QUANTITATIVE PESIN THEORY IN DIMENSION TWO

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Abstract. We develop a measure-independent, quantitative formulation of Pesin theory. We first quantify the amount of regularity exhibited at each point in the phase space with a set of explicit inequalities. Then we relate this directly to the sizes of the regular neighborhoods and the smooth norms of the corresponding regular charts. As a corollary, we establish the existence of smooth stable and center manifolds for regular points. This provides us with the technical background for the renormalization theory of Hénon-like maps developed in [CLPY1], [Y].

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1. Introduction

In 1970's, Pesin laid down the foundations for the theory of non-uniformly hyperbolic dynamical systems through a series of landmark works [Pe1],[Pe2],[Pe3]. The classical implementation of this technique reveals statistical information about the system under consideration with respect to a given invariant measure. As such, we refer to it as *measurable Pesin theory*.

In this paper, we develop a new quantitative version of this theory. We begin by formulating the relevant regularity conditions that control dominating and hyperbolic features of finite or infinite orbits under consideration (see Definition 2.3 and Section 5). We then relate this directly to the sizes of the *regular neighborhoods* and the smooth norms of the corresponding *regular charts* (see Theorem 6.1 and Lemma 6.2). As a corollary, we obtain invariant curves that approximate the local stable and center manifolds in case of finite-time regularity, and coincide with them in case of infinite-time regularity (see Theorems 6.13 and 6.16).

The machinery developed in this paper plays an integral role in our subsequent works on the renormalization theory of dissipative Hénon-like maps [CLPY1], [Y]. In the class of 2D dynamical systems that we consider, each map features a unique orbit of tangencies between strong-stable and center manifolds. We refer to this orbit as the *critical orbit*. Note that the critical orbit is highly atypical, and hence is unaccounted for by classical measurable Pesin theory. Nonetheless, it turns out that this single orbit is the primary source of non-linearity for the system, and is chiefly responsible for shaping its overall dynamics (analogously to the critical orbit of a unimodal map in 1D). Quantitative Pesin theory provides an adequate language to analyze and describe how this happens.

Although the aforementioned works are specifically about renormalization, the tools developed in this paper are quite general. We expect that they will find applications in other settings as well. In particular, we believe that they will be especially useful in the study of dissipative systems that feature tangencies between stable and center/unstable manifolds, in the spirit of the work of Benedicks and Carleson [**?**]. Furthermore, it should be commented that we restrict ourselves to the 2D case mainly for simplicity. Our quantitative approach should naturally generalize to arbitrary dimensions.

2. Derivatives in Projective Space

In this section we supply for reader's convenience some basic calculations for the first and second derivatives of the projectivization of a smooth two-dimensional map. We also define the notion of the projective attracting and repelling directions.

For $p \in \mathbb{R}^2$, denote the projective tangent space at p by $\mathbb{P}^2_p$. In this section, we write an element of $\mathbb{P}^2_p$ as

$$E^t_p := \{r(\cos t, \sin t) \mid r \in \mathbb{R}\} \quad \text{for} \quad t \in \mathbb{R}/2\pi\mathbb{Z}.$$

Let $F : \Omega \to F(\Omega)$ be a C^1-diffeomorphism on a domain $\Omega \subset \mathbb{R}^2$. For $p \in \Omega$, define $l_p : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}^+$ and $\theta_p : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}/2\pi\mathbb{Z}$ by

$$l_p(t)(\cos(\theta_p(t)), \sin(\theta_p(t))) := D_pF(\cos t, \sin t) \quad \text{for} \quad t \in \mathbb{R}/\mathbb{Z}.$$

For $n \geq 1$, the *nth projective derivative of F at E^t_p* is defined as

$$\partial^n_{\mathbb{P}} F(E^t_p) := \theta^{(n)}_p(t).$$

Also define the *growth variance of F at p* as $\max_t |l'_p(t)|$.

Proposition 2.1. *We have*

$$\partial_{\mathbb{P}} F(E^t_p) = \frac{\operatorname{Jac}_p F}{\|D_pF|_{E^t_p}\|^2}.$$

Proof. Let $v_{\max} = (\cos\alpha, \sin\alpha) \in \mathbb{P}^2_p$ be the direction of maximum expansion for D_pF. Let

$$\beta = \theta_p(\alpha).$$

Denote

$$R_t = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}.$$

Consider

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = R_\beta^{-1} D_p F R_\alpha.$$

Note that $a > b$.

For $v = (\cos t, \sin t) \in \mathbb{P}^2_p$, write

$$w = D_pF(v) = (l\cos(\tau), l\sin(\tau)).$$

Observe

$$l(\cos(\tau-\beta), \sin(\tau-\beta)) = R_\beta^{-1} w = AR_\alpha^{-1} v = (a\cos(t-\alpha), b\sin(t-\alpha)).$$

So

$$\tan(\tau-\beta) = \frac{b}{a}\tan(t-\alpha). \tag{2.1}$$

Differentiating, we obtain

$$\frac{d\tau}{dt} = \frac{b}{a}\left(\frac{\cos(\tau-\beta)}{\cos(t-\alpha)}\right)^2 = \frac{ab}{l^2}.$$

□

For $\Lambda \subset \Omega$, define the *eccentricity of F on Λ* as

$$\mathrm{ecc}_\Lambda(F) := \sup_{p\in\Lambda} \|D_pF\|\|D_{F(p)}F^{-1}\|.$$

Proposition 2.2. *The first and second projective derivatives, and the growth variance of F on Λ are uniformly bounded in terms of* $\mathrm{ecc}_\Lambda(F)$.

Proof. Consider the same set up as in the proof of Proposition 2.1. Clearly,

$$\frac{d\tau}{dt} < \frac{a}{b} < \mathrm{ecc}_\Lambda(F).$$

Differentiating (2.1) twice, we obtain

$$\begin{aligned}\frac{d^2\tau}{dt^2} &= \frac{2b}{l}\left(\frac{\frac{d\tau}{dt}\sin(\tau-\beta)\cos(t-\alpha) - \cos(\tau-\beta)\sin(t-\alpha)}{\cos^2(t-\alpha)}\right) \\ &= 2\frac{ab}{l^2}\tan(t-\alpha)\left(\frac{b^2}{l^2}-1\right) \\ &= \frac{2ab(b+l)\sin(t-\alpha)}{l^4}\frac{b-l}{\cos(t-\alpha)}.\end{aligned}$$

If $\cos(t-\alpha) > k$ for some uniform constant $k > 0$, then we have

$$\frac{d^2\tau}{dt^2} < \frac{2}{k}\frac{a}{b^3} < K(\mathrm{ecc}_\Lambda(F))^3$$

for some uniform constant $K \geq 1$.

Lastly, recall that

$$l = \sqrt{a^2\cos^2(t-\alpha) + b^2\sin^2(t-\alpha)}.$$

Plugging in, and taking limits as $t \to \alpha \pm \pi/2$, we see that

$$\left.\frac{d^2\tau}{dt^2}\right|_{t=\alpha\pm\pi/2} = 0.$$

For the growth variance, we see that

$$\begin{aligned}\|l_p'\| &= \sup_t \frac{d}{dt}\sqrt{a^2\cos^2(t) + b^2\sin^2(t)} \\ &= \sup_t \frac{(a^2+b^2)\cos t \sin t}{\sqrt{a^2\cos^2 t + b^2 \sin^2 t}} \\ &< \frac{a^2+b^2}{b}.\end{aligned}$$

□

Definition 2.3. Let $\rho, \varepsilon \in (0,1)$; $L \geq 1$ and $N, M \in \mathbb{N} \cup \{0, \infty\}$. A tangent direction $E_p \in \mathbb{P}^2_p$ at a point $p \in \Omega$ is called

- a (L, ε, ρ)-*regular projective attractor for the* N-*forward iterates of* F if
$$L^{-1}\rho^{(1+\varepsilon)n} < \partial_{\mathbb{P}} DF^n(E_p) < L\rho^{(1-\varepsilon)n} \quad \text{for} \quad 0 \leq n \leq N; \tag{2.2}$$
- a (L, ε, ρ)-*regular projective repeller for the* N-*forward iterates of* F if
$$L^{-1}\rho^{-(1-\varepsilon)n} < \partial_{\mathbb{P}} DF^n(E_p) < L\rho^{-(1+\varepsilon)n} \quad \text{for} \quad 0 \leq n \leq N; \tag{2.3}$$
- a (L, ε, ρ)-*regular projective attractor for the* M-*backward iterates of* F if
$$L^{-1}\rho^{(1+\varepsilon)n} < \partial_{\mathbb{P}} DF^{-n}(E_p) < L\rho^{(1-\varepsilon)n} \quad \text{for} \quad 0 \leq n \leq M; \tag{2.4}$$
- a (L, ε, ρ)-*regular projective repeller for the* M-*backward iterates of* F if
$$L^{-1}\rho^{-(1-\varepsilon)n} < \partial_{\mathbb{P}} DF^{-n}(E_p) < L\rho^{-(1+\varepsilon)n} \quad \text{for} \quad 0 \leq n \leq M. \tag{2.5}$$

3. Dynamics in Projective Space

Denote the projective space of $\mathbb{R}^2$ by $\mathbb{P}^2$. In this section, we write an element of $\mathbb{P}^2$ as

$$E^t := \{r(\cos t, \sin t) \mid r \in \mathbb{R}\} \quad \text{for} \quad t \in \mathbb{R}/2\pi\mathbb{Z}.$$

3.1. **Projective attractor.** In this subsection, we consider an orbit with a projective attracting direction. We construct a transverse projective repelling direction, and estimate the growth/contraction rate of the derivative of the original map along this direction.

For $0 \le n < N$ with $N \in \mathbb{N} \cup \{\infty\}$, let

$$A_n = \begin{bmatrix} a_n & c_n \\ 0 & b_n \end{bmatrix}$$

be a linear transformation with $a_n, b_n > 0$ and $c_n \in \mathbb{R}$. Suppose $\|A_n^{\pm 1}\| < \mathbf{C}$ for some uniform constant $\mathbf{C} \ge 1$. For $1 \le i \le N - n$, denote

$$A_n^i := A_{n+i-1} \cdot \ldots \cdot A_n.$$

Suppose that there exist constants $\rho, \varepsilon \in (0,1)$ and $L \ge 1$ such that

$$L^{-1}\rho^{(1+\varepsilon)n} < \partial_{\mathbb{P}} A_0^n(E^0) = \frac{b_0 \ldots b_{n-1}}{a_0 \ldots a_{n-1}} < L\rho^{(1-\varepsilon)n} \quad \text{for} \quad 0 \le n \le N. \tag{3.1}$$

Remark 3.1. Condition (3.1) means that the fixed horizontal direction E^0 exponentially attracts nearby directions under A_0^n for n sufficiently large.

Define

$$\sigma_n := \frac{\rho^{(1-\varepsilon)n}}{(b_0 \ldots b_{n-1})/(a_0 \ldots a_{n-1})} \quad \text{and} \quad \mathcal{S}_n := \begin{bmatrix} 1 & 0 \\ 0 & \sigma_n \end{bmatrix}.$$

Lemma 3.2. *For $0 \le n \le N$, we have $1 < \sigma_n < L\rho^{-2\varepsilon n}$, and*

$$\tilde{A}_n := \mathcal{S}_{n+1} \cdot A_n \cdot (\mathcal{S}_n)^{-1} = \begin{bmatrix} a_n & c_n/\sigma_n \\ 0 & \tilde{\rho} a_n \end{bmatrix} \quad \textit{where} \quad \tilde{\rho} := \rho^{1-\varepsilon}.$$

Lemma 3.3. *Consider the genuine vertical cone*

$$\nabla^{gv}(\omega) := \{(x,y) \mid x/y < \omega\} \quad \textit{with} \quad \omega := \frac{\mathbf{C}^2}{\tilde{\rho}(1-\tilde{\rho})}.$$

Then we have

$$\tilde{A}_n^{-1}\left(\nabla^{gv}(\omega)\right) \Subset \nabla^{gv}(\omega).$$

Consequently, if $N = \infty$, then there exists a unique direction $\hat{E}_0 \in \mathbb{P}^2$ such that

$$\hat{E}_n = \tilde{A}_{n-1} \cdot \ldots \cdot \tilde{A}_0(\hat{E}_0) \in \nabla^{gv}(\omega) \quad \textit{for all} \quad n \ge 0.$$

Proof. Observe that

$$\tilde{A}_n^{-1} = \begin{bmatrix} a_n^{-1} & -c_n/(\sigma_n \tilde{\rho} a_n^2) \\ 0 & \tilde{\rho}^{-1} a_n^{-1} \end{bmatrix}.$$

Let $v = (x, y)$. Then

$$\tilde{A}_n^{-1} v = (x', y') = (a_n^{-1} x - c_n y/(\sigma_n \tilde{\rho} a_n^2), y/(\tilde{\rho} a_n)).$$

We compute

$$\left|\frac{x'}{y'}\right| = \left|\left(\tilde{\rho} - \frac{c_n}{\tilde{\rho} a_n \sigma_n} \cdot \frac{y}{x}\right) \cdot \frac{x}{y}\right|.$$

If

$$\omega > \frac{|c_n|}{\tilde{\rho}(1-\tilde{\rho})a_n} > \frac{|c_n|}{\tilde{\rho}(1-\tilde{\rho})a_n\sigma_n},$$

then it follows that $|x/y| > \omega$ implies $|x'/y'| < |x/y|$. □

Proposition 3.4. *There exists a direction $\hat{E} \in \mathbb{P}^2$ such that*

$$\frac{1}{L^2\rho^{2\varepsilon n}\sqrt{1+\omega^2}} < \frac{\|A_0^n|_{\hat{E}}\|}{b_0 \dots b_{n-1}} < L\rho^{-2\varepsilon n}\sqrt{1+\omega^2} \quad \textit{for all} \quad 1 \leq n \leq N,$$

where $\omega > 0$ is the uniform constant given in Lemma 3.3. If $N = \infty$, then $\hat{E}$ is unique.

Proof. If $N = \infty$, let $\hat{E}_0$ be given by Lemma 3.3. Otherwise, let

$$\hat{E}_0 := (\tilde{A}_{N-1} \cdot \dots \cdot \tilde{A}_0)^{-1}(E^{\pi/2}).$$

Let $v_0 = (x_0, y_0) \in \hat{E}_0$ with $y_0 = 1$, and denote $v_n = (x_n, y_n) := \tilde{A}_0^n(v_0)$. By Lemma 3.3, we have $v_n \in \nabla^{gv}(\omega)$ for $0 \leq n \leq N$. Thus,

$$|y_n| \leq \|v_n\| \leq \sqrt{1+\omega^2} \cdot |y_n|.$$

Hence,

$$1 \leq \frac{\|\tilde{A}_0^n|_{\hat{E}_0}\|\|v_0\|}{\rho^{(1-\varepsilon)n}a_0 \dots a_{n-1}} \leq \sqrt{1+\omega^2}.$$

Observe that

$$1 < \frac{\|\tilde{A}_0^n|_{\hat{E}_0}\|}{\|A_0^n|_{\hat{E}_0}\|} < \sigma_n.$$

Moreover,

$$\rho^{(1-\varepsilon)n}a_0 \dots a_{n-1} = \left(\frac{\rho^{(1-\varepsilon)n}}{(b_0 \dots b_{n-1})/(a_0 \dots a_{n-1})}\right) \cdot b_0 \dots b_{n-1},$$

and the term in the bracket is bounded by

$$L^{-1} \leq \frac{\rho^{(1-\varepsilon)n}}{(b_0 \dots b_{n-1})/(a_0 \dots a_{n-1})} \leq L\rho^{-2\varepsilon n}.$$

So,

$$\left(L\sigma_n\sqrt{1+\omega^2}\right)^{-1} \leq \frac{\|A_0^n|_{\hat{E}_0}\|}{b_0 \dots b_{n-1}} \leq L\rho^{-2\varepsilon n}\sqrt{1+\omega^2}.$$

The result now follows from Lemma 3.2. □

Remark 3.5. Intuitively, Proposition 3.4 means that the existence of a "projective attractor" E^0 implies the existence of a transverse "projective repeller" $\hat{E}$ (which is unique if $N = \infty$).

3.2. **Projective repeller.** This subsection is dual to the previous one. Assuming the existence of a projective repelling direction, we establish that the orbit of any other direction is asymptotic to a projective attractor, and estimate the corresponding rate of convergence.

For $0 \le n < N$ with $N \in \mathbb{N} \cup \{\infty\}$, let

$$A_n = \begin{bmatrix} a_n & 0 \\ c_n & b_n \end{bmatrix}$$

be a linear transformation with $a_n, b_n > 0$ and $c_n \in \mathbb{R}$. Suppose $\|A_n\| < \mathbf{C}$ for some uniform constant $\mathbf{C} \ge 1$. Additionally, suppose that there exist constants $\rho, \varepsilon \in (0,1)$ and $L \ge 1$ such that

$$L^{-1}\rho^{-(1-\varepsilon)n} < \partial_{\mathbb{P}} A_0^n(E^{\pi/2}) = \frac{a_0 \dots a_{n-1}}{b_0 \dots b_{n-1}} < L\rho^{-(1+\varepsilon)n} \quad \text{for} \quad 0 \le n \le N. \tag{3.2}$$

Remark 3.6. Condition (3.2) means that the fixed vertical direction $E^{\pi/2}$ exponentially repels nearby directions under A_0^n for sufficiently large n.

Define

$$\hat{\sigma}_n := \frac{(b_0 \dots b_{n-1})/(a_0 \dots a_{n-1})}{\rho^{(1+\varepsilon)n}} \quad \text{and} \quad \tilde{\mathcal{S}}_n := \begin{bmatrix} \hat{\sigma}_n & 0 \\ 0 & 1 \end{bmatrix}.$$

Lemma 3.7. *For $1 \le n \le N$, we have $1 < \hat{\sigma}_n < L\rho^{-2\varepsilon n}$, and*

$$\hat{A}_n := \hat{\mathcal{S}}_{n+1} \cdot A_n \cdot (\hat{\mathcal{S}}_n)^{-1} = \begin{bmatrix} b_n/\hat{\rho} & 0 \\ c_n/\hat{\sigma}_n & b_n \end{bmatrix}, \quad \textit{where} \quad \hat{\rho} := \rho^{1+\varepsilon}.$$

Proposition 3.8. *Consider the genuine horizontal cone*

$$\nabla^{gh}(\hat{\omega}) := \{(x,y) \mid y/x < \hat{\omega}\} \quad \textit{where} \quad \hat{\omega} := \frac{\hat{\rho}}{1-\hat{\rho}} \cdot \mathbf{C}^2.$$

Then

$$\hat{A}\left(\nabla^{gh}(\hat{\omega})\right) \Subset \nabla^{gh}(\hat{\omega}).$$

Proof. Let $v = (x,y)$. Then

$$\hat{A}_n v = (x', y') = (b_n x/\hat{\rho}, c_n x/\hat{\sigma}_n + b_n y).$$

We compute

$$\left|\frac{y'}{x'}\right| = \left|\hat{\rho}\left(1 + \frac{c_n}{b_n\hat{\sigma}_n} \cdot \frac{x}{y}\right) \cdot \frac{y}{x}\right|.$$

If

$$\hat{\omega} > \frac{\hat{\rho}}{1-\hat{\rho}} \cdot \frac{|c_n|}{b_n} > \frac{\hat{\rho}}{1-\hat{\rho}} \cdot \frac{|c_n|}{b_n\hat{\sigma}_n}.$$

then it follows that $|y/x| > \hat{\omega}$ implies $|y'/x'| < |y/x|$. ☐

For $1 \leq n \leq N$, let

$$\mathcal{T}_n = \begin{bmatrix} 1 & 0 \\ -\tau_n & 1 \end{bmatrix}$$

be a linear transformation that maps $\tilde{E}_n := \tilde{A}_0^n(E^0)$ to E^0. Note that $|\tau_n| \leq \hat{\omega}$. Moreover, we have

$$\check{A}_n = \begin{bmatrix} b_n/\rho^{1+\varepsilon} & 0 \\ 0 & b_n \end{bmatrix} = \mathcal{T}_{n+1} \cdot \hat{A}_n \cdot (\mathcal{T}_n)^{-1}.$$

Proposition 3.9. *For $t \in (0, \pi/2]$, we have*

$$\frac{1}{KL|t|^2}\rho^{(1+3\varepsilon)n} < \partial_{\mathbb{P}} A_0^n(E^{\pi/2-t}) < \frac{KL}{|t|^2}\rho^{(1-\varepsilon)n} \quad \textit{for all} \quad 1 \leq n \leq N,$$

where $K = K(\hat{\omega}) \geq 1$ is a uniform constant.

Proof. Let $v_0 = (x_0, y_0) \in E^{\pi/2-t}$ such that $\|v_0\| = 1$. Denote

$$E^{s_n} := \check{A}_0^n(E^{\pi/2-t}) \quad \text{and} \quad v_n = (x_n, y_n) := \check{A}_0^n(v_0) \in E^{s_n}.$$

Let $0 \leq m \leq N$ be the largest number such that $s_m \leq \pi/4$. Note that

$$\rho^{(1+\varepsilon)m} \asymp |t|. \tag{3.3}$$

A straightforward computation shows that

$$\|v_n\| \asymp |b_0 \cdot \ldots \cdot b_{n-1}| \quad \text{for} \quad n \leq m. \tag{3.4}$$

Similarly, we have

$$\|v_n\| \asymp \frac{|b_0 \cdot \ldots \cdot b_{n-1}|}{\rho^{(1+\varepsilon)(n-m)}} \quad \text{for} \quad m \leq n \leq N. \tag{3.5}$$

First we establish upper bounds. For $n \leq m$:

$$\begin{aligned}
\partial_{\mathbb{P}} A_0^n(E^{s_0}) &= \partial_{\mathbb{P}}(\hat{\mathcal{S}}_n^{-1} \cdot \mathcal{T}_n^{-1} \cdot \check{A}_0^n \cdot \mathcal{T}_0 \cdot \hat{\mathcal{S}}_0)(E^{s_0}) \\
&< K \partial_{\mathbb{P}} \check{A}_0^n(E^{s_0}) \\
&= K \frac{\operatorname{Jac} \check{A}_0^n}{\|v_n\|^2} \\
&\asymp K \rho^{-(1+\varepsilon)n} \\
&< K|t|^{-2}\rho^{(1+\varepsilon)n}.
\end{aligned}$$

For $n \geq m$:

$$\begin{aligned}\partial_{\mathbb{P}} A_0^n(E^{s_0}) &< K\hat{\sigma}_n \partial_{\mathbb{P}} \check{A}_0^n(E^{s_0}) \\ &= K\hat{\sigma}_n \frac{\operatorname{Jac} \check{A}_0^n}{\|v_n\|^2} \\ &\asymp K\hat{\sigma}_n \frac{\rho^{2(1+\varepsilon)(n-m)}}{\rho^{(1+\varepsilon)n}} \\ &\asymp K\hat{\sigma}_n |t|^{-2} \rho^{(1+\varepsilon)n} \\ &< KL|t|^{-2}\rho^{(1-\varepsilon)n}.\end{aligned}$$

Next, we establish lower bounds. For $n \leq m$:

$$\begin{aligned}\partial_{\mathbb{P}} A_0^n(E^{s_0}) &> \frac{1}{K\hat{\sigma}_n} \partial_{\mathbb{P}} \check{A}_0^n(E^{s_0}) \\ &\asymp \frac{1}{K\hat{\sigma}_n} \rho^{-(1+\varepsilon)n} \\ &> \frac{1}{KL} \rho^{(-1+\varepsilon)n}.\end{aligned}$$

For $n \geq m$:

$$\begin{aligned}\partial_{\mathbb{P}} A_0^n(E^{s_0}) &> \frac{1}{K\hat{\sigma}_n} \partial_{\mathbb{P}} \check{A}_0^n(E^{s_0}) \\ &\asymp \frac{1}{K\hat{\sigma}_n} \frac{\rho^{2(1+\varepsilon)(n-m)}}{\rho^{(1+\varepsilon)n}} \\ &> \frac{1}{KL} \rho^{(1+3\varepsilon)n}.\end{aligned}$$

□

Remark 3.10. Intuitively, Proposition 3.9 means that the existence of a"projective repeller" $E^{\pi/2}$ implies that the height of any horizontal cone in the complement contracts exponentially fast.

4. Dynamics of Almost Linear Maps

In this section, we consider a bi-infinite sequence of global diffeomorphisms of $\mathbb{R}^2$ that are C^1-close to diagonal linear maps satisfying domination condition with strongly attracting vertical direction. Moreover, we assume uniform bounds on the C^r-norms of these maps. Under these circumstances, using the C^r-section theorem, we construct:

- a sequence of invariant C^r horizontal graphs that, in forward time, attracts exponentially fast in C^r-topology any sufficiently horizontal graph; and dually
- a sequence of global C^{r-1} vertical direction fields that, in backward time, attracts exponentially fast in C^{r-1}-topology any sufficiently vertical direction field.

We follow up with C^r-bounds for compositions of these maps.

Fix an integer $r \geq 1$. Let $\lambda, \eta, \rho \in (0,1)$; $0 < \mu_- < \mu_+$ and $\mathbf{C} \geq 1$ be constants such that $\eta < \lambda$ and $\lambda/\mu_- < \rho$. For $m \in \mathbb{Z}$, let $F_m : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a C^r-diffeomorphism such that

$$D_0 F_m = \begin{bmatrix} a_m & 0 \\ 0 & b_m \end{bmatrix}$$

for some $a_m, b_m > 0$, and the following properties are satisfied:

- $\mu_- + \eta < a_m < \mu_+ - \eta$ and $b_m < \lambda - \eta$;
- $\sup_p \|D_p F_m - D_0 F_m\| < \eta/2$; and
- $\|DF_m\|_{C^{r-1}} < \mathbf{C}$.

For $n \in \mathbb{N}$, denote $F^n_m := F_{m+n-1} \circ \ldots \circ F_m$.

4.1. **Applications of C^r-Section Theorem.** In this section, we formulate some classical applications of graph transform techniques and C^r-section theorem. See [Sh, Chapters 5 and 6] for more details.

Let $g : \mathbb{R} \to \mathbb{R}$ be a C^r-function. The curve

$$\Gamma_g := \{(x, g(x)) \; x \in \mathbb{R}\}$$

is the *horizontal graph of* g. For $t \geq 0$, we say that Γ_g is *t-horizontal* if $\|g'\| \leq t$. Additionally, Γ_g is *center-aligned* if $g(0) = 0$ and $g'(0) = 0$. The space of center-aligned t-horizontal graphs of C^r-functions is denoted $\mathfrak{G}^r_h(t)$. We define a metric on $\mathfrak{G}^r_h(t)$ by

$$\|\phi_1 - \phi_2\|_* := \sup_{x \neq 0} \frac{\|\phi_1(x) - \phi_2(x)\|}{\|x\|}.$$

Let $H : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^r-diffeomorphism. Suppose that there exists a C^r-function $H_*(g) : \mathbb{R} \to \mathbb{R}$ such that $H(\Gamma_g) = \Gamma_{H_*(g)} \in \mathfrak{G}^r_h$. Then $H_*(g)$ and $\Gamma_{H_*(g)}$ are referred to as the *horizontal graph transform of* g and Γ_g *by* H respectively.

Proposition 4.1 (Forward horizontal graph transform). *Suppose*

$$\frac{\lambda}{\mu_-^r} < 1. \tag{4.1}$$

Then there exists a uniform constant $\omega = \omega(\rho, \eta) > 0$ with $\omega(\rho, \eta) \to 0$ as $\eta \to 0$ such that the following holds for all $m \in \mathbb{Z}$. For $\Gamma_g \in \mathfrak{G}^r_h(\omega)$, the horizontal graph transform $(F_m)_*(g)$ is well-defined. Moreover, there exists a unique sequence $\{\Gamma_{g^*_m}\}_{m \in \mathbb{Z}} \subset \mathfrak{G}^r_h(\omega)$ such that $F_m(\Gamma_{g^*_m}) = \Gamma_{g^*_{m+1}}$, and for $\Gamma_g \in \mathfrak{G}^1_h(t)$, we have

$$\|(F^n_m)_*(g) - g^*_{m+n}\|_* < \left(\frac{\lambda}{\mu_-}\right)^n \|g - g^*_m\|_* \quad \textit{for} \quad n \in \mathbb{N}.$$

Additionally, $\|(g^*_m)''\|_{C^{r-2}} < K$ for some uniform constant $K = K(\mathbf{C}, \lambda, \mu_-, r) > 0$.

For $p \in \mathbb{R}^2$ and $u \in \mathbb{R}$, let $E^u_p \in \mathbb{P}^2_p$ be the tangent direction at p given by

$$E^u_p := \{r(u,1) \mid r \in \mathbb{R}\}.$$

Let $\xi : \mathbb{R}^2 \to \mathbb{R}$ be a C^{r-1}-map. The direction field

$$\mathcal{E}_\xi := \{E^{\xi(p)}_p \mid p \in \mathbb{R}^2\}$$

is the *vertical direction field of* ξ. For $t \geq 0$, we say that $\mathcal{E}_\xi$ is *t-vertical* if $\|\xi\| \leq t$. The space of t-vertical direction fields of C^{r-1}-maps is denoted $\mathfrak{D}\mathfrak{F}^{r-1}_v(t)$.

Let $H : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^r-diffeomorphism. Suppose that there exists a C^{r-1}-map $H^*(\xi) : \mathbb{R}^2 \to \mathbb{R}$ such that $DH^{-1}(\mathcal{E}_\xi) = \mathcal{E}_{H^*(\xi)} \in \mathfrak{D}\mathfrak{F}^{r-1}_v$. Then $H^*(\xi)$ and $\mathcal{E}_{H^*(\xi)}$ are referred to as the *vertical direction field transform of* ξ and $\mathcal{E}_\xi$ *by* H respectively.

Proposition 4.2 (Backward vertical direction field transform)**.** *Suppose*

$$\lambda \mu_+^{r-1} < 1. \tag{4.2}$$

Then there exists a uniform constant $\omega = \omega(\rho, \eta) > 0$ with $\omega(\rho, \eta) \to 0$ as $\eta \to 0$ such that the following holds for all $m \in \mathbb{Z}$. For $\mathcal{E}_\xi \in \mathfrak{D}\mathfrak{F}^0_v(\omega)$, the vertical direction field transform $(F_m)^*(\xi)$ is well-defined. Moreover, there exists a unique sequence $\{\mathcal{E}_{\xi^*_m}\}_{m\in\mathbb{Z}} \subset \mathfrak{D}\mathfrak{F}^{r-1}_v(\omega)$ such that $DF^{-1}_m(\mathcal{E}_{\xi^*_m}) = \mathcal{E}_{\xi^*_{m-1}}$, and for $\mathcal{E}_\xi \in \mathfrak{D}\mathfrak{F}^0_v(\omega)$, we have

$$\|(DF^n_{m-n})^*(\xi) - \xi^*_{m-n}\|_{C^0} < (\lambda\mu_+)^n \|\xi - \xi^*_m\|_{C^0} \quad \text{for} \quad n \in \mathbb{N}.$$

Additionally, $\|D\xi^*_m\|_{C^{r-2}} < K$ for some uniform constant $K = K(\mathbf{C}, \lambda, \mu_+, r) > 0$.

Proposition 4.3 (Rectification)**.** *Let $\omega \in (0, 1/2)$ and $K > 0$. Consider $\Gamma_g \in \mathfrak{G}^r_h(\omega)$ and $\mathcal{E}_\xi \in \mathfrak{D}\mathfrak{F}^{r-1}_v(\omega)$ such that*

- *$g(0) = 0$, $\|g'\| < \omega$ and $\|g''\|_{C^{r-2}} < K$; and*
- *$\|\xi\| < \omega$ and $\|D\xi\|_{C^{r-2}} < K$.*

Then there exists a unique C^{r-1}-chart $\Psi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that

- *$\Psi(x, g(x)) = (x, 0)$ for $x \in \mathbb{R}$; and*
- *$D\Psi(\mathcal{E}_\xi(p)) = E^0_{\Psi(p)}$.*

Moreover, we have $\|D\Psi\|_{C^{r-2}} < C$ for some uniform constant $C = C(K) > 0$.

Proof. Define

$$V(x, y) := (x, y - g(x)). \tag{4.3}$$

For $z = (x, y) \in \mathbb{R}^2$, observe that

$$D_zV(\xi(z), 1) = (\xi(z), 1 - g'(x)\xi(z)).$$

Define

$$\tilde{\xi}(z) := \frac{\xi(V^{-1}(z))}{1 - g'(x)\xi(V^{-1}(z))} \quad \text{and} \quad H(x, y) := \left(x + \int_0^y \tilde{\xi}(x, t)dt, y\right). \tag{4.4}$$

Then $\Psi := H \circ V$ gives the desired rectifying map. □

Lemma 4.4. *For $m \in \mathbb{Z}$, let Ψ_m be the chart given by Proposition 4.3, where g and ξ are taken to be g^*_m and ξ^*_m given by Propositions 4.1 and 4.2 respectively. Then $\tilde{F}_m := \Psi_m \circ F_m \circ (\Psi_m)^{-1}$ is of the form*

$$\tilde{F}_m(x, y) = (\tilde{f}_m(x), \tilde{e}_m(x, y)) \quad \textit{for} \quad (x, y) \in \mathbb{R}^2.$$

Moreover, we have $|\partial_x^{r-1}\tilde{e}_m(x, y)| < K|y|$ for $y \in \mathbb{R}$, where $K = K(\mathbf{C}, \lambda, \mu_\pm, r) > 0$ is a uniform constant.

Proof. The first claim is obvious.

Write $\Psi_m = H_m \circ V_m$, where V_m and H_m are given by (4.3) and (4.4) respectively. Let

$$\hat{F}_m(x, y) = (\hat{f}_m(x), \hat{e}_m(x, y)) := V_m \circ F_m \circ (V_m)^{-1}(x, y).$$

Then $\hat{F}_m$ is a C^r-diffeomorphism with $\|\hat{F}_m\|_{C^r} < \hat{K}$ for some uniform constant $\hat{K} = \hat{K}(\mathbf{C}, \mu_-)$. It follows immediately that for $0 \leq s < r$, we have

$$|\partial_x^s \hat{e}_m(\cdot, y)| < \hat{K}|y| \quad \text{for} \quad y \in \mathbb{R}.$$

The first component of H_m is given by

$$h_m(x, y) = h_{m,y}(x) = x + \int_0^y \tilde{\xi}_m(x, t)dt,$$

with $\|\tilde{\xi}_m\|_{C^{r-1}} < C$ for some uniform constant $C = C(\mathbf{C}) > 0$. Hence, for $2 \leq s < r$, we have

$$|\partial_x^s h_{m,y}| < C|y| \quad \text{for} \quad y \in \mathbb{R}.$$

Observe that

$$\tilde{e}_m(x, y) = \hat{e}_m(h^{-1}_{m,y}(x), y).$$

The bound on $|\partial_x^{r-1}\tilde{e}_m(\cdot, y)|$ follows. □

4.2. **C^r-convergence under graph transform.** Suppose that F_m for $m \in \mathbb{Z}$ is of the form

$$F_m(x, y) = (f_m(x), e_m(x, y)) \quad \text{for} \quad (x, y) \in \mathbb{R}^2$$

where $f_m : \mathbb{R} \to \mathbb{R}$ is a C^r-diffeomorphism, and $e_m : \mathbb{R}^2 \to \mathbb{R}$ is a C^r-map such that for all $0 \leq s \leq r$, we have

$$|\partial_x^s e_m(\cdot, y)| < \mathbf{C}|y| \quad \text{for} \quad y \in \mathbb{R}. \tag{4.5}$$

Clearly, we have

$$\mu_- < f'_m(x) < \mu_+ \quad \text{and} \quad \partial_y e_m(x, y) < \lambda < 1.$$

Let $\sigma_- < 1$ and $\sigma_+ > 1$ be constants such that $\sigma_- \leq \mu_-$ and $\sigma_+ \geq \mu_+$. For $n \in \mathbb{N}$, denote

$$F^n = (f^n, e^n) := F_{n-1} \circ \ldots \circ F_0.$$

Proposition 4.5 (Convergence of horizontal graphs)**.** *Let $g : \mathbb{R} \to \mathbb{R}$ be a C^r-map with $\|g'\|_{C^{r-1}} < \infty$. For $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, consider the graph transform $\tilde{g} := (F^n)_*(g)$. Then*

$$\|\tilde{g}'\|_{C^{r-1}} < C \left(\frac{\sigma_+}{\sigma_-}\right)^{(2r-1)n} \lambda^n (1 + \|g'\|_{C^{r-1}})^r$$

where $C = C(\mathbf{C}, \sigma_\pm, \lambda, r) \geq 1$ is a uniform constant.

Proof. Observe that

$$\tilde{g}(x) = e^n(u, g(u)) \quad \text{where} \quad u := (f^n)^{-1}(x).$$

The result follows from Propositions B.4, B.5 and B.6. □

Proposition 4.6 (Convergence of vertical direction fields)**.** *There exist uniform constants $C, \tilde{C} \geq 1$ depending only on $\mathbf{C}, \sigma_+, \lambda, r$ such that the following holds. Let $\xi : \mathbb{R}^2 \to \mathbb{R}$ be a C^{r-1}-map with $\|\xi\|_{C^{r-1}} < \infty$. For $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, consider the vertical direction transform*

$$\tilde{\xi} := (F^n)^*(\xi)|_{\mathbb{R}\times(-1,1)}.$$

Suppose

$$C\sigma_+^{rn}\lambda^n(1 + \|\xi\|_{C^{r-1}}) < \sigma_-^n.$$

Then

$$\|\tilde{\xi}\|_{C^{r-1}} < \tilde{C} \left(\frac{\sigma_+^3}{\sigma_-}\right)^{(r-1)n} \lambda^n \|\xi\|_{C^{r-1}}.$$

Proof. A straightforward computation shows that

$$\tilde{\xi} = \left(\frac{\xi \cdot \partial_y e^n}{\partial_x f^n - \xi \cdot \partial_x e^n}\right) \circ F^{n-1}.$$

Denote

$$\phi := \xi \cdot \partial_y e^n \quad \text{and} \quad \psi := \partial_x f^n - \xi \cdot \partial_x e^n.$$

Then we can write

$$\tilde{\xi} = \phi \cdot (1/\psi) \circ F^{n-1}.$$

In the following discussion, C_i with $i \in \mathbb{N}$ denotes a uniform constant. By Lemma B.2 and Proposition B.6, we have

$$\|\phi\|_{C^{r-1}} < C_1\sigma_+^{rn}\lambda^n\|\xi\|_{C^{r-1}}.$$

Similarly, Proposition B.5 implies

$$\|\psi\|_{C^{r-1}} < C_2\sigma_+^n + C_1\sigma_+^{rn}\lambda^n\|\xi\|_{C^{r-1}} < C_3\sigma_+^n.$$

Additionally, $\psi(x, y) > \sigma_-^n/2$ for all $x \in \mathbb{R}$ and $y \in (-1, 1)$. Lastly,

$$\|DF^{n-1}\|_{C^{r-2}} < C_2\sigma_+^{n-1} + C_1\sigma_+^{(r-1)(n-1)}\lambda^{n-1} < C_4\sigma_+^n.$$

The result now follows from Lemmas B.1, B.2 and B.3. □

5. Definitions of Regularity

Consider a C^1-diffeomorphism $F : \Omega \to F(\Omega)$ defined on a domain $\Omega \subset \mathbb{R}^2$. Let $\lambda, \rho, \varepsilon \in (0,1)$; $L \geq 1$ and $N, M \in \mathbb{N} \cup \{0, \infty\}$. A point $p \in \Omega$ is *N-times forward $(L, \varepsilon, \lambda, \rho)_v$-regular along* $E^{v,+}_p \in \mathbb{P}^2_p$ if, for all $1 \leq n \leq N$, we have

$$L^{-1}\lambda^{(1+\varepsilon)n} \leq \|DF^n|_{E^{v,+}_p}\| \leq L\lambda^{(1-\varepsilon)n} \tag{5.1}$$

and

$$L^{-1}\rho^{(1+\varepsilon)n} \leq \frac{\|DF^n|_{E^{v,+}_p}\|^2}{\operatorname{Jac}_p F^n} \leq L\rho^{(1-\varepsilon)n}. \tag{5.2}$$

Similarly, p is *M-times backward $(L, \varepsilon, \lambda, \rho)_v$-regular along* $E^{v,-}_p \in \mathbb{P}^2_p$ if, for all $1 \leq n \leq M$, we have

$$L^{-1}\lambda^{(1+\varepsilon)n} \leq \|DF^{-n}|_{E^{v,-}_p}\|^{-1} \leq L\lambda^{(1-\varepsilon)n} \tag{5.3}$$

and

$$L^{-1}\rho^{(1+\varepsilon)n} \leq \frac{\operatorname{Jac}_p F^{-n}}{\|DF^{-n}|_{E^{v,-}_p}\|^2} \leq L\rho^{(1-\varepsilon)n}. \tag{5.4}$$

If all four conditions (5.1) - (5.4) hold with $E^v_p := E^{v,+}_p = E^{v,-}_p$, then p is *(M, N)-times $(L, \varepsilon, \lambda, \rho)_v$-regular along* E^v_p. If, additionally, we have $M = N = \infty$, then p is *Pesin $(L, \varepsilon, \lambda, \rho)_v$-regular along* E^v_p .

We say that p is *N-times forward $(L, \varepsilon, \lambda, \rho)_h$-regular along* $E^{h,+}_p \in \mathbb{P}^2_p$ if, for all $1 \leq n \leq N$, we have

$$L^{-1}\lambda^{(1+\varepsilon)n} \leq \frac{\operatorname{Jac}_p F^n}{\|D_pF^n|_{E^{h,+}_p}\|} \leq L\lambda^{(1-\varepsilon)n} \tag{5.5}$$

and

$$L^{-1}\rho^{(1+\varepsilon)n} \leq \frac{\operatorname{Jac}_p F^n}{\|D_pF^n|_{E^{h,+}_p}\|^2} \leq L\rho^{(1-\varepsilon)n} \tag{5.6}$$

Similarly, we say that p is *M-times backward $(L, \varepsilon, \lambda, \rho)_h$-regular along* $E^{h,-}_p \in \mathbb{P}^2_p$ if, for all $1 \leq n \leq M$, we have

$$L^{-1}\lambda^{(1+\varepsilon)n} \leq \frac{\|D_pF^{-n}|_{E^{h,-}_p}\|}{\operatorname{Jac}_p F^{-n}} \leq L\lambda^{(1-\varepsilon)n} \tag{5.7}$$

and

$$L^{-1}\rho^{(1+\varepsilon)n} \leq \frac{\|D_pF^{-n}|_{E^{h,-}_p}\|^2}{\operatorname{Jac}_p F^{-n}} \leq L\rho^{(1-\varepsilon)n}. \tag{5.8}$$

If all four conditions (5.5) - (5.8) hold with $E^h_p := E^{h,+}_p = E^{h,-}_p$, then p is *(M, N)-times $(L, \varepsilon, \lambda, \rho)_h$-regular along* E^h_p. If, additionally, we have $M = N = \infty$, then p is *Pesin $(L, \varepsilon, \lambda, \rho)_h$-regular along* E^h_p.

In the above definitions, the letters v and h stand for "vertical" and "horizontal." The constants L, ε, and λ and ρ are referred to as an *irregularity factor*, a *marginal exponent* and *contraction bases* respectively. Henceforth, once the contraction bases are introduced and fixed, we will sometimes write "$(L,\varepsilon,\lambda,\rho)_{v/h}$-regular" as simply "$(L,\varepsilon)_{v/h}$-regular".

Remark 5.1. Note that if p is (M,N)-times $(L,\varepsilon,\lambda,\rho)_v$-regular along E^v_p, then E^v_p is a (L,ε,ρ)-regular projective repeller and attractor for the N-forward iterates and M-backward iterates of F respectively. Similarly, if p is (M,N)-times $(L,\varepsilon,\lambda,\rho)_h$-regular along E^h_p, then E^h_p is a (L,ε,ρ)-regular projective attractor and repeller for the N-forward iterates and M-backward iterates of F respectively.

Proposition 5.2 (Vertical forward regularity = horizontal forward regularity)**.** *There exists a uniform constant $K = K(\rho, \|F\|_{C^1}) \geq 1$ such that the following holds. Suppose p is N-times forward $(L,\varepsilon)_v$-regular along $E^v_p \in \mathbb{P}^2_p$. Let $E^h_p \in \mathbb{P}^2_p \setminus \{E^v_p\}$. If $\measuredangle(E^v_p, E^h_p) > \theta$, then the point p is N-times forward $(L_1,\varepsilon_1)_h$-regular along E^h_p, where*

$$L_1 := KL^3/\theta^2 \quad \text{and} \quad \varepsilon_1 := (3 + 2\log_\lambda \rho)\varepsilon.$$

Conversely, if p is N-times forward $(L,\varepsilon)_h$-regular along $E^h_p \in \mathbb{P}^2_p$, then there exists $E^v_p \in \mathbb{P}^2_p$ such that p is N-times forward $(L_2,\varepsilon_2)_v$-regular along E^v_p, where

$$L_2 := KL^3 \quad \text{and} \quad \varepsilon_2 := (1 + 2\log_\lambda \rho)\varepsilon.$$

Proof. Suppose that p is vertically regular along E^v_p. For $1 \leq n \leq N$, let $A_n := D_{F^n(p)}F$ and $E^{s_0} := E^h_p$, and define $\check{A}_n$ as in Subsection 3.2. We use the same set up as in the proof of Proposition 3.9. Then

$$\frac{\operatorname{Jac}_p F^n}{\|DF^n|_{E^h_p}\|} = \frac{\operatorname{Jac} A^n_0}{\|A^n_0(v)\|} = \frac{|b_0 \ldots b_{n-1}|^2}{\hat{\sigma}_n \rho^{(1+\varepsilon)n} \|A^n_0(v)\|},$$

where $v \in E^h_p$ with $\|v\| = 1$.

We first establish upper bounds. For $n \leq m$:

$$\begin{aligned}\frac{\operatorname{Jac} A^n_0}{\|A^n_0(v)\|} &< \frac{K|b_0 \ldots b_{n-1}|^2}{\rho^{(1+\varepsilon)n}\|v_n\|} \\ &\asymp \frac{K|b_0 \ldots b_{n-1}|}{\rho^{(1+\varepsilon)n}} \\ &< \frac{K}{|t|}|b_0 \ldots b_{n-1}|.\end{aligned}$$

For $n \geq m$:

$$\begin{aligned}\frac{\operatorname{Jac} A^n_0}{\|A^n_0(v)\|} &< \frac{K|b_0 \ldots b_{n-1}|}{\rho^{(1+\varepsilon)m}} \\ &\asymp \frac{K}{|t|}|b_0 \ldots b_{n-1}|.\end{aligned}$$

Next, we establish lower bounds. For $n \leq m$:

$$\frac{\operatorname{Jac} A_0^n}{\|A_0^n(v)\|} > \frac{|b_0 \dots b_{n-1}|^2}{K\hat{\sigma}_n^2 \rho^{(1+\varepsilon)n}\|v_n\|} \asymp \frac{|b_0 \dots b_{n-1}|}{K\hat{\sigma}_n^2 \rho^{(1+\varepsilon)n}} > \frac{1}{KL^2}|b_0 \dots b_{n-1}|$$

For $n \geq m$:

$$\frac{\operatorname{Jac} A_0^n}{\|A_0^n(v)\|} > \frac{|b_0 \dots b_{n-1}|}{K\hat{\sigma}_n^2 \rho^{(1+\varepsilon)m}} > \frac{1}{KL^2}\rho^{2\varepsilon n}|b_0 \dots b_{n-1}|.$$

The horizontal projective regularity is given in Proposition 3.9.

Suppose that p is horizontally regular along E^h_p. The claimed vertical regularity of p along some direction E^v_p follows immediately from Proposition 3.4. □

Proposition 5.3 (Horizontal backward regularity = vertical backward regularity)**.** *There exists a uniform constant $K = K(\rho, \|F\|_{C^1}) \geq 1$ such that the following holds. Suppose p is M-times backward $(L, \varepsilon)_h$-regular along $E^h_p \in \mathbb{P}^2_p$. Let $E^v_p \in \mathbb{P}^2_p \setminus \{E^h_p\}$. If $\measuredangle(E^h_p, E^v_p) > \theta$, then the point p is M-times backward $(L_1, \varepsilon_1)_v$-regular along E^v_p, where*

$$L_1 := KL^3/\theta^2 \quad \text{and} \quad \varepsilon_1 := (3 + 2\log_\lambda \rho)\varepsilon.$$

Conversely, if p is M-times backward $(L, \varepsilon)_v$-regular along $E^v_p \in \mathbb{P}^2_p$, then there exists $E^h_p \in \mathbb{P}^2_p$ such that p is M-times backward $(L_2, \varepsilon_2)_h$-regular along E^h_p, where

$$L_2 := KL^3 \quad \text{and} \quad \varepsilon_2 := (1 + 2\log_\lambda \rho)\varepsilon.$$

Proposition 5.4 (Pesin regularity = vertical forward regularity + horizontal backward regularity + transversality)**.** *Suppose p is N-times forward $(L, \varepsilon)_v$-regular along $E^v_p \in \mathbb{P}^2_p$ and M-times backward $(L, \varepsilon)_h$-regular along $E^h_p \in \mathbb{P}^2_p$ with $\theta := \measuredangle(E^v_p, E^h_p) > 0$. Let $\mathcal{L} \geq 1$ be the minimum value such that p is (M, N)-times $(\mathcal{L}, \bar{\varepsilon})_v$-regular along E^v_p and $(\mathcal{L}, \bar{\varepsilon})_h$-regular along E^h_p. Then we have $\underline{L}^{-1}\theta^{-2} < \mathcal{L} < \bar{L}\theta^{-2}$, where*

$$\underline{L} := KL, \quad \bar{L} := KL^3 \quad \text{and} \quad \bar{\varepsilon} := (3 + 2\log_\lambda \rho)\varepsilon$$

for some uniform constant $K = K(\rho, \|F\|_{C^1}) \geq 1$.

Proof. The upper bound follows immediately from Propositions 5.2 and 5.3. The lower bound follows from Proposition 3.9. □

Suppose $p_0 \in \Lambda$ is (M, N)-times $(L, \varepsilon)_{v/h}$-regular along $E^{v/h}_{p_0} \in \mathbb{P}^2_{p_0}$. For $-M \leq m \leq N$, denote

$$p_m := F^m(p_0) \quad \text{and} \quad E^{v/h}_{p_m} := D_{p_0}F^m(E^{v/h}_{p_0}).$$

Define the *irregularity* of p_m as the smallest value $\mathcal{L}_{p_m} \geq 1$ such that p_m is $(M+m, N-m)$-times $(\mathcal{L}_{p_m}, \varepsilon)_{v/h}$-regular along $E^{v/h}_{p_m}$.

Proposition 5.5 (Growth in irregularity)**.** *Let* $\check{\lambda} := \min\{\lambda, \rho\}$*. Then*

$$\mathcal{L}_{p_m} < L^2 \check{\lambda}^{-2\varepsilon|m|} \quad \textit{for} \quad -M \leq m \leq N.$$

Proof. Denote

$$b_i := \|D_{p_i}F|_{E^v_{p_i}}\| \quad \text{and} \quad a_i := \frac{\operatorname{Jac}_{p_i} F}{\|D_{p_i}F|_{E^v_{p_i}}\|}.$$

For $0 \leq m \leq N$ and $0 < n \leq N - m$, we have

$$L^{-1}\lambda^{(1+\varepsilon)(m+n)} \leq b_0 \dots b_{m+n-1} \leq L\lambda^{(1-\varepsilon)(m+n)}.$$

Since

$$L^{-1}\lambda^{(1+\varepsilon)m} \leq b_0 \dots b_{m-1} \leq L\lambda^{(1-\varepsilon)m},$$

it follows that

$$L^{-2}\lambda^{-2\varepsilon m}\lambda^{(1+\varepsilon)n} < b_m \dots b_{m+n-1} < L^2\lambda^{-2\varepsilon m}\lambda^{(1-\varepsilon)n}.$$

For $0 < k \leq m$, we conclude immediately that

$$L^{-2}\lambda^{2\varepsilon k}\lambda^{(1+\varepsilon)(m-k)} < b_k \dots b_{m-1} < L^2\lambda^{-2\varepsilon k}\lambda^{(1-\varepsilon)(m-k)}.$$

Lastly, for $0 < l \leq M$, we have

$$L^{-1}\lambda^{(1+\varepsilon)l} \cdot L^{-1}\lambda^{(1+\varepsilon)m} \leq b_{-l} \dots b_{-1} \cdot b_0 \dots b_{m-1} \leq L\lambda^{(1-\varepsilon)l} \cdot L\lambda^{(1-\varepsilon)m}.$$

Similar computations imply analogous bounds for

$$\frac{b_0 \dots b_{m+n-1}}{a_0 \dots a_{m+n-1}}, \quad \frac{b_m \dots b_{m+n-1}}{a_m \dots a_{m+n-1}} \quad \text{and} \quad \frac{b_{-l} \dots b_{-1} \cdot b_0 \dots b_{m-1}}{a_{-l} \dots a_{-1} \cdot a_0 \dots a_{m-1}}.$$

Thus, the claim holds for $0 \leq m \leq N$.

The proof in the case $-M \leq m < 0$ is nearly identical, and will be omitted. □

6. Linearization Along Regular Orbits

Let $r \geq 1$ be an integer, and consider a C^{r+1}-diffeomorphism $F : \Omega \to F(\Omega) \Subset \Omega$ defined on a domain $\Omega \subset \mathbb{R}^2$. Suppose a point $p_0 \in \Omega$ is (M, N)-times $(L, \varepsilon, \lambda, \rho)_v$-regular along $E^v_{p_0} \in \mathbb{P}^2_{p_0}$ for some $\lambda, \rho, \varepsilon \in (0, 1)$; $M, N, \in \mathbb{N} \cup \{\infty\}$ and $L \geq 1$. We impose the following condition on the contraction bases and marginal exponent:

$$\frac{\rho^{(r+1)-\varepsilon(r+3)}}{\lambda^{(1+\varepsilon)r}} < 1 \quad \text{and} \quad \frac{\lambda^{(1-\varepsilon)(r+1)}}{\rho^{r+\varepsilon(2-r)}} < 1 \tag{6.1}$$

6.1. **Construction of regular charts.** For $l > 0$, denote

$$\mathbb{B}(l) := (-l, l) \times (-l, l) \subset \mathbb{R}^2.$$

For $p \in \mathbb{R}^2$, let $E_p^{gv}, E_p^{gh} \in \mathbb{P}^2$ be the genuine vertical and horizontal tangent directions at p.

Theorem 6.1. *Suppose that* (6.1) *holds. Then there exists a uniform constant*

$$C = C(\|DF\|_{C^r}, \lambda^{-\varepsilon}) \geq 1$$

such that the following holds. For $-M \leq m \leq N$, *let*

$$\omega := \frac{\rho^{1-\varepsilon}}{1-\rho^{1-\varepsilon}} \cdot \|DF^{-1}\| \cdot \|DF\| \quad \text{and} \quad \mathcal{K}_m := \frac{L^3 \rho^{-2\varepsilon} \lambda^{1-\varepsilon} \|DF^{-1}\| (1+\omega)^5}{\rho^{4\varepsilon|m|} \lambda^{2\varepsilon|m|}}.$$

Define

$$l_m := \check{\lambda}(C\mathcal{K}_m)^{-1} \quad \text{and} \quad U_m := \mathbb{B}(l_m),$$

where $\check{\lambda} := \lambda^{1+\varepsilon}(1-\lambda^\varepsilon)$. *Then there exists a* C^r*-chart* $\Phi_m : (\mathcal{U}_m, p_m) \to (U_m, 0)$ *such that* $D\Phi_m(E^v_{p_m}) = E_0^{gv}$,

$$\|D\Phi_m^{-1}\|_{C^{r-1}} < C(1+\omega), \quad \|D\Phi_m\|_{C^s} < C\mathcal{K}_m^{s+1} \quad \text{for} \quad 0 \leq s < r,$$

and the map $\Phi_{m+1} \circ F|_{\mathcal{U}_m} \circ \Phi_m^{-1}$ *extends to a globally defined* C^r*-diffeomorphism* $F_m : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ *satisfying the following properties:*

i) $\|DF_m\|_{C^{r-1}} \leq \|DF\|_{C^r}$;

ii) we have

$$D_0 F_m = \begin{bmatrix} \alpha_m & 0 \\ 0 & \beta_m \end{bmatrix},$$

where

$$\alpha_m = \frac{\lambda^{1-\varepsilon_m}}{\rho^{1+3\varepsilon_m}} \quad \text{and} \quad \beta_m = \frac{\lambda^{1-\varepsilon_m}}{\rho^{2\varepsilon_m}} \quad \text{with} \quad \varepsilon_m := \operatorname{sign}(m) \cdot \varepsilon.$$

iii) $\|D_z F_m - D_0 F_m\|_{C^0} < \check{\lambda}$ *for* $z \in \mathbb{R}^2$;

iv) we have

$$F_m(x, y) = (f_m(x), e_m(x, y)) \quad \text{for} \quad (x, y) \in \mathbb{R}^2,$$

where $f_m : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ *is a* C^r*-diffeomorphism, and* $e_m : \mathbb{R}^2 \to \mathbb{R}$ *is a* C^r*-map such that for all* $0 \leq s \leq r$, *we have*

$$\partial_x^s e_{p_m}(\cdot, y) \leq \|DF\|_{C^r} |y| \quad \text{for} \quad y \in \mathbb{R}.$$

Proof. Set

$$A_m = \begin{bmatrix} a_m & 0 \\ c_m & b_m \end{bmatrix} := D_{p_m} F.$$

For $n \geq 1$, define

$$\sigma_n := \frac{\lambda^{(1-\varepsilon)n}}{b_0 \dots b_{n-1}} \quad \text{and} \quad \sigma_{-n} := \frac{b_{-1} \dots b_{-n}}{\lambda^{(1+\varepsilon)n}}.$$

For $-M \le m \le N$, let

$$\mathcal{S}_m := \begin{bmatrix} \sigma_m & 0 \\ 0 & \sigma_m \end{bmatrix} \quad \text{and} \quad \tilde{A}_m := \mathcal{S}_{m+1} \cdot A_m \cdot \mathcal{S}_m^{-1}$$

The following properties can be checked by straightforward computations:

- $\sigma_{n+1}/\sigma_n = \lambda^{1-\varepsilon}/b_n$ and $\sigma_{-n+1}/\sigma_{-n} = \lambda^{1+\varepsilon}/b_{-n}$;
- $1 \le |\sigma_m| \le L\lambda^{-2\varepsilon|m|}$; and
- we have

$$\tilde{A}_n = \begin{bmatrix} \lambda^{1-\varepsilon} a_n/b_n & 0 \\ \lambda^{1-\varepsilon} c_n/b_n & \lambda^{1-\varepsilon} \end{bmatrix} \quad \text{and} \quad \tilde{A}_{-n} = \begin{bmatrix} \lambda^{1+\varepsilon} a_{-n}/b_{-n} & 0 \\ \lambda^{1+\varepsilon} c_{-n}/b_{-n} & \lambda^{1+\varepsilon} \end{bmatrix}.$$

Define

$$\hat{\sigma}_n := \frac{b_0 \dots b_{n-1}}{\rho^{(1+\varepsilon)n} a_0 \dots a_{n-1}} \quad \text{and} \quad \hat{\sigma}_{-n} := \frac{\rho^{(1-\varepsilon)n} a_{-n} \dots a_{-1}}{b_{-n} \dots b_{-1}}.$$

For $-M \le m \le N$, let

$$\hat{\mathcal{S}}_m := \begin{bmatrix} \hat{\sigma}_m & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \hat{A}_m := \hat{\mathcal{S}}_{m+1} \cdot \tilde{A}_m \cdot \hat{\mathcal{S}}_m^{-1}.$$

The following properties can be checked by straightforward computations:

- $\hat{\sigma}_{n+1}/\hat{\sigma}_n = b_n/(\rho^{1+\varepsilon} a_n)$ and $\hat{\sigma}_{-n+1}/\hat{\sigma}_{-n} = b_{-n}/(\rho^{1-\varepsilon} a_{-n})$;
- $1 \le |\hat{\sigma}_m| \le L\rho^{-2\varepsilon|m|}$;
- we have

$$\hat{A}_n = \begin{bmatrix} \lambda^{1-\varepsilon}/\rho^{1+\varepsilon} & 0 \\ \hat{c}_n & \lambda^{1-\varepsilon} \end{bmatrix} \quad \text{and} \quad \hat{A}_{-n} = \begin{bmatrix} \lambda^{1+\varepsilon}/\rho^{1-\varepsilon} & 0 \\ \hat{c}_{-n} & \lambda^{1+\varepsilon} \end{bmatrix},$$

  where

$$\hat{c}_{\pm n} := \frac{\lambda^{1\mp\varepsilon} c_{\pm n}}{b_{\pm n} \hat{\sigma}_{\pm n}}; \quad \text{and} \tag{6.2}$$

- $|\hat{c}_m| < \lambda^{1-\varepsilon} \|DF^{-1}\| \cdot \|DF\|$.

Define

$$\bar{\mathcal{S}}_m := \begin{bmatrix} \rho^{-2\varepsilon|m|} & 0 \\ 0 & \rho^{-2\varepsilon|m|} \end{bmatrix} \quad \text{and} \quad \bar{A}_m := \bar{\mathcal{S}}_{m+1} \cdot \hat{A}_m \cdot \bar{\mathcal{S}}_m^{-1}.$$

Then

$$\bar{A}_n = \begin{bmatrix} \lambda^{1-\varepsilon}/\rho^{1+3\varepsilon} & 0 \\ \rho^{-2\varepsilon} \hat{c}_n & \lambda^{1-\varepsilon} \rho^{-2\varepsilon} \end{bmatrix} \quad \text{and} \quad \bar{A}_{-n} = \begin{bmatrix} \lambda^{1+\varepsilon}/\rho^{1-3\varepsilon} & 0 \\ \rho^{2\varepsilon} \hat{c}_{-n} & \lambda^{1+\varepsilon} \rho^{2\varepsilon} \end{bmatrix}.$$

By Proposition 3.8, there exists a genuine horizontal cone $\nabla^{gh}(\omega)$ with

$$\omega := \frac{\rho^{1-\varepsilon}}{1 - \rho^{1-\varepsilon}} \cdot \|DF^{-1}\| \cdot \|DF\| \tag{6.3}$$

such that

$$\bar{A}_m \left(\nabla^{gh}(\omega) \right) \Subset \nabla^{gh}(\omega).$$

If $M = \infty$, then by Proposition 3.3 (note that the genuine vertical direction $E^{\pi/2}$ is a projective attractor for $\{\bar{A}_{-n}\}_{n=1}^{\infty}$), there exists a unique direction $\hat{E}_m \in \nabla^{gh}(\omega)$ such that $\bar{A}_m(\hat{E}_m) = \hat{E}_{m+1}$. If $M < \infty$, define

$$\hat{E}_m := \bar{A}_{m-1} \cdot \ldots \cdot \bar{A}_{-M}(E^0) \in \nabla^{gh}(\omega),$$

where E^0 is the genuine horizontal direction. Define

$$\mathcal{T}_m = \begin{bmatrix} 1 & 0 \\ -\tau_m & 1 \end{bmatrix}$$

as the unique matrix such that $\mathcal{T}_m(\hat{E}_m) = E^0$. Note that $|\tau_m| < \omega$.

Let

$$\check{A}_m := \mathcal{T}_{m+1} \cdot \bar{A}_m \cdot \mathcal{T}_m^{-1}.$$

Then

$$\check{A}_n = \begin{bmatrix} \lambda^{1-\varepsilon}/\rho^{1+3\varepsilon} & 0 \\ 0 & \lambda^{1-\varepsilon}\rho^{-2\varepsilon} \end{bmatrix} \quad \text{and} \quad \check{A}_{-n} = \begin{bmatrix} \lambda^{1+\varepsilon}/\rho^{1-3\varepsilon} & 0 \\ 0 & \lambda^{1+\varepsilon}\rho^{2\varepsilon} \end{bmatrix}.$$

Let

$$\mathcal{Z}_m := \kappa \cdot \mathcal{T}_m \circ \bar{\mathcal{S}}_m \circ \hat{\mathcal{S}}_m \circ \mathcal{S}_m \circ C_m \tag{6.4}$$

where

$$\kappa := L\rho^{-2\varepsilon}\lambda^{1-\varepsilon}\|DF^{-1}\|(1+\omega)^4, \tag{6.5}$$

and C_m is the translation and rotation of $\mathbb{R}^2$ so that $C_m(E_0^{\pi/2}) = E_{p_m}^v$. Define

$$\check{F}_m := \mathcal{Z}_{m+1} \circ F|_{\mathcal{U}_m} \circ \mathcal{Z}_m^{-1},$$

where $\mathcal{U}_m$ is a sufficiently small neighborhood of p_m to be specified later. Observe that $D_0\check{F}_m = \check{A}_m$. We claim that for any partial derivative ∂^i of order $|i| = s$ with $2 \leq s \leq r+1$, we have $\|\partial^i \check{F}_m\| \leq \|\partial^i F\|$.

First, an elementary computation shows that

$$\max_{|i|=s} \frac{\|\partial^i \check{F}_m\|}{\|\partial^i \left(\mathcal{T}_{m+1}^{-1} \circ \check{F}_m \circ \mathcal{T}_m\right)\|} \leq (1+\omega)^{2s}. \tag{6.6}$$

Write

$$C_{m+1} \circ F \circ C_m^{-1}(x,y) = (f_m(x,y), g_m(x,y)).$$

Denote

$$\check{f}_m := \pi_h \circ \mathcal{T}_{m+1}^{-1} \circ \check{F}_m \circ \mathcal{T}_m \quad \text{and} \quad \check{g}_m := \pi_v \circ \mathcal{T}_{m+1}^{-1} \circ \check{F}_m \circ \mathcal{T}_m,$$

where

$$\pi_h(x,y) := x \quad \text{and} \quad \pi_v(x,y) := y.$$

Then

$$\check{f}_m(x,y) = \frac{\kappa\sigma_{m+1}\hat{\sigma}_{m+1}}{\rho^{2\varepsilon|m+1|}} \cdot f_m\left(\frac{\rho^{2\varepsilon|m|}x}{\kappa\sigma_m\hat{\sigma}_m}, \frac{\rho^{2\varepsilon|m|}y}{\kappa\sigma_m}\right)$$

and

$$\check{g}_m(x,y) = \frac{\kappa\sigma_{m+1}}{\rho^{2\varepsilon|m+1|}} \cdot g_m\left(\frac{\rho^{2\varepsilon|m|}x}{\kappa\sigma_m\hat{\sigma}_m}, \frac{\rho^{2\varepsilon|m|}y}{\kappa\sigma_m}\right).$$

Taking a partial derivative of order $|i| = s$ with $2 \leq s \leq r+1$, we have

$$\begin{aligned}
\|\partial^i \check{f}_m\| &\leq \frac{\kappa\sigma_{m+1}\hat{\sigma}_{m+1}}{\rho^{2\varepsilon|m+1|}} \cdot \left(\frac{\rho^{2\varepsilon|m|}}{\kappa\sigma_m}\right)^s \cdot \|\partial^i f_m\| \\
&\leq \frac{1}{\rho^{2\varepsilon}\kappa^{s-1}} \cdot \left(\frac{\sigma_{m+1}}{\sigma_m}\right) \cdot \left(\rho^{2\varepsilon|m|}\hat{\sigma}_{m+1}\right) \cdot \|\partial^i f_m\| \\
&\leq \frac{1}{\rho^{2\varepsilon}\kappa^{s-1}} \cdot \lambda^{1-\varepsilon}\|DF^{-1}\| \cdot L \cdot \|\partial^i f_m\| \\
&\leq (1+\omega)^{-4(s-1)}\|\partial^i f_m\|.
\end{aligned}$$

By (6.6), the claimed bound on the higher order partial derivatives of $\check{F}_m$ follows.

Let

$$\check{U}_m := \mathbb{B}(\check{\lambda}\|F\|_{C^2}^{-1}).$$

Then

$$\sup_z \|D_z\check{F}_m|_{\check{U}_m} - \check{A}_m\| \leq \check{\lambda}. \tag{6.7}$$

Extend $\check{F}_m|_{\check{U}_m}$ to a globally defined C^{r+1}-diffeomorphism $\tilde{F}_m : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that

- $\tilde{F}_m|_{\check{U}_m} \equiv \check{F}_m$; and
- $\tilde{F}_m|_{\mathbb{R}^2 \setminus \tilde{U}_m} \equiv \check{A}_m$, where $\tilde{U}_m$ is a suitable neighborhood of $\check{U}_m$.

Additionally, define $\tilde{F}_m$ for $m \in \mathbb{Z} \setminus [-M, N]$ by

$$\tilde{F}_m := \begin{bmatrix} \lambda/\rho & 0 \\ 0 & \lambda \end{bmatrix}.$$

Then the sequence $\{\tilde{F}_m\}_{m\in\mathbb{Z}}$ satisfies the conditions in Appendix 4.1.

Thus, the conditions given in (6.1), together with Propositions 4.1 and 4.2 imply that there exist unique sequences of horizontal graphs $\{\Gamma_{g_m^*}\}_{m\in\mathbb{Z}}$ and vertical direction fields $\{\mathcal{E}_{\xi_m^*}\}_{m\in\mathbb{Z}}$ that are invariant under $\{\tilde{F}_m\}_{m\in\mathbb{Z}}$. Applying Proposition 4.3, we obtain a sequence of C^r-charts

$$\{\Psi_m : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)\}_{m\in\mathbb{Z}}$$

such that $\|D\Psi_m\|_{C^{r-1}} < C$, and the map

$$F_m := \Psi_{m+1} \circ \tilde{F}_m \circ \Psi_m^{-1}$$

is of the form given in iii).

Finally,

$$\Phi_m := \Psi_m \circ \mathcal{Z}_m \tag{6.8}$$

gives the desired chart. □

The construction in Theorem 6.1 is referred to as *a linearization of F along the (M, N)-orbit of p_0 with vertical direction $E^v_{p_0}$*. For $-M \leq m \leq N$, we refer to l_m, $\mathcal{U}_m$, Φ_m and F_m as a *regular radius*, a *regular neighborhood*, a *regular chart* and a *linearized map at p_m* respectively.

For $p \in \mathbb{R}^2$ and $t > 0$, let

$$\mathbb{D}_p(t) := \{\|q - p\| < t\}.$$

Lemma 6.2. *For $-M \leq m \leq N$, we have*

$$\mathcal{U}_m \supset \mathbb{D}_{p_m}\left(\frac{\check{\lambda}}{C^2\mathcal{K}_m^2}\right),$$

where $\check{\lambda} \in (0,1)$ and $C, \mathcal{K}_m \geq 1$ are given in Theorem 6.1.

Proof. The result follows immediately from the fact that $U_m := \mathbb{B}(\check{\lambda}C^{-1}\mathcal{K}_m^{-1})$ and $\|D\Phi_m\| < C\mathcal{K}_m$. □

For $-M \leq m \leq N$ and $q \in \mathcal{U}_m$, write $z := \Phi_m(q)$. The *vertical/horizontal direction at q in $\mathcal{U}_m$* is defined as $E_q^{v/h} := D\Phi_m^{-1}(E_z^{gv/gh})$. By the construction of regular charts in Theorem 6.1, vertical directions are invariant under F (i.e. $DF(E_q^v) = E_{F(q)}^v$ for $q \in \mathcal{U}_m$). Note that the same is not true for horizontal directions.

Lemma 6.3. *Define*

$$\check{\alpha}_\pm := \frac{\lambda^{1\mp2\varepsilon}}{\rho^{1\pm3\varepsilon}} \quad \text{and} \quad \check{\beta}_\pm := \frac{\lambda^{1\mp2\varepsilon}}{\rho^{\pm2\varepsilon}}.$$

For $-M \leq m \leq N$ and $z \in \mathbb{R}^2$, we have

$$\check{\alpha}_- < |f_m'(z)| < \check{\alpha}_+ \quad \text{and} \quad \check{\beta}_- < |\partial_y e_m(z)| < \check{\beta}_+.$$

Proof. By a straightforward computation, we can check that

$$\lambda^{1+\varepsilon} - \check{\lambda} > \lambda^{1+2\varepsilon} \quad \text{and} \quad \lambda^{1-\varepsilon} + \check{\lambda} < \lambda^{1-2\varepsilon}.$$

The result is now an immediate consequence of Theorem 6.1 ii) and iii). □

For $0 \leq n \leq N - m$, denote

$$F_m^n(x,y) = (f_m^n(x), e_m^n(x,y)) := F_{p_{m+n-1}} \circ \ldots \circ F_m(x,y). \tag{6.9}$$

Proposition 6.4. *For $-M \leq m \leq N$ and $0 \leq n \leq N - m$, consider the C^r-diffeomorphism F_m^n given in (6.9). Let $z = (x,y) \in U_m$, and suppose that*

$$z_i = (x_i, y_i) := F_m^i(z) \in U_{m+i} \quad \text{for} \quad 0 \leq i \leq n.$$

Denote

$$D_zF_m^n =: \begin{bmatrix} \alpha_m^n(z) & 0 \\ \gamma_m^n(z) & \beta_m^n(z) \end{bmatrix}.$$

Define

$$\check{l}_h := \sup_n \frac{2n\check{\lambda}}{C\mathcal{K}_n} < \infty \quad \text{and} \quad \check{l}_v := \frac{\check{\lambda}}{C\mathcal{K}_0(1-\rho^{-2\varepsilon}\lambda^{1-2\varepsilon})}.$$

and

$$\kappa_h := \exp\left(\frac{\check{l}_h\|F\|_{C^2}}{\lambda^{1+2\varepsilon}/\rho^{1-3\varepsilon}}\right) \quad \text{and} \quad \kappa_v := \exp\left(\left(\frac{\check{l}_h}{\lambda^{1+2\varepsilon}/\rho^{1-3\varepsilon}} + \frac{\check{l}_v}{\lambda^{1+2\varepsilon}\rho^{2\varepsilon}}\right)\|F\|_{C^2}\right).$$

Then

$$\frac{1}{\kappa_h} \leq \frac{\alpha_m^n(z)}{\alpha_m^n(0)} \leq \kappa_h \quad \textit{and} \quad \frac{1}{\kappa_v} \leq \frac{\beta_m^n(z)}{\beta_m^n(0)} \leq \kappa_v.$$

Additionally, we have

$$\|\gamma_m^n\| < \check{l}_n \check{\beta}_+^{n-1} \mathcal{K}_m^{-1} \quad \textit{where} \quad \check{l}_n := \sum_{i=0}^{n-1} \check{\alpha}_+^i.$$

Proof. Observe that

$$\sum_{i=0}^{n-1} |x_i| < \check{l}_h.$$

We compute

$$\begin{aligned}\left|\log\left(\frac{(f_m^n)'(x_0)}{(f_m^n)'(0)}\right)\right| &= \sum_{i=0}^{n-1} \left|\log f'_{m+i}(x_i) - \log f'_{m+i}(0)\right| \\ &\leq \sum_{i=0}^{n-1} \left(\int_0^{x_i} \left|\frac{f''_{m+i}(x)}{f'_{m+i}(x)}\right| dx\right) \\ &\leq \frac{\check{l}_h \|F\|_{C^2}}{\lambda^{1+2\varepsilon}/\rho^{1+\varepsilon}},\end{aligned}$$

since $|f'_m(x)| > \lambda^{1+2\varepsilon}/\rho^{1+\varepsilon}$ for all $-M \leq m \leq N$.

Similarly, we have

$$\begin{aligned}\left|\log\left(\frac{\partial_y e_m^n(x_0,0)}{\partial_y e_m^n(0,0)}\right)\right| &= \sum_{i=0}^{n-1} |\log \partial_y e_{m+i}(x_i,0) - \log \partial_y e_{m+i}(0,0)| \\ &\leq \sum_{i=0}^{n-1} \left(\int_0^{x_i} \left|\frac{\partial_x \partial_y e_{m+i}(x,0)}{\partial_y e_{m+i}(x,0)}\right| dx\right) \\ &\leq \frac{\check{l}_h \|F\|_{C^2}}{\lambda^{1+2\varepsilon}/\rho^{1-3\varepsilon}},\end{aligned}$$

since $|\partial_y e_m(x,y)| > \lambda^{1+2\varepsilon}/\rho^{1-3\varepsilon}$ for all $-M \leq m \leq N$.

Comparing $\partial_y e_m^n(x_0, y_0)$ to $\partial_y e_m^n(x_0, 0)$, we first observe that

$$\sum_{i=0}^{n-1} |y_i| < \check{\lambda} C^{-1} \mathcal{K}_0^{-1} \sum_{i=0}^{n-1} \rho^{-2\varepsilon i} \lambda^{(1-2\varepsilon)i} < \check{l}_v.$$

Thus,

$$\left|\log\left(\frac{\partial_y e^n_m(x_0,y_0)}{\partial_y e^n_m(x_0,0)}\right)\right| = \sum_{i=0}^{n-1} |\log \partial_y e_{m+i}(x_i,y_i) - \log \partial_y e_{m+i}(x_i,0)|$$
$$\leq \sum_{i=0}^{n-1}\left(\int_0^{y_i}\left|\frac{\partial_y^2 e_{m+i}(x,y)}{\partial_y e_{m+i}(x,y)}\right| dy\right)$$
$$\leq \frac{\check{l}_v \|F\|_{C^2}}{\lambda^{1+2\varepsilon}\rho^{2\varepsilon}}.$$

Since $\gamma_m(\cdot,0)\equiv 0$, we have

$$|\gamma_m(\cdot,y)| < \|F\|_{C^2}|y|.$$

Denote $z_n = (x_n,y_n) := F^n_m(z)$. By Lemma 6.3, we see that

$$|y_n| < \check{\beta}_+^{n-1}|y_0| < \check{\beta}_+^{n-1}\check{\lambda}C^{-1}\mathcal{K}_m^{-1}.$$

Arguing by induction, we see that

$$|\gamma^n_m(z)| < \check{\beta}_+^{n-1}\left(\sum_{i=0}^{n-1}\check{\alpha}_+^i\right)\check{\lambda}\mathcal{K}_m^{-1}.$$

□

Proposition 6.5. *For $-M\leq m\leq N$ and $q\in\mathcal{U}_m$, we have*

$$\frac{1}{\sqrt{2}} \leq \frac{\|D\Phi_m|_{E_z^{v/h}}\|}{\|D\Phi_m|_{E_{p_m}^{v/h}}\|} \leq \sqrt{2}.$$

Proof. Recall that $\Phi_m := \Psi_m \circ \mathcal{Z}_m$, and $\operatorname{diam}(U_m) < \|\Psi_m\|_{C^2}\cdot\|\mathcal{Z}_m^{\pm 1}\|^{-1}$. Denote

$$z := \Phi_m(q) \quad\text{and}\quad \tilde{z} := \Psi_m^{-1}(z);$$

and

$$\tilde{E}^{v/h}_{\tilde{z}} := D\Psi_m^{-1}(E_z^{gv/gh}),\quad \tilde{E}^{v/h}_q := D\mathcal{Z}_m^{-1}(\tilde{E}^{v/h}_{\tilde{z}}) \quad\text{and}\quad \hat{E}^{v/h}_q := D\mathcal{Z}_m^{-1}(\tilde{E}^{gv/gh}_{\tilde{z}}).$$

We see that

$$\measuredangle(\tilde{E}^{v/h}_{\tilde{z}}, E^{gv/gh}_{\tilde{z}})\ ,\ \measuredangle(\tilde{E}^{v/h}_q, \hat{E}^{v/h}_q) < \pi/4.$$

The result follows immediately. □

Corollary 6.6. *For some $-M\leq m_0\leq N$, let $q_{m_0}\in\mathcal{U}_{m_0}$. Suppose for $m_0\leq m\leq m_1\leq N$, we have $q_m\in\mathcal{U}_m$. Let*

$$\hat{E}^h_{q_m} := DF^{m-m_0}(E^h_{q_{m_0}}).$$

Then for $m_0\leq m'\leq m_1$, we have

$$\frac{1}{2\kappa_h} \leq \frac{\|DF^{m'-m}|_{\hat{E}^h_{q_m}}\|}{\|DF^{m'-m}|_{E^h_{p_m}}\|} \leq 2\kappa_h \quad\textit{and}\quad \frac{1}{2\kappa_v} \leq \frac{\|DF^{m'-m}|_{E^v_{q_m}}\|}{\|DF^{m'-m}|_{E^v_{p_m}}\|} \leq 2\kappa_v,$$

where κ_h and κ_v are constants given in Proposition 6.4.

Proof. Denote

$$z_m := \Phi_m(q_m) \quad \text{and} \quad \hat{E}^h_{z_m} := D\Phi_m(\hat{E}^h_{q_m}).$$

By the bound on $\|\gamma^n_m\|$ given in Proposition 6.4, we see that

$$\measuredangle(\hat{E}^h_{z_m}, E^{gh}_{z_m}) \ , \ \measuredangle(\hat{E}^h_{q_m}, E^h_{q_m}) < \pi/4.$$

The result now follows from Propositions 6.4 and 6.5. □

Lemma 6.7. *Consider a matrix of the form*

$$T = \begin{bmatrix} 1 & 0 \\ \tau & 1 \end{bmatrix}$$

for some $\tau \in \mathbb{R}$. Then $\|T\| < |\tau| + 2$.

Proof. Let $v = (x, y)$ be a unit vector. Observe

$$\begin{aligned} \|Tv\|^2 &= x^2 + (\tau x + y)^2 \\ &\leq 1 + (|\tau| + 1)^2 \\ &\leq (|\tau| + 2)^2. \end{aligned}$$

□

Proposition 6.8 (Vertical alignment of forward contracting directions)**.** *Consider the constants C, ω in Theorem 6.1 and κ_h in Proposition 6.4. Let $q_0 \in \mathcal{U}_0$ and $\tilde{E}^v_{q_0} \in \mathbb{P}^2_{q_0}$. Suppose $q_i \in \mathcal{U}_i$ for $0 \leq i \leq n \leq N$, and that*

$$\nu := \|DF^n|_{\tilde{E}^v_{q_0}}\| < \frac{\lambda^{(1+4\varepsilon)n}}{\kappa_h(2+\omega)^3 C^2 L^2 \rho^{(1-7\varepsilon)n}}.$$

Denote $z_0 := \Phi_0(q_0)$ and $\tilde{E}^v_{z_0} := D\Phi_0(\tilde{E}^v_{q_0})$. Then

$$\measuredangle(\tilde{E}^v_{z_0}, E^{gv}_{z_0}) < \left(\frac{\kappa_h(1+\omega)C^2 L^2 \rho^{(1-7\varepsilon)n}}{\lambda^{(1+4\varepsilon)n}}\right) \cdot \nu.$$

Proof. Suppose towards a contradiction that for $u = (1, t) \in \tilde{E}^v_{z_0}$, we have $t \leq 2(1+\omega)$. Using Lemma 6.3 and Corollary 6.6, a straightforward computation shows that

$$\|DF^n_0|_{\tilde{E}^v_{z_0}}\| > \frac{\alpha^n_-}{4\kappa_h(\omega+2)}.$$

Following the construction in Theorem 6.1, we see that $\Phi_n := \Psi_m \circ \mathcal{Z}_m$, where $\mathcal{Z}_m$ is given in (6.4) and $\|D\Psi_m\|_{C^{r-1}} < C$. In particular, we have $\mathcal{Z}_0 = \kappa\mathcal{T}_0 \circ \mathcal{C}_0$, and

$$\|D\Phi_m\| < \frac{\kappa C L^2(1+\omega)}{\rho^{4\varepsilon n}\lambda^{2\varepsilon n}}.$$

Applying Lemma 6.7, we have

$$\nu > \|D\Phi_n\|^{-1} \cdot \|F^n_0|_{\tilde{E}^v_{z_0}}\| \cdot \|D\Phi_0^{-1}\|^{-1} > \frac{\lambda^{(1+4\varepsilon)n}}{4\kappa_h(2+\omega)^3 C^2 L^2 \rho^{(1-7\varepsilon)n}}.$$

Absorbing the constant 4 into C, this is a contradiction.

Suppose that we have $v = (t, 1) \in \tilde{E}^v_{z_0}$ with $t < 1/(2\omega + 2)$. Then

$$\measuredangle(\tilde{E}^v_{q_0}, E^v_{q_0}) \asymp t \quad \text{and} \quad \|D\mathcal{Z}_0|_{\tilde{E}^v_{q_0}}\| \asymp 1.$$

Moreover,

$$\|DF_0^n|_{\tilde{E}^v_{z_0}}\| > \frac{\alpha_-^n t}{2\sqrt{2}\kappa_h}.$$

Hence, a straightforward computation shows that

$$\nu > \frac{\lambda^{(1+4\varepsilon)n} t}{\kappa_h(1+\omega)C^2L^2\rho^{(1-7\varepsilon)n}}.$$

□

Proposition 6.9 (Horizontal alignment of backward neutral directions)**.** *Consider the constants C, ω in Theorem 6.1 and κ_v in Proposition 6.4. Let $q_0 \in \mathcal{U}_0$ and $\tilde{E}^h_{q_0} \in \mathbb{P}^2_{q_0}$. Suppose $q_{-i} \in \mathcal{U}_{-i}$ for $0 \leq i \leq n \leq M$, and that*

$$\mu := \|DF^{-n}|_{\tilde{E}^h_{q_0}}\| < \frac{\rho^{6\varepsilon n}}{\kappa_v(2+\omega)^3C^2L^2\lambda^{(1-4\varepsilon)n}}.$$

Denote

$$z_0 := \Phi_0(q_0), \quad \tilde{E}^h_{z_0} := D\Phi_0(\tilde{E}^h_{q_0}) \quad \textit{and} \quad \hat{E}^h_{z_0} := D\Phi_0 \circ F^n(E^h_{q_{-n}}).$$

Then

$$\measuredangle(\tilde{E}^h_{q_0}, \hat{E}^h_{q_0}) < \left(\frac{\kappa_v(1+\omega)C^2L^2\lambda^{(1-4\varepsilon)n}}{\rho^{6\varepsilon n}}\right) \cdot \mu.$$

6.2. **Special domains inside regular neighborhoods.** For $w, l > 0$, denote

$$\mathbb{B}(w, l) := (-w, w) \times (-l, l) \subset \mathbb{R}^2 \quad \text{and} \quad \mathbb{B}(l) := \mathbb{B}(l, l).$$

For $-M \leq m \leq N$, recall that $U_m := \mathbb{B}(l_m)$, where l_m is the regular radius given in Theorem 6.1.

Suppose that

$$\frac{\lambda^{1-4\varepsilon}}{\rho^{6\varepsilon}} < 1. \tag{6.10}$$

Let

$$e_\pm := \max\{1, \check{\alpha}_\pm^{\pm 1}\}, \tag{6.11}$$

where $\check{\alpha}_\pm$ are given in Lemma 6.3. For $0 \leq n_+ \leq N - m$ and $0 \leq n_- \leq M + m$, denote

$$k_\pm := \max\{|m|, |m \pm n_\pm|\}.$$

The $n_\pm$*-times forward/backward truncated regular neighborhood of* p_0 is defined as

$$\mathcal{U}^{n,\pm}_m := \Phi_m^{-1}\left(U^{n,\pm}_m\right) \subset \mathcal{U}_m, \quad \text{where} \quad U^{n,\pm}_0 := \mathbb{B}\left(e_\pm^{-n} l_{k_\pm}, l_m\right).$$

The purpose of truncating a regular neighborhood is to ensure that its iterated images stay inside regular neighborhoods.

Lemma 6.10. *Let $-M \le m \le N$ and $0 \le n \le N - m$. We have $F^i(\mathcal{U}_m^{n,+}) \subset \mathcal{U}_{m+i}$ for $0 \le i \le n$.*

For $\omega > 1$ and $a > 0$, define

$$T_0^\omega(a) := \{(x, y) \in U_0 \mid |y| < a|x|^\omega\}.$$

We refer to

$$\mathcal{T}_0^\omega(a) := \Phi_0^{-1}(T_0^\omega(a)) \subset \mathcal{U}_0$$

as a *ω-pinched regular neighborhood (of dilation a) at p_0*. Note that $p_0 \notin \mathcal{T}_0^\omega(a)$. For $n \ge 0$, we also denote

$$T_0^{\omega,n}(a) := T_0^\omega(a) \cap U_0^{n,-}$$

and

$$\mathcal{T}_0^{\omega,n}(a) := \Phi_0^{-1}(T_0^{\omega,n}(a)) = \mathcal{T}_0^\omega(a) \cap \mathcal{U}_0^{n,-}.$$

If the dilation a is not indicated, it is assumed to be equal to 1: e.g. $\mathcal{T}_0^\omega := \mathcal{T}_0^\omega(1)$.

The purpose of pinching a regular neighborhood is to ensure that its iterated preimages stay inside regular neighborhoods.

Lemma 6.11. *Let $0 \le n \le M$. If*

$$\omega > \frac{\log\left(\lambda^{1+4\varepsilon}\rho^{6\varepsilon}\right)}{\log\left(\lambda^{1+4\varepsilon}/\rho^{1-7\varepsilon}\right)},$$

then $F^{-i}(\mathcal{T}_0^{\omega,n}) \subset \mathcal{U}_{-i}$ for $0 \le i \le n$.

Proof. By Lemma 6.3, we see that

$$F_{-n}^n(U_{-n}^{n,+}) \supset \mathbb{B}(l_{-n}e_-^{-n}, l_{-n}\beta_+^n) \supset T_0^{\omega,n,-}.$$

$\square$

Lemma 6.12. *For $q_0 \in \mathcal{U}_0 \setminus \{p_0\}$, let $n \in \mathbb{N}$ be the largest number such that $q_0 \in \mathcal{U}_0^{n,-}$. If $q_{-n} \in \mathcal{U}_{-n}$, then $q_0 \in \mathcal{T}_0^{\omega,n}(C)$, where*

$$\omega = \max\left\{\frac{\log(\lambda^{1-2\varepsilon}/\rho^{2\varepsilon})}{\log(\lambda^{2\varepsilon}\rho^{4\varepsilon})}, \frac{\log(\lambda\rho^{2\varepsilon})}{\log(\lambda^{1+4\varepsilon}/\rho^{1-7\varepsilon})}\right\}$$

and

$$C = \max\left\{\frac{1}{\lambda^{2\varepsilon\omega}\rho^{4\varepsilon\omega}l_0^{\omega-1}}, \frac{\rho^{(1-7\varepsilon)\omega}}{\lambda^{(1+4\varepsilon)\omega}l_0^{\omega-1}}\right\}.$$

Proof. Denote $z_0 = (x_0, y_0) := \Phi_0(q_0)$. We have

$$e_-^{-(n+1)}l_{n+1} < |x_0| < e_-^{-n}l_n$$

Moreover, by Lemma 6.3, we see that $|y_0| < l_n\check{\beta}_+^n$. A straightforward computation shows that $|y_0| < C|x_0|^\omega$. $\square$

6.3. **Stable manifolds.** For $-M \leq m \leq N$, define the *local vertical* and *horizontal manifold at* p_m as

$$W^v_{\mathrm{loc}}(p_m) := \Phi_m^{-1}(\{(0,y) \in U_m\}) \quad \text{and} \quad W^h_{\mathrm{loc}}(p_m) := \Phi_m^{-1}(\{(x,0) \in U_m\})$$

respectively.

If $N = \infty$, then Proposition 3.9 implies that $E^v_{p_0}$ is the unique direction along which p_0 is infinitely forward regular (see Remark 5.1). In this case, we denote $E^{ss}_{p_0} := E^v_{p_0}$, and refer to this direction as the *strong stable direction at* p_0. Additionally, we define the *strong stable manifold of* p_0 as

$$W^{ss}(p_0) := \left\{ q_0 \in \Omega \mid \limsup_{n\to\infty} \frac{1}{n} \log \|q_n - p_n\| < (1-\varepsilon)\log\lambda \right\}.$$

Theorem 6.13 (Canonical strong stable manifold)**.** *Suppose*

$$\lambda^{1-\varepsilon} < \rho^{8\varepsilon}\lambda^{4\varepsilon} \quad \textit{and} \quad \lambda^{1-\varepsilon} < \frac{\lambda^{1+\varepsilon}}{\rho^{1-\varepsilon}}. \tag{6.12}$$

If $N = \infty$*, then*

$$W^{ss}(p_0) := \bigcup_{n=0}^{\infty} F^{-n}(W^v_{\mathrm{loc}}(p_n)).$$

Consequently, $W^{ss}(p_0)$ *is a* C^{r+1}*-smooth manifold.*

Proof. Clearly, $W^v_{\mathrm{loc}}(p_0) \subset W^{ss}(p_0)$. Let $q_0 \in W^{ss}(p_0)$. By the first inequality in (6.12) and Lemma 6.2, we see that $q_n \in \mathcal{U}_n$ for all n sufficiently large. Denote $z_n := \Phi_n(q_n)$. We see by the first inequality in (6.12) and Lemma 6.12 that either $q_n \in W^v_{\mathrm{loc}}(p_n)$, or $q_n \in \mathcal{T}^\omega_n(l_n, C)$, where $\omega > 1$, and $C > 0$ is a uniform constant. Thus, z_n converges to 0 tangentially along the horizontal direction. However, by Theorem 6.1 and Proposition 6.5, we have

$$\liminf_{n\to\infty} \frac{1}{n} \log \|q_n - p_n\| > \log\left(\frac{\lambda^{1-\varepsilon}}{\rho^{1+3\varepsilon}} \cdot \rho^{4\varepsilon}\lambda^{2\varepsilon}\right) = \log\left(\frac{\lambda^{1+\varepsilon}}{\rho^{1-\varepsilon}}\right).$$

This contradicts the second inequality in (6.12). □

6.4. **Neutral direction.** If $\rho = \lambda$, then instead of "$(L, \varepsilon, \lambda, \lambda)_{v/h}$-regular", we simply write "$(L, \varepsilon, \lambda)_{v/h}$-regular." Suppose that p_0 is (M,N)-times $(L, \varepsilon, \lambda)_v$-regular along $E^v_{p_0}$.

Proposition 6.14 (Jacobian bounds)**.** *We have*

$$L^{-3}\lambda^{(1+3\varepsilon)n} \leq \mathrm{Jac}_{p_0} F^n \leq L^3\lambda^{(1-3\varepsilon)n} \quad \textit{for} \quad 1 \leq n \leq N,$$

and

$$L^{-3}\lambda^{-(1-3\varepsilon)n} \leq \mathrm{Jac}_{p_0} F^{-n} \leq L^3\lambda^{-(1+3\varepsilon)n} \quad \textit{for} \quad 1 \leq n \leq M.$$

Proposition 6.15 (Derivative bounds)**.** *Let $C \geq 1$ and $\omega > 0$ be the constants given in Theorem 6.1. Then for $E_{p_0} \in \mathbb{P}^2_{p_0}$, we have*

$$\frac{\lambda^{(1+3\varepsilon)n}}{CL^2(1+\omega)^2} \leq \|DF^n|_{E_{p_0}}\| \leq C(1+\omega)^2\lambda^{-4\varepsilon n} \quad \textit{for} \quad 1 \leq n \leq N,$$

and

$$\frac{\lambda^{2\varepsilon n}}{CL^2(1+\omega)^2} \leq \|DF^{-n}|_{E_{p_0}}\| \leq C(1+\omega)^2\lambda^{-(1+3\varepsilon)n} \quad \textit{for} \quad 1 \leq n \leq M.$$

Proof. We prove the inequalities for the forward direction. The proof of the backward direction is similar.

Decompose $F^n = \Phi_n^{-1} \circ F_0^n \circ \Phi_0$. Let

$$\kappa := L\rho^{-2\varepsilon}\lambda^{1-\varepsilon}\|DF^{-1}\|(1+\omega)^4.$$

By (6.4), (6.8) and Theorem 6.1 ii), we see that $\|DF_0^n\| < \lambda^{-4\varepsilon n}$,

$$\|D\Phi_0\| < C\kappa \quad \text{and} \quad \|D\Phi_n^{-1}\| < \kappa^{-1}C(1+\omega).$$

After replacing C by C^2, the upper bound follows. For the lower bound, we similarly observe that $\|DF_0^n|_{E_{\Phi_0(p_0)}}\| > \lambda^{(1-3\varepsilon)n}$,

$$\|D\Phi_0|_{E_{p_0}}\| > \frac{\kappa}{C(1+\omega)} \quad \text{and} \quad \|D\Phi_n|_{E_{p_n}}\| < \kappa(1+\omega)CL^2\lambda^{-6\varepsilon n}$$

By uniformly increasing C if necessary, the lower bound also follows. □

If $M = \infty$, then Proposition 3.9 implies that $E^h_{p_0}$ is the unique direction along which p_0 is infinitely backward regular (see Remark 5.1). In this case, we denote $E^c_{p_0} := E^h_{p_0}$, and refer to this direction as the *center direction at* p_0. Moreover, we define the *(local) center manifold at* p_0 as

$$W^c(p_0) := \Phi_0^{-1}(\{(x,0) \in U_0\}).$$

Unlike strong stable manifolds, center manifolds are not canonically defined. However, the following result states that it still has a canonical jet.

Theorem 6.16 (Canonical jets of center manifolds)**.** *Suppose $M = \infty$. Let $\Gamma_0 : (-t,t) \to \mathcal{U}_0$ be a C^{r+1}-curve parameterized by its arclength such that $\Gamma_0(0) = p_0$, and for all $n \in \mathbb{N}$, we have*

$$\|DF^{-n}|_{\Gamma_0'(t)}\| < \lambda^{-\mathfrak{r}n} \quad \textit{for} \quad |t| < \lambda^{11\varepsilon n},$$

where

$$\mathfrak{r} := (1+2\varepsilon)\left(1 - \frac{r}{r+1}\cdot\frac{1+9\varepsilon}{1-\varepsilon}\right) - 2\varepsilon.$$

Then Γ_0 has a degree $r+1$ tangency with $W^c(p_0)$ at p_0.

Proof. Let $\mathcal{N}_0$ be a sufficiently small neighborhood of p_0, and denote $\gamma_0 := \Phi_0(\Gamma_0 \cap \mathcal{N}_0)$. If Γ_0 does not have a degree $r+1$ tangency with $W^c(p_0)$ at p_0, then there exists $a_0 \neq 0$ and $l_0 \in (0,1)$ such that

$$\gamma_0 = \{(x, a_0 x^{r+1}) \mid |x| < l_0\}.$$

Let $z_0 = (x_0, y_0) \in \gamma_0 \cap U_0^{n,-}$. This means

$$|x_0| < e_-^n l_{-n} = \lambda^{5\varepsilon n} \lambda^{6\varepsilon n} = \lambda^{11\varepsilon n}.$$

Denote $z_{-n} = (x_{-n}, y_{-n}) := (F_{-n}^n)^{-1}(z_0)$. By Theorem 6.1 ii), we see that

$$y_{-n} < \lambda^{-(1+3\varepsilon)n} y_0.$$

Since $z_{-n} \in U_{-n}$ if $y_{-n} < l_{-n}$, we conclude that $z_{-n} \in U_{-n}$ if

$$x_0 < \lambda^{(1+3\varepsilon)n} \lambda^{6\varepsilon n} = \lambda^{(1+9\varepsilon)n}.$$

Let $E_{z_0} \in \mathbb{P}_{z_0}^2$ be the tangent direction to γ_0 at z_0. Denote

$$E_{z_{-n}} := D(F_{-n}^n)^{-1}(E_{z_0}) \quad \text{and} \quad t_{-n} := \measuredangle(E_{z_{-n}}, E_{z_{-n}}^{gh}).$$

Note that $t_0 = (r+1)a_0 x_0^r$. Again, by Theorem 6.1 ii), we have $t_{-n} > \lambda^{-(1-\varepsilon)n} t_0$. Set $x_0 = \lambda^{(1+9\varepsilon)n}$, and let $0 \leq n_0 \leq n$ be the smallest number such that $t_{-n_0} > 1$. By a straightforward computation, we conclude that

$$n_0 \asymp \frac{r}{r+1} \cdot \frac{1+9\varepsilon}{1-\varepsilon} n.$$

Denote $q_0 := \Phi_0^{-1}(z_0)$ and $E_{q_0} := D\Phi_0^{-1}(E_{z_0})$. Using Theorem 6.1 and Propositions 6.3 and 6.5, we obtain

$$\begin{aligned} \|DF^{-n}|_{E_{q_0}}\| &> \|D\Phi_{-n}^{-1}\| \cdot \|D(F_{-n+n_0}^{n-n_0})^{-1}|_{E_{z_{n_0}}}\| \cdot \|D(F_{-n_0}^{n_0})^{-1}|_{E_{z_0}}\| \\ &> \lambda^{6\varepsilon n} \cdot \lambda^{-(1+2\varepsilon)(n-n_0)} \cdot \lambda^{-4\varepsilon} \\ &= \lambda^{\mathfrak{r} n}. \end{aligned}$$

□

7. Homogeneity

Let $r \geq 2$ be an integer, and consider a C^{r+1}-diffeomorphism $F : \Omega \to F(\Omega) \Subset \Omega$ defined on a domain $\Omega \Subset \mathbb{R}^2$. Suppose that $\Lambda \Subset \Omega$ is a compact totally invariant set for F not equal to the orbit of a periodic sink. For $\lambda, \eta \in (0,1)$, we say that F is (η, λ)-*homogeneous on* Λ if for all $p \in \Lambda$ and $E_p \in \mathbb{P}_p^2$, we have:

i) $\lambda^{1+\eta} < \|D_p F|_{E_p}\| < \lambda^{-\eta}$ and
ii) $\lambda^{1+\eta} < \operatorname{Jac}_p F < \lambda^{1-\eta}$.

Proposition 7.1. *Suppose $F|_\Lambda$ has a unique invariant probability measure μ, and that with respect to μ, the Lyapunov exponents of $F|_\Lambda$ are* $\log \lambda < 0$ *and* 0. *Then for any $\eta > 0$, there exists $N = N(\eta) \in \mathbb{N}$ such that if $n \geq N$, then the map F^n is (η, λ)-homogeneous on Λ.*

Proof. For all $p \in \Lambda$, we have

$$\frac{1}{n}\log\left(\operatorname{Jac}_p F^n\right) = \frac{1}{n}\sum_{i=0}^{n-1}\log\left(\operatorname{Jac}_{F^i(p)} F\right) \xrightarrow{n\to\infty} \int \log\left(\operatorname{Jac} F\right) d\mu = \log\lambda.$$

This implies that for $\eta_1 > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, we have

$$\left|\frac{1}{n}\log\left(\operatorname{Jac}_p F^n\right) - \log\lambda\right| < \eta_1.$$

Thus,

$$(\lambda^n)^{1+\eta_1/\log\lambda} < \operatorname{Jac}_p F^n < (\lambda^n)^{1-\eta_1/\log\lambda}.$$

Since the largest Lyapunov exponent of $F|_\Lambda$ is 0, for $\eta_2 > 0$, there exists $M \in \mathbb{N}$ such that for all $m > M$, we have

$$\frac{1}{m}\int \log\|DF^m\| d\mu < \eta_2.$$

For all $p \in \Lambda$, we have

$$\frac{1}{n}\log\|D_pF^n\| \leq \frac{1}{n}\sum_{i=0}^{n-1}\left(\frac{1}{m}\log\|D_{F^i(p)}F^m\|\right) \xrightarrow{n\to\infty} \frac{1}{m}\int \log\|DF^m\| d\mu.$$

Thus, there exists $N' \in \mathbb{N}$ such that for all $n > N'$, we have

$$\|D_pF^n\| \leq e^{\eta_2 n} = (\lambda^n)^{\eta_2/\log\lambda}.$$

Lastly, let

$$k_n := \min_{E_p \in \mathbb{P}^2_p} \|DF^n|_{E_p}\|.$$

Then

$$\operatorname{Jac}_p F^n = k_n \|D_pF^n\|.$$

The result follows. □

Suppose that F is (η, λ)-homogeneous on Λ. Let $C, D > 1$ be sufficiently large uniform constants independent of F. Let $0 < \varepsilon < \eta < 1$ be sufficiently small constants such that

$$\bar{\eta} := C\eta^{1/D} < \varepsilon \quad \text{and} \quad \bar{\varepsilon} := C\varepsilon^{1/D} < 1.$$

Homogeneity considerably simplifies the regularity conditions given in Section **??**. Let $N, M \in \mathbb{N} \cup \{0, \infty\}$ and $L \geq 1$. Then a point $p \in \Lambda$ is:

- N-times forward $(L, \varepsilon, \lambda, \lambda)_v$-regular along $E^v_p \in \mathbb{P}^2_p$ if
$$\|DF^n|_{E^v_p}\| \leq L\lambda^{(1-\varepsilon)n} \quad \text{for} \quad 1 \leq n \leq N;$$
- M-times backward $(L, \varepsilon, \lambda, \lambda)_v$-regular along E^v_p if
$$\|DF^{-n}|_{E^v_p}\| \geq L^{-1}\lambda^{-(1-\varepsilon)n} \quad \text{for} \quad 1 \leq n \leq M;$$
- N-times forward $(L, \varepsilon, \lambda, \lambda)_h$-regular along $E^h_p \in \mathbb{P}^2_p$ if
$$\|DF^n|_{E^h_p}\| \geq L^{-1}\lambda^{\varepsilon n} \quad \text{for} \quad 1 \leq n \leq N;$$

- M-times backward $(L, \varepsilon, \lambda, \lambda)_h$-regular along $E^h_p \in \mathbb{P}^2_p$ if

$$\|DF^{-n}|_{E^h_p}\| \leq L\lambda^{-\varepsilon n} \quad \text{for} \quad 1 \leq n \leq M.$$

Appendix A. Adaptation Lemma

Fix $\lambda, \varepsilon \in (0,1)$. Let $\{u_m\}_{m=-M}^{N-1}$ be a sequence of numbers with $M, N \in \mathbb{N} \cup \{0, \infty\}$. Moreover, suppose there exists $L \geq 1$ such that

$$L^{-1}\lambda^{(1+\varepsilon)n} \leq u_0 \dots u_{n-1} \leq L\lambda^{(1-\varepsilon)n} \quad \text{for} \quad 1 \leq n \leq N;$$

and

$$L^{-1}\lambda^{(1+\varepsilon)m} \leq u_{-1} \dots u_{-m} \leq L\lambda^{(1-\varepsilon)m} \quad \text{for} \quad 1 \leq m \leq M.$$

Choose $\delta \in (\varepsilon, 1)$. Define

$$\zeta_n := \frac{\lambda^{(1-\delta)n}}{u_0 \dots u_{n-1}} \quad \text{and} \quad \zeta_{-m} := \frac{u_{-1} \dots u_{-m}}{\lambda^{(1+\delta)m}}$$

We record the following useful properties, which can be checked by straightforward computations.

Lemma A.1. *For $0 \leq n < N$ and $1 \leq m \leq M$, we have*

$$\frac{\zeta_{n+1}}{\zeta_n} = \frac{\lambda^{1-\delta}}{u_n} \quad \textit{and} \quad \frac{\zeta_{-m+1}}{\zeta_{-m}} = \frac{\lambda^{1+\delta}}{u_{-m}}.$$

Moreover,

$$1 < \zeta_l < L\lambda^{-(\delta+\varepsilon)|l|} \quad \textit{for} \quad -M \leq l < N.$$

Appendix B. C^r-bounds under Compositions

Lemma B.1. [PuSh, (4)] *Let F, G be C^r-maps such that $F \circ G$ is well-defined. Then*

$$\|F \circ G\|_r \leq r^r \|F\|_r \|G\|_r^r,$$

where $\|F\|_r := \|DF\|_{C^{r-1}}$.

Lemma B.2 (Product)**.** *Let $d \in \mathbb{N}$. Consider C^r-maps $F, G : U \to \mathbb{R}$ defined on $U \subset \mathbb{R}^d$ such that $\|DF\|_{C^{r-1}}, \|DG\|_{C^{r-1}} < \infty$. Then there exists a uniform constant $C = C(r) \geq 1$ such that*

$$\|D(F \cdot G)\|_{C^{r-1}} < C\|DF\|_{C^{r-1}}\|DG\|_{C^{r-1}}.$$

Lemma B.3 (Quotient)**.** *Let $d \in \mathbb{N}$. Consider a C^r-map $F : U \to \mathbb{R}$ defined on $U \subset \mathbb{R}^d$ such that $|F(x)| > \mu > 0$ for $x \in U$, and $\|DF\|_{C^{r-1}} < \infty$. Then there exists a uniform constant $C = C(r) \geq 1$ such that*

$$\|D(1/F)\|_{C^{r-1}} < C\mu^{1-r}\|DF\|_{C^{r-1}}^{r-1}.$$

Proof. The proof is by induction on r. Suppose that an rth order partial derivative $\partial^r(1/F)(x)$ is a sum of uniform number of terms of the form $P\cdot(F(x))^{-l}$, where $l<r$, and P is a degree $k<r$ polynomial expression of partial derivatives of F up to order r.

Differentiating, we obtain

$$\frac{\partial P}{(F(x))^l}-\frac{P\cdot l\partial F(x)}{(F(x))^{l+1}}.$$

The result follows. □

Lemma B.4 (Inverse)**.** *Consider a C^r-diffeomorphism $f:\mathbb{R}\to\mathbb{R}$. Suppose $\|f'\|>\mu$ for some constant $\mu\in(0,1)$. Then there exists a uniform constant $C=C(r)\geq 1$ such that*

$$\|(f^{-1})'\|_{C^{r-1}}<C\mu^{1-2r}\|f''\|^{r-1}_{C^{r-2}}.$$

Proof. The proof is by induction on r. Let $u:=f^{-1}(x)$ for $x\in\mathbb{R}$. Note that $u'=1/(f'(u))$. The case $r=1$ follows. Suppose that $(f^{-1})^{(r)}(x)$ is a sum of uniform number of terms of the form $P\cdot(f'(u))^{-l}$, where $l<2r$, and P is a degree $k<r$ polynomial expression of $f^{(i)}(u)$ for $2\leq i\leq r$.

Differentiating, we obtain

$$\frac{P'\cdot u'}{(f'(u))^l}-\frac{P\cdot lf''(u)\cdot u'}{(f'(u))^{l+1}},$$

The result follows. □

Proposition B.5 (Compositions of 1D Diffeomorphisms)**.** *Consider a sequence $f_n:\mathbb{R}\to\mathbb{R}$ for $n\geq 0$ of C^r-diffeomorphisms. Denote $f^n:=f_{n-1}\circ\ldots\circ f_0$. Suppose $|f_n'|<\mu$ and $\|f_n'\|_{C^{r-1}}<\mathbf{C}$ for some constants $\mu,\mathbf{C}>1$. Then there exists a uniform constant $C=C(\mathbf{C},\mu,r)\geq 1$ such that*

$$\|Df^n\|_{C^{r-1}}<C\mu^n.$$

Proposition B.6 (Compositions of 2D contractions)**.** *Consider a sequence $F_n:\mathbb{R}\times(-1,1)\to\mathbb{R}\times(-1,1)$ for $n\geq 0$ of C^r-diffeomorphisms of the form*

$$F_n(x,y)=(f_n(x),e_n(x,y))\quad\textit{for}\quad(x,y)\in\mathbb{R}\times(-1,1)$$

where $f_n:\mathbb{R}\to\mathbb{R}$ is a C^r-diffeomorphism, and $e_n:\mathbb{R}\times(-1,1)\to\mathbb{R}$ is a C^r-map. For $n\in\mathbb{N}$, denote

$$F^n=(f^n,e^n):=F_{n-1}\circ\ldots\circ F_0.$$

Suppose there exist constants $\mu,\mathbf{C}>1$ and $\lambda\in(0,1)$ such that $\|f_n'\|<\mu$; $\|\partial_y e^n_m\|<\lambda^n$; $\|DF_n\|_{C^{r-1}}<\mathbf{C}$; and

$$|\partial_x^s e_m(\cdot,y)|<\mathbf{C}|y|\quad\textit{for}\quad y\in(-1,1)$$

for all $0\leq s\leq r$. Then there exists a uniform constant $C=C(\mathbf{C},\mu,\lambda,r)\geq 1$ such that

$$\|De^n\|_{C^{r-1}}=C\mu^{rn}\lambda^n.$$

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