^{v,n}_{p_M})\|_{C^{r-1}}) \leq \eta.$$

Rectifying the vertical directions near $l^{v,n_0}_{p_0}$ given by $\mathcal{E}^{v,n}_{p_0}$, we obtain the desired extension of Ψ^n.

Observe that for $0 \leq k \leq M$

$$\measuredangle(E^{v,n}_{q_k}, E^{v,n_0}_{q_k}) < \lambda^{(1-\bar{\varepsilon})(R_{n_0}-k)}.$$

It follows that

$$\frac{1}{\sqrt{2}} < \frac{\|DF^k|_{E^{v,n}_{q_0}}\|}{\|DF^k|_{E^{v,n_0}_{q_0}}\|} < \sqrt{2}.$$

Concatenating with the forward M-times $(\lambda^{-\bar{\varepsilon}M}, \varepsilon, \lambda)$-regularity of q_M, we see that

$$\frac{\lambda^{\bar{\varepsilon}M}}{\sqrt{2}L}\lambda^{(1+\varepsilon)(M+i)} \leq \|DF^{M+i}|_{E^{v,n}_{q_0}}\| \leq \sqrt{2}L\lambda^{-\bar{\varepsilon}M}\lambda^{(1-\varepsilon)(M+i)}.$$

Since n_0 is assumed to be the smallest number that satisfies (4.1), we have $\lambda^{-\bar{\varepsilon}M} < K_0$. The claimed R_n-times forward regularity of q_0 along $E^{v,n}_{q_k}$ follows.

Lastly, replacing the renormalization depth n_0 in the above argument by n, we obtain (4.3). □

Remark 4.3. In Section 6, we will show that $\hat{\mathcal{B}}^n_0$ is R_n-periodic (and hence, we may assume that $\mathcal{B}^n_0 = \hat{\mathcal{B}}^n_0$). See (6.7).

Consider C^r-curves $\Gamma_1, \Gamma_2 \subset \mathbb{R}^2$ with $|\Gamma_1| \geq |\Gamma_2|$. For $i \in \{1, 2\}$, let $\phi_{\Gamma_i} : J_i \subset \mathbb{R} \to \Gamma_i$ be a parameterization of Γ_i such that

- $|\phi'_{\Gamma_i}| \equiv 1$;
- $J_1 \supset J_2$;
- $\|\phi_{\Gamma_1}|_{J_2} - \phi_{\Gamma_2}\|_{C^r}$ is minimal.

In this case, define

$$\|\Gamma_1\|_{C^r} := \|\phi_{\Gamma_1}\| \quad \text{and} \quad \mathrm{dist}_{C^r}(\Gamma_1, \Gamma_2) := \|\phi_{\Gamma_1}|_{J_2} - \phi_{\Gamma_2}\|_{C^r}. \tag{4.4}$$

Lemma 4.4. *For $n_0 \leq n \leq N$, let l^n_0 be a full horizontal leaf in $\hat{\mathcal{B}}^n_0$. Denote $l^n_m := F^m(l^n_0)$ for $m \in \mathbb{Z}$. Then we have $\|l^n_{R_n-1}\|_{C^r} < K_0$; and*

$$\mathrm{dist}_{C^r}(l^n_{R_n-1}, l^{n+1}_{R_{n+1}-1}) < \lambda^{(1-\bar{\varepsilon})R_n}.$$

Proof. For $p_{-1} \in \mathcal{Z}_{-1} := F^{-1}(\mathcal{Z}_0)$, let

$$\{\Phi_{p_{-m}} : \mathcal{U}_{p_{-m}} \to U_{p_{-m}}\}^{R_N}_{m=1}$$

be a linearization of F along the R_N-times backward orbit of p_{-1} with vertical direction $E^v_{p_{-1}}$ (if $N = \infty$, then $R_\infty = \infty$). Let $\mathcal{V}_{-R_n}$ be the connected component of $F^{-R_n+1}(\mathcal{U}^{R_n}_{p_{-1}}) \cap \hat{\mathcal{B}}^n_0$ containing p_{-R_n}. Note that $\Psi^n|_{\mathcal{V}_{-R_n}}$ defines a chart on $\mathcal{V}_{-R_n}$, so that $\mathcal{V}_{-R_n}$ is 0-vertical in $\hat{\mathcal{B}}_0^n$. Moreover, arguing as in the proof of Proposition 4.2, we see that $\mathcal{V}_{-R_n}$ is also vertically proper in $\hat{\mathcal{B}}_0^n$.

Consider the map

$$H_n(x, y) = (h_n(x), e_n(x, y)) := \Phi_{p_{-1}} \circ F^{R_n - 1} \circ (\Psi^n)^{-1}(x, y)$$

for $(x, y) \in \Psi^n(\mathcal{V}_{-R_n})$. Denote

$$F_{p_{-n}} := \Phi_{p_{-n+1}} \circ F \circ (\Phi_{p_{-n}})^{-1}.$$

Then

$$H_n = F_{p_{-2}} \circ \ldots \circ F_{p_{-R_n}} \circ \Phi_{p_{-R_n}} \circ (\Psi^n)^{-1}. \tag{4.5}$$

By Theorem A.2, we see that

$$\|\Phi_{p_{-R_n}} \circ (\Psi^n)^{-1}\|_{C^r} < \lambda^{-\bar{\varepsilon} R_n}.$$

Applying Proposition A.12, we conclude that

$$\|e_n\|_{C^r} < \lambda^{(1-\bar{\varepsilon})R_n}. \tag{4.6}$$

The result follows. □

Proposition 4.5 (Locating the critical value)**.** *If $N = \infty$, then the following statements hold.*

i) For any point $p_0 \in \mathcal{Z}_0$, there exists a unique strong stable direction $E^{ss}_{p_0} \in \mathbb{P}^2_{p_0}$ such that

$$\|E^{v,n}_{p_0} - E^{ss}_{p_0}\| < \lambda^{(1-\bar{\varepsilon})R_n} \quad \textit{for} \quad n \geq n_0.$$

Moreover, p_0 is infinitely forward $(L, \varepsilon, \lambda)$-regular along $E^{ss}_{p_0}$.

ii) Any point $p_{-1} \in \mathcal{Z}_{-1} := F^{-1}(\mathcal{Z}_0)$ is infinitely backward $(L, \varepsilon, \lambda)$-regular along $E^v_{p_{-1}}$. Moreover, there exists a unique center direction $E^c_{p_{-1}} \in \mathbb{P}^2_{p_{-1}}$ such that

$$\|E^{h,n}_{p_{-1}} - E^c_{p_{-1}}\| < \lambda^{(1-\bar{\varepsilon})R_n} \quad \textit{for} \quad n \geq n_0.$$

iii) There exists a unique point $v_0 \in \mathcal{Z}_0$ such that

$$E^{ss}_{v_0} = DF(E^c_{v_{-1}}).$$

Moreover, the strong stable manifold $W^{ss}(v_0)$ and the center manifold $F(W^c(v_{-1}))$ have a quadratic tangency at v_0.

Proof. The first and second claim follow immediately from Propositions A.9 and A.10.

For $n \geq n_0$, recall that $\mathcal{I}^n_{R_n}$ is a vertical quadratic curve in $\mathcal{B}^n_0$. Let $v^n_0 \in \mathcal{I}^n_0$ be the unique point such that

$$E^{v,n}_{v^n_{R_n}} = DF^{R_n}(E^h_{v^n_0}).$$

By Proposition 4.2 and Lemma 4.4, we have

$$\mathrm{dist}(v^n_{R_n}, v^{n+1}_{R_{n+1}}) < \lambda^{(1-\bar{\varepsilon})R_n}.$$

Thus, there exists a unique point $v_0 \in \mathcal{Z}_0$ such that

$$\mathrm{dist}(v^n_{R_n}, v_0) < \lambda^{(1-\bar{\varepsilon})R_n} \quad \text{and} \quad \mathrm{dist}_{C^r}(\mathcal{I}^n_{R_n}, W^c(v_0)) < \lambda^{(1-\bar{\varepsilon})R_n}.$$

By (4.3), we see that $W^{ss}(v_0)$ and $W^c(v_0)$ have a quadratic tangency at v_0.

Lastly, let $\mathcal{U}_{v_0}$ be a neighborhood of v_0. Then there exists a uniform constant $k > 0$ such that for all n sufficiently large, if $p_{R_n} \in \mathcal{I}^n_{R_n} \setminus \mathcal{U}_{v_0}$ then

$$\measuredangle(E^{v,n}_{p_{R_n}}, DF^{R_n}(E^h_{p_0})) > k.$$

Thus, v_0 is the unique point in $\mathcal{Z}_0$ satisfying $E^{ss}_{v_0} = E^c_{v_0}$. □

We define the *critical value* $v_0 \in \mathcal{Z}_0$ as follows. If $N = \infty$, let v_0 be the point given in Proposition 4.5 iii). Otherwise, let v_0 be the unique point in $\mathcal{I}^N_{R_N}$ such that

$$DF^{R_N}(E^h_{v_{-R_N}}) = E^{v,N}_{v_0}$$

(recall that such a point exists since $\mathcal{I}^N_{R_N}$ is a vertical quadratic curve in $\mathcal{B}^N_0$). Define the *critical point* as $v_{-1} := F^{-1}(v_0)$.

Remark 4.6. In Section 7, we will prove that if $N = \infty$ and the combinatorics of the renormalizations of F are of bounded type, then $\mathcal{Z}_0 = \{v_0\}$.

Theorem 4.7 (Valuable charts)**.** *Let $K_0 \geq 1$ be the constant given in* (4.2)*. There exist charts*

$$\Phi_0 : (\mathcal{B}_0, v_0) \to (B_0, 0) \quad \textit{and} \quad \Phi_{-1} : (\mathcal{B}_{-1}, v_{-1}) \to (B_{-1}, 0)$$

such that

- *Φ_0 is centered at v_0 and is genuine horizontal;*
- *$\mathcal{B}_0 \supset \mathcal{B}^{n_0}_0$, $\mathcal{B}_{-1} \supset \mathcal{B}^{n_0}_{R_{n_0}-1}$ and $F(\mathcal{B}_{-1}) \Subset \mathcal{B}_0$;*
- *$\|\Phi^{\pm 1}_i\|_{C^r} < K_0$ for $i \in \{0, -1\}$; and*
- *we have*

$$\Phi_0 \circ F \circ \Phi^{-1}_{-1}(x, y) = (f_0(x) - \lambda y, x) \quad \textit{for} \quad (x, y) \in B_{-1}, \tag{4.7}$$

  where $f_0 : (\pi_h(B_{-1}), 0) \to (\pi_h(B_0), 0)$ is a C^r-map that has a unique critical point at 0 such that

$$\|f''_0\|_{C^{r-2}} < K_0 \quad \textit{and} \quad \kappa_F := \inf_{x \in \pi_h(B_{-1})} f''_0(x) > 0. \tag{4.8}$$

Moreover, the following properties hold for $n_0 \leq n \leq N$.

i) We have

$$\|\Phi_0 \circ (\Psi^n)^{-1} - \mathrm{Id}\,\|_{C^r} < \lambda^{(1-\bar\varepsilon)R_n}.$$

ii) Let

$$H_n := \Phi_{-1} \circ F^{R_n - 1} \circ (\Psi^n)^{-1}.$$

Then $H_n(x, y) = (h_n(x), e_n(x, y))$, where $h_n : I^n_0 \to h_n(I^n_0)$ is a C^r-diffeomorphism and e_n is a C^r-map such that

$$\lambda^{\bar\varepsilon R_n} < |h'_n(x)| < \lambda^{-\bar\varepsilon R_n} \quad \textit{for} \quad x \in I^n_0 \quad \textit{and} \quad \|e_n\|_{C^r} < \lambda^{(1-\bar\varepsilon)R_n}. \tag{4.9}$$

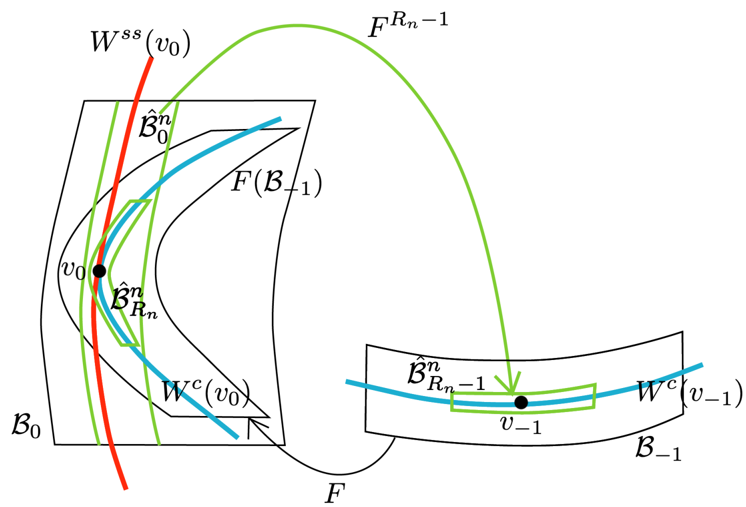

Figure 5. Geometry near the critical value v_0 and the critical point v_{-1} (if $N = \infty$). For $n \geq n_0$, we have $v_0 \in \hat{\mathcal{B}}_0^n \subset \mathcal{B}_0$ and $v_{-1} \in \hat{\mathcal{B}}_{R_n-1}^n \subset \mathcal{B}_{-1}$. There exist charts $\Phi_0 : \mathcal{B}_0 \to B_0$ and $\Phi_{-1} : \mathcal{B}_{-1} \to B_{-1}$ such that $\Phi_0 \circ F \circ \Phi_{-1}$ is Hénon-like (see (4.7)). The charts Ψ^n converges to Φ_0

Proof. For $t \geq 0$ and $X \subset \mathbb{R}^2$, denote

$$X(t) := \{p \in \mathbb{R}^2 \mid \operatorname{dist}(p, X) \leq t\}.$$

Let

$$\mathcal{B}_0 := \mathcal{B}_0^{n_0}(\lambda^{\bar{\varepsilon} R_{n_0}}) \quad \text{and} \quad \mathcal{C}_0^n := \hat{\mathcal{B}}_0^n(\lambda^{\bar{\varepsilon} R_n}) \setminus \hat{\mathcal{B}}_0^n.$$

By (4.3), there exists a C^r-diffeomorphism Φ_0 defined in a neighborhood of $\mathcal{Z}_0$ such that

$$\|\Phi_0 \circ (\Psi^n)^{-1} - \operatorname{Id}\|_{C^r} < \lambda^{(1-\bar{\varepsilon})R_n} \quad \text{for all} \quad n_0 \leq n \leq N.$$

Moreover, Φ_0 can be extended a centered chart $\Phi_0 : (\mathcal{B}_0, v_0) \to (B_0, 0)$ such that

$$\Phi_0|_{\hat{\mathcal{B}}_0^n \setminus (\hat{\mathcal{B}}_0^{n+1} \cup \mathcal{C}_0^{n+1})} = \Psi^n|_{\hat{\mathcal{B}}_0^n \setminus (\hat{\mathcal{B}}_0^{n+1} \cup \mathcal{C}_0^{n+1})}$$

and

$$\|\Phi_0 \circ (\Psi^n|_{\mathcal{C}_0^{n+1}})^{-1} - \operatorname{Id}\|_{C^r} < \lambda^{(1-\bar{\varepsilon})R_n}.$$

Let $\mathcal{I}_{-1}^h := \mathcal{I}_{R_N-1}^N \ni v_{-1}$ if $N < \infty$, and $\mathcal{I}_{-1}^h := W^c(v_{-1})$ if $N = \infty$. Observe that $F(\mathcal{I}_{-1}^h) \ni v_0$ is a vertical quadratic curve in $\mathcal{B}_0$. Denote

$$J_0^v := \pi_v \circ \Phi_0 \circ F(\mathcal{I}_{-1}^h).$$

Then there exists a C^r-map

$$f_0 : (J_0^v, 0) \to (\pi_h(B_0), 0)$$

with a unique quadratic critical point at 0 such that

$$\Phi_0 \circ F(\mathcal{I}_{-1}^h) = \{(f_0(y), y) \mid y \in J_0^v\}.$$

The C^r-bound on f_0 follows from Proposition 4.2 and Lemma 4.4.

Let $C \geq 1$ be the uniform constant given in (4.1). Let

$$D_0 := \{(f_0(y)+t, y) \in B_0 \mid |t| \leq \lambda {K_0}^{-1} \text{ and } y \in J_0^v\},$$

and

$$\mathcal{B}_{-1} := (\Phi_0 \circ F)^{-1}(D_0).$$

We define $\Phi_{-1} : (\mathcal{B}_{-1}, v_{-1}) \to (B_{-1}, 0)$ to be the unique chart satisfying

$$\Phi_0 \circ F \circ \Phi_{-1}^{-1}(x,y) = (f_0(x) - \lambda y, x) \quad \text{for} \quad (x,y) \in B_{-1}.$$

Consider the decomposition of H_n given in (4.5). The second inequality in (4.9) is given by (4.6). The upper bound in the first inequality follows immediately from Proposition A.12. For the lower bound, we observe that

$$\| \left(\Phi_{p_{-R_n}} \circ (\Psi^n)^{-1}\right)^{-1} \|_{C^1} < \lambda^{-\bar{\varepsilon} R_n}$$

by Theorem A.2. The lower bound now follows immediately from Theorem A.2 ii) and iii). □

Denote

$$I_i^{h/v} := \pi_{h/v}(B_i) \quad \text{and} \quad \mathcal{I}_i^h := \Phi_i^{-1}(I_i^h \times \{0\}) \quad \text{for} \quad i \in \{0, -1\}. \tag{4.10}$$

Observe that

$$I_0^h \Supset I_0^{n_0} \Supset I_0^{n_0+1} \Supset \ldots \quad \text{and} \quad I_{-1}^h \Supset h_{n_0}(I_0^{n_0}) \Supset h_{n_0+1}(I_0^{n_0+1}) \Supset \ldots.$$

Moreover, if $X \subset \mathcal{B}_0^n$, then (4.9) implies

$$\Phi_{-1} \circ F^{R_n - 1}(X) \subset h_n(I_0^n) \times [-\lambda^{(1-\bar{\varepsilon})R_n}, \lambda^{(1-\bar{\varepsilon})R_n}]. \tag{4.11}$$

Define $P_{-1} : (\mathcal{B}_{-1}, v_{-1}) \to (I_{-1}^h, 0)$ and $P_0^n : (\hat{\mathcal{B}}_0^n, v_0) \to (I_0^n, 0)$ for $n_0 \leq n \leq N$ by

$$P_{-1} := \pi_h \circ \Phi_{-1} \quad \text{and} \quad P_0^n := \pi_h \circ \Psi^n.$$

Denote

$$I_{R_n-1}^n := P_{-1}(\hat{\mathcal{B}}_{R_n-1}^n) = P_{-1}(\mathcal{I}_{R_n-1}^n) = h_n(I_0^n).$$

Define the *nth (valuable) projection map* $\mathcal{P}_0^n : \hat{\mathcal{B}}_0^n \to \mathcal{I}_0^n$ by

$$\mathcal{P}_0^n := (\Psi^n)^{-1} \circ \Pi_h \circ \Psi^n.$$

Observe that $\mathcal{P}_0^n|_{\mathcal{I}_0^n} = \mathrm{Id}$.

We record the following immediate consequence of Theorem A.2 and Propositions A.13 and A.14.

Lemma 4.8. *For $n_0 \leq n \leq N$, denote $\lambda_n := \lambda^{(1-\bar{\varepsilon})R_n}$. Then for $0 < t < \lambda^{-\bar{\varepsilon}R_n}$, the following statements hold.*

i) Let $\tilde{E}_{p_0} \in \mathbb{P}_{p_0}^2$ be a t-horizontal direction at $p_0 \in \hat{\mathcal{B}}_0^n$. Then $\tilde{E}_{p_{R_n-1}}$ is $(1+t)\lambda_n$-horizontal in $\mathcal{B}_{-1}$.

ii) Let $E_{p_{R_n-1}} \in \mathbb{P}_{p_{R_n-1}}^2$ be a t-vertical direction at $p_{R_n-1} \in \hat{\mathcal{B}}_{R_n-1}^n$. Then E_{p_0} is $t\lambda_n$-vertical in $\hat{\mathcal{B}}_0^n$.

iii) Let Γ_0^h be a curve that is t-horizontal in C^r in $\hat{\mathcal{B}}_0^n$. Then $F^{R_n-1}(\Gamma_0^h)$ is $(1+t)^r\lambda_n$-horizontal in C^r in $\hat{\mathcal{B}}_{-1}$.

iv) Let $\Gamma_{R_n-1}^v$ be a curve that is t-vertical in C^r in $\hat{\mathcal{B}}_{R_n-1}^n$. Then $F^{-R_n+1}(\Gamma_{R_n-1}^v)$ is $t\lambda_n$-vertical in C^r in $\hat{\mathcal{B}}_0^n$.

By Lemma 4.8 iii), $\mathcal{I}_{R_n-1}^n$ is η_n-horizontal in $\mathcal{B}_{-1}$. Thus, there exists a C^r-map $g_n : I_{R_n-1}^n \to \mathbb{R}$ with $\|g_n\|_{C^r} < \lambda_n$ such that

$$\Phi_{-1}(\mathcal{I}_{R_n-1}^n) = \{(x, g_n(x)) \mid x \in I_{R_n-1}^n\}.$$

Define $G_n : I_{R_n-1}^n \to \Phi_{-1}(\mathcal{I}_{R_n-1}^n)$ by $G_n(x) := (x, g_n(x))$. Define the *nth critical projection map* $\mathcal{P}_{-1}^n : P_{-1}^{-1}(I_{R_n-1}^n) \to \mathcal{I}_{R_n-1}^n$ by

$$\mathcal{P}_{-1}^n := \Phi_{-1}^{-1} \circ G_n \circ P_{-1}. \tag{4.12}$$

Lemma 4.9. *For $n_0 \le n \le N$, let Γ_0 be a horizontal curve in $\hat{\mathcal{B}}_0^n$. Then*

$$F^{R_n-1}|_{\Gamma_0} = (\mathcal{P}_{-1}^n|_{\Gamma_{R_n-1}})^{-1} \circ F^{R_n-1} \circ \mathcal{P}_0^n|_{\Gamma_0}.$$

Proof. Note that $\mathcal{P}_{-1}^n$ is a projection along the vertical foliation $\mathcal{F}_{-1}^v$ on $\mathcal{B}_{-1}$, and $\mathcal{P}_0^n$ is a projection along the vertical foliation on $\hat{\mathcal{B}}_0^n$ obtained by pulling back $\mathcal{F}_{-1}^v$ by F^{-R_n+1}. The claim follows immediately. □

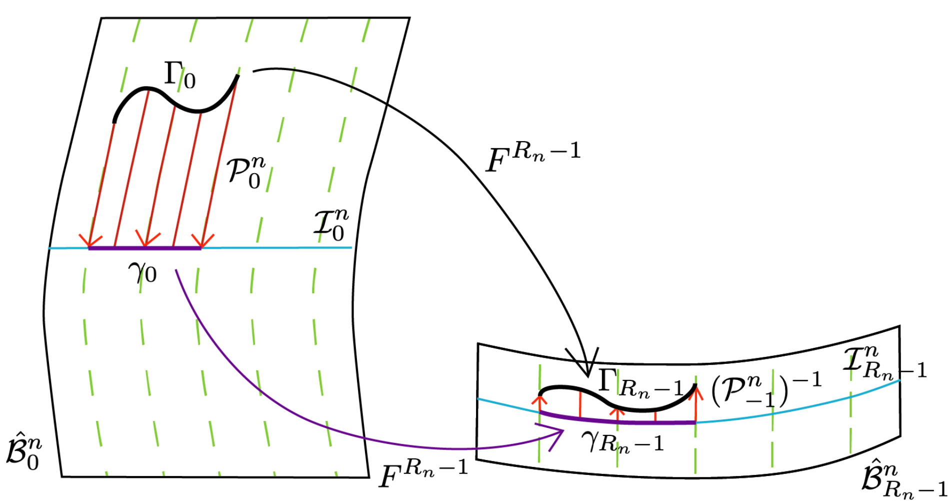

Figure 6. Projections $\mathcal{P}_0^n : \hat{\mathcal{B}}_0^n \to \mathcal{I}_0^n$ and $\mathcal{P}_{-1}^n : \hat{\mathcal{B}}_{R_n-1}^n \to \mathcal{I}_{R_n-1}^n$ near the critical value v_0 and critical point v_{-1} respectively. On any horizontal curve $\Gamma_0 \subset \hat{\mathcal{B}}_0^n$, the iterate F^{R_n-1} commutes with these projections.

We record the following consequences of Theorem 4.7.

Lemma 4.10. *Let $f_0 : I^h_{-1} \to I^h_0$ be the map with a unique critical point at 0 given in Theorem 4.7. Then*

$$\frac{\kappa_F}{2}x^2 < f_0(x) < \frac{K_0}{2}x^2 \quad \textit{and} \quad \kappa_F|x| < |f_0'(x)| < K_0|x|.$$

Lemma 4.11. *Let*

$$I^{h,\pm}_{-1} := \{x \in I^h_{-1} \mid \pm x > 0\} \quad \textit{and} \quad g_\pm := \left(f_0|_{I^{h,\pm}_{-1}}\right)^{-1}.$$

Denote $\theta := K_0/\kappa_F$. Then for $1 \leq i \leq r$, we have

$$|g^{(i)}_\pm(t)| < \frac{\bar{\theta}}{|t|^{i-1/2}} \quad \textit{for} \quad t > 0.$$

Proof. By Theorem 4.7 and Lemma 4.10, we have

$$\|f_0''\|_{C^{r-2}} < K_0, \quad t < \frac{K_0x^2}{2} \quad \text{and} \quad |f_0'(x)| > \kappa_F|x|.$$

The result now follows from Lemma D.3. □

5. Avoiding the Critical Value

For some $N \in \mathbb{N} \cup \{\infty\}$, let F be the N-times regularly Hénon-like renormalizable map considered in Section 4. Recall the constants n_0, η and K_0 given in (4.1) and (4.2), and the constant κ_F given in (4.8). Additionally, assume that $n_0 \leq N$ is the smallest number such that

$$\overline{K_1}\lambda^{\varepsilon R_{n_0}} < \eta, \tag{5.1}$$

where

$$K_1 = K_1(K_0, \kappa_F) \geq 1 \tag{5.2}$$

is a uniform constant. In this section, we show that if a (finite or infinite) orbit of a point avoids getting "too close" to the critical value v_0, then it has uniform regularity.

For $n_0 \leq n \leq N$, recall that the nth centered straightening chart $\Psi^n : \mathcal{B}^n_0 \to B^n_0$ extends vertically to a domain $\hat{\mathcal{B}}^n_0 \supset \mathcal{B}^n_0$ that is vertically proper in $\mathcal{B}_0 \supset \mathcal{B}^{n_0}_0$ (see Proposition 4.2 and Theorem 4.7). Let $z = (a, b) \in B_0 = I^h_0 \times I^v_0$. For $t \geq 0$, define

$$V_z(t) := [a - t, a + t] \times I^v_0.$$

For $p \in \hat{\mathcal{B}}^n_0$ and $t \geq 0$, denote

$$\mathcal{V}^n_p(t) := (\Psi^n)^{-1}(V_{\Psi^n(p)}(t)) \subset \hat{\mathcal{B}}^n_0.$$

We record the following immediate consequences of Lemmas 4.10 and 4.11, and (5.1).

Lemma 5.1. *For $n_0 \leq n \leq N$, let $E_{p_{-1}} \in \mathbb{P}^2_{p_{-1}}$ be a $\lambda^{\bar{\varepsilon}R_n}$-horizontal direction at $p_{-1} \in \mathcal{B}_{-1}$. If*

$$p_0 \in \hat{\mathcal{B}}^n_0 \setminus \mathcal{V}^n_{v_0}(\lambda^{\bar{\varepsilon}R_n})$$

then E_{p_0} *is* $\lambda^{-\bar{\varepsilon}R_n}$*-horizontal in* $\hat{\mathcal{B}}^n_0$*. Similarly, let* Γ_{-1} *be* $\lambda^{\bar{\varepsilon}R_n}$*-horizontal curve in* $\mathcal{B}_{-1}$*. If*

$$\Gamma_0 := F(\Gamma_{-1}) \subset \hat{\mathcal{B}}^n_0 \setminus \mathcal{V}^n_{v_0}(\lambda^{\bar{\varepsilon}R_n}) \quad \textit{with} \quad t > \lambda^{\bar{\varepsilon}R_n},$$

then Γ_0 *is* $\lambda^{-\bar{\varepsilon}R_n}$*-horizontal in* C^r *in* $\hat{\mathcal{B}}^n_0$*.*

Lemma 5.2. *For* $n_0 \leq n \leq N$*, let* $\tilde{E}_{p_0} \in \mathbb{P}^2_{p_0}$ *be a* $\lambda^{\bar{\varepsilon}R_n}$*-vertical direction at* $p_0 \in \hat{\mathcal{B}}^n_0$*. If*

$$p_0 \in \hat{\mathcal{B}}^n_{R_n} \setminus \mathcal{V}^n_{v_0}(\lambda^{\bar{\varepsilon}R_n}),$$

then $\tilde{E}_{p_0}$ *is* $\lambda^{-\bar{\varepsilon}R_n}$*-vertical in* $\mathcal{B}_{-1}$*. Similarly, let* $\tilde{\Gamma}_0$ *be* $\lambda^{\bar{\varepsilon}R_n}$*-vertical curve in* $\hat{\mathcal{B}}^n_0$*. If*

$$\tilde{\Gamma}_0 \subset \hat{\mathcal{B}}^n_{R_n} \setminus \mathcal{V}^n_{v_0}(\lambda^{\bar{\varepsilon}R_n}),$$

then $\tilde{\Gamma}_{-1} := F^{-1}(\tilde{\Gamma}_0)$ *is* $\lambda^{-\bar{\varepsilon}R_n}$*-vertical in* C^r *in* $\mathcal{B}_{-1}$*.*

Proposition 5.3. *For* $n_0 \leq n \leq N$*, let* $p_0 \in \hat{\mathcal{B}}^n_{R_n} \setminus \mathcal{V}^n_{v_0}(\lambda^{\bar{\varepsilon}R_n})$*. If* E_{p_0} *is* $\lambda^{\bar{\varepsilon}R_n}$*-vertical in* $\hat{\mathcal{B}}^n_0$*, then* $E_{p_{-R_n}}$ *is* $\lambda^{(1-\bar{\varepsilon})R_n}$*-vertical in* $\hat{\mathcal{B}}^n_0$*. Moreover,* p_{-R_n} *is* R_n*-times forward* $(CK_0, \bar{\varepsilon}, \lambda)$*-regular along* $E_{p_{-R_n}}$ *for some uniform constant* $C \geq 1$ *independent of* F*. Consequently, if* $p_{kR_n} \in \hat{\mathcal{B}}^n_0 \setminus \mathcal{V}^n_{v_0}(\lambda^{\bar{\varepsilon}R_n})$ *for all* $k \in \mathbb{N}$*, then* p_0 *is infinitely forward* $(CK_0, \bar{\varepsilon}, \lambda)$*-regular.*

Proof. Consider a linearization

$$\{\Phi_{p_{-m}} : \mathcal{U}_{p_{-m}} \to U_{p_{-m}}\}_{m=0}^{R_n}$$

of F along the R_n-backward orbit of p_0 with vertical direction

$$E^{v,n}_{p_0} := (D\Psi^n)^{-1}\left(E^{gh}_{\Psi^n(p_0)}\right).$$

Note that since (F^{R_n}, Ψ^n) is a Hénon-like return, we have

$$D\Psi^n\left(E^{v,n}_{p_{-R_n}}\right) = E^{gv}_{\Psi^n(p_{-R_n})}.$$

Denote

$$E^{h,n}_{p_{-1}} := D\Phi_{p_{-1}}\left(E^{gh}_0\right) \quad \text{and} \quad E^h_{p_{-1}} := D\Phi_{-1}\left(E^{gh}_{\Phi_{-1}(p_{-1})}\right),$$

where $\Phi_{-1} : \mathcal{U}_{-1} \to U_{-1}$ is the chart defined over the critical point given in Theorem 4.7. By Theorem A.2 ii) and (4.9), we see that

$$\|DF^{-R_n+1}|_{E^{h,n}_{p_{-1}}}\|\ ,\ \|DF^{-R_n+1}|_{E^h_{p_{-1}}}\| < \lambda^{-\bar{\varepsilon}R_n}.$$

Hence, it follows from Proposition A.10 that

$$\measuredangle(E^{h,n}_{p_{-1}}, E^h_{p_{-1}}) < \lambda^{(1-\bar{\varepsilon})R_n}.$$

Thus, by Lemma 4.10, we have

$$\measuredangle(E^{h,n}_{p_{-1}}, E_{p_{-1}}) > \lambda^{\bar{\varepsilon}R_n}.$$

For $1 \leq i \leq R_n$, denote

$$\theta_{-i} := \measuredangle(E^{gh}_0, D\Phi_{p_{-i}}(E_{p_{-i}})).$$

Choose a suitable uniform constant $c \in (0, \pi/2)$ independent of F, and let $1 \leq M \leq R_n$ be the smallest number such that $\theta_{-M} > c$. By Theorem A.2, we see that

$$\theta_{-i} > \lambda^{-(1-\bar{\varepsilon})i}\theta_{-1} > \lambda^{-(1-\bar{\varepsilon})i}\lambda^{\bar{\varepsilon}R_n}.$$

Consequently, $M < \bar{\varepsilon}R_n$. Let $M' := CM$ for some suitable uniform constant $C \geq 1$ independent of F.

By Corollary A.8, we have

$$\|DF^i|_{E_{p_{-R_n}}}\| \asymp \|DF^i|_{E^{v,n}_{p_{-R_n}}}\| \quad \text{for} \quad 0 \leq i < R_n - M' \tag{5.3}$$

By Proposition 4.2 iii), p_{-R_n} is R_n-times forward $(K_0, \varepsilon, \lambda)$-regular along $E^{v,n}_{p_{-R_n}}$. Hence, (5.3) implies that p_{-R_n} is $(R_n - M')$-times forward $(CK_0, \varepsilon, \lambda)$-regular along $E_{p_{-R_n}}$. By Proposition A.5, we have

$$\lambda^{\bar{\varepsilon}R_n} < \lambda^{(1+\bar{\varepsilon})M'} < \|DF^i|_{\tilde{E}_{p_{-M'}}}\| < \lambda^{-\bar{\varepsilon}M'} < \lambda^{-\bar{\varepsilon}R_n} \tag{5.4}$$

for any $\tilde{E}_{p_{-M'}} \in \mathbb{P}^2_{p_{-M'}}$. We conclude that for $0 \leq i < M'$, we have

$$\lambda^{\bar{\varepsilon}R_n} < \frac{\|DF^{R_n - M' + i}|_{E^{v,n}_{p_{-R_n}}}\|}{\|DF^{R_n - M' + i}|_{E_{p_{-R_n}}}\|} < \lambda^{-\bar{\varepsilon}R_n}.$$

The $(CK_0, \bar{\varepsilon}, \lambda)$ forward regularity of p_{-R_n} along $E_{p_{-R_n}}$ follows. □

Proposition 5.4. *For $n_0 \leq n \leq N$, let $p_0 \in \hat{\mathcal{B}}^n_0$. If p_0 is infinitely forward $(\bar{K}_0, \bar{\varepsilon}, \lambda)$-regular, then $W^{ss}(p_0)$ is $\lambda^{(1-\bar{\varepsilon})R_n}$-vertical and vertically proper in $\hat{\mathcal{B}}^n_0$.*

Proof. The verticality of $W^{ss}(p_0)$ follows immediately from Proposition A.9. Consider a linearization

$$\{\Phi_{p_m} : \mathcal{U}_{p_m} \to U_{p_m}\}_{m=0}^{\infty}$$

of F along the infinite forward orbit of p_0 with vertical direction $E^{ss}_{p_0}$. By Theorem A.15, we have

$$\Phi_{p_m}(W^{ss}_{\text{loc}}(p_m)) \subset \{(0, y) \in U_{p_m} \mid y \in \mathbb{R}\}. \tag{5.5}$$

Let

$$\mathcal{V}_{p_0} := \mathcal{V}^n_{p_0}(\lambda^{\bar{\varepsilon}R_n}) \subset \hat{\mathcal{B}}^n_0.$$

Arguing as in the proof of Proposition 4.2, we see that if M is the nearest integer to $R_n/2$, then

$$\Phi_{p_M}(F^M(\mathcal{V}_{p_0})) \subset (-\lambda^{\bar{\varepsilon}R_n}, \lambda^{\bar{\varepsilon}R_n}) \times (-\lambda^{(1-\bar{\varepsilon})M}, \lambda^{(1-\bar{\varepsilon})M}). \tag{5.6}$$

For $q_0 \in \mathcal{V}_{p_0}$, denote

$$\hat{E}^{v/h}_{q_m} := D(F^m \circ (\Psi^n)^{-1}) \left(E^{gv/gh}_{\Psi^n(q_0)}\right).$$

By Propositions 4.2 iii), A.5 and A.8, we have

$$\|DF^m|_{\hat{E}^v_{q_0}}\| < K_0 \lambda^{(1-\bar{\varepsilon})m} \quad \text{and} \quad \|DF^m|_{\hat{E}^h_{q_0}}\| > K_0{}^{-1} \lambda^{\bar{\varepsilon}m}.$$

Let $M_0 \leq M$ be the smallest number number such that

$$K_0 \lambda^{(1-\bar{\varepsilon})M_0} < \operatorname{diam}(\mathcal{U}_{p_{M_0}}) \asymp {K_0}^{-1} \lambda^{\bar{\varepsilon} M_0}.$$

Then it follows from Proposition A.3 that $q_m \in \mathcal{U}_{p_m}$ for all $M_0 \leq m \leq R_n$. Define

$$\tilde{E}^{v/h}_{q_m} := D\Phi^{-1}_{p_m}\left(E^{gv/gh}_{\Phi_{p_m}(q_m)}\right).$$

By Proposition A.1, p_{M_0} is infinitely forward $(\overline{K_0}\lambda^{-\bar{\varepsilon}M_0}, \varepsilon, \lambda)$-regular. Hence, by Theorem A.2 and Corollary A.8, we have

$$\left\| DF^{(M-M_0)}|_{\tilde{E}^h_{q_{M_0}}} \right\| > \lambda^{\bar{\varepsilon}M}.$$

Thus, Proposition A.10 implies that

$$\measuredangle(\tilde{E}^h_{q_M}, \hat{E}^h_{q_M}) < \lambda^{(1-\bar{\varepsilon})M}.$$

On the other hand, since $\|D\Phi^{\pm 1}_{p_M}\| < \lambda^{-\bar{\varepsilon}M}$, it follows that

$$\measuredangle(E^{ss}_{q_M}, \hat{E}^h_{q_M}) > \lambda^{\bar{\varepsilon}M}.$$

We conclude by (5.5) and (5.6) that $W^{ss}_{\mathrm{loc}}(p_M)$ is vertically proper in $F^M(\mathcal{V}_{p_0})$. $\square$

Proposition 5.5. *For $n_0 \leq n \leq N$, let $\mathcal{C}_0 \subset \mathcal{B}^n_0$ be a connected set totally invariant under F^{dR_n} for some $d \in \mathbb{N}$ such that $d\bar{\varepsilon} < 1$. Denote $\mathcal{C}_m := F^m(\mathcal{C}_0)$ for $m \in \mathbb{Z}$. If*

$$\mathcal{V}^n_{v_0}(\lambda^{\bar{\varepsilon}R_n}) \cap \mathcal{C} = \varnothing, \quad \textit{where} \quad \mathcal{C} := \bigcup_{i=0}^{d-1} \mathcal{C}_{iR_n},$$

then every point in $\mathcal{C}$ is infinitely forward regular. Moreover, there exists a chart $\Phi : \mathcal{D}_0 \to D_0$ such that the following statements hold:

i) $\mathcal{C}_0 \subset \mathcal{D}_0$;
ii) $\mathcal{D}_0$ is $\lambda^{(1-\bar{\varepsilon})R_n}$-vertical and vertically proper in $\hat{\mathcal{B}}^n_0$;
iii) for $p \in \mathcal{C}_0$, we have

$$\Phi^{-1}(\{x\} \times \pi_v(D_0)) \subset W^{ss}(p) \quad \textit{where} \quad \Phi(p) = (x, y) \in D_0; \quad \textit{and}$$

iv) the map $H := \Phi \circ F^{dR_n} \circ \Phi^{-1}$ is of the form $H(x, y) = (h(x), v(x, y))$, where $h : \pi_h(D_0) \to \pi_h(D_0)$ is a diffeomorphism.

Proof. The infinite forward regularity of any point $p \in \mathcal{C}$ is given in Proposition 5.3. Let $W^{ss}_{\mathrm{loc}}(p)$ be the connected component of $W^{ss}(p) \cap \hat{\mathcal{B}}^n_0$ containing p. Define

$$\mathcal{D} := \bigcup_{p \in \mathcal{C}} W^{ss}_{\mathrm{loc}}(p). \quad \text{and} \quad \mathcal{D}_0 := \bigcup_{p \in \mathcal{C}_0} W^{ss}_{\mathrm{loc}}(p).$$

By Proposition 5.4, the foliation of each component of $\mathcal{D}$ given by the local strong-stable manifolds is $\lambda^{(1-\bar{\varepsilon})R_n}$-vertical and vertically proper in $\hat{\mathcal{B}}^n_0$. Let $\Phi : \mathcal{D}_0 \to D_0$ be a genuine horizontal chart that rectifies this vertical foliation over $\mathcal{D}_0$. Then the map $H := \Phi \circ F^{dR_n} \circ \Phi^{-1}$ preserves the vertical foliation.

By Theorem 4.7 ii), (4.7) and the fact that

$$\mathcal{D} \cap \mathcal{V}^n_{v_0}(\lambda^{\bar{\varepsilon} R_n}) = \varnothing,$$

it follows that $h := \Pi_{1\mathrm{D}}(H)$ is a diffeomorphism. □

6. Combinatorial Structure of Renormalization

In this section, we show that at sufficiently deep renormalization depths (i.e. beyond the depth n_0 given by (4.1)), the dynamical structure of a 2D Hénon-like map closely resembles that of a 1D unimodal map. See Figure 7. In particular, we prove that topological renormalizations at these depths with combinatorics of bounded type are guaranteed to be regular Hénon-like, as long as they are not *trivial* in the sense defined below.

Following [CPTr], we say that a diffeomorphism (in any dimension) is *generalized Morse-Smale (of θ-bounded type)* for some $\theta \in \mathbb{N}$ if

- the ω-limit set of any forward orbit is a periodic point;
- the α-limit set of any backward orbit is a periodic point; and
- all periodic orbits have periods bounded by θ.

Note that a diffeomorphism of an interval to itself is generalized Morse-Smale of either 1-bounded type if orientation-preserving, or 2-bounded type if orientation-reversing. A renormalization of a map is *trivial* if the induced return map is a generalized Morse-Smale diffeomorphism of 2-bounded type.

6.1. **For unimodal maps.**

Lemma 6.1. *Consider a unimodal map $f : I \to I$ with the critical point $c \in I$. Suppose that $f''(c) > 0$. Then the following statements hold.*

i) If $f(c) > c$, then c converges to either a fixed attracting or parabolic sink.
ii) If $f^2(c) < c$, then c converges to either a fixed or 2-periodic attracting or parabolic sink.
iii) If $f^3(c) > f^2(c)$, then c converges to a fixed attracting or parabolic sink.

If none of the above cases hold, then $J := [f(c), f^2(c)]$ is the minimal invariant interval containing c.

Consider a unimodal map $f : I \to I$ with the critical point $c \in I$. For concreteness, assume that $f''(c) > 0$. For $\theta \geq 1$, we say that f has *θ-bounded kneading* if $f(c) < c$ and $f^{1+\theta}(c) < c$. Recall that f is valuably renormalizable if there exists an R-periodic interval $I^1 \subset I$ for some integer $R \geq 2$ such that $f^R(I^1)$ contains the critical value v for f. Note that in this case, f has θ-bounded kneading for some $\theta \leq R$. The *renormalization type $\tau(f)$ of f* is given by the order of points in $\{f^i(v)\}_{i=0}^{R-1} \subset I$. If f is N-times valuably renormalizable, then its *N-combinatorial type* is defined as

$$\tau_N(f) := (\tau(f), \tau(\mathcal{R}(f)), \dots, \tau(\mathcal{R}^{N-1}(f))).$$

Lemma 6.2. *Let $f : I \to I$ be a unimodal map with critical value v. If f is non-trivially topologically renormalizable with return time $R \geq 2$, then f is valuably renormalizable. In this case, $I^1 := [v, f^R(v)]$ is the minimal R-periodic interval containing v.*

6.2. **For Hénon-like maps.** For some $N \in \mathbb{N} \cup \{\infty\}$, let F be the N-times regularly Hénon-like renormalizable map considered in Section 5. Suppose that the combinatorics of renormalizations of F are of $\mathbf{b}$-bounded type for some integer $\mathbf{b} \geq 2$. Moreover, assume that ε is sufficiently small so that $\mathbf{b}\bar{\varepsilon} < 1$. By only considering every other returns if necessary, we may also assume without loss of generality that $r_n := R_{n+1}/R_n \geq 3$ for $n_0 \leq n \leq N$.

Let $z = (a, b), w = (c, d) \in B_0 = I^h_0 \times I^v_0$. Denote

$$m := \min\{a, c\} \quad \text{and} \quad M := \max\{a, c\}.$$

For $t \geq 0$, define

$$V_{[z,w]}(t) := [m - t, M + t] \times I^v_0.$$

For $n_0 \leq n \leq N$; $p, q \in \hat{\mathcal{B}}^n_0$ and $t \geq 0$, denote

$$\mathcal{V}^n_{[p,q]}(t) := (\Psi^n)^{-1}(V_{[\Psi^n(p),\Psi^n(q)]}(t)).$$

Let $s \in \{0, 1, 2\}$. For $n_0 \leq n \leq N - s$ and $k \geq -1$, denote

$$a^n_k := P^n_0(v_{kR_n}) \quad \text{and} \quad b^{n,s}_k := P^n_0(v_{kR_n+R_{n+s}}) = a^n_{k+R_{n+s}/R_n}.$$

Define

$$\hat{B}^{n,s}_{kR_n} := V_{[a^n_k, b^{n,s}_k]}(\lambda^{\bar{\varepsilon}R_n}) \quad \text{and} \quad \hat{\mathcal{B}}^{n,s}_{kR_n} := (\Psi^n)^{-1}(\hat{B}^{n,s}_{kR_n}).$$

In particular, we have

$$\hat{\mathcal{B}}^n_0 \supset \hat{\mathcal{B}}^{n,0}_0 := \mathcal{V}^n_{[v_0, v_{R_n}]}(\lambda^{\bar{\varepsilon}R_n}).$$

See Figure 7.

Lemma 6.3. *Let $n_0 \leq n \leq N$. Suppose that $F^{R_n}|_{\mathcal{B}^n_0}$ is non-trivially topologically renormalizable with combinatorics of $\mathbf{b}$-bounded type. Then*

$$\mathcal{V}^n_{v_{kR_n}}(\lambda^{\bar{\varepsilon}R_n}) \cap \mathcal{V}^n_{v_{-R_n}}(\lambda^{\bar{\varepsilon}R_n}) = \varnothing \quad \textit{for} \quad k = O(\mathbf{b}).$$

Proof. Let $\delta \in (\bar{\varepsilon}, 1)$ with $\mathbf{b}\bar{\delta} < 1$. Suppose towards a contradiction that for some $k = O(\mathbf{b})$, we have

$$V_{v_{kR_n}}(\lambda^{\bar{\delta}R_n}) \cap V_{v_{-R_n}}(\lambda^{\bar{\delta}R_n}) \neq \varnothing. \tag{6.1}$$

Without loss of generality, assume that k is the smallest number for which (6.1) holds.

For $y \in I^v_0$, consider

$$J^n_0 \subset (-\lambda^{\bar{\delta}R_n}, +\lambda^{\bar{\delta}R_n}) \quad \text{and} \quad \mathcal{J}^n_0 := \Psi^{-n}(J^n_0 \times \{y\}) \subset \mathcal{V}^n_{v_0}(\lambda^{\bar{\delta}R_n}).$$

For $i \in \mathbb{N}$, denote $\mathcal{J}^n_i := F^i(\mathcal{J}^n_0)$.

By Proposition A.5, we see that

$$|\mathcal{J}^n_{iR_n-1}| < \lambda^{-i\bar{\varepsilon}R_n}|J^n_0| < \lambda^{\underline{\delta}R_n} \quad \text{for} \quad i = O(\mathbf{b}).$$

Moreover, since

$$\mathcal{J}^n_{iR_n} \cap V_{v_{-R_n}}(\lambda^{\bar{\delta}R_n}) = \varnothing \quad \text{for} \quad i < k,$$

we can argue by induction using Lemma 4.8 iii) and Lemma 5.1 that $\mathcal{J}^n_{(k+1)R_n-1}$ is $\lambda^{(1-\bar{\varepsilon})R_n}$-horizontal in C^r in $\mathcal{B}_{-1}$. Then it follows from (6.1), (4.7) and Lemma 4.10 that

$$|P^n_0(\mathcal{J}^n_{(k+1)R_n})| < \lambda^{\underline{\delta}R_n}|\mathcal{J}^n_{(k+1)R_n-1}| < \lambda^{\underline{\delta}R_n}|J^n_0|.$$

We conclude that

$$F^{(k+1)R_n}(\mathcal{V}^n_{v_0}(\lambda^{\bar{\delta}R_n})) \Subset \mathcal{V}^n_{v_0}(\lambda^{\bar{\delta}R_n}).$$

Denote $E^h_{p_i} := DF^i(E^h_{p_0})$ for $i \in \mathbb{N}$. Arguing by induction using Lemma 4.8 i) and Lemma 5.1, we also see that $E^h_{p_{(k+1)R_n-1}}$ is $\lambda^{(1-\bar{\varepsilon})R_n}$-horizontal in $\mathcal{B}_{-1}$. Consequently, by (6.1), (4.7) and Lemma 4.10, we have

$$\measuredangle(E^h_{p_{(k+1)R_n}}, E^{v,n}_{p_{(k+1)R_n}}) < \lambda^{\underline{\delta}R_n}.$$

Hence, by Theorem 4.7 ii), it follows that

$$\|DF^{R_n}|_{E^h_{p_{(k+1)R_n}}}\| < \lambda^{\underline{\delta}R_n}.$$

Since, by Proposition A.5, we have

$$\|D_{p_0}F^{(k+1)R_n}\| < \lambda^{-\bar{\varepsilon}R_n},$$

we conclude that

$$\|D_{p_0}F^{(k+2)R_n}\| \asymp \|D_{p_0}F^{(k+2)R_n}|_{E^h_{p_0}}\| < \lambda^{-\bar{\varepsilon}R_n}\lambda^{\underline{\delta}R_n} = \lambda^{\underline{\delta}R_n}.$$

Applying Proposition A.5 again, we have

$$\|D_{p_0}F^{2(k+1)R_n}\| \leq \|D_{p_0}F^{(k+2)R_n}\| \cdot \|D_{p_{k+2(R_n)}}F^{kR_n}\| < \lambda^{\underline{\delta}R_n}\lambda^{-\bar{\varepsilon}R_n} = \lambda^{\underline{\delta}R_n}.$$

Thus, under $F^{2(k+1)R_n}$, there exists a unique fixed sink q_0 that attracts the orbit of every point in $\mathcal{V}^n_{v_0}(\lambda^{\bar{\varepsilon}R_n})$. Since $\mathcal{V}^n_{v_0}(\lambda^{\bar{\varepsilon}R_n})$ maps into itself under $F^{(k+1)R_n}$, we see that q_0 is fixed under $F^{(k+1)R_n}$.

Denote

$$\mathcal{V}^n_{v_0} := \mathcal{V}^n_{v_0}(\lambda^{\bar{\varepsilon}R_n}) \quad \text{and} \quad R_{n+1} := (k+1)R_n.$$

The set $\partial\mathcal{V}^n_{v_0} \setminus \partial\hat{\mathcal{B}}^n_0$ consists of two vertical curves $\gamma^{l,0}$ and $\gamma^{r,0}$: the former to the left and the latter to the right of v_0.

For $i \geq 0$, let $\mathcal{B}^{n+1,-i}_0$ be the component of $F^{-iR_{n+1}}(\mathcal{V}^n_{v_0}) \cap \hat{\mathcal{B}}^n_0$ that contain $\mathcal{V}^n_{v_0}$. Proceeding inductively, suppose that $\mathcal{B}^{n+1,-i}_0$ is a quadrilateral such that $\partial\mathcal{B}^{n+1,-i}_0 \setminus \partial\hat{\mathcal{B}}^n_0$ consists of two vertical curves $\gamma^{l,-i}$ and $\gamma^{r,-i}$: the former to the left of $\gamma^{l,-i+1}$ and the latter to the right of $\gamma^{r,-i+1}$. Let $y \in I^v_0$, and denote

$$\mathcal{I}^{-i,y}_0 := (\Psi^n)^{-1}(I^n_0 \times \{y\}) \cap \mathcal{B}^{n+1,-i}_0 \quad \text{and} \quad \mathcal{I}^{-i,y}_j := F^j(\mathcal{I}^{-i,y}_0) \quad \text{for} \quad j \in \mathbb{N}.$$

Arguing as above, we see that $\mathcal{I}^{-i,y}_{R_{n+1}-1}$ is $\lambda^{(1-\bar{\varepsilon})R_n}$-horizontal in $\mathcal{B}_0$, and hence, $\mathcal{I}^{-i,y}_{R_{n+1}}$ is vertical quadratic in $\hat{\mathcal{B}}^n_0$ whose endpoints are contained in $\gamma^{r,-i}$. Moreover, by

Lemma 4.8 iv) and Lemma 5.2, we also see that the induction hypothesis holds for $\mathcal{B}_0^{n+1,-i-1}$.

By Proposition 5.3, we see that as $i \to \infty$, the curves $\gamma^{l,-i}$ and $\gamma^{r,-i}$ converge to subarcs γ^l and γ^r respectively of the strong-stable manifold $W^{ss}(w_0)$ of a R_{n+1}-periodic saddle $w_0 \in \gamma^r := W^{ss}_{\mathrm{loc}}(y_0)$. Moreover, γ^l and γ^r are $\lambda^{(1-\bar{\delta})R_n}$-vertical and vertically proper in $\hat{\mathcal{B}}_0^n$. It follows that these curves bound the immediate basin $\mathcal{B}_0^{n+1} \subset \hat{\mathcal{B}}_0^n$ of q_0.

Let $R_n < \hat{R}_{n+1} \leq \mathbf{b}R_n$. We claim that any $\hat{R}_{n+1}$-periodic Jordan domain $\mathcal{D}_0^{n+1} \Subset \mathcal{B}_0^n$ induces a trivial renormalization of $F^{R_n}|_{\mathcal{B}_0^n}$.

If $\mathcal{D}_0^{n+1} \cap \mathcal{B}_0^{n+1} = \varnothing$, then by Proposition 5.5, $\mathcal{D}_0^{n+1}$ induces a trivial renormalization of $F^{R_n}|_{\mathcal{B}_0^n}$. Assume that $\mathcal{D}_0^{n+1} \cap \mathcal{B}_0^{n+1} \neq \varnothing$. Then $\mathcal{D}_0^{n+1} \ni q_0$ and $\hat{R}_{n+1} = R_{n+1}$. Define

$$\mathcal{C}_0^{n+1} := \bigcap_{i \in \mathbb{N}} F^{iR_{n+1}}(\mathcal{D}_0^{n+1}).$$

By Proposition 5.3, every point $p \in \mathcal{C}_0^{n+1}$ that does not eventually map into $\mathcal{B}_0^{n+1}$ is infinitely forward $(CK_0, \bar{\delta}, \lambda)$-regular. Let $W^{ss}_{\mathrm{loc}}(p)$ be the connected component of $W^{ss}(p) \cap \hat{\mathcal{B}}_0^n$ that contain p, which must be $\lambda^{(1-\bar{\delta})R_n}$-vertical and vertically proper in $\hat{\mathcal{B}}_0^n$ by Proposition 5.4.

Let p_0^l and p_0^r be the leftmost and the rightmost points in $\mathcal{C}_0^{n+1}$ respectively. Let $\hat{\mathcal{D}}_0^{n+1}$ be the quadrilaterals vertically proper in $\hat{\mathcal{B}}_0^n$ enclosed between $W^{ss}_{\mathrm{loc}}(p_0^l)$ and $W^{ss}_{\mathrm{loc}}(p_0^r)$, so that $\mathcal{C}_0^{n+1} \subset \hat{\mathcal{D}}_0^{n+1}$. Similarly, let $\mathcal{E}_0^{n+1}$ be the quadrilaterals vertically proper in $\hat{\mathcal{B}}_0^n$ enclosed between $W^{ss}_{\mathrm{loc}}(w_0)$ and $W^{ss}_{\mathrm{loc}}(p_0^r)$.

Observe that

$$\hat{\mathcal{D}}_{iR_n}^{n+1} \cap \mathcal{V}_{v_0}^n(\lambda^{\bar{\varepsilon}R_n}) = \varnothing \quad \text{for} \quad 1 \leq i < R_{n+1}/R_n. \tag{6.2}$$

Let $\mathcal{J}_0$ be a genuine horizontal arc contained in $\mathcal{E}_0^{n+1}$. Using (6.2), a similar argument as above shows that $\mathcal{J}_{iR_n}$ for $0 \leq i < R_{n+1}/R_n$ is $\lambda^{-\bar{\delta}R_n}$-horizontal in $\hat{\mathcal{B}}_0^n$. Consequently,

$$F^{R_{n+1}}(\mathcal{E}_0^{n+1}) \cap \mathcal{B}_0^{n+1} = \varnothing.$$

It follows that $\mathcal{E}_0^{n+1}$ is connected and $F^{R_{n+1}}(\mathcal{E}_0^{n+1}) = \mathcal{E}_0^{n+1}$. Applying Proposition 5.5, we conclude that $\mathcal{D}_0^{n+1}$ induces a trivial renormalization of $F^{R_n}|_{\mathcal{B}_0^n}$. This is a contradiction, and therefore, (6.1) cannot hold. □

Proposition 6.4. *Let $n_0 \leq n \leq N$. Suppose that $F^{R_n}|_{\mathcal{B}_0^n}$ is twice non-trivially topological renormalizable with combinatorics of $\mathbf{b}$-bounded type. Then the following statements hold:*

i) $|a_i^n - a_j^n| > \lambda^{\bar{\varepsilon}R_n}$ *for* $-1 \leq i, j \leq 2$ *with* $i \neq j$;
ii) $a_0^n = 0 < a_m^n < a_1^n$ *for* $m \in \{-1, 2\}$; *and*
iii) $F^{R_n}(\hat{\mathcal{B}}_0^{n,0}) \Subset \hat{\mathcal{B}}_0^{n,0}$.

Proof. For $0 \leq i \leq \mathbf{b}$, mapping u^n_{-1} and u^n_i by F^{R_n}, and applying Theorem 4.7, it follows from Lemma 6.3 that

$$|a^n_0 - a^n_{i+1}| > \lambda^{\bar{\varepsilon}R_n}|a^n_{-1} - a^n_i| > \lambda^{\bar{\varepsilon}R_n}. \tag{6.3}$$

Now, suppose towards a contradiction that $a^n_0 = 0 < a^n_{-1} < a^n_1$ is not true. Then we have

$$\lambda^{\bar{\varepsilon}R_n} < a^n_1 < a^n_{-1} - \lambda^{\bar{\varepsilon}R_n}.$$

Denote

$$J^n_0 := [-\lambda^{\bar{\varepsilon}R_n}, a^n_{-1} - \lambda^{\bar{\varepsilon}R_n}], \quad \check{B}^n_0 := J^n_0 \times I^v_0 \quad \text{and} \quad \check{\mathcal{B}}^n_0 := (\Psi^n)^{-1}(\check{B}^n_0).$$

By Lemma 4.8 iii) and (4.7), we see that

$$F^{R_n}(\check{\mathcal{B}}^n_0) \Subset \check{\mathcal{B}}^n_0 \setminus \mathcal{V}^n_{v_0}(\lambda^{\bar{\varepsilon}R_n}).$$

It follows from Proposition 5.5 that $\check{\mathcal{B}}^n_0$ induces a trivial renormalization of F.

Define

$$\check{\mathcal{D}}^n_0 = \check{\mathcal{D}}^{n,0}_0 := \check{\mathcal{B}}^n_0 \cup \mathcal{V}^n_{v_{-R_n}}(\lambda^{\bar{\varepsilon}R_n}).$$

Observe that $F^{R_n}(\check{\mathcal{D}}^n_0) \Subset \check{\mathcal{B}}^n_0$. For $i \in \mathbb{N}$, let

$$\mathcal{D}^{n,-i}_0 := (F^{R_n}|_{\hat{\mathcal{B}}^n_0})^{-1}(\check{\mathcal{D}}^{n,-i+1}_0).$$

Arguing as in the proof of Lemma 6.3, we see that $\mathcal{D}^{n,-i}_0$ is a quadrilateral vertically proper in $\hat{\mathcal{B}}^n_0$ that is bounded between two $\lambda^{(1-\bar{\varepsilon})R_n}$-vertical curves $\gamma^{l,-i}$ and $\gamma^{r,-i}$: the former to the left of $\gamma^{l,-i+1}$ and the latter to the right of $\gamma^{r,-i+1}$. Moreover, there exists an R_n-periodic saddle w_0 such that $\gamma^{r,-i}$ converges to the local strong-stable manifold $W^{ss}_{\mathrm{loc}}(w_0)$ as $i \to \infty$.

Write

$$\hat{\mathcal{B}}^n_0 \setminus W^{ss}_{\mathrm{loc}}(w_0) = \mathcal{D}^n_0 \sqcup \mathcal{E}^n_0,$$

where $\mathcal{D}^n_0$ is the connected component that contains $\check{\mathcal{D}}^n_0$. Observe that $F^{R_n}(\mathcal{E}^n_0) \subset \mathcal{E}^n_0$. Applying Proposition 5.5, we conclude that $\hat{\mathcal{B}}^n_0$ induces a trivial renormalization of F, which is a contradiction.

Next, suppose towards a contradiction that

$$|a^n_1 - a^n_2| < \lambda^{\bar{\varepsilon}R_n}.$$

By the R_n-times regularity of F on $\hat{\mathcal{B}}^n_0$, we conclude that

$$\|v_{kR_n} - v_{(k+1)R_n}\| < \lambda^{-\bar{\varepsilon}R_n}|a^n_1 - a^n_2| < \lambda^{\bar{\varepsilon}R_n} \quad \text{for} \quad 1 \leq k < \mathbf{b}. \tag{6.4}$$

For $l \in \{1, 2\}$, let $\mathcal{B}^{n+l}_0 \Subset \mathcal{B}^{n+l-1}_0$ be the R_{n+l}-periodic Jordan domain that induces a non-trivial renormalization of $F^{R_n}|_{\mathcal{B}^n_0}$. By Proposition 5.5, we may assume without loss of generality that a $\lambda^{\bar{\varepsilon}R_n}$-neighborhood of $\mathcal{B}^{n+l}_0$ contains v_0 (and hence, also $v_{R_{n+l}}$).

Proposition B.1 implies the existence of a saddle point $x_0 \in \mathcal{B}^{n+1}_0$ of period dR_{n+1} for some $d \leq r_{n+1}$. For $0 \leq i < R_{n+1}$, let $W^{ss}_{\mathrm{loc}}(x_{iR_n})$ be the connected component of $W^{ss}(x_{iR_n}) \cap \hat{\mathcal{B}}^n_0$ containing x_{iR_n}. Clearly,

$$W^{ss}_{\mathrm{loc}}(x_{iR_n}) \cap \mathcal{B}^{n+1}_{jR_n} = \varnothing \tag{6.5}$$

for $0 \leq j < r_n$ such that $i \neq j$. Moreover, it follows from Propositions 5.4 and 5.5 that $W^{ss}_{\mathrm{loc}}(x_{iR_n})$ is $\lambda^{(1-\bar{\varepsilon})R_n}$-vertical and vertically proper in $\hat{\mathcal{B}}^n_0$.

Mapping $v_{R_{n+1}}$ and v_0 by F^{R_n} and applying Theorem 4.7, it follows from (6.3) that

$$|a^n_1 - a^n_{r_n+1}| > \lambda^{\bar{\varepsilon}R_n}.$$

Thus, by (6.4), we have $a^n_{r_n+1} < a^n_k < a^n_1$ for $2 \leq k \leq r_n$. This contradicts (6.5). Therefore, claim i) holds.

Suppose towards a contradiction that $a^n_2 < a^n_1$ is not true. Then we have

$$a^n_1 < a^n_2 - \lambda^{\bar{\varepsilon}R_n}.$$

Let $y \in I^v_0$, and consider

$$A^n_0 := [a^n_{-1} + \lambda^{\varepsilon R_n}, a^n_1] \quad \text{and} \quad \mathcal{A}^n_0 := (\Psi^n)^{-1}(A^n_0 \times \{y\}).$$

Denote $\mathcal{A}^n_i := F^i(\mathcal{A}^n_0)$. Applying Theorem 4.7 and Lemma 5.1, we see that $\mathcal{A}^n_{R_n}$ is $\lambda^{-\bar{\varepsilon}R_n}$-horizontal in $\hat{\mathcal{B}}^n_0$. This means that $f_n := \Pi_{1\mathrm{D}}(F_n)$, where

$$F_n := \Psi^n \circ F^{R_n} \circ (\Psi^n)^{-1},$$

maps A^n_0 onto $f_n(A^n_0)$ as an orientation-preserving diffeomorphism. Moreover,

$$f_n(A^n_0) \supset [\lambda^{\bar{\varepsilon}R_n}, a^n_2 - \lambda^{(1-\bar{\varepsilon})R_n}] \Supset A^n_0.$$

Let

$$L^n_0 := \{(a^n_1, t) \in \hat{\mathcal{B}}^n_0\} \quad \text{and} \quad \mathcal{L}^n_0 := (\Psi^n)^{-1}(L^n_0).$$

For $i \in \mathbb{N}$, define

$$\mathcal{L}^n_{-i} := F^{-R_n}(\mathcal{L}^n_{-i+1} \cap \mathcal{A}^n_{R_n}).$$

Applying Lemma 5.2 and Lemma 4.8 iv) and arguing by induction, we see that $\mathcal{L}^n_{-i}$ is $\lambda^{(1-\bar{\varepsilon})R_n}$-vertical and vertically proper in $\hat{\mathcal{B}}^n_0$. Moreover, by Lemma 5.3, we see that any point $p \in \mathcal{L}^n_{-i}$ is iR_n-times forward $(CK_0, \bar{\varepsilon}, \lambda)$-regular along the tangent direction to $\mathcal{L}^n_{-i}$ at p. It follows that $\mathcal{L}^n_{-i}$ converges as $i \to \infty$ to the local strong-stable manifold of some R_n-periodic saddle z_0. Let $\mathcal{B}^{n,r}_0$ be the connected components of $\mathcal{B}^n_0 \setminus W^{ss}(z_0)$ containing v_{R_n}. It follows that $F^{R_n}(\mathcal{B}^{n,r}_0) \subset \mathcal{B}^{n,r}_0$. This is a contradiction. Claim ii) now follows.

Theorem 4.7 implies that

$$F^{R_n}(\mathcal{V}^n_{[v_0, v_{-R_n}]}(\lambda^{\bar{\varepsilon}R_n})) \Subset \mathcal{V}^n_{[v_0, v_{R_n}]}(\lambda^{\bar{\varepsilon}R_n}).$$

We similarly conclude that

$$F^{R_n}(\mathcal{V}^n_{[v_{-R_n}, v_{R_n}]}(\lambda^{\bar{\varepsilon}R_n})) \Subset \mathcal{V}^n_{[v_0, v_{2R_n}]}(\lambda^{\bar{\varepsilon}R_n}).$$

Thus, claim iii) holds. $\square$

Proposition 6.5. *Let $n_0 \leq n \leq N$. Suppose that $F^{R_n}|_{\mathcal{B}^n_0}$ is twice non-trivially topological renormalizable with combinatorics of* $\mathbf{b}$*-bounded type. Then for $m := n - s$ with $s \in \{1, 2\}$, the following statements hold:*

i) $\hat{\mathcal{B}}^{m,s}_{lR_m} \cap \hat{\mathcal{B}}^{m,s}_{kR_m} = \varnothing$ for $0 \leq l, k < R_n/R_m$ with $l \neq k$;
ii) $a^m_0 = 0 < b^m_0 < a^m_k$, $b^m_k < b^m_1 < a^m_1$ for $2 \leq k < R_n/R_m$; and

iii) $F^{R_m}(\hat{\mathcal{B}}^{m,s}_{kR_m}) \Subset \hat{\mathcal{B}}^{m,s}_{(k+1)R_m \ (\mathrm{mod}\ R_n)}$ *for* $0 \le k < R_n/R_m$.

Consequently, $\hat{\mathcal{B}}^{n,0}_0$ *is* R_n*-periodic.*

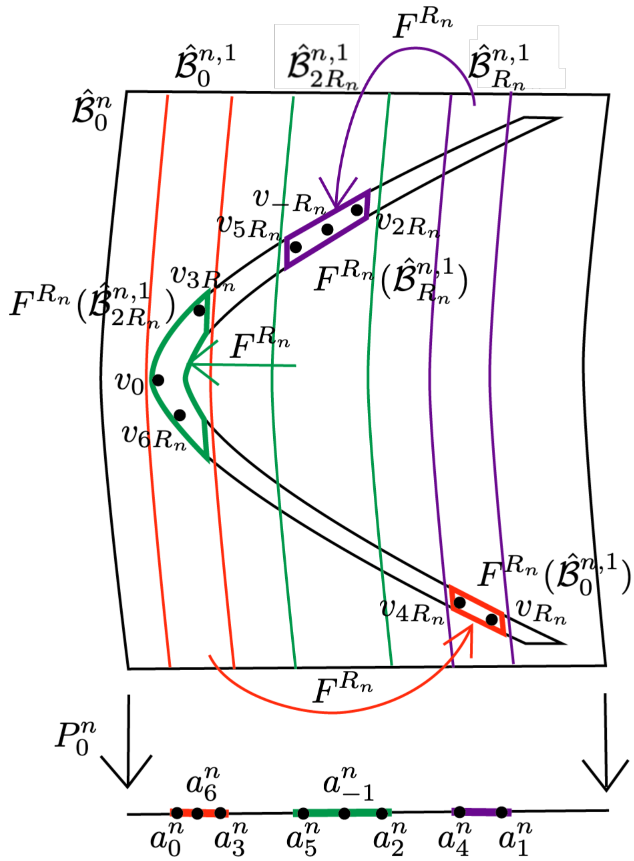

Figure 7. The combinatorial structure of the nth renormalization of F for $n \ge n_0$ (for $r_n := R_{n+1}/R_n = 3$). The R_{n+1}-periodic domains $\hat{\mathcal{B}}^{n,1}_0$, $\hat{\mathcal{B}}^{n,1}_{R_n}$ and $\hat{\mathcal{B}}^{n,1}_{2R_n}$ containing v_0, v_{R_n} and v_{2R_n} respectively are vertically proper and pairwise disjoint in $\hat{\mathcal{B}}^n_0$. Moreover, $F^{R_n}(\hat{\mathcal{B}}^{n,1}_{kR_n}) \Subset \hat{\mathcal{B}}^{n,1}_{(k+1)R_n \ (\mathrm{mod}\ R_{n+1})}$. Under the projection $P^n_0 : \hat{\mathcal{B}}^n_0 \to I^n_0 \subset \mathbb{R}$, the orbit $\{v_{kR_n}\}_{k=-1}^{r_n}$ of the critical value are mapped to $\{a^n_k\}_{k=-1}^{r_n}$.

Proof. Suppose towards a contradiction that for some $1 \le j, i < 2r_m$ with $j < i$, we have

$$|a^m_i - a^m_j| < \lambda^{\bar{\varepsilon} R_m}.$$

Applying $F^{(2r_m-i)R_m}$ to v_{iR_m} and v_{jR_m}, we see by the regularity of F that

$$\|v_{2R_{m+1}} - v_{j'R_m}\| < \lambda^{\bar{\varepsilon}R_m},$$

where $j' := j + 2r_m - i$. Note that $j' \neq 0 \pmod{r_m}$. Hence,

$$\mathcal{B}^{m+1}_{j'R_m} \cap \mathcal{B}^{m+1}_0 = \varnothing.$$

Moreover, by Proposition 6.4, we have

$$\lambda^{\bar{\varepsilon}R_m} < a^m_{2r_m} - \lambda^{\bar{\varepsilon}R_m} < a^m_{j'} < a^m_{2r_m} + \lambda^{\bar{\varepsilon}R_m} < a^m_{r_m}. \tag{6.6}$$

Proposition B.1 implies the existence of a saddle point $x_0 \in \mathcal{B}^{m+2}_0$ of period dR_{m+2} for some $d \leq r_{m+2}$. For $0 \leq k < R_{m+2}/R_m$, let $W^{ss}_{\text{loc}}(x_{kR_m})$ be the connected component of $W^{ss}(x_{kR_m}) \cap \hat{\mathcal{B}}^m_0$ containing x_{kR_m}. Then (6.6) implies that $W^{ss}_{\text{loc}}(x_{j'R_m})$ intersect $\mathcal{B}^{m+1}_0$. This is a contradiction. Claim i) follows. Then Claim ii) and iii) follow by Proposition 6.4 and Theorem 4.7 respectively. □

By Proposition 6.5, we may henceforth assume without loss of generality that for all $n_0 \leq n \leq N$ such that $F^{R_n}|_{\mathcal{B}^n_0}$ is twice non-trivially renormalizable, we have

$$\mathcal{B}^n_0 := \hat{\mathcal{B}}^{n,0}_0 := \mathcal{V}^n_{[v_0, v_{R_n}]}(\lambda^{\bar{\varepsilon}R_n}). \tag{6.7}$$

Let $0 \leq n < N$. Suppose that $F^{R_{n+1}}|_{\mathcal{B}^{n+1}_0}$ is twice non-trivially topological renormalizable with combinatorics of $\mathbf{b}$-bounded type. Consider the sequence of points

$$\{a^n_k := \pi_h \circ \Psi^n(v_{kR_n})\}_{k=0}^{r_n-1} \subset I^n_0 := \pi_h(B^n_0) \subset \mathbb{R}. \tag{6.8}$$

The *renormalization type* $\tau(\mathcal{R}^n(F))$ *of* $\mathcal{R}^n(F)$ is given by the order of points in (6.8). Additionally, the N*-renormalization type of* F is defined as

$$\tau_N(F) := (\tau(F), \tau(\mathcal{R}(F)), \ldots, \tau(\mathcal{R}^{N-1}(F))).$$

Lemma 6.6. *Let $n_0 \leq n \leq N$. Suppose that $F^{R_n}|_{\mathcal{B}^n_0}$ is twice topological renormalizable with combinatorics of* $\mathbf{b}$*-bounded type. For $s \in \{1, 2\}$ and $m := n - s$, let Γ_0 be a $\lambda^{-\bar{\varepsilon}R_m}$-horizontal curve in $\mathcal{B}^n_0$. Then the following statements hold for $1 \leq k \leq R_n/R_m$:*

i) $\Gamma_{(k-1)R_m}$ is $\lambda^{-\bar{\varepsilon}R_m}$-horizontal in $\mathcal{B}^m_0$; and

ii) Γ_{kR_m-1} is $\lambda^{(1-\bar{\varepsilon})R_m}$-horizontal in $\mathcal{B}_{-1}$.

Proof. The result is an immediate consequence of Lemmas 4.8 iii) and 5.1, and Proposition 6.5. □

Theorem 6.7 (Hénon-likeness of deep returns)**.** *Suppose that $F^{R_N}|_{\mathcal{B}^N_0}$ is three times non-trivially topologically renormalizable with combinatorics of* $\mathbf{b}$*-bounded type. Then F is $(N+1)$-times $(CK_0, \bar{\varepsilon}, \lambda)$-regularly Hénon-like renormalizable, where and $C, K_0 \geq 1$ are uniform constants (the former independent of F, and the latter given in* (4.2)*).*

Proof. For $l \in \{1, 2\}$, let $\mathcal{B}^{N+l}_0 \Subset \mathcal{B}^{N+l-1}_0$ be an R_{N+l}-periodic Jordan domain with

$$r_{N+l-1} := R_{N+l}/R_{N+l-1} \leq \mathbf{b}.$$

Define

$$\mathcal{C}_0^{N+l} := \bigcap_{i=1}^{\infty} F^{iR_{N+l}}(\mathcal{B}_0^{N+l}) \quad \text{and} \quad \mathcal{C}^{N+l} := \bigcup_{i=0}^{R_{N+l}/R_N-1} F^{iR_N}(\mathcal{C}_0^{N+l}).$$

By Proposition 5.5, we see that

$$\mathcal{V}_{v_0}^N(\lambda^{\bar{\varepsilon}R_N}) \cap \mathcal{C}^{N+l} \neq \varnothing.$$

Without loss of generality, assume that

$$\mathcal{V}_{v_0}^N(\lambda^{\bar{\varepsilon}R_N}) \cap \mathcal{C}_0^{N+l} \neq \varnothing.$$

By (4.9) and Proposition A.5, it follows that

$$\operatorname{dist}(v_{R_{N+l}}, \mathcal{C}_0^{N+l}) < \lambda^{\bar{\varepsilon}R_N}.$$

For $m \geq -1$, let

$$a_m^N := \pi_h \circ \Psi^N(v_{mR_N}).$$

Denote

$$F_{N+1} := \Psi^N \circ F^{R_{N+1}} \circ (\Psi^N)^{-1} \quad \text{and} \quad f_{N+1} := \Pi_{1\mathrm{D}}(F_{N+1}).$$

Note that $a_0^N = 0$ and $a_{r_N}^N = f_{N+1}(0)$. Moreover, by (4.9), we see that

$$|a_{(i+1)r_N}^N - f_{N+1}(a_{ir_N}^N)| < \lambda^{(1-\bar{\varepsilon})R_N} \quad \text{for} \quad i \in \mathbb{N}.$$

Define

$$J_0 := (-\lambda^{\bar{\varepsilon}R_N}, a_{r_N}^N + \lambda^{\bar{\varepsilon}R_N}) \quad \text{and} \quad \mathcal{D}_0 := (\Psi^N)^{-1}(J_0 \times I_0^v).$$

Denote $\mathcal{D}_i := F^i(\mathcal{D}_0)$ for $i \in \mathbb{N}$. By the regularity of F, we see that $\mathcal{D}_{iR_N}$ is contained in the $\lambda^{\bar{\varepsilon}R_N}$-neighborhood of $\mathcal{B}_{iR_N}^{N+1}$ for all $1 \leq i \leq \mathbf{b}$. We claim that

$$a_{-1}^N \in \pi_h \circ \Psi^N(\mathcal{D}_{(r_N-1)R_N}). \tag{6.9}$$

Suppose towards a contradiction that we have

$$a_{-1}^N \in \pi_h \circ \Psi^N(\mathcal{D}_{(\hat{r}-1)R_N}) \quad \text{for some} \quad \hat{r} < r_N. \tag{6.10}$$

Proposition B.1 implies the existence of a saddle point $q_0 \in \mathcal{B}_0^{N+1}$ of period dR_{N+1} for some $d \leq r_{N+1}$. For $0 \leq i < R_{N+1}$, let $W^{ss}_{\mathrm{loc}}(q_{iR_N})$ be the connected component of $W^{ss}(q_{iR_N}) \cap \mathcal{B}_0^N$ containing q_{iR_N}. It follows from Propositions 5.4 and 5.5 that $W^{ss}_{\mathrm{loc}}(q_{iR_N})$ is $\lambda^{(1-\bar{\varepsilon})R_N}$-vertical and vertically proper in $\mathcal{B}_0^N$.

By (6.10), either $\pi_h \circ \Psi^N(\mathcal{B}_0^{N+1})$ is contained in a $\lambda^{\bar{\varepsilon}R_N}$-neighborhood of $\pi_h \circ \Psi^N(\mathcal{B}_{\hat{r}R_N}^{N+1})$, or vice versa. In the former case, $W^{ss}(q_0)$ intersects $\mathcal{B}_{\hat{r}R_N}^{N+1}$, and in the latter case, $W^{ss}(q_{\hat{r}R_N})$ intersects $\mathcal{B}_0^{N+1}$. In either case, we have a contradiction. Hence, (6.10) does not hold.

Suppose towards a contradiction that

$$a_{-1}^N \notin \pi_h \circ \Psi^N(\mathcal{D}_{(r_N-1)R_N}) \tag{6.11}$$

For $y \in I_0^v$, let

$$\mathcal{J}_0^y := (\Psi^N)^{-1}(J_0 \times \{y\}) \quad \text{and} \quad \mathcal{J}_i^y := F^i(\mathcal{J}_0^y) \quad \text{for} \quad i \in \mathbb{N}.$$

Arguing inductively using Lemmas 4.8 iii) and 5.1, we see that

$$\mathcal{J}^y_{R_{N+1}} \cap \mathcal{V}^N_{v_0}(\lambda^{\bar{\varepsilon} R_N}) = \varnothing,$$

and $\mathcal{J}^y_{R_{N+1}}$ is $\lambda^{-\bar{\varepsilon} R_N}$-horizontal. Hence, f_{N+1} maps J_0 as a diffeomorphism onto $f_{N+1}(J_0)$.

If f_{N+1} is orientation-reversing, then $a^N_{2r_N}$ and $a^N_{r_N}$ are $\lambda^{(1-\bar{\varepsilon})R_N}$-close to the left and right endpoints of $f_{N+1}(J_0)$ respectively. Lemma 6.3 implies that $\lambda^{\bar{\varepsilon} R_N} < a^N_{2r_N}$. Thus, we see that $f_{N+1}(J_0) \Subset J_0$. We conclude that

$$F^{R_{N+1}}(\mathcal{D}_0) \Subset \mathcal{D}_0 \setminus \mathcal{V}^N_{v_0}(\lambda^{\bar{\varepsilon} R_N}).$$

Applying Proposition 5.5, it follows that $\mathcal{D}_0$ induces a trivial renormalization of $F^{R_N}|_{\mathcal{B}^N_0}$. Then by a similar argument as in the proof of Lemma 6.3, we see that $\mathcal{B}^{N+1}_0$ also induces a trivial renormalization of $F^{R_N}|_{\mathcal{B}^N_0}$. This is a contradiction.

If f_{N+1} is orientation-preserving, then $a^N_{r_N}$ and $a^N_{2r_N}$ are $\lambda^{(1-\bar{\varepsilon})R_N}$-close to the left and right endpoints of $f_{N+1}(J_0)$ respectively. Proceeding inductively on $k \in \mathbb{N}$, suppose that $\mathcal{J}_{kR_{N+1}}$ is $\lambda^{-\bar{\varepsilon} R_N}$-horizontal; and every $p_0 \in \mathcal{D}_0$ is kR_{N+1}-times forward $(CK_0, \bar{\varepsilon}, \lambda)$-times regular along

$$\hat{E}^{v,k}_{p_0} := DF^{-kR_{N+1}}(E^{v,N}_{p_{kR_{N+1}}}).$$

It follows that

$$\operatorname{dist}_{C^r}(\mathcal{J}^y_{kR_{N+1}}, \mathcal{J}^{y'}_{kR_{N+1}}) < \lambda^{(1-\bar{\varepsilon})kR_{N+1}} \quad \text{for} \quad y, y' \in I^v_0.$$

By a similar argument as above used to disprove (6.10), we see that

$$\mathcal{J}^y_{(kr_N+i)R_N} \cap \mathcal{V}^n_{v_0}(\lambda^{\bar{\varepsilon} R_N}) = \varnothing \quad \text{for} \quad i < r_N.$$

Hence, arguing inductively using Lemmas 4.8 iii) and 5.1, it follows that $\mathcal{J}^y_{(k+1)R_{N+1}-1}$ is $\lambda^{(1-\bar{\varepsilon})R_N}$-horizontal.

Let

$$\hat{\mathcal{J}}^y_k := \bigcup_{i=0}^{k-1} \mathcal{J}^y_{iR_{N+1}} \quad \text{and} \quad \hat{J}^y_k := \pi_h \circ \Psi^N(\hat{\mathcal{J}}^y_k).$$

If

$$\mathcal{J}^y_{(k+1)R_{N+1}} \cap \mathcal{V}^n_{v_0}(\lambda^{\bar{\varepsilon} R_N}) \neq \varnothing, \tag{6.12}$$

then $f_{N+1}|_{\hat{J}^y_k}$ is a C^r-map on the interval $\hat{J}^y_k$ that maps $\hat{J}_{k-1}$ as an orientation-preserving diffeomorphism to $f_{N+1}(\hat{J}_{k-1})$, and maps the unique turning point in $\hat{J}_k \setminus \hat{J}_{k-1}$ to an image that is $\lambda^{(1-\bar{\varepsilon})R_N}$-close to 0. This is clearly impossible. It follows by induction that (6.12) does not hold for all $k \in \mathbb{N}$.

Let

$$\mathcal{D} := \bigcup_{i=0}^{\infty} F^{iR_{N+1}}(\mathcal{D}_0) \quad \text{and} \quad \mathcal{A} := \bigcap_{i=0}^{\infty} F^{iR_{N+1}}(\mathcal{D})$$

Then the above observations imply that $\mathcal{A}_0$ is a totally invariant connected set disjoint from $\mathcal{V}^N_{v_0}(\lambda^{\bar{\varepsilon}R_N})$, whose basin contains $\mathcal{D}$. By a similar argument as in the proof of Lemma 6.3, we see that $\mathcal{B}^{N+1}_0$ induces a trivial renormalization of $F^{R_N}|_{\mathcal{B}^N_0}$. This is a contradiction. Hence, we conclude that (6.9) holds. Moreover, arguing as in the proofs of Propositions 6.4 and 6.5, we conclude that $\mathcal{D}_0$ is R_{N+1}-periodic.

For $p_0 \in \mathcal{D}_0$, let

$$E^{v,N+1}_{p_0} := DF^{-R_{N+1}}(E^h_{p_{R_{N+1}}}). \tag{6.13}$$

Then Proposition 5.3 implies that p_0 is R_{N+1}-times forward $(CK_0, \bar{\varepsilon}, \lambda)$-regular along $E^{v,N+1}_{p_0}$. Denote $\mathcal{B}^{N+1}_0 := \mathcal{D}_0$, and let $\Psi^{N+1} : \mathcal{B}^{N+1}_0 \to B^{N+1}_0$ be a genuine horizontal chart that rectifies the vertical direction field given by (6.13). It follows from (6.9) that

$$(F^{R_{N+1}}, \Psi^{N+1} : \mathcal{B}^{N+1}_0 \to B^{N+1}_0)$$

is a Hénon-like return.

It remains to prove that any point $p_0 \in \mathcal{B}^{N+1}_{R_{N+1}}$ is backward $(CK_0, \bar{\varepsilon}, \lambda)$-regular along $E^h_{p_0}$. By the regularity of the Nth Hénon-like return, p_0 is R_N-times backward $(L, \varepsilon, \lambda)$-regular along $E^h_{p_0}$. Proceeding by induction, suppose that for some $1 \leq l < r_{N+1}$, the point p_0 is lR_N-times backward $(CK_0, \bar{\varepsilon}, \lambda)$-regular along $E^h_{p_0}$. By Proposition A.9,

$$E^{v,N+1}_{p_{-lR_N}} := DF^{-lR_N}(E^h_{p_0})$$

is $\lambda^{(1-\bar{\varepsilon})R_N}$-vertical in $\mathcal{B}^N_0$. Arguing as in the proof of Proposition 5.3, we see that

$$\lambda^{\bar{\varepsilon}R_N} < \frac{\|DF^{-i}|_{E^{v,N+1}_{p_{-lR_N}}}\|}{\|DF^{-i}|_{E^h_{p_{-lR_n}}}\|} < \lambda^{-\bar{\varepsilon}R_N} \quad \text{for} \quad 1 \leq i \leq R_N.$$

Concatenating with the lR_N-times backward regularity of p_0, we conclude that p_0 is actually $(l+1)R_N$-times backward $(\bar{C}K_0, \bar{\varepsilon}, \lambda)$-regular along $E^h_{p_0}$ (with $\bar{\varepsilon}$ increased some uniform amount from the lth step). □

7. Critical Recurrence

Let F be the infinitely regularly Hénon-like renormalizable map with combinatorics of bounded type considered in Subsection 6.2 (with $N = \infty$). In this section, we prove the following result.

Theorem 7.1. *We have*

$$\mathcal{Z}_0 := \bigcap_{n=1}^{\infty} \mathcal{B}^n_{R_n} = \{v_0\}.$$

Consequently, the orbit of v_0 is recurrent.

Proof. Let

$$\mathcal{Y}_0 := \bigcap_{n=1}^{\infty} \mathcal{B}^n_0, \quad \mathcal{I}^\infty_0 := \mathcal{I}^h_0 \cap \mathcal{Y}_0 \quad \text{and} \quad I^\infty_0 := \pi_h \circ \Phi_0(\mathcal{I}^\infty_0).$$

Note that every point $p_0 \in \mathcal{Y}_0$ is infinitely forward $(L, \varepsilon, \lambda)$-regular. Moreover, by Proposition 5.4, $W^{ss}(p_0)$ is vertically proper in $\mathcal{B}_0^1$. Let $W^{ss}_{\text{loc}}(p_0)$ be the connected component of $W^{ss}(p_0) \cap \mathcal{B}_0^1$ containing p_0. Then we have

$$\mathcal{Y}_0 = \bigcup_{p_0 \in \mathcal{I}_0^\infty} W^{ss}_{\text{loc}}(p_0).$$

Since $\mathcal{Y}_i := F^i(\mathcal{Y}_0) \subset \mathcal{B}_i^n$ for all $n \in \mathbb{N}$ such that $0 \leq i < R_n$, we see that

$$\mathcal{Y}_i \cap \mathcal{Y}_0 = \varnothing \quad \text{for} \quad i \in \mathbb{N}.$$

We claim that $\mathcal{Y}_0 = W^{ss}_{\text{loc}}(v_0)$. Suppose towards a contradiction that this is not true. By (4.7) and (4.11), this means that there exists a uniform constant $b > 0$ such that $(0, b) \subset I_0^\infty$.

Recall that for $n \in \mathbb{N}$, the curve $\mathcal{I}^n_{R_n}$ is vertical quadratic in $\mathcal{B}_0^n$. Let $c_0^n \in \mathcal{I}_0^n$ be the unique point such that

$$E^{gv}_{\Psi^n(c^n_{R_n})} = D(\Psi^n \circ F^{R_n}) \left(E^h_{c_0^n}\right).$$

By Theorem 4.7 ii), we see that $\Phi_{-1}(\mathcal{I}^n_{R_n - 1})$ is $\lambda^{(1-\bar{\varepsilon})R_n}$-horizontal in $\mathcal{B}_{-1}$. Hence, by Theorem 4.7 i), we have

$$\|c^n_{R_n} - v_0\| < \lambda^{(1-\bar{\varepsilon})R_n}.$$

Let $M \in \mathbb{N}$ be sufficiently large so that for $n \geq M$, we have

$$P_0^n(c^n_{R_n}) < \lambda^{(1-\bar{\varepsilon})R_n} < \lambda^{\bar{\varepsilon}R_M} < b/2.$$

Note that for $0 \leq k < R_n/R_M$, we have

$$\mathcal{B}^n_{kR_M} \cap \mathcal{B}^n_0 = \varnothing.$$

Thus, applying Lemma 5.1 and proceeding by induction, we see that the curve $\mathcal{I}^n_{kR_M}$ is $\lambda^{-\bar{\varepsilon}R_M}$-horizontal in $\mathcal{B}_0^M$, and $\mathcal{I}^n_{(k+1)R_M - 1}$ is $\lambda^{(1-\bar{\varepsilon})R_M}$-horizontal in $\mathcal{B}_{-1}$.

Define $\mathcal{B}^n_{-kR_M}$ for $0 \leq k < R_n/R_M$ inductively as follows. Suppose that

- $F^{kR_M}(\mathcal{B}^n_{kR_M}) \Subset \mathcal{B}^n_0$;
- $\Psi^M(\mathcal{B}^n_{-kR_M})$ is a vertically proper quadrilateral in B_0^M, whose side boundaries are $\lambda^{(1-\bar{\varepsilon})R_M}$-vertical; and
- $\mathcal{B}^n_{-kR_M} \supset \mathcal{I}^n_{R_n - kR_M}$.

Since $\mathcal{B}_0^n$ is R_n-periodic (see Proposition 6.5 iii)), property i) implies that

$$\mathcal{B}^n_{-kR_M} \cap \mathcal{B}^n_{-iR_M} = \varnothing \quad \text{for} \quad 0 \leq i < k.$$

This, together with property ii) ensure that

$$F^{-1}(\mathcal{B}^n_{-kR_M}) \cap \mathcal{B}^M_{R_M - 1}$$

consists of exactly two connected components (unless $k = 0$, in which case there is only one connected component). Let $\mathcal{B}^n_{-kR_M - 1}$ be the component containing $\mathcal{I}^n_{R_n - kR_M - 1}$. Define

$$\mathcal{B}^n_{-(k+1)R_M} := F^{-R_M + 1}(\mathcal{B}^n_{-kR_M - 1}).$$

By Lemma 5.2, we see that

$$\partial\mathcal{B}^n_{-kR_M-1} \setminus \partial\mathcal{B}^M_{R_M-1}$$

consists of two $\lambda^{-\bar{\varepsilon}R_M}$-vertical curves $\Gamma^{n,\pm}_{-kR_M-1}$ in $\mathcal{B}_{-1}$, and

$$\Gamma^{n,\pm}_{-(k+1)R_M} := F^{-R_M+1}(\Gamma^{n,\pm}_{-kR_M-1})$$

are $\lambda^{(1-\bar{\varepsilon})R_M}$-vertical and vertically proper in $\mathcal{B}^M_0$.

Since the sets

$$\mathcal{B}^n_{-(k+1)R_M} \supset \mathcal{I}^n_{R_n-(k+1)R_M} \quad \text{for} \quad 0 \leq k < R_n/R_M$$

are disjoint, the intervals

$$I^n_{kR_M} := P^M_0(\mathcal{I}^n_{kR_M})$$

must be disjoint in I^M_0.

Consider the diffeomorphism h_M given in Theorem 4.7 ii). Define

$$g^n_k(x) := \mathcal{P}^M_0 \circ F \circ (\mathcal{P}^M_{-1}|_{\mathcal{I}^n_{(k+1)R_M-1}})^{-1}(h_M(x), 0) \quad \text{for} \quad x \in I^n_{kR_M}.$$

Since $\mathcal{I}^n_{(k+1)R_M-1}$ and $\mathcal{I}^n_{(k+1)R_M}$ are uniformly horizontal in $\mathcal{B}_{-1}$ and $\mathcal{B}_0$ respectively, it follows that $\|g^n_k\|_{C^r}$ is uniformly bounded. Moreover,

$$\sum_{k=0}^{R_n/R_M-1} |I^n_{kR_M}| < |I^M_0|.$$

Thus, we conclude from Theorem C.1 that

$$G^n := g^n_{R_n/R_M-1} \circ \ldots \circ g^n_0$$

has uniformly bounded distortion.

Let

$$I^{n+1}_{-R_n} = P^M_0(\mathcal{B}^{n+1}_{-R_n}).$$

Then $I^{n+1}_{-R_n}$ and I^{n+1}_0 are disjoint intervals in I^n_0. Moreover, $|I^{n+1}_0|$ is uniformly bounded below, while

$$|I^{n+1}_{-R_n}|, |I^{n+1}_{R_n}| \to 0 \quad \text{as} \quad n \to \infty.$$

However,

$$G^n(I^{n+1}_{-R_n}) = I^{n+1}_0 \quad \text{and} \quad G^n(I^{n+1}_0) = I^{n+1}_{R_n}.$$

This contradicts the fact that G^n has uniformly bounded distortion. The result follows. $\square$

8. A Priori Bounds

Let $r \geq 2$ be an integer, and consider a C^{r+4}-Hénon-like map $F : D \to D$. For some $N \in \mathbb{N} \cup \{\infty\}$; $L \geq 1$ and $\varepsilon, \lambda \in (0,1)$, suppose that F has N nested $(L, \varepsilon, \lambda)$-regular Hénon-like returns given by (2.8) with combinatorics of **b**-bounded type for some integer $\mathbf{b} \geq 3$. By only considering every other returns if necessary, we may also assume without loss of generality that $r_n := R_{n+1}/R_n \geq 3$ for $n_0 \leq n \leq N$. Assume that ε is sufficiently small so that $\mathbf{b}\bar{\varepsilon} < 1$. Also assume that N is sufficiently large, so that for some smallest number $0 \leq n_0 \leq N$, we have (5.1). Lastly, suppose that $F^{R_N}|_{\mathcal{B}_0^N}$ is twice non-trivially topologically renormalizable (so that Proposition 6.5 applies).

The goal of this section is to prove Theorem A stated in Section 3.

8.1. **The outline of strategy.** For $n_0 \leq n \leq N$, consider the horizontal cross-section of the nth renormalization domain $\mathcal{B}_0^n$:

$$\mathcal{I}_0^n := (\Psi^n)^{-1}(I_0^n \times \{0\}) = \mathcal{I}_0^h \cap \mathcal{B}_0^n \ni v_0.$$

See (4.10). We want to prove that $\mathrm{Dis}(F^{R_n}, \mathcal{I}_0^n)$ is uniformly bounded.

The general strategy is to reduce the 2D dynamics of F acting on $\mathcal{I}_0^n$ to a 1D mapping scheme for which standard 1D arguments can be applied to control distortion. Below we give a broad description of this 1D mapping scheme using simpler notations to better convey the main ideas. In the actual proof, the 1D scheme is derived from the 2D dynamics it is modeling, which forces the notations to become more complicated.

Fix some intervals $I_0, I_{-1} \subset \mathbb{R}$. For $1 \leq n \leq N$, let $\{A_i^n\}_{i=0}^{R_n-1}$ and $\{\breve{A}_i^n\}_{i=0}^{R_n-1}$ be collections of pairwise disjoint subintervals in I_0 and I_{-1} respectively so that $0 \in \breve{A}_{R_n-1}^n \subset I_{-1}$. Consider the following mapping scheme for $1 \leq i \leq R_n$:

- a C^2-diffeomorphism
$$\phi_i^n : A_{i-1}^n \to \breve{A}_{i-1}^n := \phi_i^n(A_{i-1}^n)$$
with uniformly bounded C^2-norm; and
- a quadratic power map
$$g_i^n : \breve{A}_{i-1}^n \to A_i^n := g_i^n(\breve{A}_{i-1}^n)$$
given by
$$g_i^n(x) = x^2 + a_i^n \quad \text{for some} \quad a_i^n \in \mathbb{R}.$$

Define

$$H_i := \phi_i^n \circ g_{i-1}^n \circ \phi_{i-1}^n \circ \ldots \circ g_1^n \circ \phi_1^n.$$

Suppose that the domains of ϕ_i^n for $1 \leq i \leq R_n$ can be extended so that $g_i^n \circ H_i$ maps a strictly larger interval $\tilde{A}_{0,i}^n \Supset A_0^n$ diffeomorphically onto an image $g_i^n \circ H_i(\tilde{A}_{0,i}^n)$ that contains the two adjacent neighbors $A_{\iota_-(i)}^n$ and $A_{\iota_+(i)}^n$ of A_i^n (or at least subintervals in $A_{\iota_\pm(i)}^n$ of commensurate lengths). Then we can apply Koebe distortion theorem (see Section C) to conclude that $H_{R_n}|_{A_0^n}$ has uniformly bounded distortion.

We now give a brief description of how the 2D dynamics of F acting on the curve $\mathcal{I}_0^n$ is reduced to the above 1D mapping scheme. The main idea is to weave into the dynamics of F systematic applications of projections near the critical value v_0. This confines the orbit of $\mathcal{I}_0^n$ to lie in a fixed union of curves $\{\mathcal{I}_i^{n_0}\}_{i=0}^{R_{n_0}-1}$. These projections are then "undone" near the critical point v_{-1} to recover the original dynamics. See the definitions of the maps H_i^n and $\hat{H}_i$, as well as Lemmas 8.2 and 8.3. See also Figure 8.

The pairwise disjointedness of the collection of images $\{\mathcal{J}_i^n\}_{i=0}^{R_n-1}$ of $\mathcal{I}_0^n$ under $\hat{H}_i$ relies on the 1D combinatorial structures of the renormalizations of the 2D map F established in Section 6. See Figure 7 and Lemma 8.10.

Contributions by quadratic power maps in the composition $\hat{H}_i$ arise in the following way. When the inverse projection is applied near the critical point v_{-1}, it is onto a nearly horizontal curve (approximating a subarc of a center manifold of v_{-1}). Under one iterate of F, this curve is mapped to a subarc of a vertical quadratic curve near the critical value v_0. Then projecting along a vertical foliation to the transverse horizontal arc $\mathcal{I}_0^{n_0} \supset \mathcal{I}_0^n$ produces the effect of applying a quadratic power map. See Proposition 8.18.

Lastly, the extension of $\hat{H}_i$ to a strictly larger domain $\tilde{\mathcal{I}}_{0,i}^n \Supset \mathcal{I}_0^n$ so that the image $\hat{H}_i(\tilde{\mathcal{I}}_{0,i}^n)$ covers (commensurate portions of) the adjacent neighbors $\mathcal{J}_{\iota_\pm(i)}^n$ of $\mathcal{J}_i^n$ is done in Propositions 8.14 and 8.17.

8.2. **The proof of Theorem A.** First, we need the following lemma (which requires the 3 additional degrees of smoothness assumed in this section). Recall that $P_0^n := \pi_h \circ \Psi^n$ for $n_0 \leq n \leq N$.

Lemma 8.1. *Let $\kappa_F, K_1 > 0$ be the constants given in Theorem 4.7 and* (5.2) *respectively. Consider a C^{r+3}-map $g : I \to \mathbb{R}$ on an interval $I \subset I_{-1}^h$ such that $\|g\|_{C^2} < \underline{\kappa_F}$. Denote $G(x) := (x, g(x))$. Then there exist $a \in I_0^h$ and a C^r-diffeomorphism $\psi : I \to \psi(I)$ with*

$$\|\psi^{\pm 1}\|_{C^r} < K_1(1 + \|g\|_{C^{r+3}})$$

such that we have

$$Q(x) := P_0^n \circ F \circ \Phi_{-1}^{-1} \circ G(x) = \kappa_F \cdot (\psi(x))^2 + a \tag{8.1}$$

where defined.

Proof. By Theorem 4.7 i), it suffices to show that there exists $\tilde{\psi}_g$ with

$$\|\tilde{\psi}^{\pm 1}\|_{C^r} < K_1(1 + \|g\|_{C^{r+3}})$$

such that

$$\tilde{Q}(x) := \pi_h \circ \Phi_0 \circ F \circ \Phi_{-1}^{-1} \circ G(x) = \kappa_F \cdot (\tilde{\psi}(x))^2 + \tilde{a}.$$

By (4.7), we have $\tilde{Q} = f_0 - \lambda \cdot g$. By the bound on $\|g\|_{C^2}$, we see that $\tilde{Q}$ has a unique critical point, $\tilde{Q}''(x)$ is bounded below by $c\kappa_F$ for some uniform constant $c > 0$, and

$$\|\tilde{Q}''\|_{C^{r+1}} < K_1(1 + \|g\|_{C^{r+3}}).$$

The result now follows from Lemma D.4. $\square$

For $n_0 \leq n \leq N$, define a sequence of maps $\{H_i^n\}_{i=0}^\infty$ as follows. First, let $H_i^{n_0} := F^i$. Proceeding inductively, suppose H_i^{n-1} is defined. Write $i = j + kR_n$ with $k \geq 0$ and $0 \leq j < R_n$. Define

$$H_i^n := H_j^{n-1} \circ \mathcal{P}_0^n \circ F^{kR_n},$$

where

$$\mathcal{P}_0^n := (\Psi^n)^{-1} \circ \Pi_h \circ \Psi^n$$

is the nth projection map near the critical value v_0. Observe that H_i^n is well-defined on $\mathcal{B}_0^n$.

Lemma 8.2. *Let $s \in \{1, 2\}$ and $n_0 \leq n \leq N - s$. Then $H_i^n|_{\mathcal{I}_1^{n+s}}$ is a diffemorphism for $0 \leq i < R_{n+s}$.*

Proof. The statement is clearly true for $n = n_0$. Suppose the statement is true for $n-1$. If $i < R_n$, then

$$H_i^n|_{\mathcal{I}_1^{n+s}} = H_i^{n-1}|_{\mathcal{I}_1^{n+s}}$$

is a diffeomorphism. Suppose the same is true for $i < (k-1)R_n$ with $2 \leq k < R_{n+s}/R_n$. Observe that

$$H_{kR_n}^n = \mathcal{P}_0^n \circ F^{kR_n}.$$

By Lemma 6.6 i), the map $\mathcal{P}_0^n|_{\mathcal{I}_{kR_n}^{n+s}}$ is a diffeomorphism. For $i = j + kR_n$ with $j < R_n$, we have

$$H_i^n := H_j^{n-1} \circ \mathcal{P}_0^n \circ F^{kR_n}.$$

Since

$$\mathcal{P}_0^n(\mathcal{I}_{kR_n}^{n+s}) \subset \mathcal{I}_0^n,$$

the result follows. $\square$

Recall the definition of $\mathcal{P}_{-1}^n$ for $n_0 \leq n \leq N$ given in (4.12).

Lemma 8.3. *For $s \in \{1, 2\}$ and $n_0 \leq n \leq N - s$, let Γ_0 be a C^r-curve which is $\lambda^{-\bar{\varepsilon}R_n}$-horizontal in $\mathcal{B}_0^{n+s}$. Then for $1 \leq k \leq R_{n+s}/R_n$, we have*

$$F^{kR_n-1}|_{\Gamma_0} = \left(\mathcal{P}_{-1}^{n_0}|_{\Gamma_{kR_n-1}}\right)^{-1} \circ H_{kR_n-1}^n|_{\Gamma_0}.$$

Proof. If $k = 1$, then the result follows immediately from Lemma 4.9. Suppose the result is true for some $n_0 \leq n < N - s$ and $1 \leq k < R_{n+s}/R_n$. By definition, we have

$$H_{(k+1)R_n-1}^n = H_{kR_n-1}^n \circ F^{R_n}.$$

If Γ_0 is a C^r-curve which is $\lambda^{-\bar{\varepsilon}R_n}$-horizontal in $\mathcal{B}_0^{n+s}$, then by Lemma 6.6 i), we see that $\Gamma_{R_n} := F^{R_n}(\Gamma_0)$ is a C^r-curve which is $\lambda^{-\bar{\varepsilon}R_n}$-horizontal in $\mathcal{B}_0^n$. Thus, by induction, we have

$$F^{kR_n-1}|_{\Gamma_{R_n}} = \left(\mathcal{P}_{-1}^{n_0}|_{\Gamma_{(k+1)R_n-1}}\right)^{-1} \circ H_{kR_n-1}^n|_{\Gamma_{R_n}}.$$

Composing on the right by $F^{R_n}|_{\Gamma_0}$, the result is true in this case.

Finally, suppose that the result is true for some $n_0 \le n < N - s$ and $k = R_{n+1}/R_n$. Let $\gamma_0 := \mathcal{P}_0^{n+1}(\Gamma_0)$. By the induction hypothesis, we have:

$$F^{R_{n+1}-1}|_{\gamma_0} = \left(\mathcal{P}_{-1}^{n_0}|_{\gamma_{R_{n+1}-1}}\right)^{-1} \circ H^n_{R_{n+1}-1}|_{\gamma_0}.$$

Applying Lemma 4.9:

$$\begin{aligned} F^{R_{n+1}-1}|_{\Gamma_0} &= \left(\mathcal{P}_{-1}^{n+1}|_{\Gamma_{R_{n+1}-1}}\right)^{-1} \circ \left(\mathcal{P}_{-1}^{n_0}|_{\gamma_{R_{n+1}-1}}\right)^{-1} \circ H^n_{R_{n+1}-1} \circ \mathcal{P}_0^{n+1}|_{\Gamma_0} \\ &= \left(\mathcal{P}_{-1}^{n_0}|_{\Gamma_{R_{n+1}-1}}\right)^{-1} \circ H^{n+1}_{R_{n+1}-1}|_{\Gamma_0}. \end{aligned}$$

□

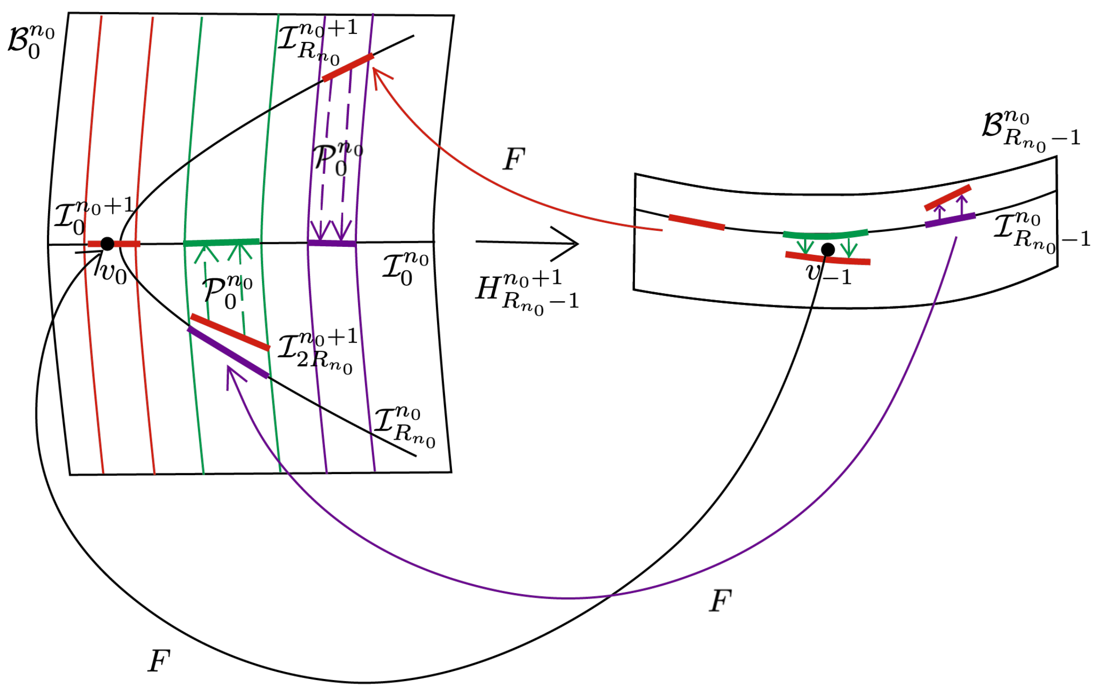

Figure 8. Visualization of the map $H_i^{n_0}$ for $0 \le i < R_{n_0+1}$ acting on the horizontal curve $\mathcal{I}_0^{n_0+1} \subset \mathcal{I}_0^{n_0}$ (for $r_{n_0} := R_{n_0+1}/R_{n_0} = 3$). The orbit of $\mathcal{I}_0^{n_0+1}$ makes returns to $\mathcal{B}_0^{n_0} \ni v_0$ under F^{kR_n} for $0 \le k < r_{n_0}$. At these moments, the projection map $\mathcal{P}_0^{n_0}$ is applied to $\mathcal{I}_{kR_{n_0}}^{n_0+1}$ to bring it down to $\mathcal{I}_0^{n_0}$. These projections are then "undone" in $\mathcal{B}_{R_{n_0}-1}^{n_0} \ni v_{-1}$ to return to $\mathcal{I}_{(k+1)R_{n_0-1}}^{n_0+1}$. For $n > n_0$, the multiple projections (at various depths) can be applied to the orbit of $\mathcal{I}_0^n$ near v_0 before they are undone near v_{-1}.

We also define another sequence of maps $\{\hat{H}_i\}_{i=0}^{R_N-1}$ as follows (if $N = \infty$, then $R_N = \infty$). If $i < 2R_{n_0}$, let $\hat{H}_i := F^i$. Otherwise, let $n_0 \leq n < N$ be the largest number such that $i \geq 2R_n$, and define $\hat{H}_i := H_i^n$. Observe that by Lemma 8.3, we have

$$\hat{H}_{R_n-1}|_{\mathcal{I}_0^n} = H_{R_n-1}^{n-1}|_{\mathcal{I}_0^n} = \mathcal{P}_{-1}^{n_0}|_{\mathcal{I}_{R_{n-1}}^n} \circ F^{R_n-1}|_{\mathcal{I}_0^n}. \tag{8.2}$$

Remark 8.4. In the definition of $\hat{H}_i := H_i^n$, we set n to be the largest number such that $i \geq 2R_n$ rather than $i \geq R_n$ for the following technical reason. Observe that

$$H_{R_n}^n(\mathcal{I}_0^{n+1}) = \mathcal{P}_0^n(\mathcal{I}_{R_n}^{n+1}).$$

The domain of $\mathcal{P}_0^n$ is equal to $\mathcal{B}_0^n$, whose right vertical boundary is $\lambda^{\bar{\varepsilon}R_n}$ distance away from $\mathcal{I}_{R_n}^{n+1}$. Hence, $H_{R_n}^n|_{\mathcal{I}_0^{n+1}}$ does not extend to a horizontal curve $\tilde{\mathcal{I}}_{0,i}^{n+1}$ substantially larger than $\mathcal{I}_0^{n+1}$ (so that its image would cover the adjacent neighbors of $H_{R_n}^n(\mathcal{I}_0^{n+1})$), since if it did, then $F^{R_n}(\tilde{\mathcal{I}}_{0,i}^{n+1})$ would lie outside of the domain $\mathcal{B}_0^n$ of $\mathcal{P}_0^n$.

The remainder of the section is devoted to the proof of the following theorem, whose corollary immediately implies Theorem A.

Theorem 8.5. *There exists a uniform constant*

$$\mathbf{K} = \mathbf{K}(L, \lambda, \varepsilon, \lambda^{1-\varepsilon}\|DF^{-1}\|, \|DF\|_{C^5}, \|F^{R_{n_0}}|_{\mathcal{B}^{n_0}}\|_{C^6}, \kappa_F) \geq 1$$

such that for all $n_0 \leq n \leq N$, we have

$$\mathrm{Dis}(\hat{H}_i, \mathcal{I}_0^n) < \mathbf{K} \quad \textit{for} \quad 0 \leq i < R_n.$$

Corollary 8.6. *For $n_0 \leq n \leq N$, let $h_n : I_0^n \to h_n(I_0^n)$ be the diffeomorphism given in Theorem 4.7 ii). Then* $\mathrm{Dis}(h_n, I_0^n) < \mathbf{K}$*, where $\mathbf{K} > 1$ is the uniform constant given in Theorem 8.5.*

Observe that any number $2R_{n_0} \leq i < R_N$ can be uniquely expressed as

$$i = j + a_{n_0}R_{n_0} + a_{n_0+1}R_{n_0+1} + \ldots + a_nR_n$$

for some $n_0 \leq n < N$, where

i) $0 \leq j < R_{n_0}$;
ii) $0 \leq a_m < r_m$ for $n_0 \leq m < n$; and
iii) $2 \leq a_n < 2r_n$.

In this case, we denote

$$i := j + [a_{n_0}, a_{n_0+1}, \ldots, a_n].$$

We extend this notation to $i < 2R_{n_0}$ by writing

$$i = j + [a_{n_0}] \quad \text{for some} \quad a_{n_0} \in \{0, 1\}$$

We record the following easy observation.

Lemma 8.7. *Let $2R_{n_0} \leq i < R_N$ be given by*

$$i = j + [a_{n_0}, \dots, a_n].$$

Then we have

$$\hat{H}_i = H_i^n = F^j \circ \left(\mathcal{P}_0^{n_0} \circ F^{a_{n_0} R_{n_0}}\right) \circ \dots \circ \left(\mathcal{P}_0^n \circ F^{a_n R_n}\right).$$

For $n_0 \leq n \leq N$, we define a collection of arcs $\{\mathcal{J}_i^n\}_{i=0}^{R_n - 1}$ by

$$\mathcal{J}_i^n := \hat{H}_i(\mathcal{I}_0^n) \quad \text{for} \quad 0 \leq i < R_n. \tag{8.3}$$

See Figure 9.

Lemma 8.8. *Let $n_0 \leq n \leq N$ and $0 \leq i < R_n$. If*

$$i = [0, \dots, 0, a_m, a_{m+1}, \dots, a_k]$$

for some $n_0 \leq m \leq k < n$, then we have $\mathcal{J}_i^n \subset \mathcal{I}_0^m$. Moreover, we have

$$\mathcal{J}_{i+l}^n = H_l^{m-1}(\mathcal{J}_i^n) \quad \textit{for} \quad 0 \leq l < R_m.$$

Proof. Observe that

$$\mathcal{P}_1^k \circ F^{a_k R_k}(\mathcal{I}_1^{k+1}) \subset \mathcal{I}_1^k.$$

By Lemma 8.7, the result follows from induction. □

Lemma 8.9. *For $n_0 \leq n \leq N$ and $0 \leq i < R_n$, we have $\mathcal{J}_i^n \subset \mathcal{I}^{n_0}_{i \pmod{R_{n_0}}}$.*

Proof. The result follows immediately from Lemma 8.8. □

Let $\Gamma : [0,1] \to \mathbb{R}^2$ be a parameterized Jordan arc. For

$$0 \leq a < b < c < d \leq 1,$$

consider the subarcs $\Gamma_1 := \Gamma(a,b)$ and $\Gamma_2 := \Gamma(c,d)$ of Γ. We denote $\Gamma_1 <_\Gamma \Gamma_2$. Let Γ_3 be another subarc of Γ. We denote $\Gamma_1 \leq_\Gamma \Gamma_3$ if either $\Gamma_1 <_\Gamma \Gamma_3$ or $\Gamma_1 = \Gamma_3$.

Henceforth, we consider $\mathcal{I}_0^{n_0}$ with parameterization given by

$$\mathcal{I}_0^{n_0}(t) := (\Psi^{n_0})^{-1}(t, 0) \quad \text{for} \quad t \in I_0^{n_0}.$$

Note that $\mathcal{I}_0^{n_0} \circ P_0^{n_0} = \mathcal{P}_0^{n_0}$. Moreover,

$$P_0^{n_0}(v_0) = 0 < P_0^{n_0}(v_{R_{n_0}}).$$

Lemma 8.10. *For $s \in \{1, 2\}$; $n_0 \leq n \leq N - s$ and $1 < k < R_{n+s}/R_n$, we have*

$$\mathcal{J}_0^{n+s} <_{\mathcal{I}_0^{n_0}} \mathcal{J}_{kR_n}^{n+s} <_{\mathcal{I}_0^{n_0}} \mathcal{J}_{R_n}^{n+s}.$$

Proof. Observe that

- For $s \in \{1, 2\}$:

$$\mathcal{J}_{R_n}^{n+s} = H_{R_n}^{n-1}(\mathcal{I}_0^{n+s}) = \mathcal{P}_0^{n-1} \circ F^{R_n}(\mathcal{I}_0^{n+s}).$$

- For $1 < k < sr_n$:

$$\mathcal{J}_{kR_n}^{n+s} = H_{kR_n}^n(\mathcal{I}_0^{n+s}) = \mathcal{P}_0^n \circ F^{kR_n}(\mathcal{I}_0^{n+s}).$$

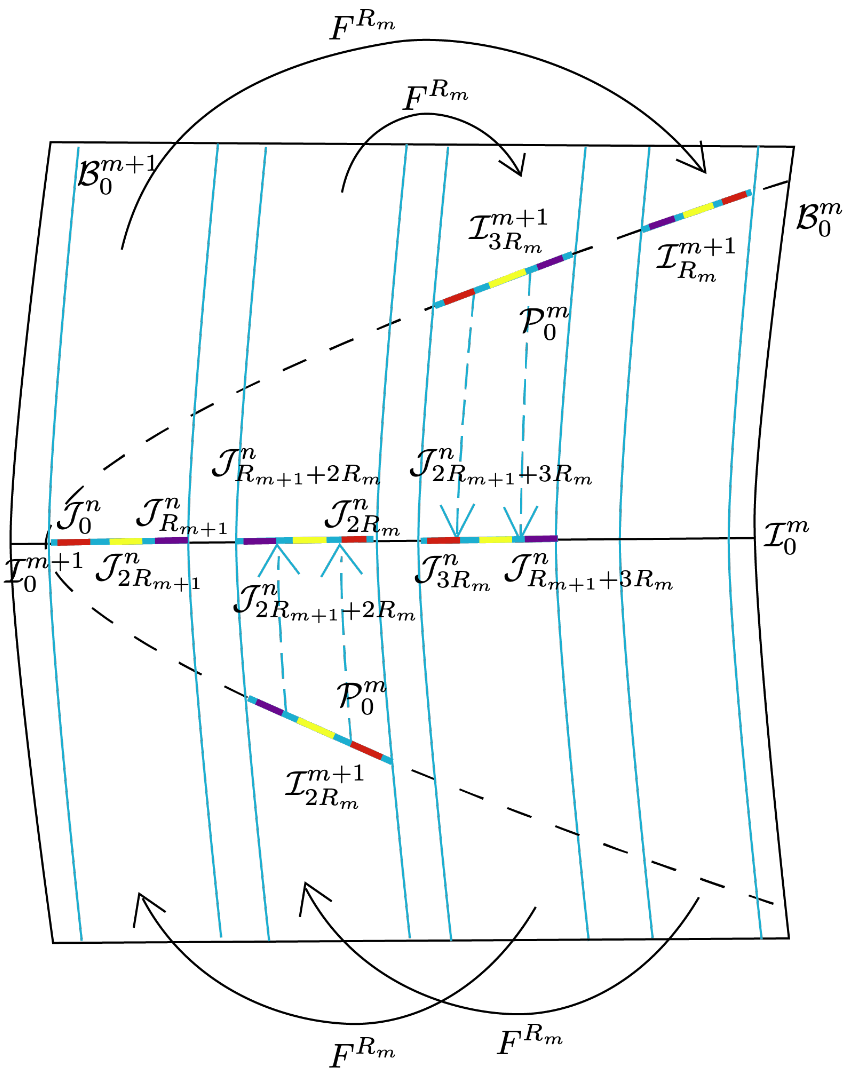

Figure 9. Arcs $\mathcal{J}_i^n := \hat{H}_i(\mathcal{I}_0^n)$ with $0 \leq i < R_n$ that are contained in $\mathcal{I}_0^m$ for some $m < n$. For $0 \leq k < r_{m+1}$, we have $\mathcal{J}_{kR_{m+1}}^n \subset \mathcal{I}_0^{m+1}$. For $2 \leq l < r_m$, we have $\mathcal{J}_{kR_{m+1}+lR_m}^n = \mathcal{P}_0^m \circ F^{R_m}(\mathcal{J}_{kR_{m+1}}^n)$.

In the case $s = 1$, and the case $s = 2$ and $1 < k < 2r_n$ follow immediately from Proposition 6.5.

Replacing n by $n+1$ and applying the above conclusion, we see that for $1 < l < r_{n+1}$:

$$\mathcal{J}_0^{n+2} <_{\mathcal{I}_0^{n_0}} \mathcal{J}_{lR_{n+1}}^{n+2} <_{\mathcal{I}_0^{n_0}} \mathcal{J}_{R_{n+1}}^{n+2}.$$

Note that for $2 < k < r_n$:

$$\mathcal{J}_{lR_{n+1}+kR_n}^{n+2} = H_{kR_n}^n|_{\mathcal{I}_0^{n+1}}(\mathcal{J}_{lR_{n+1}}^{n+2}).$$

The result now follows from Lemma 8.2. □

Let $\Gamma_0 : [0, |\Gamma_0|] \to \mathbb{R}^2$ be a C^1-curve parameterized by its arclength. Let $\Gamma_1 := \Gamma_0(l, |\Gamma_0| - l)$ for some $0 < l < |\Gamma_0|/2$ be a subarc of Γ_0. We denote $\Gamma_1 = \Gamma_0[-l]$ and $\Gamma_0 = \Gamma_1[+l]$. Let Γ_2 be a C^1-curve such that $\Gamma_1 \subset \Gamma_2 \subset \Gamma_0$. We denote

$$\Gamma_0\{-l\} = \Gamma_2 = \Gamma_1\{+l\}.$$

Lastly, if Γ_3 and Γ_4 are C^1-curves in $\mathbb{R}^2$ and we have $\Gamma_3[-l] \subset \Gamma_4 \subset \Gamma_3[+l]$, then we denote $\Gamma_4 = \Gamma_3\{\sim l\}$. See Figure 10. These notations can be extended to intervals in $\mathbb{R}$ in the obvious way.

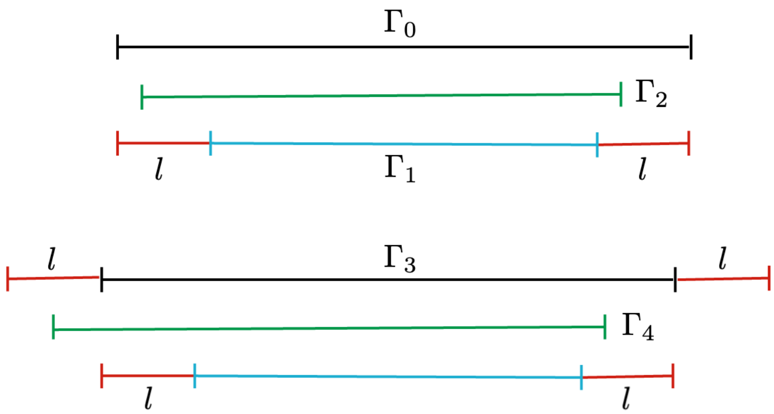

Figure 10. Illustration of the relations between $\Gamma_0 = \Gamma_1[+l]$, $\Gamma_1 = \Gamma_0[-l]$, $\Gamma_0\{-l\} = \Gamma_2 = \Gamma_1\{+l\}$ (above); and Γ_3 and $\Gamma_4 = \Gamma_3\{\sim l\}$ (below).

Let $n_0 < n \leq N$, and consider the collection of arcs $\{\mathcal{J}_i^n\}_{i=0}^{R_n-1}$. By Lemma 8.9 and Lemma 8.10, for $2R_{n_0} \leq i < R_n$, there exist unique numbers $0 \leq \iota_-^n(i), \iota_+^n(i) < R_n$ such that

$$\iota_\pm^n(i) = i \ (\text{mod } R_{n_0}),$$

and the arcs $\mathcal{J}_{\iota_-^n(i)}^n$ and $\mathcal{J}_{\iota_+^n(i)}^n$ are the two nearest neighbors of $\mathcal{J}_i^n$ (one on each side) in $\mathcal{I}_{i \,(\text{mod } R_{n_0})}^{n_0}$. Define $\hat{\mathcal{J}}_i^n$ as the convex hull of $\mathcal{J}_{\iota_-^n(i)}^n \cup \mathcal{J}_i^n \cup \mathcal{J}_{\iota_+^n(i)}^n$ in $\mathcal{I}_{i \,(\text{mod } R_{n_0})}^{n_0}$.

We also define a subarc $\tilde{\mathcal{J}}_i^n$ of $\mathcal{I}_{i \,(\text{mod } R_{n_0})}^{n_0}$ containing $\mathcal{J}_i^n$ as follows. Write

$$i = j + [a_{n_0}, a_{n_0+1}, \dots, a_m]$$

for some $n_0 \leq m < n$. If $m < n - 1$, define

$$\tilde{\mathcal{J}}_i^n := \hat{\mathcal{J}}_i^n[+\lambda^{\bar{\varepsilon} R_m}].$$

Otherwise, define

$$\tilde{\mathcal{J}}_i^n := \hat{\mathcal{J}}_i^n[-\lambda^{\bar{\varepsilon} R_{n-1}}].$$

Proposition 8.11. *There exists a uniform constant $K > 0$ such that for $n_0 \leq n \leq N$, we have*
$$\sum_{i=2R_{n_0}}^{R_n-1} |\tilde{\mathcal{J}}_i^n| < K.$$

Proof. Observe that
$$\sum_{i=2R_{n_0}}^{R_n-1} |\tilde{\mathcal{J}}_i^n| < \sum_{i=2R_{n_0}}^{R_n-1} |\hat{\mathcal{J}}_i^n| + \sum_{m=n_0}^{n-1} 2R_{m+1}\lambda^{\bar{\varepsilon}R_m}.$$
By Lemma 8.10, the maximum number of overlaps among arcs in $\{\hat{\mathcal{J}}_i^n\}_{2R_{n_0}}^{R_n-1}$ is three. Hence, the above sum has a uniform upper bound. □

Lemma 8.12. *For $n_0 \leq n \leq N$, let $\Gamma_0 \subset \mathcal{I}_0^n$ be an arc. Then we have*
$$K_0{}^{-1}\lambda^{\bar{\varepsilon}i} < \frac{|H_i^n(\Gamma_0)|}{|\Gamma_0|} < K_0\lambda^{-\bar{\varepsilon}i} \quad \textit{for} \quad 0 \leq i < R_n,$$
where $K_0 \geq 1$ is the uniform constant given in (4.2).

Proof. For $p_0 \in \Gamma_0$, let $E_{p_0} \in \mathbb{P}^2_{p_0}$ be the direction tangent to Γ_0 at p_0. Note that p_0 is R_n-times forward $(L, \varepsilon, \lambda)$-regular along $E^v_{p_0}$. Thus, by Theorems 4.7 and A.2, and Corollary A.8, we have
$$K_0{}^{-1}\lambda^{\bar{\varepsilon}l} < \|DF^l|_{E_{p_0}}\| < K_0\lambda^{-\bar{\varepsilon}l} \quad \text{for} \quad 0 \leq l < R_n.$$

By Proposition 6.5 and Lemma 6.6 i), the curve $\Gamma_{kR_m} := F^{kR_m}(\Gamma_0)$ for $0 \leq k < r_m$ is $\lambda^{-\bar{\varepsilon}R_m}$ horizontal in $\mathcal{B}^m_0$. Hence, by Theorem 4.7, we see that
$$K_0{}^{-1}\lambda^{\bar{\varepsilon}R_m} < \|D\mathcal{P}^m_0|_{E_{p_{kR_m}}}\| < K_0.$$

Write
$$i = j + [a_{n_0}, \ldots, a_m]$$
for some $n_0 \leq m < n$. Then by Lemma 8.7 we have
$$H_i^n = F^j \circ \mathcal{P}_0^{n_0} \circ F^{a_{n_0}R_{n_0}} \circ \ldots \circ \mathcal{P}_0^m \circ F^{a_mR_m}.$$
Concatenating the previous estimates, we obtain the desired result. □

Lemma 8.13. *For $s \in \{1, 2\}$; $n_0 \leq n \leq N - s$ and $2 \leq k < 2r_n$, let $X_{-1} \subset \mathcal{B}^n_{R_n-1}$ be a set such that*
$$\mathcal{P}^{n_0}_{-1}(X_{-1}) = \mathcal{J}^{n+s}_{kR_n-1}.$$
Then
$$\mathcal{P}^n_0 \circ F(X_{-1}) = \mathcal{J}^{n+s}_{kR_n}\{\sim \lambda^{(1-\bar{\varepsilon})R_n}\}.$$

Proof. By Lemma 8.3, we have
$$\mathcal{I}^{n+s}_{kR_n-1} = \left(\mathcal{P}^{n_0}_{-1}|_{\mathcal{I}^{n+s}_{kR_n-1}}\right)^{-1}(\mathcal{J}^{n+s}_{kR_n-1}) = \left(\mathcal{P}^{n_0}_{-1}|_{\mathcal{I}^{n+s}_{kR_n-1}}\right)^{-1} \circ \mathcal{P}^{n_0}_{-1}(X_{-1}).$$

Since

$$\mathcal{J}^{n+s}_{kR_n} = \mathcal{P}^n_0 \circ F(\mathcal{I}^{n+s}_{kR_n-1}),$$

the claim follows from (4.7) and (4.11). □

Proposition 8.14. *For $n_0 \leq n \leq N-2$ and $2R_n \leq i < 2R_{n+1}$, there exists an arc $\mathcal{K}_{0,i}$ containing $\mathcal{I}^{n+2}_0$ such that the following properties are satisfied.*

i) We have $\mathcal{K}_{0,i} \supset \mathcal{K}_{0,i+1}$.

ii) The map $\hat{H}_i|_{\mathcal{K}_{0,i}}$ is a diffeomorphism.

iii) We have $\hat{H}_i(\mathcal{K}_{0,i}) \supset \tilde{\mathcal{J}}^{n+1}_i$.

iv) Denote $\mathcal{K}_i := F^i(\mathcal{K}_{0,i})$. Then for $2 < k \leq 2r_n$, the arc $\mathcal{K}_{kR_n-1}$ is $\lambda^{(1-\bar{\varepsilon})R_n}$-horizontal in $\mathcal{B}_{-1}$, and

$$\mathcal{K}_{kR_n} \subset \mathcal{B}^n_{R_n} \setminus \mathcal{V}_{v_0}(\lambda^{\bar{\varepsilon}R_n}).$$

Proof. We first extend $\mathcal{I}^{n_0+1}_{2n_0-1}$ to an arc $\mathcal{K}_{2R_{n_0}-1} \subset \mathcal{B}_{-1}$ such that $\mathcal{K}_{2R_{n_0}-1}$ is $\lambda^{(1-\bar{\varepsilon})R_{n_0}}$-horizontal in $\mathcal{B}_{-1}$, and the curve $\mathcal{K}_{2R_{n_0}} := F(\mathcal{K}_{2R_{n_0}-1})$ maps diffeomorphically onto $\mathcal{I}^{n_0}_0 \setminus \mathcal{V}_{v_0}(\lambda^{\bar{\varepsilon}R_{n_0}})$ under $\mathcal{P}^{n_0}_0|_{\mathcal{K}_{2R_{n_0}}}$. We define

$$\mathcal{K}_{0,2R_{n_0}} := F^{-2R_{n_0}}(\mathcal{K}_{2R_{n_0}}).$$

Proceeding by induction, suppose the result holds for $i \leq (k-1)R_n$ with $2 < k \leq 2r_n$. For $0 \leq l < R_n$, define

$$\mathcal{K}_{0,(k-1)R_n+l} := \mathcal{K}_{0,(k-1)R_n}.$$

Observe that

$$\hat{H}_{(k-1)R_n+l} = H^n_l \circ F^{(k-1)R_n}.$$

Thus, property ii) follows from Lemma 8.2; property iii) follows from Lemmas 8.8 and 8.12; and property iv) for $\mathcal{K}_{kR_n-1}$ follows from Lemma 6.6 ii).

If $k < 2r_n$, then define $\mathcal{K}_{kR_n}$ to be the component of $F(\mathcal{K}_{kR_n-1}) \setminus \mathcal{V}_{v_0}(\lambda^{\bar{\varepsilon}R_n})$ containing $\mathcal{I}^{n+2}_{kR_n}$. By Lemma 6.6 i), $\mathcal{K}_{kR_n}$ maps injectively under $\mathcal{P}^n_0$. Lastly, property iii) follows from Lemma 8.13.

If $k = 2r_n$, then define $\mathcal{K}_{2R_{n+1}}$ to be the component of

$$F(\mathcal{K}_{2R_{n+1}-1}) \cap \left(\mathcal{B}^{n+1}_0 \setminus \mathcal{V}_{v_0}(\lambda^{\bar{\varepsilon}R_{n+1}})\right)$$

containing $\mathcal{I}^{n+3}_{2R_{n+1}}$. Properties ii) and iii) for $\mathcal{K}_{2R_{n+1}}$ can be checked similarly as above. □

By Lemma 8.10, for $n_0 \leq n \leq N-2$, there exists a unique number $2 \leq \chi_n < r_n$ such that

$$\mathcal{J}^{n+1}_0 <_{\mathcal{I}^{n_0}_0} \mathcal{J}^{n+1}_{\chi_n R_n} \leq_{\mathcal{I}^{n_0}_0} \mathcal{J}^{n+1}_{kR_n} \quad \text{for all} \quad 1 \leq k < r_n.$$

After relabelling $\iota^n_\pm$ if necessary, the following results hold.

Lemma 8.15. *Let $n_0 \leq n \leq N-2$. Then*

$$\iota^{n+1}_+(i) = i + \chi_n R_n \quad \textit{for} \quad 2R_{n_0} \leq i < R_n.$$

Proof. The claim follows immediately from Lemmas 8.2 and 8.8. □

Lemma 8.16. *Let $n_0+2 \le n \le N$. For $1 \le m \le n-2$ and $2 \le k < 2r_m$, there exists $1 \le i < 2r_m$ such that*

$$\iota_-^n(kR_m) = \iota_-^{m+2}(kR_m) = iR_m.$$

Proof. By Lemmas 8.10, 8.2 and 8.8, we see that the extremal intervals in $\mathcal{J}^{m+1}_{lR_m}$ for $0 \le l < r_m$ are $\mathcal{J}^n_{lR_m}$ and $\mathcal{J}^n_{lR_m+R_{m+1}}$. Moreover, by Lemma 8.15, we have

$$\mathcal{J}^n_{\iota_+^n(lR_m+jR_{m+1})} \subset \mathcal{J}^{m+1}_{lR_m} \quad \text{for} \quad j \in \{0,1\}.$$

The claim follows. □

Proposition 8.17. *For $n_0+2 \le n \le N$ and $2R_{n_0} \le i < R_n$, there exists an arc $\tilde{\mathcal{I}}^n_{0,i}$ such that the following conditions hold for all $2R_{n_0} \le j \le i$.*

i) We have $\mathcal{I}^n_0 \subset \tilde{\mathcal{I}}^n_{0,i} \subset \mathcal{K}_{0,i}$.
ii) Denote

$$\tilde{\mathcal{J}}^n_{j,i-j} := \hat{H}_j(\tilde{\mathcal{I}}^n_{0,i}).$$

Then we have

$$\tilde{\mathcal{J}}^n_{j,i-j} \subset \tilde{\mathcal{J}}^n_j \quad \textit{and} \quad \tilde{\mathcal{J}}^n_{i,0} \supset \tilde{\mathcal{J}}^n_i.$$

Proof. First consider the case when $i < 2R_{n-1}$. Proceeding by induction, suppose that the result is true for $j \le kR_m$ with $n_0 \le m \le n-2$ and $2 \le k < 2r_m$. Then the result holds for $kR_m < j < (k+1)R_m$ by Lemmas 8.2 and 8.8.

Note that we have,

$$\mathcal{P}^m_0(\mathcal{K}_{kR_m}) \supset \tilde{\mathcal{J}}^{m+2}_{kR_m} \supset \mathcal{J}^{m+2}_{\iota_-^{m+2}(kR_m)} \cup \mathcal{J}^{m+2}_{kR_m} \cup \mathcal{J}^{m+2}_{\iota_+^{m+2}(kR_m)},$$

where by Lemmas 8.15 and 8.16, we have

$$\mathcal{J}^{m+2}_{\iota_-^{m+2}(kR_m)} = \mathcal{J}^{m+2}_{\iota_-^{n}(kR_m)} \supset \mathcal{J}^{n}_{\iota_-^{n}(kR_m)} \quad \text{and} \quad \mathcal{J}^{m+2}_{kR_m} \supset \mathcal{J}^n_{kR_m} \cup \mathcal{J}^n_{\iota_+^n(kR_m)}.$$

Hence, there exists an arc $\mathcal{I}'_{kR_m} \subset \mathcal{K}_{kR_m}$ such that

$$\mathcal{P}^m_0(\mathcal{I}'_{kR_m}) = \tilde{\mathcal{J}}^{m+2}_{kR_m}.$$

By Lemmas 8.12 and 8.3, we have

$$\mathcal{P}^{n_0}_{-1} \circ F^{R_m-1}(\mathcal{I}'_{kR_m}) = \hat{\mathcal{J}}^{m+2}_{(k+1)R_m-1}[+\lambda^{\bar{\varepsilon}R_m}].$$

Thus, by Lemmas 8.13 and 8.15, we see that

$$\mathcal{P}^m_0 \circ F^{R_m}(\mathcal{I}'_{kR_m}) \supset \hat{\mathcal{J}}^{m+2}_{(k+1)R_m},$$

and hence, the result holds for $j = (k+1)R_m$.

Next, consider the case when $i \ge 2R_{n-1}$. For $j < 2R_{n-1}$, the result follows by the same argument as in the previous case. Proceeding by induction, suppose that the result is true for $j \le kR_{n-1}$ with $2 \le k < r_{n-1}$. Then the result holds for $kR_{n-1} < j < (k+1)R_{n-1}$ by Lemmas 8.2, 8.8 and Lemma 8.12.

Similar to the previous case, there exists an arc $\mathcal{I}'_{kR_{n-1}} \subset \mathcal{K}_{kR_{n-1}}$ such that

$$\mathcal{P}^{n-1}_0(\mathcal{I}'_{kR_{n-1}}) \supset \tilde{\mathcal{J}}^n_{kR_{n-1}}$$

and

$$\mathcal{P}^{n_0}_{-1} \circ F^{R_{n-1}-1}(\mathcal{I}'_{kR_{n-1}}) = \hat{\mathcal{J}}^{m+2}_{(k+1)R_{n-1}-1}[-\lambda^{\bar{\varepsilon}R_n}].$$

Let $\mathcal{I}''_{(k+1)R_{n-1}}$ be the connected component of

$$F(\mathcal{I}'_{(k+1)R_{n-1}}) \setminus \mathcal{V}_{v_0}(\lambda^{\bar{\varepsilon}R_n})$$

containing $\mathcal{I}^n_{(k+1)R_{n-1}}$. By Lemma 8.13, we have

$$\mathcal{P}^{n-1}_0(\mathcal{I}''_{(k+1)R_{n-1}}) \supset \hat{\mathcal{J}}^n_{(k+1)R_{n-1}}[-\lambda^{\bar{\varepsilon}R_n}].$$

Thus, the result holds for $j = (k+1)R_{n-1}$. □

Let $i \geq 2R_{n_0}$ be a number given by

$$i = [0, \dots, 0, a_m, a_{m+1}, \dots, a_k]$$

for some $n_0 \leq m \leq k$ so that $a_m > 0$. Denote

$$\hat{m}(i) := m, \quad \hat{k}(i) := k \quad \text{and} \quad \hat{a}(i) := a_m.$$

We extend this notation to the case when $i = a_{n_0}R_{n_0}$ with $a_{n_0} \in \{0,1\}$ by letting

$$\hat{m}(i) := 1, \quad \hat{k}(i) := 1 \quad \text{and} \quad \hat{a}(i) := a_{n_0}.$$

Proposition 8.18. *Let $n_0 \leq n \leq N$ and $i = j + sR_{n_0}$ with $0 \leq j < R_{n_0}$ and $0 \leq s < R_n/R_{n_0}$. For $0 \leq l \leq s$, denote*

$$\hat{m}_l := \hat{m}(lR_{n_0}), \quad \hat{k}_l := \hat{k}(lR_{n_0}) \quad \textit{and} \quad \hat{a}_l := \hat{a}(lR_{n_0}).$$

If $\hat{m}_l = \hat{k}_l$, let

$$\check{\mathcal{I}}^n_l := F^{lR_{n_0}-1}(\tilde{\mathcal{I}}^n_{0,i}).$$

Otherwise, let

$$\check{\mathcal{I}}^n_l := \mathcal{I}^{\hat{m}_l+1}_{\hat{a}_l R_{\hat{m}_l}-1}.$$

Then $\check{\mathcal{I}}^n_l$ is $\lambda^{(1-\bar{\varepsilon})R_{\hat{m}_l}}$-horizontal. Moreover, define

$$\check{H}_l := \mathcal{P}^{\hat{m}_l}_0 \circ F \circ \left(\mathcal{P}^{n_0}_{-1}|_{\check{\mathcal{I}}^n_l}\right)^{-1} \circ F^{R_{n_0}-1}|_{\mathcal{I}^{n_0}_0}.$$

Then we have

$$\hat{H}_i|_{\tilde{\mathcal{I}}^n_{0,i}} = F^j|_{\mathcal{I}^{n_0}_0} \circ \check{H}_s \circ \ldots \circ \check{H}_4 \circ \check{H}_3 \circ \mathcal{P}^{n_0}_0 \circ F^{2R_{n_0}}|_{\tilde{\mathcal{I}}^n_{0,i}}.$$

Proof. We proceed by induction. Clearly, the result is true for $i < 2R_{n_0}$. Suppose that the result is true for all $i' < i$.

First, suppose $i = 2R_{k+1}$ for some $n_0 \leq k+1 < n$. Denote

$$\Gamma_d := F^d(\tilde{\mathcal{I}}^n_{0,i}) \quad \text{for} \quad 0 \leq d \leq i.$$

By Lemma 8.7:

$$\hat{H}_{2R_{k+1}}|_{\Gamma_0} = \mathcal{P}^{k+1}_0 \circ F^{2R_{k+1}} = \mathcal{P}^{k+1}_0 \circ F \circ F^{R_k-1} \circ F^{(2r_k-1)R_k}|_{\Gamma_0}. \tag{8.4}$$

By Proposition 8.14 iv), $\Gamma_{(2r_k-1)R_k}$ is $\lambda^{-\bar{\varepsilon}R_k}$-horizontal in $\mathcal{B}_0^k$. So it follows from Lemma 4.9 that

$$F^{R_k-1}|_{\Gamma_{(2r_k-1)R_k}} = \left(\mathcal{P}_{-1}^{n_0}|_{\Gamma_{2R_{k+1}-1}}\right)^{-1} \circ F^{R_k-1} \circ \mathcal{P}_0^k|_{\Gamma_{(2r_k-1)R_k}}.$$

Note that

$$\hat{H}_{(2r_k-1)R_k} = H^k_{(2r_k-1)R_k} = \mathcal{P}_0^k \circ F^{(2r_k-1)R_k}.$$

Substituting into (8.4), we obtain

$$\hat{H}_{2R_{k+1}}|_{\Gamma_0} = \mathcal{P}_0^{k+1} \circ F \circ \left(\mathcal{P}_{-1}^{n_0}|_{\Gamma_{2R_{k+1}-1}}\right)^{-1} \circ F^{R_k-1} \circ \hat{H}_{(2r_k-1)R_k}|_{\Gamma_0}.$$

By Lemma 8.3, we have

$$F^{R_k-1}|_{\mathcal{I}_0^k} = \left(\mathcal{P}_{-1}^{n_0}|_{\mathcal{I}^k_{R_k-1}}\right)^{-1} \circ H^k_{R_k-1}|_{\mathcal{I}_0^k}.$$

Thus, we conclude:

$$\hat{H}_{2R_{k+1}}|_{\Gamma_0} = \mathcal{P}_0^{k+1} \circ F \circ \left(\mathcal{P}_{-1}^{n_0}|_{\Gamma_{2R_{k+1}-1}}\right)^{-1} \circ H^k_{R_k-1}|_{\mathcal{I}_0^k} \circ \hat{H}_{(2r_k-1)R_k}|_{\Gamma_0}.$$

We can apply the induction hypothesis to decompose $\hat{H}_{(2r_k-1)R_k}$ into factors of the form $\check{H}_l$. Observe that for

$$e_0 := (2r_k-1)R_k < e < 2R_{k+1},$$

we have

$$\hat{m}(e) = \hat{m}(e-e_0) < \hat{k}(e) \leq k \quad \text{and} \quad \hat{a}(e) = \hat{a}(e-e_0).$$

Hence, we can also apply the induction hypothesis to $H^k_{R_k-1}|_{\mathcal{I}_1^k}$ to decompose them into factors of the form $\check{H}_l$. The claim follows.

Next, suppose that $i = a_kR_k$ for some $n_0 \leq k < n$ and $a_k \geq 3$. Proceeding in the same way as in the previous case, we obtain (in place of (8.4)):

$$\hat{H}_i|_{\Gamma_0} = \mathcal{P}_0^k \circ F^{a_kR_k} = \mathcal{P}_0^k \circ F \circ F^{R_k-1} \circ F^{(a_k-1)R_k}|_{\Gamma_0}.$$

The rest of the argument is identical *mutatis mutandis*.

Lastly, suppose that

$$i = j + [a_{n_0}, \dots, a_k]$$

for some $n_0 < k < n$ such that

$$\hat{m}(i) < k = \hat{k}(i) < n.$$

Then

$$\hat{H}_i = H^{k-1}_{i-a_kR_k} \circ \mathcal{P}_0^k \circ F^{a_kR_k} = H^{k-1}_{i-a_kR_k}|_{\mathcal{I}_0^k} \circ \hat{H}_{a_kR_k}.$$

Applying the induction hypothesis to $\hat{H}_{a_kR_k}$ and $H^{k-1}_{i-a_kR_k}|_{\mathcal{I}_0^k}$ and arguing as above, we obtain the desired result. $\square$

Let $G : U \to G(U)$ be a C^1-diffeomorphism defined on a domain $U \subset \mathbb{R}^2$. For a C^1-curve $\Gamma \subset U$, we define the *cross-ratio distortion* $\mathrm{CrD}(G, \Gamma)$ *of* G *on* Γ as the cross-ratio distortion of

$$G_\Gamma := \phi_{G(\Gamma)}^{-1} \circ G \circ \phi_\Gamma,$$

where ϕ_Γ and $\phi_{G(\Gamma)}$ are parameterizations of Γ and $G(\Gamma)$ by their respective arclengths (see Section C).

Proposition 8.19. *Let $n_0 \leq n \leq N$ and $1 \leq i < R_n$. Then there exists a uniform constant $\nu > 0$ such that the maps $\hat{H}_i$ and $\hat{H}_{R_n-1} \circ \hat{H}_i^{-1}$ have ν-bounded cross-ratio distortion on $\tilde{\mathcal{I}}^n_{0,i}$ and $\hat{H}_i(\tilde{\mathcal{I}}^n_{0,R_n-1})$ respectively.*

Proof. Consider the decomposition of $\hat{H}_i$ given in Proposition 8.18:

$$\hat{H}_i|_{\tilde{\mathcal{I}}^n_{0,i}} = F^j|_{\mathcal{I}^{n_0}_0} \circ \check{H}_s \circ \ldots \circ \check{H}_3 \circ \mathcal{P}^{n_0}_0 \circ F^{2R_{n_0}}|_{\tilde{\mathcal{I}}^n_{0,i}}.$$

Denote

$$\mathcal{J} := \mathcal{P}^{n_0}_0 \circ F^{2R_{n_0}}(\tilde{\mathcal{I}}^n_{0,i}) \quad \text{and} \quad \check{H} := \check{H}_s \circ \ldots \circ \check{H}_3.$$

To prove the cross-ratio distortion bound for $\hat{H}_i$, it suffices to prove it for $\check{H}$ on $\mathcal{J}$.

The maps

$$\phi_0 := (P^{n_0}_0|_{\mathcal{I}^{n_0}_0})^{-1} : I^{n_0}_0 \to \mathcal{I}^{n_0}_0 \quad \text{and} \quad \phi_{-1} := (P_{-1}|_{\mathcal{I}^{n_0}_{R_{n_0}-1}})^{-1} : I^{n_0}_{R_{n_0}-1} \to \mathcal{I}^{n_0}_{R_{n_0}-1}$$

give parameterizations of $\mathcal{I}^{n_0}_0$ and $\mathcal{I}^{n_0}_{R_{n_0}-1}$. Denote

$$J_2 := \phi_0^{-1}(\mathcal{J}) \quad \text{and} \quad h_1 := \phi_{-1}^{-1} \circ F^{R_{n_0}-1}|_{\mathcal{I}^{n_0}_0} \circ \phi_0.$$

For $3 \leq l \leq s$, let

$$H_l := \phi_0^{-1} \circ \check{H}_l \circ \ldots \circ \check{H}_3 \circ \phi_0;$$

and

$$J'_l := h_1(J_{l-1}) \quad \text{and} \quad J_l := H_l(J_2).$$

By Proposition 8.18 and Lemma 8.1, there exist a diffeomorphism $\psi_l : J'_l \to \psi_l(J'_l)$ and a constant $a_l \in \mathbb{R}$ such that

$$H_l(x) = (\psi_l \circ h_1 \circ H_{l-1}(x))^2 + a_l.$$

By (C.2) and Lemma C.2, we see that

$$\mathrm{CrD}(\check{H}, \mathcal{J}) \asymp \mathrm{CrD}(H_s, J_2) > \left(\prod_{l=2}^{s-1} \mathrm{CrD}(h_1, J_l)\right) \cdot \left(\prod_{l=3}^{s} \mathrm{CrD}(\psi_l, J'_l)\right).$$

Note that the diffeomorphisms h_1 and $\{\psi_l\}_{l=3}^s$ have uniformly bounded second derivatives. Moreover, Propositions 8.11 and 8.17 implies that the total length of $\{J_l, J'_l\}_{l=3}^s$ is uniformly bounded. The bound on the cross ratio distortion of $\hat{H}_i$ now follows from Lemma C.3.

Now, consider the decomposition of $\hat{H}_{R_n-1}$ on $\tilde{\mathcal{I}}^n_{0,R_n-1}$:

$$\hat{H}_{R_n-1}|_{\tilde{\mathcal{I}}^n_{0,R_n-1}} = F^{R_{n_0}-1}|_{\mathcal{I}^{n_0}_0} \circ \check{H}_S \circ \ldots \circ \check{H}_3 \circ \mathcal{P}^{n_0}_0 \circ F^{2R_{n_0}}|_{\tilde{\mathcal{I}}^n_{0,R_n-1}},$$

where $S := R_n/R_{n_0} - 1$. The same argument as above implies the bound on the cross ratio distortion of

$$\hat{H}_{R_n-1} \circ \hat{H}_i^{-1}|_{\mathcal{I}} = F^{R_{n_0}-1}|_{\mathcal{I}_0^{n_0}} \circ \check{H}_S \circ \ldots \circ \check{H}_{S-s} \circ F^{R_{n_0}-1-j}|_{\mathcal{I}}$$

on $\mathcal{I} := \hat{H}_i(\tilde{\mathcal{I}}^n_{0,R_n-1})$. □

Proof of Theorem 8.5. Consider the arcs $\{\mathcal{J}_i^n\}_{i=0}^{R_n-1}$. There exists $2R_{n_0} \leq i_1 < R_n$ such that

$$|\mathcal{J}^n_{\iota^n_+(i_1)}| \, , \, |\mathcal{J}^n_{\iota^n_-(i_1)}| > k|\mathcal{J}^n_{i_1}|$$

for some uniform constant $k > 0$. By Proposition 8.17, there exists an arc $\tilde{\mathcal{I}}^n_{0,i_1} \supset \mathcal{I}^n_0$ which is mapped diffeomorphically onto $\tilde{\mathcal{J}}^n_{i_1}$ by $\hat{H}_{i_1}$.

Recall that the nearest neighbor of $\mathcal{I}^n_0$ in $\mathcal{I}^{n_0}_0$ is given by $\mathcal{J}^n_{\chi_{n-1}R_{n-1}}$. Let $\hat{\mathcal{I}}^n_0$ be the convex hull of $\mathcal{I}^n_0 \cup \mathcal{J}^n_{\chi_{n-1}R_{n-1}}$. Then

$$(\tilde{\mathcal{I}}^n_{0,i_1} \cap \mathcal{I}^{n_0}_0) \setminus \mathcal{I}^n_0 \subset \hat{\mathcal{I}}^n_0 \setminus \mathcal{I}^n_0.$$

Hence, Proposition 8.19 and Theorem C.4 imply

$$\left|\hat{\mathcal{I}}^n_0 \setminus \mathcal{I}^n_0\right| > k\,|\mathcal{I}^n_0|\,.$$

By Lemma 8.13, we conclude that the two components of $\tilde{\mathcal{J}}^n_{R_n-1} \setminus \mathcal{J}^n_{R_n-1}$ have lengths greater than $k\left|\mathcal{J}^n_{R_n-1}\right|$. By Proposition 8.17, $\hat{H}_{R_n-1}$ maps $\tilde{\mathcal{I}}^n_{0,R_n-1} \supset \mathcal{I}^n_0$ diffeomorphically onto $\tilde{\mathcal{J}}^n_{R_n-1}$. The result now follows from Proposition 8.19 and Theorem C.4. □

9. Uniform C^1-Bounds

9.1. **For unimodal maps.** Let $s \geq 1$ be an integer, and consider a normalized C^{s+3}-unimodal map $f : I \to I \in \mathfrak{U}^{s+3}$. Recall that this means $f'(0) = 0$ and $f''(0) = 2$. Let ψ_f be the C^s-diffeomorphism given in Lemma D.4 so that $f(x) = 2(\psi_f(x))^2$. An elementary computation shows that

$$2 = f''(0) = 2(\psi_f'(0))^2.$$

Hence, $\psi_f'(0) = 1$.

For $K \geq 1$, we say that f has *K-bounded non-linearity* if

$$\sup_{x,y\in I} \frac{\psi_f'(x)}{\psi_f'(y)} \leq K. \tag{9.1}$$

Denote the space of all normalized C^{s+3}-unimodal maps with K-bounded non-linearity by $\mathfrak{U}^{s+3}(K)$. Observe that if $f : I \to I$ is in $\mathfrak{U}^{s+3}(K)$, then

$$K^{-1} \leq |\psi_f'(x)| \leq K \quad \text{for} \quad x \in I. \tag{9.2}$$

Lemma 9.1. *Let $f : I \to I \in \mathfrak{U}^4(K)$ for some $K \geq 1$. Then $|I| < 2|K|^2$. Consequently, we have $\|f'\| < C$ for some uniform constant $C = C(K) \geq 1$.*

Proof. By (9.2), we see that $|\psi_f(I)| > K^{-1}|I|$. If the length of an interval is bigger than $2|K|^2$, then $|f(I)| > (2|K|)^2 > 4|K|^2$. Thus, iterated images of I under f become unbounded. This is a contradiction.

We compute

$$f'(x) = 2\psi_f(x)\psi_f'(x).$$

The result follows. $\square$

Lemma 9.2. *Let $f : I \to I$ be in $\mathfrak{U}^4(K)$ for some $K \geq 1$, and let $n \in \mathbb{N}$. If the critical orbit of f does not converge to an n-periodic sink, then there exists a uniform constant $\rho_n = \rho_n(K) > 0$ such that $|f^n(0)| > \rho_n$. In particular, $|I| > \rho_1$.*

Proof. By Lemma 9.1, there exists a uniform constant $C = C(K) \geq 1$ such that $\|f'\| < C$. For $n \in \mathbb{N}$, let

$$l_n := \frac{1}{4K^2C^n} \quad \text{and} \quad J_n := (-l_n, l_n).$$

Observe that

$$|f^n(J_n)| < C^n|\psi_f(J_n)|^2 < \frac{1}{16K^2C^n}.$$

Hence, if $f^n(0) \in (-l_n/8, l_n/8)$, then $f^n(J_n) \Subset J_n$. The result now follows from

$$|(f^n)'(x)| < 2|\psi_f(x)||\psi_f'(x)|C^n < 1/2 \quad \text{for} \quad x \in J_n.$$

$\square$

Proposition 9.3. *Let $f : I \to I$ be in $\mathfrak{U}^4(K)$ for some $K \geq 1$. Suppose that f is non-trivially renormalizable, so that there exists an R-periodic interval I^1 such that $f^R(I^1)$ contains the critical value v for f. Denote by c^1 the critical point for $f^R|_{I^1}$. Assume that $\mathcal{R}_{1\mathrm{D}}(f)$ has χ-bounded kneading for some $\chi \geq R$. Let J be a connected component of $I \setminus \{f^i(c^1)\}_{i=0}^{3R+1}$. Then we have $|J| > \rho$, where $\rho = \rho(K, \chi) \in (0, 1)$ is a uniform constant.*

Proof. By Lemma 6.2, we have $I^1 := [v, f^R(v)] \ni c^1$. Denote $I^1_i := f^i(I^1)$ for $0 \leq i < R$. The fact that we have $|f^{l+i}(v) - f^i(c^1)| > \rho$ for for $l \in \{0, R\}$ and $1 \leq i < R$ follows from Lemmas 9.1 and 9.2.

By the assumption on bounded kneading, there exists a smallest integer $r_1 \leq \chi$ such that $f^{(r_1+1)R}(c^1) < c^1$. By Lemma 9.2, there exists a uniform constant $\rho_1 = \rho_1(K, \chi) > 0$ such that

$$v < f^{r_1R}(v) < f^R(v) - \rho_1. \tag{9.3}$$

Let $L_i := [f^{iR}(v), f^{(i-1)R}(v)]$ for $2 \leq i \leq r_1$. If $r_1 = 2$, then it follows from (9.3) that $|L_2| > \rho$. If $r_1 > 2$, then observe that f^R maps L_i diffeomorphically onto L_{i+1} for $i < r_1$. By Lemma 9.1, $\|(f^R)'\| < C$ for some uniform constant $C \geq 1$, and

$$L_2 \sqcup L_3 \sqcup \ldots \sqcup L_{r_1} \supset [c^1, f^R(v)].$$

This implies that $|L_2| > \rho'$ for some uniform constant ρ'.

Let J_0 be the gap between I^1_k and I^1_l with $0 \leq k < l < R$. If $J_m := f^m(J_0)$ with $m = \bar{\chi}$ maps onto an interval I^1_i for some $0 \leq i < R$, then by Lemma 9.1 and

Lemma 9.2, we have $|J_0| > C^{-m}\tilde{\rho}$ for some uniform constants $C = C(K) \geq 1$ and $\tilde{\rho} = \tilde{\rho}(K, \chi) > 0$. Thus, we may assume, after replacing J_0 with J_R if necessary, that $\partial J_0 \ni f^{k+R}(v)$.

Map J_0 by $f^{r_1 R-k-R}$. Since

$$I^1_{l+r_1R-k-R} \cap I^1_0 = \varnothing,$$

the image J_{r_1R-k-R} of the gap must contain $(c^1, f^R(v))$. The result now follows from (9.3). ☐

9.2. **For Hénon-like maps.** For an integer $r \geq 2$ and a constant $K \geq 1$, let $\mathfrak{HL}^{r+2}(K)$ be the space of all normalized C^{r+2}-Hénon-like maps whose 1D profiles are contained in $\mathfrak{U}^{r+2}(K)$. Additionally, for $\beta \in (0,1)$, let $\mathfrak{HL}^{r+2}_\beta(K)$ be the set of all Hénon-like maps in $\mathfrak{HL}^{r+2}(K)$ that are β-thin in C^{r+2}.

For some $N \in \mathbb{N} \cup \{\infty\}$, let F be the N-times regularly Hénon-like renormalizable C^{r+4}-map with combinatorics of $\mathbf{b}$-bounded type considered in Section 8. If $N < \infty$, suppose that $F^{R_N}|_{\mathcal{B}^N_0}$ is twice topologically renormalizable with combinatorics of $\mathbf{b}$-bounded type.

Let $\mathbf{K} \geq 1$ be the uniform constant given in Theorem 8.5. Assume that $n_0 \leq N$ is the smallest number such that

$$\overline{K_2}\lambda^{\varepsilon R_{n_0}} < 1, \tag{9.4}$$

where

$$K_2 = K_2(\mathbf{K}, \mathbf{b}) \geq 1 \tag{9.5}$$

is a uniform constant.

For $n_0 \leq n \leq N$, denote

$$I^n_0 := \pi_h(B^n_0) \quad \text{and} \quad \mathcal{I}^n_0 := (\Psi^n)^{-1}(I^n_0 \times \{0\}).$$

Define

$$\tilde{F}_n := \Psi^n \circ F^{R_n} \circ (\Psi^n)^{-1} \quad \text{and} \quad \tilde{f}_n := \Pi_{1\mathrm{D}}(\tilde{F}_n).$$

Proposition 9.4. *There exists a uniform constant $K = K(\mathbf{K}) \geq 1$ such that for all $n_0 \leq n \leq N$, we have $\|\tilde{F}_n\|_{C^1} < K$.*

Proof. The result follows immediately from Corollary 8.6 and Lemma 9.1. ☐

Proposition 9.5. *For $n_0 \leq n \leq N$, we have*

$$\|\tilde{f}^i_n - \Pi_{1\mathrm{D}}(\tilde{F}^i_n)\|_{C^0} < \lambda^{(1-\bar{\varepsilon})R_n} \quad \textit{for} \quad i = O(\mathbf{b}).$$

Proof. Denote $\Pi_h(x,y) := (x, 0)$. It suffices to show that

$$\|(\tilde{F}_n \circ \Pi_h)^i - \tilde{F}^i_n \circ \Pi_h\|_{C^0} < \lambda^{(1-\bar{\varepsilon})R_n}.$$

By Proposition 9.4, $\|\tilde{F}_n\|_{C^1}$ is uniformly bounded. Moreover, by Theorem 4.7 ii), we have

$$\|\tilde{F}_n - \tilde{F}_n \circ \Pi_h\|_{C^{r+3}} < \lambda^{(1-\bar{\varepsilon})R_n}.$$

The result now follows from Lemma D.1. ☐

Proposition 9.6. *For $n_0 \leq n < N$ and $-r_n \leq k \leq 2r_n$, denote*

$$u_k^n := \Psi^n(v_{kR_n}) \quad \textit{and} \quad a_k^n := \pi_h(u_k^n).$$

Let J be a connected component of $I_0^n \setminus \{a_k^n\}_{k=-r_n}^{2r_n}$. Then we have $|J| > \rho|I_0^n|$, where $\rho = \rho(\mathbf{K}, \mathbf{b}) \in (0,1)$ is a uniform constant.

Proof. Let v_0 be the critical value of F defined in Section 4. Denote $u_0^n := \Psi^n(v_0)$. Note that u_0^n is a point of tangency between foliation by vertical quadratic curves $\lambda^{(1-\bar{\varepsilon})R_n}$-close to the image of the horizontal foliation by F_n, and foliation by $\lambda^{(1-\bar{\varepsilon})R_n}$-vertical curves. On the other hand, $(f_n(0), 0)$ is the unique tangency between the image curve under the degenerate Hénon map $\iota(f_n)$, and the genuine vertical foliation. Since

$$\|F_n - \iota(f_n)\|_{C^{r+3}} < \lambda^{(1-\bar{\varepsilon})R_n},$$

we see that

$$|\pi_h(u_0^n) - f_n(0)| < \lambda^{(1-\bar{\varepsilon})R_n}.$$

The result now follows from Propositions 9.3 and 9.5. □

Theorem 9.7. *For $n_0 \leq n \leq N$, there exist $\sigma_n \asymp |I_0^n|^{1/2}$; $\tau_n \in I_0^n$ and a C^{r+3}-diffeomorphism ϕ_n defined on the interval $\pi_v(B_0^n)$ such that for*

$$\mathcal{S}^n(x,y) := (\sigma_n^{-2}x + \tau_n, \sigma_n^{-1}y + \tau_n),$$

and

$$\mathcal{Y}^n(x,y) = (x, \phi_n(y)) \quad \textit{and} \quad \Phi^n := (\mathcal{Y}^n)^{-1} \circ \Psi^n,$$

the following statements hold.

i) There exist uniform constants $0 < \rho_1 < \rho_2 < 1$ depending only on $\mathbf{K}$ and $\mathbf{b}$ such that $\rho_1^n < \sigma_n < \rho_2^n$.

ii) The distortion of ϕ_n is bounded by $\mathbf{K}$, and $\|\phi_n^{\pm 1}\|_{C^1} < K$, where $K = K(\mathbf{K}) \geq 1$ is a uniform constant.

iii) We have

$$\mathcal{R}^n(F) := \mathcal{S}^n \circ \Phi^n \circ F^{R_n} \circ (\mathcal{S}^n \circ \Phi^n)^{-1} \in \mathfrak{HL}_{\lambda_n}^{r+3}(\mathbf{K}),$$

where $\lambda_n := \lambda^{(1-\bar{\varepsilon})R_n}$.

Proof. Consider the maps

$$F_0 := \Phi_0 \circ F \circ \Phi_{-1}^{-1} \quad \text{and} \quad H_n := \Phi_{-1} \circ F^{R_n - 1} \circ (\Psi^n)^{-1}.$$

By Theorem 4.7, we have

$$F_0(x,y) = (f_0(x) - \lambda y, x) \quad \text{and} \quad H_n(x,y) = (h_n(x), e_n(x,y)),$$

where f_0 is a map with a unique critical point at 0 with $f_0''(0) > 0$; h_n is a diffeomorphism; and e_n is a map such that $\|e_n\|_{C^{r+3}} < \lambda^{(1-\bar{\varepsilon})R_n}$. Moreover, Corollary 8.6 states that h_n has $\mathbf{K}$-bounded distortion.

The map $\tilde{F}_n := (\Psi^n) \circ F^{R_n - 1} \circ (\Psi^n)^{-1}$ is of the form

$$\tilde{F}_n(x,y) = (g_n(x,y), h_n(x)),$$

where $g_n(\cdot, y)$ for $y \in \pi_v(\mathcal{B}_0^n)$ is a unimodal map. By Theorem 4.7 i), we see that

$$\|g_n(\cdot, y) - f_0 \circ h_n(\cdot)\|_{C^{r+3}} < \lambda^{(1-\bar{\varepsilon})R_n}.$$

We claim that $|h_n(I_0^n)|^2 \asymp |I_0^n|$. Write $I_0^n = [a_n, b_n]$ and $h_n(I_0^n) = [\alpha_n, \beta_n]$. For $-r_n \leq k \leq 2r_n$, denote

$$u_k^n := \Psi^n(v_{kR_n}) \quad \text{and} \quad a_k^n := \pi_h(u_k^n).$$

Recall that a_n and b_n are $\lambda^{\bar{\varepsilon}R_n}$-close to $a_0^n = 0$ and $a_{r_n}^n$ respectively, and that $a_{-1}^n \in I_0^n$. Additionally, observe that $g_n(0)$, $g_n(a_{-1}^n)$ and $g_n(a_{r_n}^n)$ are $\lambda^{(1-\bar{\varepsilon})R_n}$-close to $a_{r_n}^n$, 0 and $a_{2r_n}^n$ respectively. By Proposition 9.6, $|a_{r_n}^n|$ and $|a_{2r_n}^n|$ are commensurate with $|I_0^n|$. The claim now follows from the fact that $f_0(\alpha_n)$ and $f_0(\beta_n)$ are $\lambda^{(1-\bar{\varepsilon})R_n}$-close to $g_n(a_n)$ and $g_n(b_n)$ respectively.

Define

$$\check{\mathcal{Y}}^n(x, y) := (x, h_n(y)) \quad \text{and} \quad \check{\Phi}^n := (\check{\mathcal{Y}}^n)^{-1} \circ \Psi^n.$$

It is easy to check that

$$\check{F}_n := \check{\Phi}^n \circ F^{R_n} \circ (\check{\Phi}^n)^{-1}$$

is a Hénon-like map. Denote $\check{f}_n := \Pi_{1\mathrm{D}}(\check{F}_n)$, and let

$$\check{S}^n(x) = \sigma_n^2 x + \tau_n$$

be the unique orientation-preserving affine map on $\mathbb{R}$ such that $\check{S}^n \circ \check{f}_n \circ (\check{S}^n)^{-1} \in \mathfrak{U}^{r+3}(\mathbf{K})$.

Lemma 9.2 implies that $\sigma_n^2 \asymp |I_0^n|$. Property i) now follows from Proposition 9.6. Let

$$\phi_n(y) := \sigma_n^{-1} h_n(y).$$

Properties ii) and iv) follow immediately. Lastly, since h_n has bounded distortion, and $|h_n(I_0^n)| \asymp |I_0^n|^{1/2} \asymp \sigma_n$, it follows that $\|\phi_n^{\pm 1}\|_{C^1}$ is uniformly bounded. Thus, we also have Property iii). □

10. Preservation of Regularity

For $N \in \mathbb{N}$, let F be the N-times $(L, \varepsilon, \lambda)$-regularly Hénon-like renormalizable map with combinatorics of $\mathbf{b}$-bounded type considered in Subsection 9.2 (with $n_0 \leq N$ satisfying (9.4)). For $n_0 \leq n \leq N$, we have by Theorem 9.7:

$$F_n := \mathcal{R}^n(F) = \mathcal{S}^n \circ \Phi^n \circ F^{R_n} \circ (\mathcal{S}^n \circ \Phi^n)^{-1},$$

where $\|(\Phi^n)^{\pm 1}\|_{C^1} < K$ for some uniform constant $K = K(\mathbf{K}) \geq 1$, and

$$\mathcal{S}^n(x, y) := (\sigma_n^{-2}(x + \tau_n), \sigma_n^{-1}(y + \tau_n))$$

for some $\sigma_n \in (0, 1)$ and $|\tau_n| \asymp \sigma_n^2$. Moreover, by Proposition 9.6, there exist uniform constants $0 < \rho_1 < \rho_2 < 1$ depending only on $\mathbf{K}$ and $\mathbf{b}$ such that $\rho_1^n < \sigma_n < \rho_2^n$. Let $f_n := \Pi_{1\mathrm{D}}(F_n)$.

For $p \in \mathcal{B}_0^n$, let

$$E_p^{v,n} := (D\Phi^n)^{-1}(E_z^{gv}) \quad \text{where} \quad z := \Phi^n(p).$$

Moreover, since Φ^n is genuinely horizontal, we have

$$E_p^{gh} = (D\Phi^n)^{-1}(E_z^{gh}).$$

Denote $P_0^n := \pi_h \circ \Phi^n$.

Lemma 10.1. *For $n_0 \leq n < N$ and $0 \leq k < r_n$, let $p \in \mathcal{B}_{kR_n}^{n+1} \subset \mathcal{B}_0^n$. Then we have*

$$\frac{1}{K} < \|DF^{iR_n}|_{E_p^{gh}}\| < \|DF^{iR_n}\| < K \quad \text{for} \quad 0 \leq i \leq r_n - k,$$

where $K = K(\mathbf{K}, \mathbf{b}) \geq 1$ is a uniform constant.

Proof. The upper bound follows immediately from Proposition 9.4. Denote $z := \mathcal{S}^n \circ \Phi^n(p)$, and $z_i = (x_i, y_i) := F_n^i(z)$. Propositions 6.5 and 9.6 imply that for $0 \leq j < r_n - k - 1$, there exists a uniform constant $\rho = \rho(\mathbf{K}, \mathbf{b}) \in (0, 1)$ such that $|x_j| > \rho$. Thus, $|f_n'(x_j)|$ is uniformly bounded below.

By the thinness of F_n, we see that for all $w = (u, v)$ in the domain of F_n, we have

$$\|D(\pi_h \circ F_n)|_{E_w^{gh}}\| = |f_n'(u)| + O(\lambda^{(1-\bar{\varepsilon})R_n}).$$

Let $h_n := \psi_{f_n}$ be the diffeomorphism given in Lemma D.4. Then

$$\|DF_n|_{E_w^{gh}}\| > c|h_n'(u)|$$

for some uniform constant $c > 0$. Let

$$\alpha_j := |f_n'(x_j)| - \lambda^{(1-\bar{\varepsilon})R_n} \quad \text{for} \quad 0 \leq j < r_n - k - 1,$$

and $\alpha_{r_n-k-1} := c|h_n'(x_{r_n-k-1})|$. Then we have

$$\|D(\pi_h \circ F_n^i)|_{E_{z_0}^{gh}}\| \geq \alpha_0 \dots \alpha_{i-1}.$$

The desired lower bound follows. □

Lemma 10.2. *For $n_0 \leq n \leq N$, let $p_0 \in \mathcal{B}_0^n$. Then*

$$\frac{1}{K^{n-n_0}} < \|DF^T|_{E_{p_0}^{gh}}\| < K^{n-n_0} \quad \text{for} \quad 0 \leq T < R_n,$$

where $K = K(\mathbf{K}, \mathbf{b}) \geq 1$ is a uniform constant.

Proof. Write

$$T = t_0 + t_{n_0} R_{n_0} + \dots + t_{n-1} R_{n-1}$$

with $0 \leq t_0 < R_{n_0}$ and $0 \leq t_k < r_k$ for $n_0 \leq k < n$. Denote

$$T_k := t_{k+1} R_{k+1} + \dots + t_{n-1} R_{n-1}.$$

Then clearly, we have

$$K^{-1} \cdot d_{n_0} \cdot \ldots \cdot d_{n-1} \leq \|DF^T|_{E_{p_0}^{gh}}\| \leq K \cdot D_{n_0} \cdot \ldots \cdot D_{n-1},$$

where

$$D_k := \|D_{p_{T_k}} F^{t_k R_k}\| \quad \text{and} \quad d_k := \left\| D\left(P_0^k \circ F^{t_k R_k}\right)|_{E_{p_{T_k}}^{gh}} \right\|$$

for $n_0 \leq k < n$. The result now follows from Lemma 10.1. □

Let v_0 be the critical value of F. For $k \geq -r_n$, denote

$$u^n_k := \mathcal{S}^n \circ \Phi^n(v_{kR_n}) \quad \text{and} \quad a^n_k := \pi_h(u^n_k).$$

Consider an increasing sequence of renormalization depths

$$n_0 \leq n_1 < n_2 < \ldots < n_k \leq N.$$

We say that this sequence is *tempered* if for $1 \leq i < k$, we have

$$\rho_1^{n_{i+1}-n_i} > \lambda^{\bar{\varepsilon} R_{n_i}},$$

where ρ_1 is given in Proposition 9.6.

Lemma 10.3. *Consider a tempered sequence $\{n_i\}_{i=1}^k$. Let*

$$S = s_1 R_{n_1} + \ldots + s_{k-1} R_{n_{k-1}} \quad \text{and} \quad \hat{S} := S + s_k R_{n_k};$$

where $1 \leq s_i < r_{n_i}$ for $1 \leq i \leq k$. For $p_0 \in \mathcal{B}^{n_k+1}_{R_{n_k+1}}$, define

$$\hat{z} = (\hat{x}, \hat{y}) := \mathcal{S}^{n_k} \circ \Phi^{n_k}(p_{S-\hat{S}}) \quad \text{and} \quad \hat{E}_{\hat{z}} := D(\mathcal{S}^{n_k} \circ \Phi^{n_k} \circ F^S)(E^{gh}_{p_{-\hat{S}}}).$$

Then $\hat{E}_{\hat{z}}$ is θ-horizontal for some uniform constant $\theta = \theta(\mathbf{K}, \mathbf{b}) \geq 1$. Moreover, we have

$$\frac{1}{K^k} < \left\| DF^{\hat{S}}|_{E^{gh}_{p_{-\hat{S}}}} \right\| < K^k \tag{10.1}$$

where $K = K(\mathbf{K}, \mathbf{b}) \geq 1$ is a uniform constant.

Proof. Proceeding by induction on k, suppose the result holds for $k' < k$. We first show that $\hat{E}_{\hat{z}}$ is uniformly horizontal. Denote

$$S' := s_1 R_{n_1} + \ldots + s_{k-2} R_{n_{k-2}};$$

and for $0 \leq i \leq s_{k-1}$,

$$z_i = (x_i, y_i) := \mathcal{S}^{n_{k-1}} \circ \Psi^{n_{k-1}}(p_{(i-s_{k-1})R_{n_{k-1}}})$$

and

$$\hat{E}_{z_i} := D(\mathcal{S}^{n_{k-1}} \circ \Phi^{n_{k-1}} \circ F^{S'+iR_{n_{k-1}}})(E^{gh}_{p_{-S}}).$$

Then by Propositions 6.5 and 9.6, it follows that for $0 \leq j < s_{k-1}$

$$|x_j - a_0^{n_{k-1}}| > \rho_1.$$

Thus, $\hat{E}_{z_j}$ is $(1/\rho_1)$-horizontal by Lemma 5.1. Propositions 6.5 and 9.6 also imply that

$$|\hat{x} - a_0^{n_k}| > \rho_1.$$

Since

$$z_{s_{k-1}} = \mathcal{S}^{n_{k-1}} \circ \Phi^{n_{k-1}} \circ (\mathcal{S}^{n_k} \circ \Phi^{n_k})^{-1}(\hat{z}),$$

it follows from Theorem 9.7 that

$$\Delta := |x_{s_{k-1}} - a_0^{n_{k-1}}| > K_0{}^{-1} \rho_1^{n_k - n_{k-1}} > \lambda^{\bar{\varepsilon} R_{n_{k-1}}}.$$

Thus, by Lemma 5.1, $\hat{E}_{z_{s_{k-1}}}$ is $O(1/\Delta)$-horizontal. Under

$$D(\mathcal{S}^{n_k} \circ \Phi^{n_k} \circ (\mathcal{S}^{n_{k-1}} \circ \Phi^{n_{k-1}})^{-1}),$$

the distance $\Delta = |x_{s_{k-1}} - a_0^{n_{k-1}}|$ is rescaled to $\rho_1 = |\hat{x} - a_0^{n_k}|$. We conclude that

$$\hat{E}_{\hat{z}} = D(\mathcal{S}^{n_k} \circ \Phi^{n_k} \circ (\mathcal{S}^{n_{k-1}} \circ \Phi^{n_{k-1}})^{-1})(\hat{E}_{z_{s_{k-1}}})$$

is θ-horizontal for some uniform constant $\theta \geq 1$.

Since $\hat{E}_{\hat{z}}$ is uniformly horizontal, we see that $\|D\pi_h|_{\hat{E}_{\hat{z}}}\| > K^{-1}$. Thus, by Lemma 10.1, we see that

$$\left\|DF^{s_k}_{n_k}|_{\hat{E}_{\hat{z}}}\right\| > K^{-1}.$$

By the induction hypothesis, we have

$$\left\|DF^{S}|_{E^{gh}_{p_{-\hat{S}}}}\right\| > K^{-(k-1)}.$$

Concatenating the above two inequalities, (10.1) follows. □

Theorem 10.4. *Fix $\delta \in (\bar{\varepsilon}, 1)$ such that $\mathbf{b}\bar{\delta} < 1$. Then there exists a uniform constant $\mathbf{L} = \mathbf{L}(\mathbf{K}, \mathbf{b}) \geq 1$ such that the following holds. For $m \in \mathbb{N} \cup \{\infty\}$, suppose that F_{n_0} is $(m+2)$-times topologically renormalizable with combinatorics of $\mathbf{b}$-bounded type. Then F has $n_0 + m$ nested $(\mathbf{L}, \delta, \lambda)$-regular Hénon-like returns.*

Proof. Proceeding by induction, suppose that for $n_0 \leq M < n_0 + m$, the map F has M nested $(\mathbf{L}, \delta, \lambda)$-regular Hénon-like returns

$$\{(F^{R_n}, \Phi^n : \mathcal{B}^n_0 \to B^n_0)\}_{n=1}^M.$$

By Theorem 6.7, F has a $(\overline{\mathbf{L}}, \bar{\delta}, \lambda)$-regular Hénon-like return

$$(F^{R_{M+1}}, \Phi^{M+1} : \mathcal{B}^{M+1}_0 \to B^{M+1}_0).$$

We claim that this return is $(\mathbf{L}, \delta, \lambda)$-regular.

Let $p_0 \in \mathcal{B}^{M+1}_0$ and

$$E^{v/h}_{p_0} := (D\Phi^{M+1})^{-1}(E^{gv/gh}_{\Phi^{M+1}(p_0)}).$$

Let $R_{n_0} \leq T < R_{M+1}$. Write

$$T = t_0 + t_1 R_{n_1} + \ldots + t_k R_{n_k},$$

with $0 \leq t_0 < R_{n_0}$; $n_{i-1} \leq n_i \leq M$ and $1 \leq t_i < r_{n_i}$ for $1 \leq i \leq k$. Lemma 10.2 implies that

$$\frac{1}{K^k} < \|DF^T|_{E^h_{p_0}}\| < K^k$$

By (9.4), we have $K < \lambda^{-\varepsilon R_{n_i}}$. Together with Proposition A.4, this implies that p_0 is R_{M+1}-times forward $(K, \varepsilon, \lambda)$-regular horizontally along $E^h_{p_0}$. By Proposition A.17, it follows that p_0 is R_{M+1}-times forward $(\mathbf{L}, \delta, \lambda)$-regular (vertically) along $E^v_{p_0}$.

Let $q_0 \in \mathcal{B}^{M+1}_{R_{M+1}}$ and $S = lR_{n_0}$ for some $1 \leq l < R_{M+1}/R_{n_0}$. Write

$$S = s_1 R_{n_1} + \ldots + s_k R_{n_k},$$

where $1 \leq s_i < r_{n_i}$ for $1 \leq i \leq k$. Let $1 \leq m \leq k$ be the smallest number such that $\{n_i\}_{i=m}^k$ is tempered. Denote

$$S' := s_m R_{n_m} + s_{m+1} R_{n_{m+1}} + \ldots + s_k R_{n_k},$$

and let

$$\hat{E}_{q_{-1}} := DF^{S'-1}(E^{gh}_{q_{-S'}}).$$

By Lemmas 4.8 i) and 10.3, we see that $\hat{E}_{q_{-1}}$ is $\lambda^{(1-\bar{\varepsilon})R_{n_k}}$-horizontal in $\mathcal{U}_{-1}$, and

$$K^{-(k-m)} < \|DF^{-S'+1}|_{\hat{E}_{q_{-1}}}\| < K^{k-m}. \tag{10.2}$$

Let

$$E^v_{q_{-1}} := DF^{-1}(E^h_{q_0}) = D\Phi_{-1}^{-1}(E^{gv}_{\Phi_{-1}(q_{-1})}).$$

Since $\|\Phi_{-1}^{\pm 1}\|_{C^1} < K_0$ by Theorem 4.7, we have

$$K_0{}^{-1} \leq \frac{\|DF^{-S'+1}|_{E^v_{q_{-1}}}\| \cdot \|DF^{-S'+1}|_{\hat{E}_{q_{-1}}}\|}{\operatorname{Jac}_{p_{-1}} F^{-S'+1}} \leq K_0. \tag{10.3}$$

Substituting in Proposition A.4 and (10.2), we obtain

$$\bar{L}^{-1} K^{-(k-m)} \lambda^{-(1-\bar{\varepsilon})S'} < \|DF^{-S'+1}|_{E^v_{q_{-1}}}\| < \bar{L} K^{k-m} \lambda^{-(1+\bar{\varepsilon})S'}.$$

By (9.4), we have $\bar{L}K < \lambda^{-\varepsilon R_{n_i}}$, and hence,

$$K\lambda^{-(1-\bar{\varepsilon})S'} < \|DF^{-S'+1}|_{E^v_{q_{-1}}}\| < K\lambda^{-(1+\bar{\varepsilon})S'}. \tag{10.4}$$

Denote

$$\hat{E}^v_{q_{-S'}} := DF^{-S'+1}(E^v_{q_{-1}}).$$

By Proposition A.5, we have

$$\bar{K}^{-1} \lambda^{\bar{\varepsilon}(S-S')} < \|DF^{-(S-S')}|_{\hat{E}^v_{q_{-S'}}}\| < \bar{K}\lambda^{-(1+\bar{\varepsilon})(S-S')}. \tag{10.5}$$

Since the gap between n_{m-1} and n_m is not tempered, we have

$$\rho_1^{n_m} < \rho_1^{n_m - n_{m-1}} < \lambda^{\bar{\varepsilon} R_{n_{m-1}}}.$$

Denote $\omega := \log_\lambda \rho_1$. Then $n_m > (\bar{\varepsilon}/\omega) R_{n_{m-1}}$. There exists a uniform constant $\hat{R} = \hat{R}(\varepsilon, \omega, \mathbf{b}) \in \mathbb{N}$ such that for all $R \geq \hat{R}$, we have

$$2^{(\varepsilon/\omega)R} > \frac{\mathbf{b}}{\varepsilon} R.$$

By uniformly increasing n_0 if necessary, we may assume that $R_{n_0} \geq R$, so that

$$\bar{\varepsilon} S' > \bar{\varepsilon} R_{n_m} > \bar{\varepsilon} 2^{n_m} > \bar{\varepsilon} 2^{(\bar{\varepsilon}/\omega) R_{n_{m-1}}} > \mathbf{b} R_{n_{m-1}} > S - S'.$$

Therefore,

$$\|DF^{-(S-S')}|_{\hat{E}^v_{q_{-S'}}}\| > \bar{K}^{-1} \lambda^{\bar{\varepsilon}(S-S')} > \bar{K}^{-1} \lambda^{\bar{\varepsilon} S'} \lambda^{-(1-\bar{\varepsilon})(S-S')}. \tag{10.6}$$

Concatenating (10.4) with (10.5) and (10.6), we conclude by Proposition A.4 that q_0 is R_{M+1}-times backward $(\mathbf{L}, \delta, \lambda)$-regular (vertically) along $E^h_{q_0}$. $\square$

11. Realization of Renormalization Combinatorics

11.1. **For unimodal maps.** Consider a C^2-unimodal map f with critical point c. For concreteness, assume $f''(c) > 0$. For $\eta > 0$, we say that f has *η-gap* if $f(c) < c - \eta$, and *double η-gap* if $f(c) < f^2(c) < c - \eta$. By Lemma 6.1, if f has double η-gap for some $\eta > 0$, then c converges to a sink of period 1 or 2. Lastly, for $\chi \in \mathbb{N}$, we say that f has *(η, χ)-kneading* if

$$f_{a_2}^{1+\chi}(c) + \eta < c < f_{a_2}^{1+i}(c) - \eta \quad \text{for} \quad 1 \leq i < \chi.$$

Let $\mathfrak{I} \subset \mathbb{R}$ be an interval, and consider a C^1-smoothly parameterized family $\mathfrak{f} = \{f_a\}_{a \in \mathfrak{I}}$ of C^2-unimodal maps (i.e. f_a depends C^1-smoothly on the parameter a). For $\eta > 0$ and $\chi \geq 2$, we say that $\mathfrak{f}$ is *(η, χ)-full* if the following conditions hold.

- For all $a \in \mathfrak{I}$, the map f_a has η-gap and χ-bounded kneading.
- There exists $a_1 \in \mathfrak{I}$ such that f_{a_1} has double η-gap.
- There exists $a_2 \in \mathfrak{I}$ such that f_{a_2} has (η, χ)-kneading.

Recall the definition of renormalization type $\tau(f)$ of a valuably renormalizable unimodal map f given in Subsection 6.1.

Proposition 11.1. *Consider a C^1-smoothly parameterized family $\mathfrak{f} = \{f_a\}_{a \in \mathfrak{I}} \subset \mathfrak{U}^2(K)$ for some $K \geq 1$. Suppose that $\mathfrak{f}$ is $(\eta_0, \mathbf{b})$-full for some $\eta_0 > 0$ and $\mathbf{b} \geq 2$. Then for any $\mathbf{b}$-bounded renormalization type T, there exist a uniform constant $\eta_1 = \eta_1(K, \mathbf{b}) > 0$ and an interval $\mathfrak{I}_1 \subset \mathfrak{I}$ such that $\tau(f_a) = T$ for $a \in \mathfrak{I}_1$, and $\mathfrak{f}_1 := \{\mathcal{R}_{1\text{D}}(f_a)\}_{a \in \mathfrak{I}_1} \subset \mathfrak{U}^2$ is $(\eta_1, \mathbf{b})$-full.*

Proof. The Intermediate Value Theorem by Milnor-Thurston [MiTh] implies that there exist a parameter interval $\hat{\mathfrak{I}}_1 \subset \mathfrak{I}$ such that $\tau(f_a) = T$ for $a \in \mathfrak{I}_1$, and $\hat{\mathfrak{f}}_1 := \{\mathcal{R}_{1\text{D}}(f_a)\}_{a \in \hat{\mathfrak{I}}_1}$ is full.

Clearly, there exist $K' = K'(K, \mathbf{b}) \geq 1$ such that $\hat{\mathfrak{f}}_1 \subset \mathfrak{U}^2(K')$. Let $\mathfrak{I}_1'$ be a maximal subinterval of $\hat{\mathfrak{I}}_1$ such that for $a \in \mathfrak{I}_1'$, the critical point 0 of $\mathcal{R}_{1\text{D}}(f_a)$ does not converge to a fixed attracting sink. Then by Lemma 9.2, we see that there exists a uniform constant $\eta_1 = \eta_1(K') > 0$ such that f_a has η_1-gap. Moreover, observe that there exist $a_1 \in \partial\mathfrak{I}_1'$ such that the critical point 0 of $\mathcal{R}_{1\text{D}}(f_{a_1})$ converges to a fixed parabolic sink of flip type. By decreasing η_1 a uniform amount if necessary, we see that $\mathcal{R}_{1\text{D}}(f_{a_1})$ has double η_1-gap.

Applying the Intermediate Value Theorem again, we can restrict $\mathfrak{I}_1'$ to a smaller subinterval such that for $a \in \mathfrak{I}_1$, the map $\mathcal{R}_{1\text{D}}(f_a)$ has $\mathbf{b}$-bounded kneading. Moreover, the endpoints of $\mathfrak{I}_1$ are a_1 and a_2, so that for $\hat{f}_{a_2} := \mathcal{R}_{1\text{D}}(f_{a_2})$, we have

$$\hat{f}_{a_2}^{1+\chi}(0) < 0 < \hat{f}_{a_2}^{1+i}(0) \quad \text{for} \quad 1 \leq i < \chi,$$

and 0 does not converge to a sink of period less than χ. By decreasing η_1 a uniform amount if necessary, it follows from Lemma 9.2 that $\hat{f}_{a_2}$ has (η_1, χ)-kneading. $\square$

11.2. **For Hénon-like maps.** For $N \in \mathbb{N}$, let F be the N-times $(L, \varepsilon, \lambda)$-regularly Hénon-like renormalizable map with combinatorics of $\mathbf{b}$-bounded type considered in Subsection 9.2. Let $\eta_1 = \eta_1(\mathbf{K}, \mathbf{b}) > 0$ be the constant given in Proposition 11.1. We assume that N is sufficiently large, so that for some $0 \leq n_0 \leq N$, we have (in addition to (9.4)):

$$\lambda^{\varepsilon R_{n_0}} < c\eta_1 \tag{11.1}$$

for some sufficiently small constant $c \in (0, 1)$ independent of F.

For $0 \leq n \leq N$, let $F_n := \mathcal{R}^n(F)$ and $f_n := \Pi_{1\mathrm{D}}(F_n)$. Denote the domain of F_n by D^n. Recall the definition of the renormalization type $\tau(F_n)$ of F_n given in Subsection 6.2 (see (6.8)).

Proposition 11.2. *Suppose that f_N is valuably renormalizable with return time $r_N \leq$ $\mathbf{b}$, and that $\mathcal{R}_{1\mathrm{D}}(f_N)$ has η_1-gap and $\mathbf{b}$-bounded kneading. Then F is $(N+1)$-times Hénon-like renormalizable, and $\tau(F_N) = \tau(f_N)$.*

Proof. Let J_0 be the r_N-periodic interval of f_N containing the critical value $f_N(0)$. Denote the critical point of $g := f_N^{r_N}|_{J^{N+1}}$ by c. By Lemma 6.2, we can assume that

$$J_0 := [g(c), c] \cup [c, g^2(c)].$$

Denote $J_i := g^i(J_0)$ for $0 \leq i < r_N$. We claim that J_i and J_j for $i \neq j$ are uniformly far apart. By Lemma 9.1, there exists a uniform constant $\rho_1 > 0$ such that if $g^2(c) <$ $c+\rho_1$, then $g^3(c) < c - \rho_1$. Considering the two cases $g^2(c) < c+\rho_1$ and $g^2(c) > c+\rho_1$ separately, and arguing as in the proof of Proposition 9.3, the claim follows.

For $0 \leq i < r_N$, let $\tilde{J}_i$ be an interval that compactly contains J_i, and the components of $\tilde{J}_i \setminus J_i$ have lengths commensurate to $\lambda^{\bar{\varepsilon} R_n}$. Define

$$W_i := \tilde{J}_i \times \pi_v(D^N).$$

Observe that $W_i \cap W_j = \varnothing$ if $i \neq j$. Moreover, by Proposition 9.5, it follows that we have $F_n(W_i) \Subset W_{i+1 \ (\mathrm{mod}\ r_n)}$ for $0 \leq i < r_n$.

Let

$$\mathcal{W}_i := (\mathcal{S}^N \circ \Phi^N)^{-1}(W_i).$$

Denote $R_{N+1} := r_N R_N$. By Theorem 9.7, we see that if $i > 0$, then

$$\mathcal{W}_i \cap \mathcal{V}^N_{v_0}(\rho^N_1) = \varnothing$$

for some uniform constant $\rho_1 \in (0, 1)$. Using Lemmas 4.8 iii) and 5.1 and proceeding by induction, one can show that under $F^{R_{N+1}}$, horizontal foliation of $\mathcal{W}_0$ maps to a foliation by vertical quadratic curves in $\mathcal{W}_0$. Similarly, using Lemmas 4.8 iv) and 5.2 and proceeding by induction, one can show that under $F^{-R_{N+1}}$, horizontal foliation of $F^{R_{N+1}}(\mathcal{W}_0)$ maps to a $\lambda^{(1-\bar{\varepsilon})R_N}$-vertical foliation of $\mathcal{W}_0$. Let Ψ^{N+1} be a genuine horizontal chart that rectifies this vertical foliation. Then it follows immediately that $(F^{R_{N+1}}, \Psi^{N+1})$ is a Hénon-like return. $\square$

Proof of Theorem E. Let $\mathfrak{F} = \{F_a\}_{a\in\mathfrak{I}} \subset \mathfrak{HL}^6$ be a C^1-smoothly parameterized family of Hénon-like maps that satisfy the following properties. For $a \in \mathfrak{I}$, the map F_a is n_0-times $(L, \varepsilon, \lambda)$-regularly Hénon-like renormalizable with combinatorics of $\mathbf{b}$-bounded type, where n_0 is sufficiently large so that (9.4) and (11.1) are satisfied.

For some $N \geq n_0$, suppose that F_a has N Hénon-like returns $\{(F_a^{R_n}, \Phi_a^n)\}_{n=1}^N$ with combinatorics of $\mathbf{b}$-bounded type. For $n_0 \leq n \leq N$, Theorem 10.4 implies that (F^{R_n}, Φ^n) is $(\mathbf{L}, \delta, \lambda)$-regular for some uniform constants $\mathbf{L} \geq 1$ and $\delta \in (\bar{\varepsilon}, 1)$ with $\mathbf{b}\bar{\delta} < 1$. Moreover, by Theorem 9.7, we have $\mathcal{R}^n(F_a) \in \mathfrak{HL}^5_{\lambda_n}(\mathbf{K})$ with $\lambda_n := \lambda^{(1-\bar{\delta})R_n}$.

Proceeding by induction, suppose that there exists an interval of parameters $\mathfrak{I}_N \subset \mathfrak{I}$ such that the following properties hold.

- For $a \in \mathfrak{I}_N$, the map F_a is N-times Hénon-like renormalizable with combinatorics of $\mathbf{b}$-bounded type.
- For all $a, a' \in \mathfrak{I}_N$, we have $\tau_N(F_a) = \tau_N(F_{a'})$.
- Denote $F_{N,a} := \mathcal{R}^N(F_a)$ and $f_{N,a} := \Pi_{1\mathrm{D}}(F_{N,a})$. Then $\mathfrak{f}_N := \{f_{N,a}\}_{a\in\mathfrak{I}_N}$ forms a $(\eta_1/2, \mathbf{b})$-full C^1-smoothly parameterized family.

Let T be a renormalization type with return time $r_n \leq \mathbf{b}$. Lemma 11.1 implies that there exists an interval $\mathfrak{I}_{N+1} \subset \mathfrak{I}_N$ such that $\tau(f_{N,a}) = T$ for $a \in \mathfrak{I}_{N+1}$, and $\{\mathcal{R}_{1\mathrm{D}}(f_{N,a})\}_{a\in\mathfrak{I}_{N+1}}$ is $(\eta_1, \mathbf{b})$-full. By Proposition 11.2, F_a is $(N+1)$-times Hénon-like renormalizable, and $\tau(F_{N,a}) = \tau(f_{N,a})$. Moreover, we see from Proposition 9.5 that

$$\|f_{N,a}^{r_N} - \Pi_{1\mathrm{D}}(F_{N,a}^{r_N})\|_{C^0} < \lambda^{(1-\bar{\delta})R_N} < c\eta_1.$$

It follows that $\{\Pi_{1\mathrm{D}} \circ \mathcal{R}^{N+1}(F_a)\}_{a\in\mathfrak{I}_{N+1}}$ is $(\eta_1/2, \mathbf{b})$-full. □

12. Uniform C^r-Bounds

Let F be the infinitely regularly Hénon-like renormalizable map with combinatorics of $\mathbf{b}$-bounded type considered in Subsection 9.2 (with $N = \infty$). For $n \geq n_0$, denote

$$\tilde{F}_n = p\mathcal{R}^n(F) := \Psi^n \circ F^{R_n} \circ (\Psi^n)^{-1} \quad \text{and} \quad \tilde{f}_n := \Pi_{1\mathrm{D}}(F_n).$$

By Corollary 8.6, there exists a uniform constant $\mathbf{K} \geq 1$ such that $\tilde{f}_n$ has $\mathbf{K}$-bounded non-linearity.

Consider the arcs

$$\mathcal{I}_0^n := (\Psi^n)^{-1}(I_0^n \times \{0\}) = \mathcal{I}_0^h \cap \mathcal{B}_0^n \ni v_0$$

and $\mathcal{I}_i^n := F^i(\mathcal{I}_0^n)$ for $i \in \mathbb{N}$. Let $\{\mathcal{J}_i^n\}_{i=0}^{R_n-1}$ be the collection of arcs given in (8.3). Recall that for $n_0 \leq m \leq n$; $0 \leq k < R_n/R_m$ and $0 \leq i < R_m$, we have

$$\mathcal{J}_0^n := \mathcal{I}_0^n, \quad \mathcal{J}_{kR_m}^n \subset \mathcal{J}_0^m \quad \text{and} \quad \mathcal{J}_{i+kR_m}^n = \hat{H}_i(\mathcal{J}_{kR_m}^n). \tag{12.1}$$

Moreover, $\{\mathcal{J}_i^n\}_{i=0}^{R_n-1}$ is pairwise disjoint by Lemma 8.10.

The map

$$\phi_0 := P_0|_{\mathcal{I}_0^h} : \mathcal{I}_0^h \to I_0^h$$

gives a parameterization of $\mathcal{I}_0^h$ by its arclength. For $n \geq n_0$ and $0 \leq l < R_n/R_{n_0}$, let

$$J_{lR_{n_0}}^n := \phi_0(\mathcal{J}_{lR_{n_0}}^n).$$

Observe that $\{J^n_{lR_{n_0}}\}_{l=0}^{R_n/R_{n_0}-1}$ is a pairwise disjoint set of intervals contained in $\mathbb{R}$. Moreover,

$$J^{n+1}_{kR_n} = \Pi_{1\mathrm{D}} \circ \tilde{F}^k_n(J^{n+1}_0) \quad \text{for} \quad 0 \leq k < r_n. \tag{12.2}$$

Let $\gamma \subset \Gamma$ be C^1-curves in $\mathbb{R}^2$. We say that γ is *commensurable with* Γ if $|\gamma| \asymp |\Gamma|$.

Proposition 12.1. *Let $n \geq n_0$ and $0 \leq i < R_n$. Then any arc $\mathcal{J}^{n+1}_{i+kR_n}$ for some $0 \leq k < r_n$, or any component of*

$$\mathcal{J}^n_i \setminus \bigcup_{k=0}^{r_n-1} \mathcal{J}^{n+1}_{i+kR_n}$$

is commensurable with $\mathcal{J}^n_i$. Consequently, there exists a uniform constant $\rho \in (0,1)$ such that

$$\sum_{i=0}^{R_n-1} |\mathcal{J}^n_i| < O(\rho^n).$$

Proof. Denote the critical point of $\tilde{f}_n$ by c^n. Then by Proposition 9.3, we see that each component of

$$J^n_0 \setminus \bigcup_{k=0}^{3r_n+1} \tilde{f}^k_n(c^n)$$

is commensurate with J^n_0. Thus, by Proposition 9.5 and (12.2), this implies the result in the case $i = 0$. The case $0 < i < R_n$ then follows immediately from Theorem 8.5 and (12.1). □

The map

$$\phi_{-1} := P_{-1}|_{\mathcal{I}^{n_0}_{R_{n_0}-1}} : \mathcal{I}^{n_0}_{R_{n_0}-1} \to I^{n_0}_{R_{n_0}-1}$$

gives a parameterization of $\mathcal{I}^{n_0}_{R_{n_0}-1}$. Denote

$$J^n_{lR_{n_0}-1} := \phi_{-1}(\mathcal{J}^n_{lR_{n_0}-1}) \quad \text{for} \quad 1 \leq l \leq R_n/R_{n_0}.$$

Observe that $\{J^n_{lR_{n_0}-1}\}_{l=1}^{R_n/R_{n_0}}$ is a pairwise disjoint set of intervals contained in $\mathbb{R}$. Define

$$\gamma^n_{-1} := \bigcup_{l=3}^{R_n/R_{n_0}-1} J^n_{lR_{n_0}-1} \subset I^h_{-1} \quad \text{and} \quad \gamma^n_0 := \bigcup_{l=3}^{R_n/R_{n_0}-1} J^n_{lR_{n_0}} \subset I^h_0. \tag{12.3}$$

Proposition 8.18 gives the following decomposition of $\hat{H}_{R_n-1}$:

$$\hat{H}_{R_n-1}|_{\mathcal{I}^n_0} = F^{R_{n_0}-1}|_{\mathcal{I}^{n_0}_0} \circ \check{H}_{\frac{R_n}{R_{n_0}}-1} \circ \ldots \circ \check{H}_3 \circ \mathcal{P}^{n_0}_0 \circ F^{2R_{n_0}}|_{\mathcal{I}^n_0}.$$

where for $3 \leq l < R_n/R_{n_0}$, we have

$$\check{H}_l := \mathcal{P}^{\hat{m}_l}_0 \circ F \circ \left(\mathcal{P}^{n_0}_{-1}|_{\check{\mathcal{I}}^n_l}\right)^{-1} \circ F^{R_{n_0}-1}|_{\mathcal{I}^{n_0}_0}.$$

Define

$$\Gamma^n_{-1} := \bigcup_{l=3}^{R_n/R_{n_0}-1} \check{\mathcal{I}}^n_l \subset \mathcal{U}_{-1} \subset \mathbb{R}^2.$$

Lemma 12.2. *For $n \in \mathbb{N}$ and $3 \leq l < R_n/R_{n_0}$, the map P_{-1} restricts to a diffeomorphism from $\check{\mathcal{I}}^n_l$ to $J^n_{lR_{n_0}-1}$ (and hence, also from Γ^n_{-1} to γ^n_{-1}). Define*

$$g^n_{-1} := \pi_v \circ \Phi_{-1} \circ (P_{-1}|_{\Gamma^n_{-1}})^{-1}.$$

Then

$$\|g^n_{-1}|_{(-t,t)}\|_{C^r} = O(t^{1/\bar{\varepsilon}}).$$

Proof. The first claim follows immediately from Proposition 8.18.

Observe that $\hat{m}_l$ is the largest integer such that

$$\{0\} \cup J^n_{lR_{n_0}-1} \subset J^{\hat{m}_l}_{R_{\hat{m}_l}-1}.$$

Moreover,

$$J^n_{lR_{n_0}-1} \subset J^{\hat{m}_l+1}_{\hat{a}_l R_{n_0}-1} \quad \text{and} \quad 0 \notin J^{\hat{m}_l+1}_{\hat{a}_l R_{n_0}-1}.$$

By Proposition 8.18, $\check{\mathcal{I}}^n_l$ is $\lambda^{(1-\bar{\varepsilon})R_{\hat{m}_l}}$-horizontal. Additionally, by Proposition 12.1, we have

$$\mathrm{dist}(0, \check{I}^n_l) \asymp \rho^{\hat{m}_l}$$

for some uniform constant $\rho \in (0,1)$. The estimate on g^n_{-1} follows. □

Let $G : \mathcal{I} \to \mathcal{J}$ be a C^1-diffeomorphism between two C^1-curves $\mathcal{I}, \mathcal{J} \subset \mathbb{R}^2$. Define the *zoom-in operator* $\mathbf{Z}$ by

$$\mathbf{Z}(G)(t) := |\mathcal{J}|^{-1} \cdot \phi_{\mathcal{J}}^{-1} \circ G \circ \phi_{\mathcal{I}}(|\mathcal{I}|t),$$

where $\phi_{\mathcal{I}} : [0, |\mathcal{I}|] \to \mathcal{I}$ is the parameterization of $\mathcal{I}$ by its arclength (and $\phi_{\mathcal{J}}$ similarly defined). Note that $\mathbf{Z}(G) : [0,1] \to [0,1]$.

This rest of this section is devoted to proving the following theorem.

Theorem 12.3. *There exists a universal constant $K > 0$ such that for all $n \geq n_0$ sufficiently large and $1 \leq i < R_n$, we have*

$$\|\mathbf{Z}(\hat{H}_i|_{\mathcal{I}^n_0})\|_{C^r} < K.$$

Define

$$\mathbf{q}(x) := \mathrm{sign}(x)x^2.$$

Denote $\check{I}^h_0 := \mathbf{q}^{-1}(I^h_0)$. For $n \geq n_0$ and $0 \leq l < R_n/R_{n_0}$, let $\check{J}^n_{lR_{n_0}} := \mathbf{q}^{-1}(J^n_{lR_{n_0}})$. The proof of Theorem 12.3 relies on the following key result.

Proposition 12.4. *Let $n \in \mathbb{N}$. There exists a C^r-diffeomorphism $\check{h}^n : I^h_0 \to \check{I}^h_0$ with*

$$\|(\check{h}^n)^{\pm 1}\|_{C^r} = O(1)$$

such that for $1 \leq l \leq R_n/R_{n_0}$, we have

$$\phi_0 \circ \hat{H}_{lR_{n_0}} \circ \phi_0^{-1}|_{I^n_0} = (\mathbf{q}^n_l \circ \check{h}^n_l) \circ \ldots \circ (\mathbf{q}^n_2 \circ \check{h}^n_2) \circ (\mathbf{q}^n_1 \circ \check{h}^n_1),$$

where $\check{h}^n_l : J^n_{(l-1)R_{n_0}} \to \check{J}^n_{lR_{n_0}}$ *and* $\mathbf{q}^n_l : \check{J}^n_{lR_{n_0}} \to J^n_{lR_{n_0}}$ *are diffeomorphisms given by*

$$\check{h}^n_l := \check{h}^n|_{J^n_{(l-1)R_{n_0}}} \quad \textit{and} \quad \mathbf{q}^n_l := \mathbf{q}|_{\check{J}^n_{lR_{n_0}}}. \tag{12.4}$$

Lemma 12.5. *For $n \in \mathbb{N}$ and $3 \leq l < R_n/R_{n_0}$, we have*

$$P_0^{\hat{m}_l} \circ F \circ (\mathcal{P}^{n_0}_{-1}|_{\check{\mathcal{I}}^n_l})^{-1} \circ F^{R_{n_0}-1} \circ \phi_0^{-1}|_{J^n_{(l-1)R_{n_0}}} = \mathbf{q}^n_l \circ \check{h}^n_l(x),$$

where $\check{h}^n_l$ and $\mathbf{q}^n_l$ are as defined in (12.4).

Proof. Define $\check{\gamma}^n_0 := \mathbf{q}^{-1}(\gamma^n_0)$, where γ^n_0 is given in (12.3). By Lemmas 8.1 and 12.2, there exists a C^r-diffeomorphism $\psi^n_{-1,0} : \gamma^n_{-1} \to \check{\gamma}^n_0$ with

$$\|(\psi^n_{-1,0})^{\pm 1}\|_{C^r} = O(1)$$

such that

$$P_0^{\hat{m}_l} \circ F \circ \Phi^{-1}_{-1} \circ G^n_{-1}|_{\check{I}^n_l} = \mathbf{q} \circ \psi^n_{-1,0}|_{\check{I}^n_l},$$

where $G^n_{-1}(x) := (x, g^n_{-1}(x))$. Precomposing with $P_{-1} \circ F^{R_{n_0}-1} \circ \phi_0^{-1}|_{J^n_{(l-1)R_{n_0}}}$ gives the desired result. $\square$

Proof of Theorem 12.3. For $1 \leq l < R_n/R_{n_0}$, let $n_0 \leq \hat{m}_l \leq n$ be the largest integer such that

$$\{0\} \cup \check{J}^n_{lR_{n_0}} \subset \check{J}^{\hat{m}_l}_{R_{\hat{m}_l}}.$$

Denote $\mathbb{L}^n_m := \{1 \leq l < R_n/R_{n_0} \mid \hat{m}_l = m\}$. Then $l \in \mathbb{L}^n_m$ if and only if

$$\check{J}^n_{lR_{n_0}} \subset \check{J}^m_{R_m} \quad \text{and} \quad \check{J}^n_{lR_{n_0}-1} \cap \check{J}^{m+1}_{R_{m+1}} = \varnothing.$$

Note that

$$\bigcup_{m=n_0}^{n} \mathbb{L}^n_m = \{1 \leq l < R_n/R_{n_0}\}.$$

Let $U^m_{R_m}$ be the component of $\check{J}^m_{R_m} \setminus \check{J}^{m+1}_{R_{m+1}}$ contained in $\mathbb{R}^-$. Applying Proposition 12.1 and Lemma D.5 to $\mathbf{Z}\left(\mathbf{q}|_{U^m_{R_m}}\right)$, we see that

$$\sum_{l \in \mathbb{L}^n_m} \|\mathbf{Z}(\mathbf{q}^n_l) - \mathrm{Id}\,\|_{C^r} = O(\rho^m)$$

for some uniform constant $\rho \in (0,1)$. The result now follows from Proposition 12.1, Proposition 12.4, and Lemmas D.5 and D.6. $\square$

Theorem 12.6. *For all $n \in \mathbb{N}$ sufficiently large, we have $\|\mathcal{R}^n(F)\|_{C^r} = O(1)$.*

Proof. By Theorem 12.3 and (8.2), we see that

$$\|\Pi_{1\mathrm{D}} \circ \mathcal{R}^n(F)\|_{C^r} = O(1).$$

Since $\mathcal{R}^n(F)$ is a $\lambda^{(1-\bar{\varepsilon})R_n}$-thin Hénon-like map, the result follows. $\square$

13. Exponentially Small Pieces

Let F be the infinitely regularly Hénon-like renormalizable map considered in Section 12. The goal of this section is to prove Theorem B.

For any integer $l \geq 2$, we have

$$dR_{n_0} = a_1 R_{n_1} + \ldots + a_k R_{n_k}, \tag{13.1}$$

for some

- $n_0 \leq n_1 < \ldots < n_k$,
- $1 \leq a_m < r_{n_k}$ for $1 \leq m < k$, and
- $2 \leq a_k < 2r_{n_k}$.

Define

$$\hat{\mathcal{H}}_{lR_{n_0}} := F^{a_1 R_{n_1}} \circ \mathcal{P}_0^{n_2} \circ F^{a_2 R_{n_2}} \circ \ldots \circ \mathcal{P}_0^{n_k} \circ F^{a_k R_{n_k}} \circ \mathcal{P}_0^{n_k},$$

where $\mathcal{P}_0^n : \mathcal{B}_0^n \to \mathcal{I}_0^n$ for $n \geq n_0$ is the projection map onto $\mathcal{I}_0^n$ given by

$$\mathcal{P}_0^n := (\Psi^n)^{-1} \circ \Pi_h \circ \Psi^n.$$

Denote $\hat{m}(lR_{n_0}) := n_1$ and $\hat{k}(lR_{n_0}) := n_k$. Recall the definition of $\hat{H}_i$ given in Section 8. Then we have

$$\mathcal{P}_0^{\hat{m}(lR_{n_0})} \circ \hat{\mathcal{H}}_{lR_{n_0}} = \hat{H}_{lR_{n_0}} \circ \mathcal{P}_0^{\hat{k}(lR_{n_0})}. \tag{13.2}$$

Lemma 13.1. *Let $i = lR_{n_0}$ for some $l \geq 2$. Then for $n \geq \hat{k}(i)$, we have*

$$\|\hat{\mathcal{H}}_i - F^i|_{\mathcal{B}_0^{\hat{k}(i)}}\|_{C^0} < \lambda^{(1-\bar{\varepsilon})R_{\hat{m}(i)}}.$$

Proof. By Theorem 4.7 and Proposition 9.4, $\|(\Psi^m)^{\pm 1}\|_{C^{r+3}}$ and $\|\tilde{F}_m\|_{C^1}$ are uniformly bounded. Moreover, by Theorem 4.7 ii), we have

$$\|\tilde{F}_m - \tilde{F}_m \circ \Pi_h\|_{C^{r+3}} < \lambda^{(1-\bar{\varepsilon})R_m}. \tag{13.3}$$

Let $i = lR_{n_0}$ be given by (13.1) with $\hat{k}(i) = n_k < n$. Note that

$$F^{R_{n_k}} = (\Psi^{n_k})^{-1} \circ \tilde{F}_{n_k} \circ \Psi^{n_k}$$

and

$$\hat{\mathcal{H}}_{R_{n_k}} = F^{R_{n_k}} \circ \mathcal{P}_0^{n_k} = (\Psi^{n_k})^{-1} \circ \left(\tilde{F}_{n_k} \circ \Pi_h\right) \circ \Psi^{n_k}.$$

Hence, we see by (13.3) and Lemma D.1 that

$$\|\hat{\mathcal{H}}_{R_{n_k}} - F^{R_{n_k}}|_{\mathcal{B}_0^{n_k}}\|_{C^0} < \lambda^{(1-\bar{\varepsilon})R_{n_k}}.$$

Moreover,

$$\hat{\mathcal{H}}_{a_k R_{n_k}} = \left((\Psi^{n_k})^{-1} \circ \tilde{F}_{n_k}^{a_k - 1} \circ \Psi^{n_k}\right) \circ \hat{\mathcal{H}}_{R_{n_k}}$$

and

$$F^{a_k R_{n_k}} = \left((\Psi^{n_k})^{-1} \circ \tilde{F}_{n_k}^{a_k - 1} \circ \Psi^{n_k}\right) \circ F^{R_{n_k}}$$

Thus, another application of Lemma D.1 imply

$$\|\hat{\mathcal{H}}_{a_k R_{n_k}} - F^{a_k R_{n_k}}|_{\mathcal{B}_0^{n_k}}\|_{C^0} < \lambda^{(1-\bar{\varepsilon})R_{n_k}}.$$

Proceeding by induction, suppose that

$$\|\hat{\mathcal{H}}_{i_{j+1}} - F^{i_{j+1}}|_{\mathcal{B}_0^{n_k}}\|_{C^0} < \lambda^{(1-\bar{\varepsilon})R_{n_{j+1}}}.$$

where $1 \leq j < k$ and

$$i_{j+1} := a_{n_{j+1}} R_{n_{j+1}} + \ldots + a_{n_k} R_{n_k}.$$

Write

$$\hat{\mathcal{H}}_{i_j} = (\Psi^{n_j})^{-1} \circ \tilde{F}_{n_j}^{a_{n_j}-1} \circ \left(\tilde{F}_{n_j} \circ \Pi_h\right) \circ \Psi^{n_j} \circ \hat{\mathcal{H}}_{i_{j+1}}$$

and

$$F^{i_j}|_{\mathcal{B}_0^{n_k}} = (\Psi^{n_j})^{-1} \circ \tilde{F}_{n_j}^{a_{n_j}-1} \circ \tilde{F}_{n_j} \circ \Psi^{n_j} \circ F^{i_{j+1}}|_{\mathcal{B}_0^{n_k}}.$$

Applying Lemma D.1, the result follows. □

Lemma 13.2. *There exists a uniform constant $\rho \in (0,1)$ such that*

$$\sum_{l=2}^{R_n/R_{n_0}} \operatorname{diam}(\hat{\mathcal{H}}_{lR_{n_0}}(\mathcal{I}_0^n)) = O(\rho^n).$$

Proof. For $2 \leq l \leq R_n/R_{n_0}$, consider the curve $\check{\mathcal{I}}_l^n \subset \mathcal{U}_{-1}$ given in Proposition 8.18. By (13.2), we have

$$\hat{\mathcal{H}}_{lR_{n_0}}(\mathcal{I}_0^n) = F \circ \left(\mathcal{P}_{-1}^{n_0}|_{\check{\mathcal{I}}_l^n}\right)^{-1} \circ F^{R_{n_0}-1} \circ \hat{H}_{(l-1)R_{n_0}}(\mathcal{I}_0^n).$$

Thus, $\{\hat{\mathcal{H}}_{lR_{n_0}}(\mathcal{I}_0^n)\}_{l=2}^{R_n/R_{n_0}}$ is the image of

$$\{\mathcal{J}_{lR_{n_0}}^n := \hat{H}_{lR_{n_0}}(\mathcal{I}_0^n)\}_{l=1}^{R_n/R_{n_0}-1}$$

under

$$G_n := F \circ \left(\mathcal{P}_{-1}^{n_0}|_{\Gamma_{-1}^n}\right)^{-1} \circ F^{R_{n_0}-1},$$

where

$$\Gamma_{-1}^n := \bigcup_{l=2}^{R_n/R_{n_0}} \check{\mathcal{I}}_l^n.$$

Since Γ_{-1}^n is uniformly horizontal, $\|G_n\|_{C^r} = O(1)$. The result now follows from Proposition 12.1. □

Theorem 13.3. *There exists a uniform constant $\tilde{\rho} \in (0,1)$ such that for $n \in \mathbb{N}$, we have*

$$\sum_{i=0}^{R_n-1} \operatorname{diam}(F^i(\mathcal{B}_{R_n}^n)) = O(\tilde{\rho}^n).$$

Proof. Choose $n_0 < m < n$ to be determined later. By Lemma 13.1, we see that for $1 \leq l < R_n/R_m$, we have

$$\operatorname{diam}(F^{lR_m}(\mathcal{B}_{R_n}^n)) < \operatorname{diam}(\hat{\mathcal{H}}_{lR_m}(\mathcal{I}_0^n)) + \lambda^{(1-\bar{\varepsilon})R_{\hat{m}(lR_m)}}.$$

Thus, by Lemma 13.2, we have

$$\sum_{l=0}^{R_n/R_m-1} \operatorname{diam}(F^{lR_m}(\mathcal{B}^n_{R_n})) = O(\rho^n) + \frac{R_n}{R_m}\lambda^{(1-\bar{\varepsilon})R_m}.$$

For m sufficiently large, the expression on the right is bounded by $O(\rho_1^n)$ for some uniform constant $\rho_1 \in (\rho, 1)$.

Let $i = j + a_0 R_{n_0} + \ldots + a_{m-1}R_{m-1} + lR_m$ with $0 \leq j < R_{n_0}$; $0 \leq a_k < r_k$ for $n_0 \leq k < m$, and $1 \leq l < R_n/R_m$. We can write

$$F^{i-lR_m} = F^j \circ (\Psi^{n_0})^{-1} \circ \tilde{F}^{a_0}_{n_0} \circ \Psi^{n_0} \circ \ldots \circ (\Psi^{m-1})^{-1} \circ \tilde{F}^{a_{m-1}}_{m-1} \circ \Psi^{m-1}.$$

By Theorem 4.7 and Proposition 9.4, we see that

$$\|F^{i-lR_m}\|_{C^1} < K^m$$

for some uniform constant $K \geq 1$. Hence,

$$\sum_{i=0}^{R_n-1} \operatorname{diam}(F^i(\mathcal{B}^n_{R_n})) = R_m K^m \sum_{l=0}^{R_n/R_m-1} \operatorname{diam}(F^{lR_m}(\mathcal{B}^n_{R_n})) = O(R_m K^m \rho_1^n).$$

For n/m sufficiently large, the expression on the right is bounded by $O(\tilde{\rho}^n)$ for some uniform constant $\tilde{\rho} \in (\rho_1, 1)$. □

Observe that Theorem B is an immediate consequence of Theorem 13.3.

14. Regular Unicriticality

Let F be the infinitely regularly Hénon-like renormalizable map considered in Section 12. Recall that the renormalization limit set of F is given by

$$\Lambda_F := \bigcap_{n=1}^{\infty} \bigcup_{i=0}^{R_n-1} \mathcal{B}^n_{R_n+i}.$$

By Theorem B, Λ_F supports a unique invariant probability measure μ given by the counting measure:

$$\mu(\mathcal{B}^n_i) = 1/R_n \quad \text{for} \quad n, i \in \mathbb{N}.$$

Proposition 14.1 (Proposition 8.3 [CLPY1])**.** *With respect to μ, the Lyapunov exponents of F on Λ_F are 0 and $\log \lambda_\mu < 0$ for some $\lambda_\mu \in (0, 1)$.*

Proposition 14.2 (Proposition 4.1 [CLPY1])**.** *For any $\eta > 0$, there exist uniform constants $N_\eta \in \mathbb{N}$ and $C_\eta \geq 1$ such that the following holds. Let $p \in \mathcal{B}^n_k$ and $E_p \in \mathbb{P}^2_p$, where $n \geq N_\eta$ and $k \geq 0$. Then for all $i \in \mathbb{N}$, we have:*

$$C_\eta^{-1}\lambda_\mu^{(1+\eta)i} < \|DF^i|_{E_p}\| < C_\eta \lambda_\mu^{-\eta i} \tag{14.1}$$

and

$$C_\eta^{-1}\lambda_\mu^{(1+\eta)i} < \operatorname{Jac}_p(F^i) < C_\eta \lambda_\mu^{(1-\eta)i}. \tag{14.2}$$

For $p \in \mathcal{B}_0^n$, define

$$E_p^{v,n} := D(\Psi^n)^{-1}(E^{gv}_{\Psi^n(p)})$$

and

$$E_p^h := D(\Psi^n)^{-1}(E^{gh}_{\Psi^n(p)}) = D(\Phi_0)^{-1}(E^{gh}_{\Phi_0(p)}).$$

Theorem 14.3. *For any $\delta > 0$, there exists $L_\delta \geq 1$ such that for all $n \in \mathbb{N}$, the nth Hénon-like return (F^{R_n}, Ψ^n) is $(L_\delta, \delta, \lambda_\mu)$-regular.*

Proof. Choose $\eta \in (0, \underline{\delta})$. It suffices to show the result for $n \geq \max\{n_0, N_\eta\}$, where N_η is given in Proposition 14.2. Let $p_0 \in \mathcal{B}_0^n$. By Lemma 10.2 and (14.2), we see that p_0 is R_n-times forward $(O(1), \bar{\eta}, \lambda_\mu)$-regular horizontally along $E^h_{p_0}$. The required forward $(L_\delta, \delta, \lambda_\mu)$-regularity (vertically) along

$$E^{v,n}_{p_0} := (D\Psi^n)^{-1}(E^{gv}_{\Psi^n(p_0)})$$

now follows from Proposition A.17.

Let $q_0 := p_{R_n}$. We claim that q_{-1} is backward regular (vertically) along

$$E^v_{q_{-1}} := (D\Phi_{-1})^{-1}(E^{gv}_{\Phi_{-1}(q_{-1})}).$$

We argue similarly as in the proof of Theorem 10.4.

Let $S = lR_{n_0}$ for some $1 \leq l < R_n/R_{n_0}$. Write

$$S = s_1 R_{n_1} + \ldots + s_k R_{n_k},$$

where $1 \leq s_i < r_{n_i}$ for $1 \leq i \leq k$. Let $1 \leq m \leq k$ be the smallest number such that $\{n_i\}_{i=m}^k$ is tempered. Denote

$$S' := s_{n_m} R_{n_m} + s_{n_{m+1}} R_{n_{m+1}} + \ldots + s_{n_k} R_{n_k},$$

and let

$$\hat{E}_{q_{-1}} := DF^{S'-1}(E^{gh}_{q_{-S'}}).$$

Subbing in (10.2) and (14.2) (instead of Proposition A.4) into (10.3), we obtain

$$K^{-(k-m)} C_\eta^{-1} \lambda_\mu^{(1+\eta)S'} < \|DF^{-S'+1}|_{E^v_{q_{-1}}}\| < K^{k-m} C_\eta \lambda_\mu^{(1-\eta)S'}.$$

Then following the same argument as in the proof of Theorem 10.4 (but using (14.1) and (14.2) instead of Proposition A.5 and Proposition A.4 respectively), we conclude that q_{-1} is $(R_n - 1)$-times backward $(L_\delta, \delta, \lambda_\mu)$-regular (vertically) along $E^v_{q_{-1}}$. $\square$

Recall that by Theorem 7.1, we have

$$\bigcap_{n=1}^{\infty} \mathcal{B}^n_{R_n} = \{v_0\}.$$

Theorem 14.4. *The orbit $\{v_m\}_{m \in \mathbb{Z}}$ is a regular quadratic critical orbit.*

Proof. By Theorem 14.3, v_0 is infinitely forward and backward $(L_\delta, \delta, \lambda_\mu)$-regular along $E^*_{v_0} = E^{ss}_{v_0} = E^c_{v_0}$ for all $\delta > 0$. Thus, $\{v_m\}_{m \in \mathbb{Z}}$ is a regular critical orbit. The quadratic tangency of $W^{ss}(v_0)$ and $W^c(v_0)$ at v_0 is given in Proposition 4.5 iii). $\square$

14.1. **Critical cover.** Let $\delta = \bar{\varepsilon}$ for some $\varepsilon \in (0,1)$. Choose $\eta \in (0, \underline{\varepsilon})$. Proposition 14.2 and Theorem 14.3 imply that by replacing F with $\mathcal{R}^{n_0}(F)$ for some $n_0 \in \mathbb{N}$ sufficiently large, we may henceforth assume the following.

- Conditions (4.1) and (9.4) hold with $\lambda = \lambda_\mu$ and $n_0 = 0$.
- The map F is *η-homogeneous*: for all $p \in \mathcal{B}$ and $E_p \in \mathbb{P}^2_p$, we have
$$\lambda_\mu^{1+\eta} < \|DF|_{E_p}\| < \lambda_\mu^{-\eta} \quad \text{and} \quad \lambda_\mu^{1+\eta} < \operatorname{Jac}_p F < \lambda_\mu^{1-\eta}.$$
- For $n \in \mathbb{N}$, the nth Hénon-like return (F^{R_n}, Ψ^n) is $(1, \eta, \lambda_\mu)$-regular.

Denote $\varepsilon' := (1+\bar{\varepsilon})\varepsilon > \varepsilon$. For $z = (a,b) \in B_0^n$ and $t \geq 0$, let
$$V_z(t) := [a-t, a+t] \times I_0^v.$$
For $p \in \mathcal{B}_0^n$ and $t \geq 0$, let
$$\mathcal{V}_p^n(t) := (\Psi^n)^{-1}(V_{\Psi^n(p)}(t)).$$
Lastly, for $t > 0$ and $p \in \mathbb{R}^2$, denote
$$\mathbb{D}_p(t) := \{q \in \mathbb{R}^2 \mid \operatorname{dist}(q,p) < t\}.$$

We now show that F is (δ, ε)-regularly unicritical on Λ_F (see Definition 2.4). First, we need to define a suitable cover of the iterated preimages of critical value v_0. For $n \in \mathbb{N}$ and $1 \leq i < r_n$, let $\mathcal{C}^n$ be the connected component of
$$\mathcal{B}^n_{R_n} \cap \mathcal{V}^n_{v_{-R_n}}(\lambda_\mu^{\varepsilon' R_n})$$
containing v_{-R_n}. Define $\mathcal{C}_i^n := F^i(\mathcal{C}^n)$ for $0 \leq j < R_n$, and
$$\mathbf{C}^N := \bigcup_{n=1}^{N+1} \bigcup_{i=0}^{R_n - 1} \mathcal{C}_i^n.$$
Note that $\{v_{-i}\}_{i=1}^{R_{N+1}} \subset \mathbf{C}^N$.

Proposition 14.5. *We have* $\operatorname{diam}(\mathcal{C}_i^n) < \lambda_\mu^{\varepsilon R_n}$. *Consequently,*
$$\mathbf{C}^N \subset \bigcup_{i=1}^{R_{N+1}} \mathbb{D}_{v_{-i}}(\lambda_\mu^{\varepsilon i}).$$

Proof. By Theorem 4.7 ii), $\mathcal{B}^n_{R_n}$ is a $\lambda_\mu^{(1-\bar{\varepsilon})R_n}$-thick strip around the curve $F^{R_n}(\mathcal{I}_0^n)$, which is vertical quadratic in $\mathcal{B}_0^n$ with the vertical tangency $\lambda_\mu^{(1-\bar{\eta})R_n}$-close to v_0. By Proposition 6.5, we have
$$\mathcal{V}_{v_{-R_n}}(\lambda_\mu^{\bar{\eta} R_n}) \cap \mathcal{V}_{v_0}(\lambda_\mu^{\bar{\eta} R_n}) = \varnothing.$$
By Lemma 5.1, the connected component Γ^n of the curve
$$\mathcal{I}^n_{R_n} \cap \mathcal{V}_{v_{-R_n}}(\lambda_\mu^{\bar{\eta} R_n})$$
is $\lambda^{-\bar{\eta} R_n}$-horizontal in $\mathcal{B}_0^n$. Consequently,
$$\operatorname{diam}(\mathcal{C}^n) \asymp |\Gamma^n| < \lambda^{-\bar{\eta} R_n} \lambda^{\varepsilon' R_n}.$$

Then by η-homogeneity of F, we have

$$\operatorname{diam}(\mathcal{C}^n_i) < \lambda^{-\bar{\eta}i} \operatorname{diam}(\mathcal{C}^n)$$

for $0 \leq i < R_n$. The result follows. $\square$

14.2. **Forward regularity away from the critical cover.** For all $p \in \Lambda_F \setminus \{v_0\}$, there exists a unique number $d_p \geq 0$ such that $p \in \mathcal{B}^{d_p}_0 \setminus \mathcal{B}^{d_p+1}_0$. Define $\operatorname{depth}(p) := d_p$. If $p = v_0$, define $\operatorname{depth}(p) = \infty$. Let $p_0 \in \Lambda_F$. For $N \in \mathbb{N}$, let $0 \leq S \leq N$ be the largest number satisfying

$$d = \operatorname{depth}(p_S) \geq \operatorname{depth}(p_i) \quad \text{for} \quad 0 \leq i \leq N.$$

Define the *valuable moment* and the *valuable depth of the N-times forward orbit of p_0* as

$$\operatorname{vm}(p_0, N) := S \quad \text{and} \quad \operatorname{vd}(p_0, N) := d$$

respectively.

Lemma 14.6. *Let $p_0 \in \Lambda_F$ and $N \in \mathbb{N}$. Denote $S := \operatorname{vm}(p_0, N)$ and $d := \operatorname{vd}(p_0, N)$. Write*

$$S = s_0 R_0 + s_1 R_1 + \ldots + s_d R_d,$$

where $0 \leq s_i < r_i$ for $0 \leq i \leq d$. If $p_0 \setminus \mathbf{C}^d$, then for $0 \leq n \leq d$ and $0 \leq s \leq s_n$, we have

$$p_{S_{n-1}+sR_n} \notin \mathcal{V}^n_{v_0}(\lambda_\mu^{\bar{\varepsilon}R_n}) \quad \textit{where} \quad S_{n-1} := s_0 R_0 + \ldots + s_{n-1} R_{n-1}.$$

Proof. If $q_0 \in \Lambda_F \cap \mathcal{V}^n_{v_0}(\lambda^{\bar{\varepsilon}R_n})$, then it follows from Theorem 4.7 ii) and η-homogeneity that $q_{-R_{n+1}} \in \mathcal{C}^{n+1}$. Thus, if $p_{S'} \in \mathcal{V}^n_{v_0}(\lambda_\mu^{\bar{\varepsilon}R_n})$, where $S' := S_{n-1} + sR_n$, then $p_{-R_{n+1}+S'} \in \mathcal{C}^{n+1}$. Therefore,

$$p_0 \in \mathcal{C}^{n+1}_{R_{n+1}-S'} \subset \mathbf{C}^n \subset \mathbf{C}^d.$$

This is a contradiction. $\square$

Lemma 14.7. *Denote*

$$\varepsilon_i = (1 + \bar{\varepsilon})^i \bar{\varepsilon} \quad \textit{for} \quad i \geq 0.$$

Let $q_0 \in \mathcal{B}^n_0$ and $E_{q_0} \in \mathbb{P}^2_{q_0}$. If

$$\measuredangle(E_{q_0}, E^{v,n}_{q_0}) > \lambda_\mu^{\varepsilon_1 R_n},$$

then

$$\|DF^{R_n}|_{E_{q_0}}\| > \lambda_\mu^{\varepsilon_2 R_n}.$$

Moreover, if $q_{R_n} \notin \mathcal{V}^n_{v_0}(\lambda_\mu^{\varepsilon_0 R_n})$, then

$$\measuredangle(E_{q_{R_n}}, E^{v,n}_{q_{R_n}}) > \lambda_\mu^{\varepsilon_1 R_n}.$$

Proof. The estimate on $\|DF^{R_n}|_{E_{q_0}}\|$ follows immediately from the $(1, \eta, \lambda_\mu)$-regularity of the Hénon-like return (F^{R_n}, Ψ^n). The estimate on $\measuredangle(E_{q_{R_n}}, E^{v,n}_{q_{R_n}})$ follows immediately from Lemma 5.1. $\square$

Lemma 14.8. *For $n, k \in \mathbb{N}$, let $q_0 \in \mathcal{B}_0^{n+k}$ and $E_{q_0} \in \mathbb{P}^2_{q_0}$. If*

$$R_n \geq \bar{\varepsilon} R_{n+k} \quad \text{and} \quad \measuredangle(E_{q_0}, E^{v,n+k}_{q_0}) > \lambda_\mu^{\bar{\varepsilon} R_{n+k}},$$

then

$$\|DF^{R_n}|_{E_{q_0}}\| > \lambda_\mu^{\bar{\varepsilon} R_n} \quad \text{and} \quad \measuredangle(E_{q_{R_n}}, E^{v,n}_{q_{R_n}}) > \lambda_\mu^{\bar{\eta} R_n}.$$

Proof. Observe that

$$\bar{\eta} R_n > \bar{\eta}\bar{\varepsilon} R_{n+k} = \bar{\varepsilon} R_{n+k}.$$

So

$$\lambda_\mu^{\bar{\eta} R_n} < \lambda_\mu^{\bar{\varepsilon} R_{n+k}}.$$

By Theorem 4.7 i), we have

$$\measuredangle(E^{v,n+k}_{q_0}, E^{v,n}_{q_0}) < \lambda_\mu^{(1-\bar{\eta})R_n}.$$

Hence,

$$\measuredangle(E_{q_0}, E^{v,n}_{q_0}) > \lambda_\mu^{\bar{\varepsilon} R_{n+k}} - \lambda_\mu^{(1-\bar{\eta})R_n} > \lambda_\mu^{\bar{\eta} R_n} - \lambda_\mu^{(1-\bar{\eta})R_n} = \lambda_\mu^{\bar{\eta} R_n}.$$

Since $\mathrm{depth}(q_{R_n}) < n$, we have $q_{R_n} \notin \mathcal{V}^n_{v_0}(\lambda_\mu^{\bar{\eta} R_n})$ by Proposition 6.4. The result then follows from Lemma 5.1. □

Theorem 14.9. *Let $p_0 \in \Lambda_F$ and $N \in \mathbb{N}$. Define*

$$\hat{E}_{p_i} := D(F^i \circ \Phi_0^{-1})(E^{gh}_{p_0}) \quad \text{for} \quad i \geq 0.$$

If $p_0 \notin \mathbf{C}^d$ with $d := \mathrm{vd}(p_0, N)$, then

$$\|DF^N|_{\hat{E}_{p_0}}\| > \lambda_\mu^{\bar{\varepsilon} N}.$$

Proof. Write

$$S := \mathrm{vm}(p_0, N) = s_0 R_0 + \ldots + s_{d_{\mathrm{in}}} R_{d_{\mathrm{in}}}$$

with $0 \leq s_n < r_n$ for $0 \leq n \leq d_{\mathrm{in}} \leq d$. Using Lemmas 14.6 and 14.7, and arguing inductively, we see that

$$\|DF^S|_{\hat{E}_{p_0}}\| > \lambda_\mu^{\bar{\varepsilon} S}, \quad p_S \notin \mathcal{V}^{d_{\mathrm{in}}}_{v_0}(\lambda_\mu^{\bar{\eta} R_{d_{\mathrm{in}}}}) \quad \text{and} \quad \measuredangle(\hat{E}_{p_S}, E^{v,d_{\mathrm{in}}}_{p_S}) > \lambda_\mu^{\bar{\eta} R_{d_{\mathrm{in}}}}.$$

Let

$$T := N - S = t_0 R_0 + \ldots + t_{d_{\mathrm{out}}} R_{d_{\mathrm{out}}}$$

with $0 \leq t_n < r_n$ for $0 \leq n \leq d_{\mathrm{out}} < d$. If $d_{\mathrm{out}} \geq d_{\mathrm{in}}$, then

$$p_S \notin \mathcal{V}^{d_{\mathrm{out}}}_{v_0}(\lambda_\mu^{\bar{\eta} R_{d_{\mathrm{out}}}}) \subset \mathcal{V}^{d_{\mathrm{in}}}_{v_0}(\lambda_\mu^{\varepsilon R_{d_{\mathrm{in}}}}) \quad \text{and} \quad \measuredangle(\hat{E}_{p_S}, E^{v,d_{\mathrm{out}}}_{p_S}) > \lambda_\mu^{\bar{\eta} R_{d_{\mathrm{out}}}}.$$

Thus, by Lemma 14.6, we have

$$\|DF^{t_{d_{\mathrm{out}}} R_{d_{\mathrm{out}}}}|_{\hat{E}_{p_S}}\| > \lambda_\mu^{\bar{\varepsilon} t_{d_{\mathrm{out}}} R_{d_{\mathrm{out}}}}.$$

Denote

$$T_n := t_0 R_0 + \ldots + t_n R_n \quad \text{and} \quad 0 \leq n \leq d_{\mathrm{out}}.$$

Note that $T_n < R_{n+1} \leq \mathbf{b} R_n$.

If $d_{\text{out}} < d_{\text{in}}$, let $\check{d} := d_{\text{out}}$, and denote $t_{d_{\text{in}}} := s_{d_{\text{in}}}$. Otherwise, let $\check{d} < d_{\text{out}}$ be the largest integer such that $t_{\check{d}} > 0$. Proceeding by induction, suppose for some $n \leq \check{d}$ with $t_n > 0$, we have

$$\|DF^{N-T_n}|_{\hat{E}_{p_0}}\| > \lambda_\mu^{\bar{\varepsilon}(N-T_n)} \quad \text{and} \quad \measuredangle(\hat{E}_{p_{N-T_n}}, E^{v,n+k}_{p_{N-T_n}}) > \lambda_\mu^{\bar{\eta}R_{n+k}},$$

where $k > 0$ is the smallest number such that $t_{n+k} > 0$.

If $R_n \geq \bar{\varepsilon}R_{n+k}$, then Lemma 14.8 implies that

$$\|DF^{t_nR_n}|_{\hat{E}_{p_{N-T_n}}}\| > \lambda_\mu^{\bar{\varepsilon}t_nR_n} \quad \text{and} \quad \measuredangle(\hat{E}_{p_{N-T_{n-1}}}, E^{v,n}_{p_{N-T_{n-1}}}) > \lambda_\mu^{\bar{\eta}R_n}.$$

If $R_n < \bar{\varepsilon}R_{n+k}$, then by η-homogeneity, we have

$$\|DF^N|_{\hat{E}_{p_0}}\| > \lambda_\mu^{(1+\eta)T_n}\|DF^{N-T_{n+k}}|_{\hat{E}_{p_0}}\| > \lambda_\mu^{\bar{\varepsilon}R_{n+k}}\lambda_\mu^{\bar{\varepsilon}(N-T_{n+k})} > \lambda_\mu^{\bar{\varepsilon}N}.$$

□

15. Renormalization Convergence

15.1. **For unimodal maps.** Let $r \geq 3$ be an integer. Consider a C^r-unimodal map $f : I \to I$ with the critical value $v \in I$. For an integer $0 \leq s \leq r$ and a number $t > 0$, the t-neighborhood of f with respect to the C^s-topology is denoted $\mathfrak{N}^s(f, t)$.

Lemma 15.1. *For $K \geq 1$ and $\mathbf{b} \geq 2$, there exists a uniform constant $t_0 = t_0(K, \mathbf{b}) > 0$ such that the following holds. Let $f \in \mathfrak{U}^r(K)$. Suppose f is non-trivially renormalizable with return time $R \leq \mathbf{b}$, and $\mathcal{R}_{1\text{D}}(f)$ has $\mathbf{b}$-bounded kneading. If $\tilde{f} \in \mathfrak{N}^s(f,t) \cap \mathfrak{U}^2$ with $2 \leq s < r$ and $t \in [0, t_0]$, then $\tilde{f}$ is valuably renormalizable with $\tau(\tilde{f}) = \tau(f)$. Moreover,*

$$\|\mathcal{R}_{1\text{D}}(f) - \mathcal{R}_{1\text{D}}(\tilde{f})\|_{C^s} < Ct,$$

where $C \geq 1$ is a uniform constant depending only on K, $\mathbf{b}$ and $\|f\|_{C^{s+1}}$.

Proof. The renormalizability of $\tilde{f}$ such that $\tau(\tilde{f}) = \tau(f)$ follows immediately from Lemma 6.2 and Proposition 9.3.

Denote the critical points of f and $\tilde{f}$ by $c = 0$ and $\tilde{c}$ respectively. Define

$$I^1 := [f(c), f^R(c)] \quad \text{and} \quad \tilde{I}^1 := [\tilde{f}(\tilde{c}), \tilde{f}^R(\tilde{c})];$$

and $f_1 := f^R|_{I^1}$ and $\tilde{f}_1 := \tilde{f}^R|_{\tilde{I}^1}$. Let S and $\tilde{S}$ be the unique orientation-preserving affine maps on $\mathbb{R}$ such that $S \circ f_1 \circ S^{-1}, \tilde{S} \circ \tilde{f}_1 \circ \tilde{S}^{-1} \in \mathfrak{U}^s$.

By Lemma D.1, we see that

$$\|f_1 - \tilde{f}_1\|_{C^s} < Ct.$$

This implies immediately that $\|S - \tilde{S}\| < Ct$. The result follows. □

Consider the full renormalization attractor $\mathfrak{A}$ contained in the space $\mathfrak{U}^\omega$ of analytic unimodal maps. For an integer $\mathbf{b} \geq 2$, the compact invariant subset of $\mathfrak{A}$ consisting of all infinitely renormalizable unimodal maps with combinatorics of $\mathbf{b}$-bounded type is denoted $\mathfrak{A}_{\mathbf{b}}$.

The following is a consequence of the fact that $\mathfrak{A}_{\mathbf{b}}$ is a hyperbolic attractor for the renormalization operator $\mathcal{R}_{1\mathrm{D}}$ acting on $\mathfrak{U}^3$.

Lemma 15.2. *Let $r \geq 3$ and $N \in \mathbb{N}$ be integers, and let $K \geq 1$ be a number. Suppose $f \in \mathfrak{U}^r$ is N-times valuably renormalizable. Then for any $f^* \in \mathfrak{A}_{\mathbf{b}}$ with $\tau_N(f) = \tau_N(f^*)$, we have:*

$$\|\mathcal{R}^n_{1\mathrm{D}}(f) - \mathcal{R}^n_{1\mathrm{D}}(f^*)\|_{C^r} = C\rho^n \|f - f^*\|_{C^r} \quad \textit{for} \quad 1 \leq n < N/2,$$

where $\rho = \rho(\mathbf{b}) \in (0,1)$ is a universal constant and $C \geq 1$ is a uniform constant depending only on $\mathbf{b}$ and $\|f\|_{C^r}$.

15.2. **For Hénon-like maps.** Let F be the infinitely regularly Hénon-like renormalizable C^{r+4}-map considered in Section 12. For $n \geq n_0$, denote $F_n := \mathcal{R}^n(F)$ and $f_n := \Pi_{1\mathrm{D}}(F_n)$. By Theorem 9.7, we have $F_n \in \mathfrak{HL}^{r+3}_{\lambda_n}(\mathbf{K})$, where $\mathbf{K} \geq 1$ is a uniform constant, and $\lambda_n := \lambda^{(1-\bar{\varepsilon})R_n}$. Moreover, by Theorem 12.6, $\|F_n\|_{C^r}$ is uniformly bounded.

Proposition 15.3 (Shadowing Lemma)**.** *For $N \in \mathbb{N}$, there exists $n_1 = n_1(N) \in \mathbb{N}$ such that for all $n \geq n_1$, the map f_n is N-times valuably renormalizable with $\tau_N(f_n) = \tau_N(F_n)$. Moreover, we have*

$$\|f_{n+k} - \mathcal{R}^k_{1\mathrm{D}}(f_n)\|_{C^{r-1}} < C^k \lambda^{(1-\bar{\varepsilon})R_n} \quad \textit{for} \quad 1 \leq k \leq N$$

for some uniform constant $C \geq 1$.

Proof. First, consider the case when $N = 1$. The renormalizability of f_n so that $\tau(f_n) = \tau(F_n)$ follows immediately from Lemma 6.2, and Propositions 9.5 and 9.6.

Note that

$$\|F_n - F_n \circ \Pi_h\|_{C^r} < \lambda^{(1-\bar{\varepsilon})R_n}.$$

Since $\|F_n\|_{C^r}$ is uniformly bounded, Lemma D.1 implies that

$$\|F_n^{r_n} - (F_n \circ \Pi_h)^{r_n}\|_{C^{r-1}} < C\lambda^{(1-\bar{\varepsilon})R_n}$$

for some uniform constant $C \geq 1$. Thus,

$$\|\Pi_{1\mathrm{D}}(F_n^{r_n}) - f_n^{r_n}\|_{C^{r-1}} < C\lambda^{(1-\bar{\varepsilon})R_n}.$$

It follows that if S and $\tilde{S}$ are the unique orientation-preserving affine maps on $\mathbb{R}$ such that $S \circ \Pi_{1\mathrm{D}}(F_n^{r_n}) \circ S^{-1}, \tilde{S} \circ f_n^{r_n} \circ \tilde{S}^{-1} \in \mathfrak{U}^s$, then

$$\|S - \tilde{S}\| < C\lambda^{(1-\bar{\varepsilon})R_n}.$$

Thus,

$$\|f_{n+1} - \mathcal{R}_{1\mathrm{D}}(f_n)\|_{C^{r-1}} < C\lambda^{(1-\bar{\varepsilon})R_n}.$$

Proceeding inductively, suppose that the result is true for all $1 \leq k < N$. In particular, we have

$$\|f_{n+N-1} - \mathcal{R}^{N-1}_{1\mathrm{D}}(f_n)\|_{C^{r-1}} < C^{N-1}\lambda^{(1-\bar{\varepsilon})R_n}.$$

By the above argument, f_{n+N-1} is valuably renormalizable so that

$$\tau(f_{n+N-1}) = \tau(F_{n+N-1}).$$

If n_1 is sufficiently large, it follows from Lemma 15.1 that $\mathcal{R}_{1\mathrm{D}}^{N-1}(f_n)$ is also valuably renormalizable, and

$$\tau(\mathcal{R}_{1\mathrm{D}}^{N-1}(f_n)) = \tau(f_{n+N-1}).$$

For $m \in \mathbb{N}$, we have

$$\|f_{n+m} - \mathcal{R}_{1\mathrm{D}}(f_{n+m-1})\|_{C^{r-1}} < \lambda^{(1-\bar{\varepsilon})R_{n+m}}.$$

Applying Lemma 15.1 $0 \leq k < N$ times, we obtain

$$\|\mathcal{R}_{1\mathrm{D}}^{k}(f_{n+m}) - \mathcal{R}_{1\mathrm{D}}^{k+1}(f_{n+m-1})\|_{C^{r-1}} < C^k \lambda^{(1-\bar{\varepsilon})R_{n+m}}.$$

Thus,

$$\begin{aligned}
\|f_{n+N} - \mathcal{R}_{1\mathrm{D}}^{N}(f_n)\|_{C^{r-1}} &\leq \sum_{k=0}^{N-1} \|\mathcal{R}_{1\mathrm{D}}^{k}(f_{n+N-k}) - \mathcal{R}_{1\mathrm{D}}^{k+1}(f_{n+N-(k+1)})\|_{C^{r-1}} \\
&< \sum_{k=0}^{N-1} C^k \lambda^{(1-\bar{\varepsilon})R_{n+N-k}} \\
&< O(C^N \lambda^{(1-\bar{\varepsilon})R_n}).
\end{aligned}$$

□

Proof of Theorem C. Statements i), ii) and iii) are given by Theorem 4.7. Statement iv) is given by Theorem 12.6. Hence, it remains to prove Statement v).

Suppose $r \geq 4$. Let $f^* \in \mathfrak{A}_{\mathbf{b}}$ so that

$$\tau_\infty(f^*) = \tau_\infty(F) := [\tau(f_0), \tau(f_1), \ldots].$$

Denote $f_n^* := \mathcal{R}_{1\mathrm{D}}^n(f^*)$ for $n \geq 0$.

Consider the constants $C \geq 1$ and $\rho \in (0,1)$ given in Lemma 15.2. Choose $N \in \mathbb{N}$ sufficiently large so that $C\rho^N < \tilde{\rho} < 1$. Let $n_1 = n_1(2N) \in \mathbb{N}$ be the number given in Proposition 15.3. Then for all $n \geq n_1$, we have

$$\begin{aligned}
\|f_{n+N} - f_{n+N}^*\|_{C^{r-1}} &\leq \|f_{n+N} - \mathcal{R}_{1\mathrm{D}}^N(f_n)\|_{C^{r-1}} + \|\mathcal{R}_{1\mathrm{D}}^N(f_n) - \mathcal{R}_{1\mathrm{D}}^N(f_n^*)\|_{C^{r-1}} \\
&\leq O(\lambda^{(1-\bar{\varepsilon})R_n}) + \tilde{\rho}\|f_n - f_n^*\|_{C^{r-1}} \\
&< \tilde{\rho}'\|f_n - f_n^*\|_{C^{r-1}},
\end{aligned}$$

for some uniform constant $\tilde{\rho}' \in (0,1)$. □

Appendix A. Quantitative Pesin Theory

In this section, we summarize the results in [CLPY2]. Let $r \geq 2$ be an integer, and consider a C^{r+1}-diffeomorphism $F : \mathcal{B} \to F(\mathcal{B}) \Subset \mathcal{B}$, where $\mathcal{B} \subset \mathbb{R}^2$ is a bounded domain. Let $\lambda, \varepsilon \in (0,1)$ with $\bar{\varepsilon} < 1$.

Let $p_0 \in \mathcal{B}$ and $E^v_{p_0} \in \mathbb{P}^2_{p_0}$. For $m \in \mathbb{Z}$, decompose the tangent space at p_m as

$$\mathbb{P}^2_{p_m} = (E^v_{p_m})^\perp \oplus E^v_{p_m}.$$

In this decomposition, we have

$$D_{p_m}F =: \begin{bmatrix} \alpha_m & 0 \\ \zeta_m & \beta_m \end{bmatrix},$$

where $\alpha_m, \beta_m > 0$ and $\zeta_m \in \mathbb{R}$.

For some $M, N \in \mathbb{N} \cup \{0, \infty\}$ and $L \geq 1$, suppose for $s \in \{0,1\}$, we have

$$L^{-1}\lambda^{(1+\varepsilon)n} \leq \frac{\beta_0 \dots \beta_{n-1}}{(\alpha_0 \dots \alpha_{n-1})^s} \leq L\lambda^{(1-\varepsilon)n} \quad \text{for} \quad 1 \leq n \leq N,$$

and

$$L^{-1}\lambda^{(1+\varepsilon)n} \leq \frac{\beta_{-n} \dots \beta_{-1}}{(\alpha_{-n} \dots \alpha_{-1})^s} \leq L\lambda^{(1-\varepsilon)n} \quad \text{for} \quad 1 \leq n \leq M.$$

Then we say that p_0 is *(M,N)-times $(L, \varepsilon, \lambda)$-regular along $E^v_{p_0}$*.

Proposition A.1 (Growth in irregularity)**.** [CLPY2, Proposition 5.5] *For $-M \leq m \leq N$, let $\mathcal{L}_{p_m} \geq 1$ be the minimum value such that p_m is $(M+m, N-m)$-times $(\mathcal{L}_{p_m}, \varepsilon, \lambda)$-regular along $E^v_{p_m}$. Then*

$$\mathcal{L}_{p_m} < \bar{L}\lambda^{-\bar{\varepsilon}|m|}.$$

A.1. **Linearization.** For $w, l > 0$, denote

$$\mathbb{B}(w,l) := (-w,w) \times (-l,l) \subset \mathbb{R}^2 \quad \text{and} \quad \mathbb{B}(l) := \mathbb{B}(l,l).$$

Theorem A.2 (Regular charts)**.** [CLPY2, Theorem 6.1] *There exists a uniform constant*

$$C = C(\|DF\|_{C^r}, \lambda^{-\varepsilon}) \geq 1$$

such that the following holds. For $-M \leq m \leq N$, let

$$\omega := \frac{\lambda^{1-\varepsilon}}{1-\lambda^{1-\varepsilon}} \cdot \|DF^{-1}\| \cdot \|DF\| \quad \textit{and} \quad \mathcal{K}_{p_m} := \bar{L}(1+\omega)^5\|DF^{-1}\|\lambda^{1-\bar{\varepsilon}|m|}.$$

Define

$$U_{p_m} := \mathbb{B}(l_{p_m}) \quad \textit{where} \quad l_{p_m} := \lambda^{1+\bar{\varepsilon}}(C\mathcal{K}_{p_m})^{-1}.$$

Then there exists a C^r-chart $\Phi_{p_m} : (\mathcal{U}_{p_m}, p_m) \to (U_{p_m}, 0)$ such that $D\Phi_{p_m}(E^v_{p_m}) = E^{gv}_0$,

$$\|D\Phi^{-1}_{p_m}\|_{C^{r-1}} < C(1+\omega), \qquad \|D\Phi_{p_m}\|_{C^s} < C\mathcal{K}^{s+1}_{p_m} \quad \textit{for} \quad 0 \leq s < r,$$

and the map $\Phi_{p_{m+1}} \circ F|_{\mathcal{U}_{p_m}} \circ \Phi^{-1}_{p_m}$ extends to a globally defined C^r-diffeomorphism $F_{p_m} : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ satisfying the following properties:

i) $\|DF_{p_m}\|_{C^{r-1}} \leq \|DF\|_{C^r}$*;*

ii) we have

$$D_0 F_{p_m} = \begin{bmatrix} a_m & 0 \\ 0 & b_m \end{bmatrix}, \quad \text{where} \quad \lambda^{\bar{\varepsilon}} < a_m < \lambda^{-\bar{\varepsilon}} \quad \text{and} \quad \lambda^{1+\bar{\varepsilon}} < b_m < \lambda^{1-\bar{\varepsilon}}.$$

iii) $\|D_z F_{p_m} - D_0 F_{p_m}\|_{C^0} < \lambda^{1+\bar{\varepsilon}}$ *for* $z \in \mathbb{R}^2$*;*

iv) we have

$$F_{p_m}(x, y) = (f_{p_m}(x), e_{p_m}(x, y)) \quad \text{for} \quad (x, y) \in \mathbb{R}^2,$$

where $f_{p_m} : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ *is a* C^r*-diffeomorphism, and* $e_{p_m} : \mathbb{R}^2 \to \mathbb{R}$ *is a* C^r*-map such that for all* $0 \leq s \leq r$*, we have*

$$\partial_x^s e_{p_m}(\cdot, y) \leq \|DF\|_{C^r} |y| \quad \text{for} \quad y \in \mathbb{R}.$$

The construction in Theorem A.2 is referred to as *a linearization of* F *along the* (M, N)*-orbit of* p_0 *with vertical direction* $E^v_{p_0}$. For $-M \leq m \leq N$, we refer to l_{p_m}, $\mathcal{U}_{p_m}$, Φ_{p_m} and F_{p_m} as a *regular radius*, a *regular neighborhood*, a *regular chart* and a *linearized map at* p_m respectively.

For $p \in \mathbb{R}^2$ and $t > 0$, let

$$\mathbb{D}_p(t) := \{\|q - p\| < t\}.$$

Lemma A.3. [CLPY2, Lemma 6.2] *For* $-M \leq m \leq N$*, we have*

$$\mathcal{U}_{p_m} \supset \mathbb{D}_{p_m}\left(\frac{\lambda^{1+\bar{\varepsilon}}}{C^2 \mathcal{K}^2_{p_m}}\right),$$

where $C, \mathcal{K}_{p_m} \geq 1$ *are given in Theorem A.2.*

A.2. C^1**-estimates.**

Proposition A.4 (Jacobian bounds)**.** [CLPY2, Proposition 6.14] *We have*

$$\bar{L}^{-1} \lambda^{(1+\bar{\varepsilon})n} \leq \operatorname{Jac}_{p_0} F^n \leq \bar{L} \lambda^{(1-\bar{\varepsilon})n} \quad \text{for} \quad 1 \leq n \leq N,$$

and

$$\bar{L}^{-1} \lambda^{-(1-\bar{\varepsilon})n} \leq \operatorname{Jac}_{p_0} F^{-n} \leq \bar{L} \lambda^{-(1+\bar{\varepsilon})n} \quad \text{for} \quad 1 \leq n \leq M.$$

Proposition A.5 (Derivative bounds)**.** [CLPY2, Proposition 6.15] *Let* $C \geq 1$ *and* $\omega > 0$ *be the uniform constants given in Theorem A.2. For* $E_{p_0} \in \mathbb{P}^2_{p_0}$*, we have*

$$\frac{\lambda^{(1+\bar{\varepsilon})n}}{C \bar{L} (1 + \omega)^2} \leq \|DF^n|_{E_{p_0}}\| \leq C(1 + \omega)^2 \lambda^{-\bar{\varepsilon} n} \quad \text{for} \quad 1 \leq n \leq N,$$

and

$$\frac{\lambda^{\bar{\varepsilon} n}}{C \bar{L} (1 + \omega)^2} \leq \|DF^{-n}|_{E_{p_0}}\| \leq C(1 + \omega)^2 \lambda^{-(1+\bar{\varepsilon})n} \quad \text{for} \quad 1 \leq n \leq M.$$

Consider the sequence of linearized maps $\{F_{p_m}\}_{-M}^N$ given in Theorem A.2. For $1 \leq n \leq N - m$, we denote

$$F^n_{p_m} = (f^n_{p_m}, e^n_{p_m}) := F_{p_{m+n-1}} \circ \ldots \circ F_{p_{m+1}} \circ F_{p_m}. \tag{A.1}$$

Proposition A.6. [CLPY2, Proposition 6.4] *For* $-M \leq m \leq N$ *and* $0 \leq n \leq N - m$, *consider the* C^r*-diffeomorphism* $F^n_{p_m}$ *given in* (A.1). *Let* $z = (x, y) \in U_{p_m}$, *and suppose that*

$$z_i = (x_i, y_i) := F^i_{p_m}(z) \in U_{p_{m+i}} \quad \text{for} \quad 0 \leq i \leq n.$$

Denote

$$D_z F^n_{p_m} =: \begin{bmatrix} a^n_m(z) & 0 \\ c^n_m(z) & b^n_m(z) \end{bmatrix}.$$

Define

$$\check{l}_h := \sup_n n\lambda^{\bar{\varepsilon}n} < \infty \quad \text{and} \quad \check{l}_v := (1 - \lambda^{1-\bar{\varepsilon}})^{-1};$$

and

$$\chi_h := \exp\left(\frac{\check{l}_h \|F\|_{C^3}}{\lambda^{\bar{\varepsilon}}}\right) \quad \text{and} \quad \chi_v := \exp\left(\frac{(\check{l}_h + \check{l}_v)\|F\|_{C^3}}{\lambda^{\bar{\varepsilon}}}\right).$$

Then

$$\frac{1}{\chi_h} \leq \frac{a^n_m(z)}{a^n_m(0)} \leq \chi_h, \quad \frac{1}{\chi_v} \leq \frac{b^n_m(z)}{b^n_m(0)} \leq \chi_v \quad \text{and} \quad \|\gamma^n_m\| < \lambda^{(1-\bar{\varepsilon})n}.$$

For $-M \leq m \leq N$ and $q \in \mathcal{U}_{p_m}$, write $z := \Phi_{p_m}(q)$. The *vertical/horizontal direction at* q *in* $\mathcal{U}_{p_m}$ is defined as $E^{v/h}_q := D\Phi^{-1}_{p_m}(E^{gv/gh}_z)$. By the construction of regular charts in Theorem A.2, vertical directions are invariant under F (i.e. $DF(E^v_q) = E^v_{F(q)}$ for $q \in \mathcal{U}_{p_m}$). Note that the same is not true for horizontal directions.

Proposition A.7. [CLPY2, Proposition 6.5] *For* $-M \leq m \leq N$ *and* $q \in \mathcal{U}_{p_m}$, *we have*

$$\frac{1}{\sqrt{2}} \leq \frac{\|D\Phi_{p_m}|_{E^{v/h}_z}\|}{\|D\Phi_{p_m}|_{E^{v/h}_{p_m}}\|} \leq \sqrt{2}.$$

Corollary A.8. [CLPY2, Corollary 6.6] *For some* $-M \leq m_0 \leq N$, *let* $q_{m_0} \in \mathcal{U}_{p_{m_0}}$. *Suppose for* $m_0 \leq m \leq m_1 \leq N$, *we have* $q_m \in \mathcal{U}_{p_m}$. *Let*

$$\hat{E}^h_{q_m} := DF^{m-m_0}(E^h_{q_{m_0}}).$$

Then for $m_0 \leq m' \leq m_1$, *we have*

$$\frac{1}{2\chi_h} \leq \frac{\|DF^{m'-m}|_{\hat{E}^h_{q_m}}\|}{\|DF^{m'-m}|_{E^h_{p_m}}\|} \leq 2\chi_h \quad \text{and} \quad \frac{1}{2\chi_v} \leq \frac{\|DF^{m'-m}|_{E^v_{q_m}}\|}{\|DF^{m'-m}|_{E^v_{p_m}}\|} \leq 2\chi_v,$$

where χ_h *and* χ_v *are constants given in Proposition A.6.*

Proposition A.9 (Vertical alignment of forward contracting directions)**.** [CLPY2, Proposition 6.8] *Let* $q_0 \in \mathcal{U}_{p_0}$ *and* $\tilde{E}^v_{q_0} \in \mathbb{P}^2_{q_0}$. *Suppose* $q_i \in \mathcal{U}_{p_i}$ *for* $0 \leq i \leq n \leq N$, *and that*

$$\nu := \|DF^n|_{\tilde{E}^v_{q_0}}\| < \frac{\lambda^{\bar{\varepsilon}n}}{\chi_h(2+\omega)^3\bar{C}},$$

where $C, \omega, \chi_h \geq 1$ *are uniform constants given in Theorem A.2 and Proposition A.6. Denote* $z_0 := \Phi_{p_0}(q_0)$ *and* $\tilde{E}^v_{z_0} := D\Phi_{p_0}(\tilde{E}^v_{q_0})$*. Then*

$$\measuredangle(\tilde{E}^v_{z_0}, E^{gv}_{z_0}) < \chi_h(1+\omega)\bar{C}\lambda^{-\bar{\varepsilon}n}\nu.$$

Proposition A.10 (Horizontal alignment of backward neutral directions)**.** [CLPY2, Proposition 6.9] *Let* $q_0 \in \mathcal{U}_{p_0}$ *and* $\tilde{E}^h_{q_0} \in \mathbb{P}^2_{q_0}$*. Suppose* $q_{-i} \in \mathcal{U}_{p_{-i}}$ *for* $0 \leq i \leq n \leq M$*, and that*

$$\mu := \|DF^{-n}|_{\tilde{E}^h_{q_0}}\| < \frac{1}{\chi_v(2+\omega)^3\bar{C}\lambda^{(1-\bar{\varepsilon})n}}.$$

Denote

$$z_0 := \Phi_{p_0}(q_0), \quad \tilde{E}^h_{z_0} := D\Phi_{p_0}(\tilde{E}^h_{q_0}) \quad \textit{and} \quad \hat{E}^h_{z_0} := D\Phi_{p_0} \circ F^n(E^h_{q_{-n}}).$$

Then

$$\measuredangle(\tilde{E}^h_{q_0}, \hat{E}^h_{q_0}) < \chi_v(1+\omega)\bar{C}\lambda^{(1-\bar{\varepsilon})n} \cdot \mu.$$

The *n-times truncated regular neighborhood of* p_0 is defined as

$$\mathcal{U}^n_{p_m} := \Phi^{-1}_{p_m}\left(U^n_{p_m}\right) \subset \mathcal{U}_{p_m}, \quad \text{where} \quad U^n_{p_0} := \mathbb{B}\left(\lambda^{\bar{\varepsilon}n} l_{p_m}, l_{p_m}\right).$$

The purpose of truncating a regular neighborhood is to ensure that its iterated images stay inside regular neighborhoods.

Lemma A.11. [CLPY2, Lemma 6.10] *Let* $-M \leq m \leq N$ *and* $0 \leq n \leq N - m$*. We have* $F^i(\mathcal{U}^n_{p_m}) \subset \mathcal{U}_{p_{m+i}}$ *for* $0 \leq i \leq n$*.*

Proposition A.12. [CLPY2, Propositions B.5 and B.6] *There exists a uniform constant* $K = K(\|DF\|_{C^r}, \lambda, \varepsilon, r) \geq 1$ *such that the following result holds. For* $-M \leq m \leq N$ *and* $0 \leq n \leq N - m$*, consider the* C^r*-maps* $f^n_{p_m}$ *and* $e^n_{p_m}$ *given in* (A.1)*. Then we have*

$$\|Df^n_{p_m}\|_{C^{r-1}} < K\lambda^{-\bar{\varepsilon}n} \quad \textit{and} \quad \|De^n_{p_m}\|_{C^{r-1}} < K\lambda^{(1-\bar{\varepsilon})n}.$$

A.3. C^r**-estimates.** Let $g : \mathbb{R} \to \mathbb{R}$ be a C^r-function. The curve

$$\Gamma_g := \{(x, g(x))\ x \in \mathbb{R}\}$$

is the *horizontal graph of* g. Let $H : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^r-diffeomorphism. Suppose that there exists a C^r-function $H_*(g) : \mathbb{R} \to \mathbb{R}$ such that $H(\Gamma_g) = \Gamma_{H_*(g)}$. Then $H_*(g)$ and $\Gamma_{H_*(g)}$ are referred to as the *horizontal graph transform of* g and Γ_g *by* H respectively.

Proposition A.13 (C^r-convergence of horizontal graphs)**.** [CLPY2, Proposition 4.5] *Let* $g : \mathbb{R} \to \mathbb{R}$ *be a* C^r*-map with* $\|g'\|_{C^{r-1}} < \infty$*. For* $-M \leq m \leq N$ *and* $1 \leq n \leq N - m$*, consider the graph transform* $\tilde{g} := (F^n_{p_m})_*(g)$*. Then*

$$\|\tilde{g}'\|_{C^{r-1}} < C\lambda^{(1-\bar{\varepsilon})n}(1 + \|g'\|_{C^{r-1}})^r$$

where $C = C(\mathbf{C}, \lambda, \varepsilon, r) \geq 1$ *is a uniform constant.*

For $p \in \mathbb{R}^2$ and $u \in \mathbb{R}$, let $E^u_p \in \mathbb{P}^2_p$ be the tangent direction at p given by

$$E^u_p := \{r(u, 1) \mid r \in \mathbb{R}\}.$$

Let $\xi : \mathbb{R}^2 \to \mathbb{R}$ be a C^{r-1}-map. The direction field

$$\mathcal{E}_\xi := \{E^{\xi(p)}_p \mid p \in \mathbb{R}^2\}$$

is the *vertical direction field of* ξ. Let $H : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^r-diffeomorphism. Suppose that there exists a C^{r-1}-map $H^*(\xi) : \mathbb{R}^2 \to \mathbb{R}$ such that $DH^{-1}(\mathcal{E}_\xi) = \mathcal{E}_{H^*(\xi)}$. Then $H^*(\xi)$ and $\mathcal{E}_{H^*(\xi)}$ are referred to as the *vertical direction field transform of* ξ and $\mathcal{E}_\xi$ *by* H respectively.

Proposition A.14 (Backward vertical direction field transform)**.** [CLPY2, Proposition 4.6] *There exist uniform constants* $C, \tilde{C} \geq 1$ *depending only on* $\mathbf{C}, \lambda, \varepsilon, r$ *such that the following holds. Let* $\xi : \mathbb{R}^2 \to \mathbb{R}$ *be a* C^{r-1}*-map with* $\|\xi\|_{C^{r-1}} < \infty$*. For* $-M \leq m < N$ *and* $0 \leq n \leq M + m$*, consider the vertical direction transform*

$$\tilde{\xi} := (F^n_{p_m})^*(\xi)|_{\mathbb{R}\times(-1,1)}.$$

Suppose

$$C\lambda^{(1-\bar{\varepsilon})n}(1 + \|\xi\|_{C^{r-1}}) < 1.$$

Then

$$\|\tilde{\xi}\|_{C^{r-1}} < \tilde{C}\lambda^{(1-\bar{\varepsilon})n}\|\xi\|_{C^{r-1}}.$$

A.4. **Stable and center manifolds.** For $-M \leq m \leq N$, define the *local vertical* and *horizontal manifold at* p_m as

$$W^v_{\mathrm{loc}}(p_m) := \Phi^{-1}_{p_m}(\{(0, y) \in U_{p_m}\}) \quad \text{and} \quad W^h_{\mathrm{loc}}(p_m) := \Phi^{-1}_{p_m}(\{(x, 0) \in U_{p_m}\})$$

respectively.

If $N = \infty$, then Proposition A.9 implies that $E^v_{p_0}$ is the unique direction along which p_0 is infinitely forward regular. In this case, we denote $E^{ss}_{p_0} := E^v_{p_0}$, and refer to this direction as the *strong stable direction at* p_0. Additionally, we define the *strong stable manifold of* p_0 as

$$W^{ss}(p_0) := \left\{ q_0 \in \Omega \mid \limsup_{n\to\infty} \frac{1}{n} \log \|q_n - p_n\| < (1 - \varepsilon) \log \lambda \right\}.$$

Theorem A.15 (Canonical strong stable manifold)**.** [CLPY2, Theorem 6.13] *If* $N = \infty$*, then*

$$W^{ss}(p_0) := \bigcup_{n=0}^{\infty} F^{-n}(W^v_{\mathrm{loc}}(p_n)).$$

Consequently, $W^{ss}(p_0)$ *is a* C^{r+1}*-smooth manifold.*

If $M = \infty$, then Proposition A.10 implies that $E^h_{p_0}$ is the unique direction along which p_0 is infinitely backward regular. In this case, we denote $E^c_{p_0} := E^h_{p_0}$, and refer to this direction as the *center direction at* p_0. Moreover, we define the *(local) center manifold at* p_0 as

$$W^c(p_0) := \Phi_{p_0}^{-1}(\{(x,0) \in U_{p_0}\}).$$

Unlike strong stable manifolds, center manifolds are not canonically defined. However, the following result states that it still has a canonical jet.

Theorem A.16 (Canonical jets of center manifolds)**.** [CLPY2, Theorem 6.16] *Suppose* $M = \infty$. *Let* $\Gamma_0 : (-t,t) \to \mathcal{U}_{p_0}$ *be a* C^{r+1}*-curve parameterized by its arclength such that* $\Gamma_0(0) = p_0$, *and for all* $n \in \mathbb{N}$, *we have*

$$\|DF^{-n}|_{\Gamma_0'(t)}\| < \lambda^{-\frac{(1-\bar{\varepsilon})n}{r+1}} \quad \textit{for} \quad |t| < \lambda^{\varepsilon n}.$$

Then Γ_0 *has a degree* $r+1$ *tangency with* $W^c(p_0)$ *at* p_0.

A.5. **Horizontal regularity.** We say that $p \in \mathcal{B}$ is N*-times forward horizontally* (L,ε,λ)*-regular along* $E_p^{h,+} \in \mathbb{P}_p^2$ if, for $s \in \{1,2\}$, we have

$$L^{-1}\lambda^{(1+\varepsilon)n} \leq \frac{\operatorname{Jac}_p F^n}{\|D_pF^n|_{E_p^{h,+}}\|^s} \leq L\lambda^{(1-\varepsilon)n} \quad \text{for} \quad 1 \leq n \leq N. \tag{A.2}$$

Similarly, we say that p is M*-times backward horizontally* (L,ε,λ)*-regular along* $E_p^{h,-} \in \mathbb{P}_p^2$ if, for $s \in \{1,2\}$, we have

$$L^{-1}\lambda^{(1+\varepsilon)n} \leq \frac{\|D_pF^{-n}|_{E_p^{h,-}}\|^s}{\operatorname{Jac}_p F^{-n}} \leq L\lambda^{(1-\varepsilon)n} \quad \text{for} \quad 1 \leq n \leq M. \tag{A.3}$$

If (A.2) and (A.3) hold with $E_p^h := E_p^{h,+} = E_p^{h,-}$, then p is (M,N)*-times horizontally* (L,ε,λ)*-regular along* E_p^h.

Proposition A.17 (Horizontal vs vertical forward regularity)**.** [CLPY2, Proposition 5.2] *If* p *is* N*-times forward horizontally* (L,ε,λ)*-regular along* $E_p^h \in \mathbb{P}_p^2$, *then there exists* $E_p^v \in \mathbb{P}_p^2$ *such that* p *is* N*-times forward* $(\bar{L},\bar{\varepsilon},\lambda)$*-regular along* E_p^v.

Proposition A.18 (Horizontal vs vertical backward regularity)**.** [CLPY2, Proposition 5.3] *Suppose* p *is* M*-times backward horizontally* (L,ε,λ)*-regular along* $E_p^h \in \mathbb{P}_p^2$. *Let* $E_p^v \in \mathbb{P}_p^2 \setminus \{E_p^h\}$. *If* $\measuredangle(E_p^h, E_p^v) > \theta$, *then the point* p *is* M*-times backward* $(\bar{L}/\theta^2, \bar{\varepsilon}, \lambda)$*-regular along* E_p^v.

Appendix B. Classification of Fixed Points

Let $F : \mathcal{B} \to F(\mathcal{B}) \Subset \mathcal{B}$ be a dissipative diffeomorphism defined on a Jordan domain $\mathcal{B} \subset \mathbb{R}^2$. Suppose that $q_0 \in \mathcal{B}$ is an isolated fixed point for F, and that $\lambda_-, \lambda_+ \in \mathbb{R}$ with $0 < |\lambda_-| < |\lambda_+|$ are the eigenvalues of $D_{q_0}F$. If $|\lambda_+| \geq 1$, then q_0 has a well-defined invariant local center manifold $W^c_{\text{loc}}(q_0)$. In this case, we classify q_0 as:

- a *saddle with reflection* if the branches of $W^c_{\text{loc}}(q_0) \setminus \{q_0\}$ alternate and are repelling;
- a *saddle with no reflection* if both branches of $W^c_{\text{loc}}(q_0) \setminus \{q_0\}$ are fixed and repelling;

- a *saddle-node* if both branches of $W^c_{\text{loc}}(q_0) \setminus \{q_0\}$ are fixed and one is repelling while the other is attracting.

The *index of* q_0, denoted $\text{Index}(q_0)$, is defined as the winding number of the vector field

$$\Delta_{q_0} F(p) := F(p - x_0) - (p - q_0),$$

and can be determined based on the type of q_0 as follows:

- $\text{Index}(q_0) = 1$ if q_0 is a sink or a saddle with reflection;
- $\text{Index}(q_0) = 0$ if q_0 is a saddle-node; or
- $\text{Index}(q_0) = -1$ if q_0 is a saddle with no reflection.

Proposition B.1. *Let $F : \mathcal{B} \to F(\mathcal{B}) \Subset \mathcal{B}$ be a dissipative diffeomorphism defined on a Jordan domain $\mathcal{B} \subset \mathbb{R}^2$. Suppose that there exists an R-periodic Jordan subdomain $\mathcal{B}^1 \Subset \mathcal{B}$ for some integer $R \geq 2$. Then there exists a r-periodic saddle point in $\mathcal{B}$ for some integer r that divides R.*

Proof. If F^R has a non-isolated fixed point q_0, then q_0 must have an indifferent eigenvalue, and hence is a type of saddle.

Suppose that all fixed points of F^R are isolated. By the classical Lefschetz formula, when the number of fixed points is finite, the sum of the index of the fixed points in the disc is equal to 1. Observe that F has at least one fixed point and one R-periodic orbit in $\mathcal{B}$. Hence, not all fixed points of F^R can be sinks. □

Appendix C. Distortion Theorems for 1D Maps

In this section, we summarize some of the techniques in 1D dynamical systems used to control distortion. See [dMvS] for complete details.

Let $f : I \to f(I)$ be a C^1-diffeomorphism on an interval $I \subset \mathbb{R}$. For $J \subset I$, the *distortion of f on J* is defined as

$$\text{Dis}(f, J) := \sup_{x,y \in J} \frac{|f'(x)|}{|f'(y)|}.$$

We denote $\text{Dis}(f) := \text{Dis}(f, I)$. For $K \geq 1$, we say that f has *K-bounded distortion on J* if

$$\text{Dis}(f, J) \leq K.$$

Clearly, if $g : I' \to g(I')$ is another C^1-diffeomorphism on an interval $I' \supset f(J)$, then we have

$$\text{Dis}(g \circ f, J) \leq \text{Dis}(g, f(J)) \cdot \text{Dis}(f, J). \tag{C.1}$$

Theorem C.1 (Denjoy Lemma). *Let $f : I \to I$ be a C^r-map on an interval $I \subset \mathbb{R}$. Then there exists a uniform constant $K > 0$ such that if $f^n|_J$ is a diffeomorphism on a subinterval $J \subset I$ for some $n \in \mathbb{N}$, then*

$$\log(\text{Dis}(f^n, J)) \leq K \sum_{i=0}^{n-1} |f(J)|.$$

C.1. **Cross Ratios.** Let $J \Subset I \subset \mathbb{R}$ be bounded open intervals. The complement $I \setminus \overline{J}$ consists of two intervals L and R. The *cross-ratio of J in I* is given by

$$\mathrm{Cr}(I, J) := \frac{|I||J|}{|L||R|}.$$

For $\tau > 0$, we say that I *contains a τ-scaled neighborhood of J* if

$$|L|, |R| > \tau |J|.$$

Let $f : I \to f(I)$ be a homeomorphism. The *cross-ratio distortion under f of J in I* is given by

$$\mathrm{CrD}(f, I, J) := \frac{\mathrm{Cr}(f(I), f(J))}{\mathrm{Cr}(I, J)}.$$

Clearly, if $g : f(I) \to g \circ f(I)$ is another homeomorphism, then

$$\mathrm{CrD}(g \circ f, I, J) = \mathrm{CrD}(g, f(I), f(J)) \cdot \mathrm{CrD}(f, I, J). \tag{C.2}$$

For $\nu > 0$, we say that f has *ν-bounded cross-ratio distortion on I* if

$$\mathrm{CrD}(f, I', J) > \nu$$

for all bounded open intervals $J \Subset I' \subset I$.

Lemma C.2. *For $\alpha > 1$, let $P_\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ be an α-power map such that*

$$P_\alpha(x) = x^\alpha \quad \text{for} \quad x \in \mathbb{R}^+.$$

Then $P_\alpha|_{\mathbb{R}^+}$ has negative Schwarzian derivative. Consequently, $P_\alpha|_{\mathbb{R}^+}$ has 1-bounded cross-ratio distortion on $\mathbb{R}_+$.

Lemma C.3. *Let $I \subset \mathbb{R}$ be a bounded open interval, and let $f : I \to f(I)$ be a C^1-diffeomorphism with K-bounded distortion on I for some $K > 0$. Then there exists a uniform constant $\nu = \nu(K) > 0$ such that f has ν-bounded cross-ratio distortion on I.*

Theorem C.4 (Koebe distortion theorem)**.** *Let $J \Subset I \subset \mathbb{R}$ be bounded open intervals, and let $f : I \to f(I)$ be a C^1-diffeomorphism with ν-bounded cross-ratio distortion on I for some $\nu > 0$. If $f(I)$ contains a τ-scaled neighborhood of $f(J)$, then there exists a uniform constant $K = K(\nu, \tau) > 0$ depending only on ν and τ such that f has K-bounded distortion on J.*

Appendix D. Compositions of Nearby Maps

Lemma D.1. *Let $d \in \mathbb{N}$. Consider maps $H_0, \tilde{H}_0 : U \to V$ and $H_1, \tilde{H}_1 : V \to \mathbb{R}^d$ defined on domains $U, V \subset \mathbb{R}^d$. Suppose $H_0, \tilde{H}_0, \tilde{H}_1$ are C^{r-1} and H_1 is C^r; and*

$$\|H_i - \tilde{H}_i\|_{C^{r-1}} < \delta \quad \text{for} \quad i \in \{0, 1\}.$$

Then we have

$$\|H_1 \circ H_0 - \tilde{H}_1 \circ \tilde{H}_0\|_{C^{r-1}} < \delta P(\|H_1\|_{C^r}, \|\tilde{H}_0\|_{C^{r-1}}),$$

where P is a two-variable homogeneous polynomial of degree r.

Proof. Let $d_i := H_i - \tilde{H}_i$. A straightforward computation shows that

$$\begin{aligned} H_1 \circ H_0 &= H_1 \circ (\tilde{H}_0 + d_0) \\ &= H_1 \circ \tilde{H}_0 + O(\|DH_1 \circ \tilde{H}_0\| \|d_0\|) \\ &= \tilde{H}_1 \circ \tilde{H}_0 + d_1 \circ \tilde{H}_0 + O(\|DH_1 \circ \tilde{H}_0\| \|d_0\|). \end{aligned}$$

The result follows. □

Lemma D.2. [PuSh, (4)] *Let F, G be C^r-maps such that $F \circ G$ is well-defined. Then*

$$\|F \circ G\|_r \leq r^r \|F\|_r \|G\|_r^r,$$

where $\|F\|_r := \|DF\|_{C^{r-1}}$.

Lemma D.3. [CLPY2, Lemma B.4] *Consider a C^r-diffeomorphism $f : \mathbb{R} \to \mathbb{R}$. Suppose $\|f'\| > \mu$ for some constant $\mu \in (0, 1)$. Then there exists a uniform constant $C = C(r) \geq 1$ such that*

$$\|(f^{-1})'\|_{C^{r-1}} < C\mu^{1-2r} \|f''\|_{C^{r-2}}^{r-1}.$$

Lemma D.4. *For $r \geq 4$, let $f : I \to f(I)$ be a C^r-map defined on an interval $0 \in I \subset \mathbb{R}$ such that $f(0) = 0 = f'(0)$ and $f''(0) = \kappa > 0$. Then there exists a C^r-diffeomorphism $\psi_f : I \to \psi_f(I)$ such that $f(x) = \kappa \cdot (\psi_f(x))^2$, and $\|\psi_f^{\pm 1}\|_{C^{r-3}} < C$ for some uniform constant $C = C(\|f\|_{C^r}, \kappa, r) > 0$.*

Proof. In the proof, let $K_i > 0$ for $i \in \mathbb{N}$ be uniform constants that depend only on $\|f\|_{C^r}$, κ and r.

Write

$$\kappa^{-1} f(x) - x^2 = h(x) + \sum_{i=3}^{r} a_i x^i,$$

where

$$\lim_{x \to 0} \frac{h(x)}{x^r} = 0 \quad \text{and} \quad \|h^{(r)}\| < K_1.$$

Consequently,

$$\|h^{(i)}\| < K_2 |x|^{r-i} \quad \text{for} \quad 0 \leq i \leq r. \tag{D.1}$$

Define

$$\psi_f(x) := x\sqrt{1 + g(x)} \quad \text{where} \quad g(x) := \frac{\kappa^{-1} f(x) - x^2}{x^2}.$$

Let $J := \{|x| < 1/K_3\}$. Observe that $f(x) > 1/K_4$ for $x \in I \setminus J$. Thus, applying Lemma D.2, we have

$$\|(\psi_f|_{I \setminus J})^{\pm 1}\|_{C^r} < \left\| \left(\sqrt{f|_{I \setminus J}} \right)^{\pm 1} \right\|_{C^r} < K_5.$$

Let $\hat{h}(x) := h(x)/x^2$. We claim that that $\hat{h}^{(k)}(x)$ with $k \leq r - 3$ is a sum of a uniform number of terms of the form

$$c \frac{h^{(i)}(x)}{x^{2+k-i}} \tag{D.2}$$

for some coefficient $c \in \mathbb{R}$ independent on f and $i \leq k$. Proceeding by induction, suppose that this is true for $k < r-3$. Differentiating, (D.2), we obtain

$$c\frac{h^{(i+1)}(x)}{x^{2+k-i}} + (2+k-i)c\frac{h^{(i)}(x)}{x^{2+k-i+1}}.$$

The claim follows. Hence, by (D.1), we conclude that

$$|\hat{h}^{(k)}(x)|,\ |g^{(k)}(x)| < K_5|x| \quad \text{for} \quad 0 \leq k \leq r-3.$$

In particular, $\|g|_J\| \ll 1$.

A simple computation shows that $\|\psi_f|_J\|_{C^{r-3}} < K_6$, and $|\psi_f'(x)| > c$ for $x \in J$, where $c > 0$ is an independent constant. Applying Lemma D.3 to obtain the required bound for the inverse of ψ_f, the result follows. □

Let $g : I \to J$ be a C^1-diffeomorphism between two intervals $I, J \subset \mathbb{R}^2$. Define the *zoom-in operator* $\mathbf{Z}$ by

$$\mathbf{Z}(g)(t) := |J|^{-1} \cdot g(|I|t).$$

Note that $\mathbf{Z}(g) : [0,1] \to [0,1]$.

Lemma D.5. [AvdMMa, Lemma 5] *Let $\phi : U \to \phi(U)$ be a C^r-diffeomorphism defined on a domain $U \subset \mathbb{R}$. Then there exists a uniform constant*

$$K = K(\|\phi\|_{C^r}, \|\phi''/\phi'\|_{C^0}) \geq 1$$

such that for any interval $I \subset U$, we have

$$\|\mathbf{Z}(\phi|_I) - \operatorname{Id}\|_{C^r} \leq K|I|.$$

Lemma D.6. [AvdMMa, Lemma 6] *For $1 \leq i \leq n$, let $\phi_i : [0,1] \to [0,1]$ be a C^r-diffeomorphism such that*

$$\sum_{i=1}^{n} \|\phi_i - \operatorname{Id}\|_{C^r} = O(1).$$

Then

$$\|\phi_n \circ \ldots \circ \phi_1\|_{C^r} = O(1).$$

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