ON REGULAR HÉNON-LIKE RENORMALIZATION

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1. Introduction

1.1. **Renormalization of unimodal maps.** Let $I \subset \mathbb{R}$ be an interval. A C^2-map $f : I \to I$ is *unimodal* if it has a unique critical point $c \in I$, which of quadratic type: i.e. $f'(c) = 0$ and $f''(c) \neq 0$. Denote the critical value of f by $v := f(c)$. We say that f is *normalized* if $c = 0$ and $v = 1$. Let $\gamma \in \{r, \omega\}$, where $r \geq 2$ is an integer. The space of normalized C^γ-unimodal maps is denoted $\mathfrak{U}^\gamma$.

Model examples of unimodal maps are given by real quadratic polynomials, which, after normalization, can be represented by the following one parameter family of maps:

$$\mathfrak{Q} := \{f_a(x) := 1 - ax^2 \mid a \in \mathbb{R}\}.$$

This is referred to as the *quadratic family.*

A unimodal map $f : I \to I$ is *topologically renormalizable* if there exists R-periodic subinterval $I^1 \subset I$ such that

$$f^i(I^1) \cap I^1 = \varnothing \quad \text{for} \quad 1 \leq i < R \quad \text{and} \quad f^R(I^1) \Subset I^1.$$

We say that f is *valuably renormalizable* if $f^R(I^1)$ contains the critical value v.

If f is valuably renormalizable, then the *pre-renormalization of f*

$$p\mathcal{R}_{1\mathrm{D}}(f) := f^R|_{I^1}$$

is also unimodal. Let $c^1 \in I^1$ be the unique critical point of $p\mathcal{R}_{1\mathrm{D}}(f)$. We define the renormalization of f to be

$$\mathcal{R}_{1\mathrm{D}}(f) := S \circ p\mathcal{R}_{1\mathrm{D}}(f) \circ S^{-1},$$

where S is the unique affine map such that $S(v) = 1$ and $S(c^1) = 0$. Observe that $\mathcal{R}_{1\mathrm{D}}(f) \in \mathfrak{U}^\gamma$.

1.2. **Hénon-like maps.** Let $B := I \times I \subset \mathbb{R}^2$ be a square, where $0 \in I \subset \mathbb{R}$ is an interval. A C^2-diffemorphism $F : B \to F(B) \Subset B$ is *Hénon-like* if F is of the form

$$F(x, y) = (f(x, y), x) \quad \text{for} \quad (x, y) \in B,$$

and for any $y \in I$, the map $f(\cdot, y) : I \to I$ is a unimodal map. We say that F is *normalized* if $f(\cdot, 0)$ is normalized. The set of normalized C^γ-Hénon-like maps is denoted $\mathfrak{HL}^\gamma$.

For $\beta \in (0, 1]$, we say that F is *β-thin (in C^γ)* if

$$\|\partial_y f\|_{C^{\gamma-1}} < \beta.$$

The space of β-thin Hénon-like maps in $\mathfrak{HL}^\gamma$ is denoted $\mathfrak{HL}^\gamma_\beta$. In particular, if $F \in \mathfrak{HL}^\gamma_1$, then F is dissipative (i.e. $\|\operatorname{Jac} F\| < 1$). We say that a β-thin Hénon-like map is *perturbative* if $\beta \ll 1$.

Model examples of Hénon-like maps are given by the following two parameter family of maps:

$$\mathfrak{H} := \{F_{a,b}(x, y) := (1 - ax^2 - by, x) \mid a, b \in \mathbb{R}\}.$$

This is referred to as the *Hénon family.* A straightforward computation shows that

$$\operatorname{Jac} F_{a,b} \equiv b,$$

and for $b \neq 0$, the map $F_{a,b}$ has a polynomial inverse (and hence, is a diffeomorphism).

For any 1D map $g : I \to I$, define a degenerate 2D map $\iota(g) : I \times \mathbb{R} \to I \times \mathbb{R}$ by

$$\iota(g)(x, y) := (g(x), x).$$

Let

$$\pi_h(x, y) := x \quad \text{and} \quad \pi_v(x, y) := y.$$

For any 2D map $G : B \to B$, define its *1D profile* $\Pi_{1\mathrm{D}}(G) : I \to I$ by

$$\Pi_{1\mathrm{D}}(G)(x) := \pi_h \circ G(x, 0).$$

Note that we have $\Pi_{1\mathrm{D}} \circ \iota(g) = g$.

The space of degenerate C^γ-Hénon-like maps is given by $\mathfrak{HL}^\gamma_0 := \iota(\mathfrak{U}^\gamma)$. Observe that $\Pi_{1\mathrm{D}}(\mathfrak{HL}^\gamma) = \mathfrak{U}^\gamma$.

1.3. **Topological renormalization of 2D maps.** Let $F : \Omega \to F(\Omega) \Subset \Omega$ be a continuous map defined on a Jordan domain $\Omega \subset \mathbb{R}^2$. We say that F is *topologically renormalizable* if there exists an R-periodic Jordan domain $\mathcal{B} \Subset \Omega$ for some integer $R \geq 2$.

Let $N \in \mathbb{N} \cup \{\infty\}$. If F is N-times renormalizable, then there exist sequences of nested Jordan domains and natural numbers:

$$\Omega =: \mathcal{B}^0 \Supset \ldots \Supset \mathcal{B}^N \quad \text{and} \quad 1 =: R_0 < \ldots < R_N$$

such that for $1 \leq n \leq N$, the domain $\mathcal{B}^n$ is R_n-periodic. If there exists a uniform constant $\mathbf{b} \geq 2$ such that

$$r_n := R_n / R_{n-1} \leq \mathbf{b} \quad \text{for} \quad 1 \leq n \leq N, \tag{1.1}$$

then the return times $\{R_n\}_{n=1}^N$ are said to be *of (*$\mathbf{b}$*-)bounded type*. If $N = \infty$, then the induced *renormalization limit set of* F is defined as

$$\Lambda_F := \bigcap_{n=1}^{\infty} \bigcup_{i=R_n}^{2R_n-1} F^i(\mathcal{B}^n). \tag{1.2}$$

1.4. **Hénon-like renormalization.** For $z \in \mathbb{R}^2$, let $E^{gv}_z, E^{gh}_z \in \mathbb{P}^2_z$ denote the genuine vertical and horizontal directions at z respectively.

A *(*C^r*-)chart* is a C^r-diffeomorphism $\Psi : \mathcal{D} \to D$ from a quadrilateral $\mathcal{D} \subset \mathbb{R}^2$ to a rectangle $D = I \times J \subset \mathbb{R}^2$, where $I, J \subset \mathbb{R}$ are intervals. The *vertical/horizontal direction* $E^{v/h}_p \in \mathbb{P}^2_p$ *at* $p \in \mathcal{D}$ *(associated to* Ψ*)* is given by

$$E^{v/h}_p := D\Psi^{-1}\left(E^{gv/gh}_{\Psi(p)}\right).$$

The chart Ψ is said to be *genuine vertical/horizontal* if $E^{v/h}_p = E^{gv/gh}_p$ for all $p \in \mathcal{D}$. A chart $\tilde{\Psi} : \mathcal{D} \to \tilde{D} := \tilde{I} \times \tilde{J}$ is said to be *vertically/horizontally equivalent to* Ψ if $\tilde{\Psi} \circ \Psi^{-1}$ is genuine vertical/horizontal. If $\tilde{\Psi}$ is both vertically and horizontal equivalent to Ψ, then we simply say that $\tilde{\Psi}$ is *equivalent to* Ψ.

Consider a C^r-Hénon-like map $F : B \to B$ defined on a square $B := I \times I \ni 0$. Let $v \in I$ be the critical value of the unimodal map $\Pi_{1\mathrm{D}}(F)$. We say that F is *Hénon-like renormalizable* if there exists an R-periodic quadrilateral $(v, 0) \in \mathcal{B}^1 \subset B$ for some integer $R \geq 2$, and a genuine horizontal chart $\Psi : \mathcal{B}^1 \to B^1 := I^1 \times I^1$ for some interval $0 \in I^1 \subset \mathbb{R}$ such that $\pi_v \circ \Psi(\cdot, 0) \equiv 0$, and the *pre-renormalization of* F:

$$p\mathcal{R}(F) := \Psi \circ F^R|_{\mathcal{B}^1} \circ \Psi^{-1}$$

is Hénon-like. Then (F^R, Ψ) is referred to as a *Hénon-like return of* F.

Denote the critical point and the critical value of $\Pi_{1\mathrm{D}} \circ p\mathcal{R}(F)$ by $c^1, v^1 \in I^1$ respectively, and let $\mathcal{S} : \mathbb{R}^2 \to \mathbb{R}^2$ be the affine map given by

$$\mathcal{S}(x, y) := \sigma^{-1}(x - c^1, y) \quad \text{where} \quad \sigma := v^1 - c^1.$$

Define the *renormalization of* F as

$$\mathcal{R}(F) := \mathcal{S} \circ \Psi \circ F^R|_{\mathcal{B}^1} \circ (\mathcal{S} \circ \Psi)^{-1}.$$

Observe that $\mathcal{R}(F) \in \mathfrak{H}\mathfrak{L}^r$

1.5. **Regular Hénon-like returns.** Consider a C^r-diffeomorphism $F : \Omega \to F(\Omega) \Subset \Omega$ defined on a Jordan disk $\Omega \Subset \mathbb{R}^2$. Let $\lambda, \varepsilon \in (0, 1)$; $L \geq 1$ and $N \in \mathbb{N} \cup \{0, \infty\}$. A point $p \in \Omega$ is *N-times forward $(L, \varepsilon, \lambda)$-regular along $E^+_p \in \mathbb{P}^2_p$* if for $s \in \{-r, r-1\}$, we have

$$L^{-1}\lambda^{(1+\varepsilon)n} \leq \frac{(\mathrm{Jac}_p\, F^n)^s}{\|DF^n|_{E^+_p}\|^{s-1}} \leq L\lambda^{(1-\varepsilon)n} \quad \text{for all} \quad 1 \leq n \leq N. \tag{1.3}$$

Similarly, p is *N-times backward $(L, \varepsilon, \lambda)$-regular along $E^-_p \in \mathbb{P}^2_p$* if for $s \in \{-r, r-1\}$, we have

$$L^{-1}\lambda^{-(1-\varepsilon)n} \leq \frac{(\mathrm{Jac}_p\, F^{-n})^s}{\|DF^{-n}|_{E^-_p}\|^{s-1}} \leq L\lambda^{-(1+\varepsilon)n} \quad \text{for all} \quad 1 \leq n \leq N. \tag{1.4}$$

The constants L, ε and λ are referred to as an *irregularity factor*, a *marginal exponent* and a *contraction base* respectively.

There exists a uniform constant $\varepsilon_0 \in (0, 1)$ independent of F such that if (1.3) (or (1.4) resp.) holds with $\varepsilon \leq \varepsilon_0$, then the local dynamics of F near the forward (or backward resp.) orbit of p can be linearized up to the Nth iterate (see Theorem A.2). If $N = \infty$, this implies in particular that p has a well-defined strong-stable manifold $W^{ss}(p)$ (or center manifold $W^c(p)$ resp.), which is C^r-smooth and tangent to E^{ss}_p (or E^c_p resp.). It should be noted that the center manifold at an infinitely backward regular point p is not uniquely defined. However, its C^r-jet at p is unique (see Proposition A.12).

Definition 1.1. A Hénon-like return $(F^R, \Psi : \mathcal{B}^1 \to B^1)$ is said to be *$(L, \varepsilon, \lambda)$-regular* if the following conditions hold.

- For any $p \in \mathcal{B}^1$, we have $\measuredangle(E^v_p, E^h_p) > 1/L$, where
$$E^{v/h}_p := D\Psi^{-1}\left(E^{gv/gh}_{\Psi(p)}\right).$$
- Every $p \in \mathcal{B}^1$ is R-times forward $(L, \varepsilon, \lambda)$-regular along E^v_p.
- Every $q \in F^R(\mathcal{B}^1) \Subset \mathcal{B}^1$ is R-times backward $(L, \varepsilon, \lambda)$-regular along E^h_q.

In this case, we say that F is *$(L, \varepsilon, \lambda)$-regular Hénon-like renormalizable.*

Example 1.2. let $f : I \to I$ be a valuably renormalizable unimodal map. pre-renormalization $p\mathcal{R}(f) := f^R|_{I^1}$ is the first return map of f on an R-periodic interval $I^1 \Subset I$ containing the critical value v. Then for any $\varepsilon > 0$, there exists $\lambda > 0$ such that any C^r-diffeomorphism of the form

$$F(x, y) = (f(x) + e(x, y), x)$$

with $\|e\|_{C^r} < \lambda$ has a $(1, \varepsilon, \lambda)$-regular Hénon-like return $(F^R, \Psi : \mathcal{B}^1 \to B^1)$, with $\mathcal{B}^1$ $\lambda^{1-\varepsilon}$-close in Hausdorff topology to $I^1 \times I^1$ and Ψ $\lambda^{1-\varepsilon}$-close in C^r-topology to the identity.

For $N \in \mathbb{N} \cup \{\infty\}$, we say that $F : \Omega \to \Omega$ is *N-times Hénon-like renormalizable* if there exist a nested sequence of quadrilaterals $\{\mathcal{B}^n\}_{n=1}^N$ contained in Ω, and a sequence of horizontally equivalent C^r-charts:

$$\Psi^n : \mathcal{B}^n \to B^n = I^n \times I^n \subset \mathbb{R}^2 \quad \text{for} \quad 1 \leq n \leq N$$

such that (F^{R_n}, Ψ^n) is a Hénon-like return of F. In this case, we say that the sequence of Hénon-like returns is *nested*.

The *nth pre-renormalization of F* is defined as

$$F_n = p\mathcal{R}^n(F) := \Psi^n \circ F^{R_n}|_{\mathcal{B}^n} \circ (\Psi^n)^{-1}.$$

Let $f_n : I^n \to I^n$ be the unimodal map given by $f_n := \Pi_{1\text{D}}(F_n)$. Denote the critical point and the critical value of f_n by $c^n, v^n \in I^n$ respectively.

Let $\mathcal{S}^n : \mathbb{R}^2 \to \mathbb{R}^2$ be the affine map given by

$$\mathcal{S}^n(x, y) := \sigma_n^{-1}(x - c^n, y) \quad \text{where} \quad \sigma_n := v^n - c^n.$$

The nth *renormalization of F* is given by

$$\mathcal{R}^n(F) := \mathcal{S}^n \circ \Psi^n \circ F^{R_n}|_{\mathcal{B}^n} \circ (\mathcal{S}^n \circ \Psi^n)^{-1}.$$

Suppose that there exist constants $\lambda, \varepsilon_0 \in (0, 1)$ and $L \geq 1$ such that the Hénon-like returns $\{(F^{R_n}, \Psi^n)\}_{n=1}^N$ are $(L, \varepsilon_0, \lambda)$-regular. Then we say that F is *N-times $(L, \varepsilon_0, \lambda)$-regular Hénon-like renormalizable.*

Assume additionally that the return times $\{R_n\}_{n=1}^N$ are of $\mathbf{b}$-bounded type for some integer $\mathbf{b} \geq 2$. For many of our results, the specific values of L, λ and ε_0 are not so important, as long as ε_0 is sufficiently small to compensate for the size of $\mathbf{b}$. That is, we have

$$\mathbf{b}\overline{\varepsilon_0} < 1, \tag{1.5}$$

where $\overline{\varepsilon_0} := \varepsilon_0^d$ for some suitably small uniform constant $d \in (0, 1)$ independent of F. In this case, we sometimes simply say that F is *N-times regular Hénon-like renormalizable* without specifying the constants of regularity.

Theorem A. *Let $r \geq 2$ be an integer, and consider a C^r-diffeomorphism $F : \Omega \to F(\Omega) \Subset \Omega$ defined on a Jordan disk $\Omega \Subset \mathbb{R}^2$. Given constants $\mathbf{b} \in \mathbb{N}$, $L \geq 1$, $\lambda \in (0, 1)$ and $\varepsilon_0 \in (0, 1)$ satisfying* (1.5)*, there exists a uniform constant $\mathbf{N} \in \mathbb{N}$ depending only on $\|F\|_{C^2}$, λ and L such that the following holds. Suppose that F is infinitely topologically renormalizable with return times of $\mathbf{b}$-bounded type. If the first $\mathbf{N}$ renormalizations are $(L, \varepsilon_0, \lambda)$-regular Hénon-like, then F is infinitely regular Hénon-like renormalizable.*

Theorem B. *Let $r \geq 2$ be an integer, and consider a C^r-diffeomorphism $F : \Omega \to F(\Omega) \Subset \Omega$ defined on a Jordan domain $\Omega \Subset \mathbb{R}^2$. Suppose that F is infinitely regular Hénon-like renormalizable with return times of bounded type. Then the Hausdorff dimension of the induced renormalization limit set Λ_F is less than 1. Consequently, Λ_F is totally disconnected, minimal, and supports a unique invariant probability measure μ.*

1.6. **Regular unicriticality.** Consider a C^r-diffeomorphism $F : \Omega \to F(\Omega) \Subset \Omega$ defined on a Jordan domain $\Omega \Subset \mathbb{R}^2$. Suppose that F is infinitely renormalizable, and is uniquely ergodic on the induced renormalization limit set Λ_F given by (1.2). Then with respect to the unique invariant probability measure μ, the Lyapunov exponents of F are 0 and $\log \lambda_\mu < 0$ for some $\lambda_\mu \in (0, 1)$ (see [CLPY]). By Oseledets theorem, μ-a.e. point $p \in \Lambda_F$ has strong-stable and center directions $E^{ss}_p, E^c_p \in \mathbb{P}^2_p$ such that

$$\lim_{n\to+\infty} \frac{1}{n} \log \|DF^n|_{E^{ss}_p}\| = \log \lambda_\mu \tag{1.6}$$

and

$$\lim_{n\to+\infty} \frac{1}{n} \log \|DF^{-n}|_{E^c_p}\| = 0. \tag{1.7}$$

Let $\varepsilon > 0$. Since $F|_{\Lambda_F}$ is uniquely ergodic, (1.6) ((1.7) resp.) implies that p is infinitely forward (backward resp.) $(L, \varepsilon, \lambda_\mu)$-regular for some $L = L(p, \varepsilon) \geq 1$ (see [CLPY]).

If $p \in \Lambda_F$ satisfies (1.6) and (1.7) with

$$E^*_p := E^{ss}_p = E^c_p,$$

then $\{F^m(p)\}_{m\in\mathbb{Z}}$ is referred to as a *regular critical orbit.* Note that in this case, the local strong-stable manifold $W^{ss}_{\text{loc}}(p)$ and the center manifold $W^c(p)$ form a tangency at p. If this tangency is quadratic, then $\{F^m(p)\}_{m\in\mathbb{Z}}$ is referred to as a *regular quadratic critical orbit.*

For $t > 0$ and $p \in \mathbb{R}^2$, we denote the ball

$$\mathbb{D}_p(t) := \{q \in \mathbb{R}^2 \mid \operatorname{dist}(q, p) < t\}.$$

Definition 1.3. For $0 < \varepsilon < \delta < 1$, we say that F is (δ, ε)*-regularly unicritical on the limit set* Λ_F if the following conditions hold.

i) There is a regular quadratic critical orbit point $v \in \Lambda_F$ (referred to as the *critical value*).
ii) For all $t > 0$, there exists $L(t) \geq 1$ such that for any $N \in \mathbb{N}$, if

$$p \in \Lambda_F \setminus \bigcup_{n=0}^{N-1} \mathbb{D}_{F^{-n}(v)}(t\lambda_\mu^{\varepsilon n}), \tag{1.8}$$

then p is N-times forward $(L(t), \delta, \lambda_\mu)$-regular.

When δ and ε are implicit, we simply say that F is *regularly unicritical on* Λ_F.

In [CLPY], we prove that if F infinitely topologically renormalizable (with return times not necessarily of bounded type), and is regular unicritical on the induced renormalization limit set, then the renormalizations of F are eventually regular henon-like.

Theorem C. *Let $r \geq 2$ be an integer, and consider a C^r-diffeomorphism $F : \Omega \to F(\Omega) \Subset \Omega$ defined on a Jordan domain $\Omega \Subset \mathbb{R}^2$. Suppose for some $L \geq 1$; $\lambda, \varepsilon_0 \in (0, 1)$*

and $\mathbf{b} \geq 2$ *satisfying* (1.5)*, the map* F *has infinite nested* $(L, \varepsilon_0, \lambda)$*-regular Hénon-like returns:*

$$\{(F^{R_n}, \Psi^n : \mathcal{B}^n \to B^n)\}_{n=1}^{\infty}$$

with return times of $\mathbf{b}$*-bounded type. Then for any* $\varepsilon > 0$*, there exists* $L_\varepsilon \geq 1$ *such that for all* $n \in \mathbb{N}$*, the Hénon-like return* (F^{R_n}, Ψ^n) *is* $(L_\varepsilon, \varepsilon, \lambda_\mu)$*-regular. Moreover,* F *is* $(\varepsilon, \varepsilon^d)$*-regularly unicritical on the induced renormalization limit set* Λ_F*, where* $d \in (0, 1)$ *is some suitably small uniform constant independent of* F*. Lastly, we have*

$$\bigcap_{n=1}^{\infty} F^{R_n}(\mathcal{B}^n) = \{v\},$$

where $v \in \Lambda_F$ *is the regular quadratic critical value.*

1.7. **Renormalization convergence.** The 1D Renormalization $\mathcal{R}_{1\mathrm{D}}$ defined in Subsection 1.1 can be viewed as an operator acting on the Banach space $\mathfrak{U}^\gamma$ of unimodal maps. In [L], Lyubich shows that $\mathcal{R}_{1\mathrm{D}}$ restricted to $\mathfrak{U}^\omega$ is an analytic operator that has a hyperbolic attractor $\mathfrak{A} \subset \mathfrak{U}^\omega$ with exactly one unstable dimension. This attractor is referred to as the *full renormalization horseshoe*.

Given an integer $\mathbf{b} \geq 2$, we identify the compact invariant subset $\mathfrak{A}_{\mathbf{b}}$ of $\mathfrak{A}$ that consist of maps of $\mathbf{b}$-bounded type. In [dFdMPi], de Faria-de Melo-Pinto show that for the renormalization operator $\mathcal{R}_{1\mathrm{D}}$ acting on the more general space $\mathfrak{U}^3 \supset \mathfrak{U}^\omega$, the set $\mathfrak{A}_{\mathbf{b}}$ remains a hyperbolic attractor with one unstable dimension.

Theorem D. *Let* $r \geq 2$ *be an integer, and consider a* C^r*-diffeomorphism* $F : \Omega \to F(\Omega) \Subset \Omega$ *defined on a Jordan domain* $\Omega \Subset \mathbb{R}^2$*. Suppose for some* $L \geq 1$*;* $\lambda \in (0, 1)$*;* $\varepsilon \in (0, \varepsilon_0]$ *and* $\mathbf{b} \geq 2$ *satisfying* (1.5)*, the map* F *has infinite nested* $(L, \varepsilon, \lambda)$*-regular Hénon-like returns:*

$$\{(F^{R_n}, \Psi^n : \mathcal{B}^n \to B^n)\}_{n=1}^{\infty}$$

with return times of $\mathbf{b}$*-bounded type. Then, after replacing* $\{\Psi^n\}_{n=1}^{\infty}$ *if necessary, the following statements hold for all* $n \in \mathbb{N}$*:*

i) $\|(\Psi^n)^{\pm 1}\|_{C^r} < \bar{L}$ *and* $\|\Psi^{n+1} - \Psi^n|_{\mathcal{B}^{n+1}}\|_{C^r} < \bar{L}\lambda^{(1-\bar{\varepsilon})R_n}$*;*
ii) $\mathcal{R}^n(F)$ *is a* δ_n*-thin* C^r*-Hénon-like map with* $\delta_n < \bar{L}\lambda^{(1-\bar{\varepsilon})R_n}$*; and*
iii) $\|\mathcal{R}^n(F)\|_{C^r} = O(1)$ *if* n *is sufficiently large;*

where $\bar{L} := KL^D > L$ *and* $\bar{\varepsilon} := \varepsilon^{1/D} > \varepsilon$ *for some uniform constants* $K > 1$ *(dependent only on* $\|F\|_{C^r}$*) and* $D > 1$ *(independent of* F*).*

If, additionally, we have $r \geq 4$*, then there exists a real analytic unimodal map* $f_* \in \mathfrak{A}_{\mathbf{b}}$ *and a universal constant* $\rho = \rho(\mathbf{b}) \in (0, 1)$ *such that*

$$\|\Pi_{1\mathrm{D}} \circ \mathcal{R}^n(F) - \mathcal{R}^n_{1\mathrm{D}}(f_*)\|_{C^{r-1}} = O(\rho^n) \quad \text{for} \quad n \in \mathbb{N}.$$

1.8. **Conventions.** Unless otherwise specified, we adopt the following conventions.

Any diffeomorphism on a domain in $\mathbb{R}^2$ is assumed to be orientation-preserving. The projective tangent space at a point $p \in \mathbb{R}^2$ is denoted by $\mathbb{P}^2_p$.

We typically denote constants by $K \geq 1$, $k > 0$ (and less frequently $C \geq 1$, $c > 0$). Given a number $\kappa > 0$, we use $\bar{\kappa}$ to denote any number that satisfy

$$\kappa < \bar{\kappa} < C\kappa^D$$

for some universal constants $C > 1$ and $D > 1$ (if $\kappa > 1$) or $D \in (0,1)$ (if $\kappa < 1$) independent of the considered map. We allow $\bar{\kappa}$ to absorb any uniformly bounded coefficient or power. So for example, if $\bar{\kappa} > 1$, then we may write

$$\text{“ } 10\bar{\kappa}^5 = \bar{\kappa} \text{ ”.}$$

Similarly, we use $\underline{\kappa}$ to denote any number that satisfy

$$c\kappa^d < \underline{\kappa} < \kappa$$

for some uniform constants $c \in (0,1)$ and $d \in (0,1)$ (if $\kappa > 1$) or $d > 1$ (if $\kappa < 1$) independent of the map. As before, we allow $\underline{\kappa}$ to absorb any uniformly bounded coefficient or power. So for example, if $\underline{\kappa} > 1$, then we may write

$$\text{“ } \tfrac{1}{3}\underline{\kappa}^{1/4} = \underline{\kappa} \text{ ”.}$$

These notations apply to any positive real number: e.g. $\bar{\varepsilon} > \varepsilon$, $\underline{\delta} < \delta$, $\bar{L} > L$, etc.

Note that $\bar{\kappa}$ can be much larger than κ (similarly, $\underline{\kappa}$ can be much smaller than κ). Sometimes, we may avoid the $\bar{\kappa}$ or $\underline{\kappa}$ notation when indicating numbers that are somewhat or very close to the original value of κ. For example, if $\kappa \in (0,1)$ is a small number, then we may denote $\kappa' := (1-\bar{\kappa})\kappa$. Then $\underline{\kappa} \ll \kappa' < \kappa$.

For any set $X_m \subset \Omega$ with a numerical index $m \in \mathbb{Z}$, we denote

$$X_{m+l} := F^l(X_m)$$

for all $l \in \mathbb{Z}$ for which the right-hand side is well-defined. Similarly, for any direction $E_{p_m} \in \mathbb{P}^2_{p_m}$ at a point $p_m \in \Omega$, we denote

$$E_{p_{m+l}} := DF^l(E_{p_m}).$$

We use n, m, i, j to denote integers (and less frequently l, k). Typically (but not always), $n \in \mathbb{N}$ and $m \in \mathbb{Z}$. We sometimes use $l > 0$ for positive geometric quantities (such as length). The letter i is never the imaginary number.

We typically use N, M to indicate fixed integers (often related to variables n, m).

We use calligraphic font $\mathcal{U}, \mathcal{T}, \mathcal{I}$, etc, for objects in the phase space and regular fonts U, T, I, etc, for corresponding objects in the linearized/uniformized coordinates. A notable exception is for the invariant manifolds W^{ss}, W^c.

We use p, q to indicate points in the phase space, and z, w for points in linearized/uniformized coordinates.

2. Chart Relations

Let $\Psi : \mathcal{B} \to B$ be a C^r-chart. A *vertical leaf in* $\mathcal{B}$ is a curve l^v such that

$$l^v \subseteq \Psi^{-1}(\{a\} \times \pi_v(B)) \quad \text{for some} \quad a \in \pi_h(B).$$

If the above containment is an equality, then l^v is said to be *full.* A *(full) horizontal leaf* l^h *in* $\mathcal{B}$ is defined analogously.

Let $p \in \mathcal{B}$ and $E_p \in \mathbb{P}^2$. Denote

$$z := \Psi(p) \quad \text{and} \quad E_z := D\Psi(E_p).$$

For $t > 0$, the direction E_p is said to be *t-vertical in* $\mathcal{B}$ if

$$\frac{\measuredangle(E_z, E_z^{gv})}{\measuredangle(E_z, E_z^{gh})} < t.$$

A *t-horizontal direction in* $\mathcal{B}$ is analogously defined.

A C^0-curve $\Gamma^v \subset \mathcal{B}$ is said to be *vertical in* $\mathcal{B}$ if $\Psi(\Gamma^v)$ is a vertical graph in B in the usual sense. That is, there exists an interval $I^v \subseteq \pi_v(B)$ and a map $g_v : I^v \to \pi_h(B)$ such that

$$\Psi(\Gamma^v) = \mathcal{G}^v(g_v) := \{(g_v(y), y) \mid y \in I^v\}.$$

If $I^v = \pi_v(B)$, then Γ^v is said to be *vertically proper in* $\mathcal{B}$. If Γ^v is C^2, and g_v has a unique critical point $c \in I^v$ of quadratic type ($g_v'(c) = 0$ and $g_v''(c) \neq 0$), then Γ^v is a *vertical quadratic curve in* $\mathcal{B}$. If Γ^v is C^r, and $\|g_v'\|_{C^{r-1}} \leq t$ for some $t \geq 0$, then we say that Γ^v is *t-vertical in* $\mathcal{B}$. Note that Γ^v is a (vertically proper) 0-vertical curve if and only if it is a (full) vertical leaf.

Let $\mathcal{E}^v : \mathcal{B} \to T^1(\mathcal{B})$ be the C^{r-1}-unit vector field given by

$$\mathcal{E}^v(p) := D\Psi^{-1}(E^{gv}_{\Psi(p)}).$$

A C^{r-1}-unit vector field $\tilde{\mathcal{E}}^v : \mathcal{U} \to T^1(\mathcal{U})$ defined on a domain $\mathcal{U} \subset \mathcal{B}$ is said to be *t-vertical in* $\mathcal{B}$ for some $t \geq 0$ if $\|\tilde{\mathcal{E}}^v - \mathcal{E}^v\|_{C^{r-1}} \leq t$.

Let $\tilde{\Psi} : \tilde{\mathcal{B}} \to \tilde{B}$ be another chart with $\tilde{\mathcal{B}} \subset \mathcal{B}$. We define the following relations between Ψ and $\tilde{\Psi}$.

- We say that $\tilde{\mathcal{B}}$ is *vertically proper in* $\mathcal{B}$ if every full vertical leaf in $\tilde{\mathcal{B}}$ is vertically proper in $\mathcal{B}$.
- We say that Ψ and $\tilde{\Psi}$ are *horizontally equivalent on* $\tilde{\mathcal{B}}$ if every horizontal leaf in $\tilde{\mathcal{B}}$ is a horizontal leaf in $\mathcal{B}$.
- For $t \geq 0$, we say that $\tilde{\mathcal{B}}$ *is t-vertical in* $\mathcal{B}$ if Ψ and $\tilde{\Psi}$ are horizontally equivalent, and the unit vector field given by

$$\tilde{\mathcal{E}}^v(p) := D\tilde{\Psi}^{-1}(E^{gv}_{\tilde{\Psi}(p)}) \quad \text{for} \quad p \in \tilde{\mathcal{B}}$$

  is t-vertical in $\mathcal{B}$.
- We say that Ψ and $\tilde{\Psi}$ are *equivalent on* $\tilde{\mathcal{B}}$ if $\tilde{\mathcal{B}}$ is 0-vertical in $\mathcal{B}$.

Let $\hat{\Psi} : \hat{\mathcal{B}} \to \hat{B}$ be a chart satisfying the following properties.

- We have $0 \in \hat{B}$.

- Let
$$\mathcal{I}^h(t) := \hat{\Psi}^{-1}(t,0) \quad \text{for} \quad t \in \pi_h(\hat{B}),$$
and
$$\mathcal{I}^v(s) := \hat{\Psi}^{-1}(0,s) \quad \text{for} \quad s \in \pi_v(\hat{B}).$$
Then $\|(\mathcal{I}^{h/v})'\| \equiv 1$.

In this case, we say that $\hat{\Psi}$ is *centered (at $\hat{\Psi}^{-1}(0)$)*.

A C^0-curve $\Gamma^h \subset \hat{\mathcal{B}}$ is said to be *horizontal in $\hat{\mathcal{B}}$* if $\hat{\Psi}(\Gamma^h)$ is the horizontal graph in $\hat{B}$ of a map $g_h : I^h \to \pi_v(\hat{B})$ defined on an interval $I^h \subset \pi_h(\hat{B})$. If Γ^h is C^r, then we say that Γ^h is *t-horizontal in $\hat{\mathcal{B}}$* if $\|g_h\|_{C^r} \leq t$. In particular, Γ^h is 0-horizontal in $\hat{\mathcal{B}}$ if and only if Γ^h is a subarc of the full horizontal leaf containing $\hat{\Psi}^{-1}(0)$.

Lemma 2.1. *Let $\Psi : \mathcal{B} \to B$ be a chart. For any point $q \in \mathcal{B}$, there exists a unique chart $\hat{\Psi} : (\mathcal{B}, q) \to (\hat{B}, 0)$ centered at q such that $\hat{\Psi}$ and Ψ are equivalent on $\mathcal{B}$.*

3. The Critical Value

3.1. **The set up.** Let $r \geq 2$ be an integer, and consider a C^r-diffeomorphism $F : \Omega \to F(\Omega)$ defined on a domain $\Omega \subset \mathbb{R}^2$. For simplicity, we assume that $\|F\|_{C^r}$ is uniformly bounded:
$$\|F\|_{C^r} = O(1). \tag{3.1}$$
Denote $\mathcal{B}_0^0 := \Omega$ and $R_0 := 1$. For $1 \leq n \leq N \leq \infty$, suppose there exist an R_n-periodic quadrilateral $\mathcal{B}_0^n \Subset \mathcal{B}_0^{n-1}$ with
$$r_{n-1} := R_n/R_{n-1} \geq 2,$$
and a C^r-chart $\Psi^n : \mathcal{B}_0^n \to B_0^n$ such that $\{(F^{R_n}, \Psi^n)\}_{n=1}^N$ is a (possibly infinite) sequence of nested Hénon-like returns of F. Furthermore, assume that the sequence of returns is $(L, \varepsilon, \lambda)$-regular for some $\lambda, \varepsilon \in (0,1)$ and $L \geq 1$ such that $\bar{\varepsilon} < 1$. Lastly, suppose that N is sufficiently large, so that by replacing (F^{R_1}, Ψ^1) with $(F^{R_{n_1}}, \Psi^{n_1})$ for some $n_1 \leq N$, we may assume additionally that:
$$\bar{L}\lambda^{(1-\bar{\varepsilon})R_1} < \rho, \tag{3.2}$$
where $\rho \in (0,1)$ is a suitably small universal constant.

Remark 3.1. In Sections 3 and 4, we do not assume that the sequence of Hénon-like returns of F is necessarily of bounded type.

3.2. **Locating the critical value.** For $i \in \mathbb{Z}$, denote $\mathcal{B}_i^n := F^i(\mathcal{B}_0^n)$. Observe that $\mathcal{B}_{R_{n+1}}^{n+1} \Subset \mathcal{B}_{R_n}^n$. Let
$$\mathcal{Z}_0 := \bigcap_{n=1}^N \mathcal{B}_{R_n}^n.$$
Let $v_0 \in \mathcal{Z}_0$ be a point to be specified later (as the *critical value of F*). By Lemma 2.1, we may assume that Ψ^n for all $1 \leq n \leq N$ is centered at v_0. Define
$$I_0^n := \pi_h(B_0^n) \quad \text{and} \quad \mathcal{I}_0^n := (\Psi^n)^{-1}(I_0^n \times \{0\}).$$

Then it follows that $I_0^n \Subset I_0^1$ and $\Psi^n|_{\mathcal{I}_0^n} = \Psi^1|_{\mathcal{I}_0^n}$. Denote $\mathcal{I}_i^n := F^i(\mathcal{I}_0^n)$ for $i \geq 0$.

For $p_0 \in \mathcal{B}_0^n$, write $z_0 := \Psi^n(p_0)$, and let

$$E_{p_0}^h := D(\Psi^n)^{-1}(E_{z_0}^{gh}) \quad \text{and} \quad E_{p_0}^{v,n} := D(\Psi^n)^{-1}(E_{z_0}^{gv}).$$

Additionally, let

$$E_{p_{R_n-1}}^{h,n} := DF^{R_n-1}(E_{p_0}^h) \quad \text{and} \quad E_{p_{R_n-1}}^v := DF^{-1}(E_{p_{R_n}}^h) = DF^{R_n-1}(E_{p_0}^{v,n}).$$

By increasing L by a uniform amount (depending only on DF) if necessary, we may assume that every $q \in \mathcal{B}_{R_n-1}^n$ is $(R_n - 1)$-times backward $(L, \varepsilon, \lambda)$-regular along E_q^v.

Proposition 3.2. *After replacing the charts $\{\Psi^n\}_{n=1}^N$ if necessary, the following properties hold. For $1 \leq n \leq N$, the domain $\mathcal{B}_0^n$ of the chart Ψ^n is vertically proper and ρ-vertical in $\mathcal{B}_0^1$. Moreover, we have*

$$\|\Psi^{n+1} - \Psi^n|_{\mathcal{B}_0^{n+1}}\|_{C^r} < \lambda^{(1-\bar{\varepsilon})R_n}. \tag{3.3}$$

Proof. For $p_0 \in \mathcal{B}_0^n$, let

$$\{\Phi_{p_m} : \mathcal{U}_{p_m} \to U_{p_m}\}_{m=0}^{R_n}$$

be a linearization of F along the R_n forward orbit of p_0 with vertical direction $E_{p_0}^{v,n}$. Let $\mathcal{E}_{p_m}^{v,n} : \mathcal{U}_{p_m} \to T^1(\mathcal{U}_{p_m})$ be the C^{r-1}-unit vector field given by $\mathcal{E}_{p_m}^{v,n}(q) \in E_q^{v,n}$ for $q \in \mathcal{U}_{p_m}$.

Let $l_{p_0}^{v,1}$ be the full vertical leaf in $\mathcal{B}_0^1$ containing p_0. For $q_0 \in l_{p_0}^{v,1}$, let

$$\{\Phi_{q_m} : \mathcal{U}_{q_m} \to U_{q_m}\}_{m=0}^{R_1}$$

be a linearization of F along the R_1 forward orbit of q_0 with vertical direction $E_{q_0}^{v,1}$.

Let M be a nearest integer to $R_1/2$. Since ρ is sufficiently small, it follows from (3.2), Theorem A.2, and Propositions A.5 and A.3 that

$$\check{\mathcal{U}}_{q_M} := F^M(\mathcal{U}_{q_0}^{\bar{\varepsilon}M}) \subset \mathcal{U}_{p_M}.$$

By Proposition A.1, q_M and p_M are M-times forward $(\bar{L}\lambda^{-\bar{\varepsilon}M}, \varepsilon, \lambda)_v$-regular along $E_{q_M}^{v,1}$ and $E_{p_M}^{v,n}$ respectively. Hence, Proposition A.8 implies that $\mathcal{E}_{p_M}^{v,n}|_{\check{\mathcal{U}}_{q_M}}$ is t-vertical in $\mathcal{U}_{q_m}$ for some $t > 0$ uniformly small. Thus, we may extend $\mathcal{E}_{p_0}^{v,n}$ to $\mathcal{U}_{q_0}^{\bar{\varepsilon}M}$ as

$$\mathcal{E}_{p_0}^{v,n}|_{\mathcal{U}_{q_0}^{\bar{\varepsilon}M}} := DF_*^{-M}(\mathcal{E}_{p_M}^{v,n}|_{\check{\mathcal{U}}_{q_M}}).$$

Then we have $\|\mathcal{E}_{p_0}^{v,n} - \mathcal{E}_{q_0}^{v,1}\|_{C^1} \leq \rho$. Rectifying the vertical directions near $l_{p_0}^{v,1}$ given by $\mathcal{E}_{p_0}^{v,n}$, we obtain the desired extension of Ψ^n.

Replacing the renormalization depth 1 in the above argument by n, we obtain (3.3). □

Consider C^r-curves $\Gamma_1, \Gamma_2 \subset \mathbb{R}^2$ with $|J_1| \geq |J_2|$. For $i \in \{1, 2\}$, let $\phi_{\Gamma_i} : J_i \subset \mathbb{R} \to \Gamma_i$ be a parameterization of Γ_i such that

- $|\phi'_{\Gamma_i}| \equiv 1$;
- $J_1 \supset J_2$;
- $\|\phi_{\Gamma_1}|_{J_2} - \phi_{\Gamma_2}\|_{C^r}$ is minimal.

In this case, define

$$\mathrm{dist}_{C^r}(\Gamma_1, \Gamma_2) := \|\phi_{\Gamma_1}|_{J_2} - \phi_{\Gamma_2}\|_{C^r}.$$

Lemma 3.3. *For $1 \leq n \leq N$, let l_0^n be a full horizontal leaf in $\mathcal{B}_0^n$. Then we have*

$$\mathrm{dist}_{C^r}(l^n_{R_n-1}, l^{n+1}_{R_{n+1}-1}) < \lambda^{(1-\bar{\varepsilon})R_n}.$$

Proof. For $p_{-1} \in \mathcal{Z}_{-1} := F^{-1}(\mathcal{Z}_0)$, let

$$\{\Phi_{p_{-m}} : \mathcal{U}_{p_{-m}} \to U_{p_{-m}}\}_{m=1}^{R_N}$$

be a linearization of F along the R_N-times backward orbit of p_{-1} with vertical direction $E^v_{p_{-1}}$ (if $N = \infty$, then $R_\infty = \infty$). Let $\mathcal{V}_{-R_n}$ be the connected component of $F^{-R_n+1}(\mathcal{U}^{\bar{\varepsilon}R_n}_{p_{-1}}) \cap \mathcal{B}^n_0$ containing p_{-R_n}. Note that $\Psi^n|_{\mathcal{V}_{-R_n}}$ defines a chart on $\mathcal{V}_{-R_n}$, so that $\mathcal{V}_{-R_n}$ is 0-vertical in $\mathcal{B}^n_0$. Moreover, arguing as in the proof of Proposition 3.2, we see that $\mathcal{V}_{-R_n}$ is also vertically proper in $\mathcal{B}^n_0$. Hence, by Theorem A.2 and Proposition A.5, the curve $l^n_{R_n-1}$ is $\lambda^{(1-\bar{\varepsilon})R_n}$-horizontal in $\mathcal{U}_{p_{-1}}$. The result follows. □

Proposition 3.4. *If $N = \infty$, then the following statements hold.*

i) For any point $p_0 \in \mathcal{Z}_0$, there exists a unique strong stable direction $E^{ss}_{p_0} \in \mathbb{P}^2_{p_0}$ such that

$$\|E^{v,n}_{p_0} - E^{ss}_{p_0}\| < \lambda^{(1-\bar{\varepsilon})R_n} \quad \text{for} \quad n \in \mathbb{N}.$$

Moreover, p_0 is infinitely forward $(L, \varepsilon, \lambda)$-regular along $E^{ss}_{p_0}$.

ii) Any point $p_{-1} \in \mathcal{Z}_{-1}$ is infinitely backward $(L, \varepsilon, \lambda)$-regular along $E^v_{p_{-1}}$. Moreover, there exists a unique center direction $E^c_{p_{-1}} \in \mathbb{P}^2_{p_{-1}}$ such that

$$\|E^{h,n}_{p_{-1}} - E^c_{p_{-1}}\| < \lambda^{(1-\bar{\varepsilon})R_n} \quad \text{for} \quad n \in \mathbb{N}.$$

iii) There exists a unique point $v_0 \in \mathcal{Z}_0$ such that

$$E^{ss}_{v_0} = DF(E^c_{v_{-1}}).$$

Moreover, the strong stable manifold $W^{ss}(v_0)$ and the center manifold $F(W^c(v_{-1}))$ have a quadratic tangency at v_0.

Proof. The first and second claim follow immediately from Propositions A.8 and A.9.

For $n \in \mathbb{N}$, let l^n_0 be a full horizontal leaf in $\mathcal{B}^n_0$. Recall that $l^n_{R_n}$ is a vertical quadratic curve in $\mathcal{B}^n_0$. Let $v^n_0 \in l^n_0$ be the unique point such that

$$E^{v,n}_{v^n_{R_n}} = DF^{R_n}(E^h_{v^n_0}).$$

By Lemma 3.3, we have

$$\mathrm{dist}(v^n_{R_n}, v^{n+1}_{R_{n+1}}) < \lambda^{(1-\bar{\varepsilon})R_n}.$$

Thus, there exists a unique point $v_0 \in \mathcal{Z}_0$ such that

$$\mathrm{dist}(v^n_{R_n}, v_0)\ ,\ \mathrm{dist}_{C^r}(l^n_{R_n}, W^c(v_0)) < \lambda^{(1-\bar{\varepsilon})R_n}.$$

By (3.3), we see that $W^{ss}(v_0)$ and $W^c(v_0)$ have a quadratic tangency at v_0.

Lastly, let $\mathcal{U}_{v_0}$ be a neighborhood of v_0. Then there exists a uniform constant $k > 0$ such that for all n sufficiently large, if $p_{R_n} \in l^n_{R_n} \setminus \mathcal{U}_{v_0}$ then

$$\measuredangle(E^{v,n}_{p_{R_n}}, DF^{R_n}(E^h_{p_0})) > k.$$

Thus, v_0 is the unique point in $\mathcal{Z}_0$ satisfying $E^{ss}_{v_0} = E^c_{v_0}$. □

We define the *critical value* $v_0 \in \mathcal{Z}_0$ as follows. If $N = \infty$, let v_0 be the point given in Proposition 3.4 iii). Otherwise, let v_0 be the unique point in $\mathcal{I}^N_{R_N}$ such that

$$DF^{R_N}(E^h_{v_{-R_N}}) = E^{v,N}_{v_0}$$

(recall that $\mathcal{I}^N_{R_N}$ is a vertical quadratic curve in $\mathcal{B}^N_0$). Define the *critical point* as $v_{-1} := F^{-1}(v_0)$.

Remark 3.5. In fact, we will show that if $N = \infty$, then $\mathcal{Z}_0 = \{v_0\}$ (see Theorem 4.7).

Theorem 3.6 (Valuable charts)**.** *There exist charts*

$$\Phi_0 : (\mathcal{B}_0, v_0) \to (B_0, 0) \quad \textit{and} \quad \Phi_{-1} : (\mathcal{B}_{-1}, v_{-1}) \to (B_{-1}, 0)$$

with

$$\mathcal{B}_0 \supset \mathcal{B}^1_0, \quad \mathcal{B}_{-1} \supset \mathcal{B}^1_{R_1-1} \quad \textit{and} \quad F(\mathcal{B}_{-1}) \Subset \mathcal{B}_0;$$

and

$$\|\Phi_i^{\pm 1}\|_{C^r} < \bar{L} \quad \textit{for} \quad i \in \{0, -1\};$$

such that

$$\Phi_0 \circ F \circ \Phi_{-1}^{-1}(x, y) = (f_0(x) - \lambda y, x) \quad \textit{for} \quad (x, y) \in B_{-1} \tag{3.4}$$

for some C^r-unimodal interval map

$$f_0 : (\pi_h(B_{-1}), 0) \to (\pi_h(B_0), 0)$$

with a unique critical point at 0 with $f_0''(0) < 0$. Moreover, the following properties hold for $1 \leq n \leq N$.

i) Let $p_0 \in \mathcal{B}^n_0$. Then

$$D\Phi_0(E^h_{p_0}) = E^{gh}_{\Phi_0(p_0)} \quad \textit{and} \quad D\Phi_{-1}(E^v_{p_{R_n-1}}) = E^{gv}_{\Phi_{-1}(p_{R_n-1})}.$$

ii) We have $\Psi^n|_{\mathcal{I}^n_0} = \Phi_0|_{\mathcal{I}^n_0}$.

iii) We have

$$\|\Psi^n \circ (\Phi_0|_{\mathcal{B}^n_0})^{-1} - \mathrm{Id}\,\|_{C^r} < \lambda^{(1-\bar{\varepsilon})R_n}.$$

iv) Let

$$H_n := \Phi_{-1} \circ F^{R_n-1} \circ (\Psi^n)^{-1}.$$

Then $H_n(x, y) = (h_n(x), e_n(x, y))$, where $h_n : I^n_0 \to h_n(I^n_0)$ is a C^r-diffeomorphism and e_n is a C^r-map such that

$$\inf_{x \in I^n_0} |h_n'(x)| > \bar{L}^{-1} \lambda^{\bar{\varepsilon} R_n} \quad \textit{and} \quad \|e_n\|_{C^r} < \lambda^{(1-\bar{\varepsilon})R_n}. \tag{3.5}$$

Proof. For $t \geq 0$ and $X \subset \mathbb{R}^2$, denote

$$X(t) := \{p \in \mathbb{R}^2 \mid \operatorname{dist}(p, X) \leq t\}.$$

Let

$$\mathcal{B}_0 := \mathcal{B}_0^1(\lambda^{\bar{\varepsilon} R_1}) \quad \text{and} \quad \mathcal{C}_0^n := \mathcal{B}_0^n(\lambda^{\bar{\varepsilon} R_n}) \setminus \mathcal{B}_0^n.$$

By (3.3), there exists a C^r-diffeomorphism Φ_0 defined in a neighborhood of $\mathcal{Z}_0$ such that

$$\|\Psi^n|_{\mathcal{Z}_0} - \Phi_0\|_{C^r} < \lambda^{(1-\bar{\varepsilon})R_n} \quad \text{for all} \quad 1 \leq n \leq N.$$

Moreover, Φ_0 can be extended a centered chart $\Phi_0 : (\mathcal{B}_0, v_0) \to (B_0, 0)$ such that

$$\Phi_0|_{\mathcal{B}_0^n \setminus (\mathcal{B}_0^{n+1} \cup \mathcal{C}_0^{n+1})} = \Psi^n|_{\mathcal{B}_0^n \setminus (\mathcal{B}_0^{n+1} \cup \mathcal{C}_0^{n+1})}$$

and

$$\|\Phi_0|_{\mathcal{C}_0^{n+1}} - \Psi^n|_{\mathcal{C}_0^{n+1}}\|_{C^r} < \lambda^{(1-\bar{\varepsilon})R_n}.$$

Let $\mathcal{I}_{-1}^h := W^c(v_{-1})$. Observe that $F(\mathcal{I}_{-1}^h)$ is a vertical quadratic curve in $\mathcal{B}_0$. Hence, there exists a C^r-unimodal interval map

$$f_0 : (\pi_h(B_{-1}), 0) \to (\pi_h(B_0), 0)$$

with a unique quadratic critical point at 0 such that

$$\Phi_0 \circ F(\mathcal{I}_{-1}^h) = \{(f_0(y), y) \mid y \in \pi_v(B_0)\}.$$

For some $l_{-1} = \bar{L}^{-1}$, let

$$D_0 := \{(f_0(y) + t, y) \in B_0 \mid |t| \leq \lambda l_{-1} \text{ and } y \in \pi_v(B_0)\},$$

and

$$\mathcal{B}_{-1} := (\Phi_0 \circ F)^{-1}(D_0).$$

We define $\Phi_{-1} : (\mathcal{B}_{-1}, v_{-1}) \to (B_{-1}, 0)$ to be the unique chart satisfying

$$\Phi_0 \circ F \circ \Phi_{-1}^{-1}(x, y) = (f_0(x) - \lambda y, x) \quad \text{for} \quad (x, y) \in B_{-1}.$$

Claims i), ii) and iii) follow immediately.

The second inequality in (3.5) follows from Lemma 3.3. Hence, for $p_0 \in \mathcal{B}_0^n$, we have

$$\|DF^{R_n - 1}|_{E_{p_0}^{v,n}}\| = \|\Phi_{-1}^{-1} \circ H_n \circ \Psi^n|_{E_{p_0}^{v,n}}\| < \bar{L}\|H_n|_{E_{\Psi^n(p_0)}^{gv}}\| < \bar{L}\lambda^{(1-\bar{\varepsilon})R_n}.$$

By regularity of the Hénon-like return (F^{R_n}, Ψ^n), we have

$$\measuredangle(E_{p_0}^{v,n}, E_{p_0}^h) > L^{-1}.$$

This implies that

$$\operatorname{Jac}_{p_0} F^{R_n - 1} < \bar{L}\|DF^{R_n - 1}|_{E_{p_0}^{v,n}}\| \cdot \|DF^{R_n - 1}|_{E_{p_0}^h}\|.$$

Thus, (1.3) imply that

$$\bar{L}\lambda^{(1-\bar{\varepsilon})R_n}\|DF^{R_n - 1}|_{E_{p_0}^h}\|^{r-1} > \bar{L}^{-1}\lambda^{(1+\varepsilon)R_n}.$$

The first inequality in (3.5) follows. □

Remark 3.7. In Theorem 7.7, we show that if $N = \infty$ and the return times are of bounded type, then the first inequality in (3.5) can be improved to

$$\inf_{x \in I_0^n} |h_n'(x)| > \mathbf{k}$$

for some uniform constant $\mathbf{k} > 0$.

For $i \in \{0, -1\}$, denote

$$I_i^{h/v} := \pi_{h/v}(B_i) \quad \text{and} \quad \mathcal{I}_i^h := \Phi_i^{-1}(I_i^h \times \{0\}). \tag{3.6}$$

Observe that

$$I_0^h \Supset I_0^1 \Supset I_0^2 \Supset \dots \quad \text{and} \quad I_{-1}^h \Supset h_1(I_0^1) \Supset h_2(I_0^2) \Supset \dots .$$

Moreover, if $X \subset \mathcal{B}_0^n$, then (3.5) implies

$$\Phi_{-1} \circ F^{R_n - 1}(X) \subset h_n(I_0^n) \times [-\lambda^{(1-\bar{\varepsilon})R_n}, \lambda^{(1-\bar{\varepsilon})R_n}]. \tag{3.7}$$

3.3. **Horizontal projections.** For $1 \leq n \leq N$, define $P_{-1} : (\mathcal{B}_{-1}, v_{-1}) \to (I_{-1}^h, 0)$ and $P_0^n : (\mathcal{B}_0^n, v_0) \to (I_0^n, 0)$ by

$$P_{-1} := \pi_h \circ \Phi_{-1} \quad \text{and} \quad P_0^n := \pi_h \circ \Psi^n.$$

Denote

$$I_{R_n-1}^n := P_{-1}(\mathcal{B}_{R_n-1}^n) = P_{-1}(\mathcal{I}_{R_n-1}^n) = h_n(I_0^n),$$

where h_n is given in Theorem 3.6 iv). Define $\mathcal{P}_0^n : \mathcal{B}_0^n \to \mathcal{I}_0^n$ by

$$\mathcal{P}_0^n(p) := (\Psi^n)^{-1} \left(P_0^n(p), 0\right) \quad \text{for} \quad p \in \mathcal{B}_0^n.$$

Observe that $\mathcal{P}_0^n|_{\mathcal{I}_0^n} = \mathrm{Id}$.

We record the following immediate consequences of Theorem 3.6.

Lemma 3.8. *For $1 \leq n \leq N$, let $p_0, q_0 \in \mathcal{B}_0^n$ be two points such that*

$$|P_0^n(p_0) - P_0^n(q_0)| > \lambda^{\bar{\varepsilon} R_n}.$$

Then we have

$$|P_{-1}(p_{R_n-1}) - P_{-1}(q_{R_n-1})| > \lambda^{\bar{\varepsilon} R_n}.$$

If, additionally, we have

$$P_0^n(p_{R_n}) \, , \; P_0^n(q_{R_n}) < -\lambda^{\bar{\varepsilon} R_n},$$

then

$$|P_0^n(p_{R_n}) - P_0^n(q_{R_n})| > \lambda^{\bar{\varepsilon} R_n}.$$

Lemma 3.9. *For $1 \leq n \leq N$, denote $\rho_n := \lambda^{(1-\bar{\varepsilon})R_n}$. Let $0 < t < \lambda^{-\bar{\varepsilon} R_n}$. Then the following statements hold.*

i) Let $\tilde{E}_{p_0} \in \mathbb{P}_{p_0}^2$ be a t-horizontal direction at $p_0 \in \mathcal{B}_0^n$. Then $\tilde{E}_{p_{R_n-1}}$ is $(1+t)\rho_n$-horizontal in $\mathcal{B}_{-1}$.

ii) Let $E_{p_{R_n-1}} \in \mathbb{P}_{p_{R_n-1}}^2$ be a t-vertical direction at $p_{R_n-1} \in \mathcal{B}_{R_n-1}^n$. Then E_{p_0} is $t\rho_n$-vertical in $\mathcal{B}_0^n$.

iii) Let Γ_0^h be a t-horizontal curve in $\mathcal{B}_0^n$. Then $\Gamma_{R_n-1}^h$ is $(1+t)\rho_n$-horizontal in $\mathcal{B}_{-1}$.

iv) Let $\Gamma^v_{R_n-1}$ be a t-vertical curve in $\mathcal{B}^n_{R_n-1}$. Then Γ^v_0 is $t\rho_n$-vertical in $\mathcal{B}^n_0$.

By Lemma 3.9 iii), $\mathcal{I}^n_{R_n-1}$ is ρ_n-horizontal in $\mathcal{B}_{-1}$. Thus, there exists a C^r-map $g_n : I^n_{R_n-1} \to \mathbb{R}$ with $\|g_n\|_{C^r} < \rho_n$ such that

$$\Phi_{-1}(\mathcal{I}^n_{R_n-1}) = \{(x, g_n(x)) \mid x \in I^n_{R_n-1}\}.$$

Define $G_n : I^n_{R_n-1} \to \Phi_{-1}(\mathcal{I}^n_{R_n-1})$ by $G_n(x) := (x, g_n(x))$. Define the *nth critical projection map* $\mathcal{P}^n_{-1} : P^{-1}_{-1}(I^n_{R_n-1}) \to \mathcal{I}^n_{R_n-1}$ by

$$\mathcal{P}^n_{-1} := \Phi^{-1}_{-1} \circ G_n \circ P_{-1}.$$

Lemma 3.10. *For $1 \leq n \leq N$, let Γ_0 be a horizontal curve in $\mathcal{B}^n_0$. Then*

$$F^{R_n-1}|_{\Gamma_0} = (\mathcal{P}^n_{-1}|_{\Gamma_{R_n-1}})^{-1} \circ F^{R_n-1} \circ \mathcal{P}^n_0|_{\Gamma_0}.$$

Proof. Note that $\mathcal{P}^n_{-1}$ is a projection along the vertical foliation $\mathcal{F}^v_{-1}$ on $\mathcal{B}_{-1}$, and $\mathcal{P}^n_0$ is a projection along the vertical foliation on $\mathcal{B}^n_0$ obtained by pulling back $\mathcal{F}^v_{-1}$ by F^{-R_n+1}. The claim follows immediately. □

Lemma 3.11. *There exists a uniform constant $k > 0$ such that the following holds. Let $g : I \to \mathbb{R}$ be a C^r-map on an interval $I \subset I^h_{-1}$ such that $\|g\|_{C^r} < k$. Denote $G(x) := (x, g(x))$. Then there exist $a \in I^h_0$ and a C^r-diffeomorphism $\psi_g : I \to \psi_g(I)$ with $\|\psi_g^{\pm 1}\|_{C^r} = O(1)$ such that we have*

$$Q(x) := P^n_0 \circ F \circ \Phi^{-1}_{-1} \circ G(x) = a - (\psi_g(x))^2 \tag{3.8}$$

where defined.

4. Avoiding the Critical Value

For $N \in \mathbb{N} \cup \{\infty\}$, let F be the N-times regular Hénon-like renormalizable diffeomorphism considered in Subsection 3.1. Suppose that N is sufficiently large, so that by replacing (F^{R_1}, Ψ^1) with $(F^{R_{n_1}}, \Psi^{n_1})$ for some $n_1 \leq N$, we may assume that:

$$\bar{L}\lambda^{\varepsilon R_1} < \rho, \tag{4.1}$$

where $\rho \in (0, 1)$ is a suitably small universal constant. Note that (4.1) is a stronger condition than (3.2).

Let $z = (a, b)$ and $w = (c, d)$ with $a, c \in \mathbb{R}$ and $b, d \in I^v_0$. Denote

$$m := \min\{a, c\} \quad \text{and} \quad M := \max\{a, c\}.$$

For $t \geq 0$, define

$$V_z(t) := [a - t, a + t] \times I^v_0 \quad \text{and} \quad V_{[z,w]}(t) := [m - t, M + t] \times I^v_0,$$

where I^v_0 is given in (3.6). If $V_{\Psi^n(p)}(t) \subset B^n_0$ for some $1 \leq n \leq N$; $p \in \mathcal{B}^n_0$ and $t \geq 0$, then we denote

$$\mathcal{V}^n_p(t) := (\Psi^n)^{-1}(V_{\Psi^n(p)}(t)).$$

We record the following two immediate consequences of Theorem 3.6.

Lemma 4.1. *For $1 \leq n \leq N$, let $E_{p_{-1}} \in \mathbb{P}^2_{p_{-1}}$ be a $\lambda^{\bar{\varepsilon}R_n}$-horizontal direction at $p_{-1} \in \mathcal{B}_{-1}$. If*

$$p_0 \in \mathcal{B}^n_0 \setminus \mathcal{V}^n_{v_0}(t) \quad \text{with} \quad t > \lambda^{\bar{\varepsilon}R_n},$$

then E_{p_0} is $O(1/t)$-horizontal in $\mathcal{B}^n_0$.

Similarly, let Γ_{-1} be $\lambda^{\bar{\varepsilon}R_n}$-horizontal curve in $\mathcal{B}_{-1}$. If

$$\Gamma_0 \subset \mathcal{B}^n_0 \setminus \mathcal{V}^n_{v_0}(t) \quad \text{with} \quad t > \lambda^{\bar{\varepsilon}R_n},$$

then Γ_0 is $O(1/t)$-horizontal in $\mathcal{B}^n_0$.

Lemma 4.2. *For $1 \leq n \leq N$, let $\tilde{E}_{p_0} \in \mathbb{P}^2_{p_0}$ be a $\lambda^{\bar{\varepsilon}R_n}$-vertical direction at $p_0 \in \mathcal{B}^n_0$. If*

$$p_0 \in \mathcal{B}^n_{R_n} \setminus \mathcal{V}^n_{v_0}(t) \quad \text{with} \quad t > \lambda^{\bar{\varepsilon}R_n},$$

then $\tilde{E}_{p_0}$ is $O(1/t)$-vertical in $\mathcal{B}_{-1}$.

Similarly, let $\tilde{\Gamma}_0$ be $\lambda^{\bar{\varepsilon}R_n}$-vertical curve in $\mathcal{B}^n_0$. If

$$\tilde{\Gamma}_0 \subset \mathcal{B}^n_{R_n} \setminus \mathcal{V}^n_{v_0}(t) \quad \text{with} \quad t > \lambda^{\bar{\varepsilon}R_n},$$

then $\tilde{\Gamma}_{-1}$ is $O(1/t)$-vertical in $\mathcal{B}_{-1}$.

Proposition 4.3. *For $1 \leq n \leq N$, let $p_0 \in \mathcal{B}^n_{R_n} \setminus \mathcal{V}^n_{v_0}(\lambda^{\bar{\varepsilon}R_n})$. If E_{p_0} is $\lambda^{\bar{\varepsilon}R_n}$-vertical in $\mathcal{B}^n_0$, then $E_{p_{-R_n}}$ is $\lambda^{(1-\bar{\varepsilon})R_n}$-vertical in $\mathcal{B}^n_0$. Moreover, p_{-R_n} is R_n-times forward $(\bar{L}, \bar{\varepsilon}, \lambda)$-regular along $E_{p_{-R_n}}$.*

Proof. Consider a linearization

$$\{\Phi_{p_{-m}} : \mathcal{U}_{p_{-m}} \to U_{p_{-m}}\}_{m=0}^{R_n}$$

of F along the R_n-backward orbit of p_0 with vertical direction

$$E^{v,n}_{p_0} := (D\Psi^n)^{-1}\left(E^{gh}_{\Psi^n(p_0)}\right).$$

Note that since (F^{R_n}, Ψ^n) is a Hénon-like return, we have

$$D\Psi^n\left(E^{v,n}_{p_{-R_n}}\right) = E^{gv}_{\Psi^n(p_{-R_n})}.$$

Denote

$$E^{h,n}_{p_{-1}} := D\Phi_{p_{-1}}\left(E^{gh}_0\right) \quad \text{and} \quad E^h_{p_{-1}} := D\Phi_{-1}\left(E^{gh}_{\Phi_{-1}(p_{-1})}\right),$$

where $\Phi_{-1} : \mathcal{U}_{-1} \to U_{-1}$ is the chart defined over the critical point given in Theorem 3.6. By Theorem A.2 ii) and (3.5), we see that

$$\|DF^{-R_n+1}|_{E^{h,n}_{p_{-1}}}\|\ ,\ \|DF^{-R_n+1}|_{E^h_{p_{-1}}}\| > \bar{L}^{-1}\lambda^{\bar{\varepsilon}R_n}.$$

Hence, it follows from Proposition A.9 that

$$\measuredangle(E^{h,n}_{p_{-1}}, E^h_{p_{-1}}) < \bar{L}\lambda^{(1-\bar{\varepsilon})R_n}.$$

Thus, by (3.4), we have

$$\measuredangle(E^{h,n}_{p_{-1}}, E_{p_{-1}}) > \bar{L}^{-1}\lambda^{\bar{\varepsilon}R_n}.$$

For $1 \leq i \leq R_n$, denote

$$\theta_{-i} := \measuredangle(E^{gh}_0, D\Phi_{p_{-i}}(E_{p_{-i}})).$$

Choose a suitable uniform constant $c \in (0, \pi/2)$ independent of F, and let $1 \leq M \leq R_n$ be the smallest number such that $\theta_{-M} > c$. By Theorem A.2 and Proposition A.5, we see that

$$\theta_{-i} > \lambda^{-(1-\bar{\varepsilon})i}\theta_{-1} > \bar{L}^{-1}\lambda^{-(1-\bar{\varepsilon})i}\lambda^{\bar{\varepsilon}R_n}.$$

Consequently,

$$M < \bar{\varepsilon}R_n - \frac{\log \bar{L}}{\log \lambda} = \bar{\varepsilon}R_n,$$

where in the last equality, we used (4.1). Let $M' := CM$ for some suitable uniform constant $C \geq 1$ independent of F.

By Proposition A.5, we have

$$\|DF|_{E_{p_{-R_n+i}}}\| \asymp \|DF^i|_{E^{v,n}_{p_{-R_n+i}}}\| \quad \text{for} \quad 0 \leq i < R_n - M' \tag{4.2}$$

Denote

$$F^i_{-j} := \Phi_{p_{-j+i}} \circ F^i \circ (\Phi_{p_{-j}})^{-1}.$$

By Proposition A.4, we have

$$\lambda^{\bar{\varepsilon}R_n} < \lambda^{(1+\bar{\varepsilon})M'} < \|DF^i_{-M'}|_{\tilde{E}_{p_{-M'}}}\| < \lambda^{-\bar{\varepsilon}M'} < \lambda^{-\bar{\varepsilon}R_n} \tag{4.3}$$

for any $\tilde{E}_{p_{-M'}} \in \mathbb{P}^2_{p_{-M'}}$. Since $\|\Phi^{\pm 1}_{p_{-i}}\|_{C^1} < \bar{L}\lambda^{-\bar{\varepsilon}i}$, we conclude that for $0 \leq i < M'$, we have

$$\lambda^{\bar{\varepsilon}R_n} < \frac{\|DF^{R_n-M'+i}|_{E^{v,n}_{p_{-R_n}}}\|}{\|DF^{R_n-M'+i}|_{E_{p_{-R_n}}}\|} < \lambda^{-\bar{\varepsilon}R_n}.$$

The $(\bar{L}, \bar{\varepsilon}, \lambda)$ forward regularity of p_{-R_n} along $E_{p_{-R_n}}$ follows. □

Proposition 4.4. *For $1 \leq n \leq N$, let $p_0 \in \mathcal{B}^n_0$. If p_0 is infinitely forward $(\bar{L}, \bar{\varepsilon}, \lambda)$-regular, then $W^{ss}(p_0)$ is $\lambda^{(1-\bar{\varepsilon})R_n}$-vertical and vertically proper in $\mathcal{B}^n_0$.*

Proof. The verticality of $W^{ss}(p_0)$ follows immediately from Proposition A.8. Consider a linearization

$$\{\Phi_{p_m} : \mathcal{U}_{p_m} \to U_{p_m}\}_{m=0}^{\infty}$$

of F along the infinite forward orbit of p_0 with vertical direction $E^{ss}_{p_0}$. Recall that

$$\Phi_{p_m}(W^{ss}_{\text{loc}}(p_m)) \subset \{(0, y) \in U_{p_m} \mid y \in \mathbb{R}\}. \tag{4.4}$$

Let

$$\mathcal{V}_{p_0} := \mathcal{V}^n_{p_0}(\lambda^{\bar{\varepsilon}R_n}).$$

Arguing as in the proof of Proposition 3.2, we see that if M is the nearest integer to $R_n/2$, then

$$\Phi_{p_M}(F^M(\mathcal{V}_{p_0})) \subset (-\lambda^{\bar{\varepsilon}R_n}, \lambda^{\bar{\varepsilon}R_n}) \times (-\lambda^{(1-\bar{\varepsilon})M}, \lambda^{(1-\bar{\varepsilon})M}). \tag{4.5}$$

For $q_0 \in \mathcal{V}_{p_0}$, denote

$$\hat{E}^{v/h}_{q_0} := (D\Psi^n)^{-1}(E^{gv/gh}_{\Psi^n(q_0)}).$$

The forward regularity of q_0, Theorem A.2 and Proposition A.5 imply that

$$\|DF^m|_{\hat{E}^h_{q_0}}\| < \bar{L}\lambda^{(1-\bar{\varepsilon})m}. \quad \text{and} \quad \|DF^m|_{\hat{E}^h_{q_0}}\| > \bar{L}^{-1}\lambda^{\bar{\varepsilon}m}.$$

Thus, follows from Proposition A.3 that $q_m \in \mathcal{U}_{p_m}$ for all m sufficiently large so that

$$\bar{L}\lambda^{(1-\bar{\varepsilon})m} < \bar{L}^{-1}\lambda^{\bar{\varepsilon}m}.$$

We conclude by (4.4), (4.5) and Proposition A.9 that $W^{ss}_{\text{loc}}(p_M)$ is vertically proper in $F^M(\mathcal{V}_{p_0})$. The result follows. □

Proposition 4.5. *For $1 \leq n \leq N$, let $\mathcal{C}_0 \subset \mathcal{B}^n_0$ be a totally invariant connected set under F^{dR_n} with $2 \leq d \leq \mathbf{b}$. If*

$$\mathcal{V}^n_{v_0}(\lambda^{\bar{\varepsilon}R_n}) \cap \mathcal{C} = \varnothing, \quad \textit{where} \quad \mathcal{C} := \bigcup_{i=0}^{d-1} \mathcal{C}_{iR_n},$$

then either $\mathcal{C}_0$ is a singleton, or it contains a sink.

Proof. Let $\mathcal{E}^v : \mathcal{B}^n_0 \to T^1\mathcal{B}^n_0$ be a C^{r-1}-unit vector field such that

$$\mathcal{E}^v(p) \in (D\Psi^n)^{-1}(E^{gv}_{\Psi^n(p)}) \quad \text{for} \quad p \in \mathcal{B}^n_0.$$

For $i \in \mathbb{N}$, define

$$\mathcal{E}^{-i} := (F^{iR_n})^*(\mathcal{E}^v|_{\mathcal{C}}).$$

For $p \in \mathcal{C}$, let $E^{-i}_p \in \mathbb{P}^2_p$ be the direction containing $\mathcal{E}^{-i}(p)$. By Proposition 4.3, p is iR_n-times forward $(\bar{L}, \bar{\varepsilon}, \lambda)$-regular along E^{-i}_p. Thus, it follows from Proposition A.8 that E^{-i}_p converges super-exponentially fast to E^{ss}_p along which p is infinitely forward $(\bar{L}, \bar{\varepsilon}, \lambda)$-regular.

Let $W^{ss}_{\text{loc}}(p)$ be the connected component of $W^{ss}(p) \cap \mathcal{B}^n_0$ containing p. Define

$$\mathcal{V}_{\mathcal{C}_0} := \bigcup_{p \in \mathcal{C}_0} W^{ss}_{\text{loc}}(p).$$

By Proposition 4.4, the foliation of $\mathcal{V}_{\mathcal{C}_0}$ given by $\{W^{ss}_{\text{loc}}(p)\}_{p\in\mathcal{C}}$ is $\lambda^{(1-\bar{\varepsilon})R_n}$-vertical and vertically proper in $\mathcal{B}^n_0$. Let

$$\Psi_{\mathcal{C}_0} : \mathcal{V}_{\mathcal{C}_0} \to V_{\mathcal{C}_0} := I_{\mathcal{C}_0} \times I^v_0$$

be the genuine horizontal chart that rectifies this vertical foliation.

Consider the map

$$H := \Psi_{\mathcal{C}_0} \circ F^{dR_n} \circ (\Psi_{\mathcal{C}_0})^{-1}.$$

By (3.7), (3.4) and the fact that

$$\mathcal{V}_{\mathcal{C}_0} \cap \mathcal{V}^n_{v_0}(\lambda^{\bar{\varepsilon}R_n}) = \varnothing,$$

it follows that $\Pi_{1\text{D}}(H)$ is a homeomorphism. If $\mathcal{C}_0$ is not a singleton, then $\Pi_{1\text{D}}(H)$ is a map on a closed interval, which immediately implies that it has a sink. □

Proposition 4.6. *For $1 \leq n \leq N$ and $m \geq -1$, denote*

$$u^n_m := \Psi^n(v_{mR_n}) \in B^n_0 \quad \textit{and} \quad a^n_m := \pi_h(u^n_m).$$

If v_{kR_n} does not converge to a sink as $k \to \infty$, then the following statements hold.

i) For $i \geq 0$ such that $i = O(1)$, we have

$$|a^n_i - a^n_{-1}| > \lambda^{\bar\varepsilon R_n}.$$

ii) We have $a^n_1 < a^n_{-1} < a^n_0 = 0$.

Proof. Let $\delta \in (\bar\varepsilon, 1)$ with $\bar\delta < 1$. Suppose towards a contradiction that

$$V_{u^n_i}(\lambda^{\bar\delta R_n}) \cap V_{u^n_{-1}}(\lambda^{\bar\delta R_n}) \neq \varnothing. \tag{4.6}$$

Without loss of generality, assume that $i \geq 0$ is the smallest number for which (4.6) holds.

For $y \in I^v_0$, consider

$$J^n_0 \subset (-\lambda^{\bar\delta R_n}, \lambda^{\bar\delta R_n}) \quad \text{and} \quad \mathcal{J}^n_0 := \Psi^{-n}(J^n_0 \times \{y\}) \subset \mathcal{V}^n_{v_0}(\lambda^{\bar\delta R_n}).$$

By Propositions A.4 and A.5, and (4.1), we see that

$$|\mathcal{J}^n_{iR_n-1}| < \lambda^{-\bar\varepsilon R_n}|J^n_0| < \lambda^{\underline{\delta} R_n}.$$

Moreover, since

$$\mathcal{J}^n_{jR_n} \cap V_{u^n_{-1}}(\lambda^{\bar\delta R_n}) = \varnothing \quad \text{for} \quad 0 \leq j < i,$$

we can argue by induction using Lemma 3.9 iii) and Lemma 4.1 that $\mathcal{J}^n_{iR_n-1}$ is $\lambda^{(1-\bar\varepsilon)R_n}$-horizontal in $\mathcal{B}_{-1}$. Then it follows from (4.6) and (3.4) that

$$|P^n_0(\mathcal{J}^n_{iR_n})| < \lambda^{\underline{\delta} R_n}|\mathcal{J}^n_{iR_n-1}| < \lambda^{\underline{\delta} R_n}|J^n_0|.$$

We conclude that

$$F^{iR_n}(\mathcal{V}^n_{v_0}(\lambda^{\bar\delta R_n})) \Subset \mathcal{V}^n_{v_0}(\lambda^{\bar\delta R_n}).$$

By Propositions A.4 and A.5, and (4.1), we see that for $p_0 \in \mathcal{J}^n_0$:

$$\|DF^{iR_n}|_{E^h_{p_0}}\| < \lambda^{-\bar\varepsilon R_n}.$$

Arguing by induction using Lemma 3.9 i) and Lemma 4.1, we also see that $E^h_{p_{iR_n-1}}$ is $\lambda^{(1-\bar\varepsilon)R_n}$-horizontal in $\mathcal{B}_{-1}$. Consequently, by (4.6) and (3.4), we have

$$\measuredangle(DF^{iR_n}(E^h_{p_0}), E^{v,n}_{p_{iR_n}}) < \lambda^{\underline{\delta} R_n}.$$

It follows by Proposition A.5 that

$$\|D_{p_0}F^{2iR_n}\| < \lambda^{\underline{\delta} R_n}.$$

We conclude that $\mathcal{V}^n_{v_0}(\lambda^{\bar\varepsilon R_n})$ is contained in an $2iR_n$-periodic sink. This is a contradiction.

Suppose towards a contradiction that $a^n_1 < a^n_{-1} < 0$ is not true. Denote

$$\check{B}^n_0 := [a^n_{-1} + \lambda^{\bar\varepsilon R_n}, -\lambda^{\bar\varepsilon R_n}] \times I^v_0.$$

Let $K_0^n := \{(t,0) \in \check{B}_0^n\}$. By Lemma 3.9 and (3.4), we see that K_0^n maps injectively into itself under the map $P_0^n \circ F^{R_n} \circ (\Psi^n)^{-1}$. Consequently, v_0 must converge to an R^n-periodic sink. This is a contradiction. □

Theorem 4.7 (Critical Recurrence)**.** *Suppose that $N = \infty$. Then*

$$\mathcal{Z}_0 := \bigcap_{n=1}^{\infty} \mathcal{B}_{R_n}^n = \{v_0\}.$$

Consequently, the orbit of v_0 is recurrent.

Proof. Let

$$\mathcal{Y}_0 := \bigcap_{n=1}^{\infty} \mathcal{B}_0^n, \quad \mathcal{I}_0^\infty := \mathcal{I}_0^1 \cap \mathcal{Y}_0 \quad \text{and} \quad I_0^\infty := \pi_h \circ \Phi_0(\mathcal{I}_0^\infty).$$

Note that every point $p_0 \in \mathcal{Y}_0$ is infinitely forward $(L, \varepsilon, \lambda)$-regular. Moreover, by Proposition 3.2, $W^{ss}(p_0)$ is vertically proper in $\mathcal{B}_0^1$. Hence, we have

$$\mathcal{Y}_0 = \bigcup_{p_0 \in \mathcal{I}_0^\infty} (W^{ss}(p_0) \cap \mathcal{B}_0^1).$$

We claim that $\mathcal{Y}_0 = W^{ss}(v_0) \cap \mathcal{B}_0^1$.

Recall that for $n \in \mathbb{N}$, the curve $\mathcal{I}_{R_n}^n$ is vertical quadratic in $\mathcal{B}_0^n$. Let $v_0^n \in \mathcal{I}_0^n$ be the unique point such that

$$E_{v_{R_n}^n}^{v,n} = DF^{R_n}(E_{v_0^n}^h).$$

Denote

$$a_0 := \pi_h \circ \Phi_0(v_0) \quad \text{and} \quad a_n := P_0^n(v_{R_n}^n).$$

By (3.3) and Lemma 3.3, we have

$$|P_0^n(v_0) - a_0| \ , \ |a_n - a_0| < \lambda^{(1-\bar{\varepsilon})R_n}.$$

Assume the correct orientation of Ψ^n so that we have $P_0^n(p_{R_n}) \leq a_n$ for $p_0 \in \mathcal{I}_0^n$. Suppose towards a contradiction that there exists a uniform constant $b > 0$ such that $(a_0 - b, a_0) \subset I_0^\infty$.

Let $M \in \mathbb{N}$ be sufficiently large so that for $n \geq M$, we have

$$a_0 - b/2 < a_0 - \lambda^{\bar{\varepsilon}R_M} < a_n.$$

Using induction and Lemma 4.1, we see that for $0 \leq k < R_n/R_M$, the curve $\mathcal{I}_{kR_M}^n$ is $O(1)$-horizontal in $\mathcal{B}_0^n$, and $\mathcal{I}_{(k+1)R_M-1}^n$ is $\lambda^{(1-\bar{\varepsilon})R_M}$-horizontal in $\mathcal{B}_{-1}$.

We define $\mathcal{B}_{-kR_M}^n$ with $0 \leq k < R_n/R_M$ inductively as follows. Let $\mathcal{B}_{-kR_M-1}^n$ be the connected component of

$$F^{-1}(\mathcal{B}_{-kR_M}^n) \cap \mathcal{B}_{R_M-1}^M$$

containing $\mathcal{I}_{R_n-kR_M-1}^n$, and let

$$\mathcal{B}_{-(k+1)R_M}^n := F^{-R_M+1}(\mathcal{B}_{-kR_M-1}^n).$$

Using induction and Lemma 4.2, we see that

$$\partial \mathcal{B}^n_{-kR_M-1} \setminus \partial \mathcal{B}^M_{R_M-1}$$

consists of two $O(1)$-vertical curves $\Gamma^{n,\pm}_{-kR_M-1}$ in $\mathcal{B}_{-1}$, and

$$\Gamma^{n,\pm}_{-(k+1)R_M} := F^{-R_M+1}(\Gamma^{n,\pm}_{-kR_M-1})$$

are $\lambda^{(1-\bar{\varepsilon})R_M}$-vertical in $\mathcal{B}^M_0$. We conclude that for $0 \leq k < R_n/R_M$, the sets

$$\mathcal{B}^n_{-(k+1)R_M} \supset \mathcal{I}^n_{R_n-(k+1)R_M}$$

are disjoint. Hence,

$$I^n_{kR_M} := P^M_0(\mathcal{I}^n_{kR_M})$$

are disjoint intervals in I^M_0.

Consider the following map

$$g^n_k := \mathcal{P}^M_0 \circ F \circ (\mathcal{P}^M_{-1}|_{\mathcal{I}^n_{(k+1)R_M-1}})^{-1} \circ F^{R_M-1}|_{I^n_{kR_M}}.$$

Since $\mathcal{I}^n_{(k+1)R_M-1}$ and $\mathcal{I}^n_{(k+1)R_M}$ are uniformly horizontal in $\mathcal{B}_{-1}$ and $\mathcal{B}_0$ respectively, it follows that $\|g^n_k\|_{C^r} = O(1)$. Moreover,

$$\sum_{k=0}^{R_n/R_M-1} |I^n_{kR_M}| < |I^M_0| = O(1),$$

and thus, we conclude from Theorem B.1 that

$$G^n := g^n_{R_n/R_M-1} \circ \ldots \circ g^n_0$$

has uniformly bounded distortion.

Let

$$I^{n+1}_{-R_n} = P^M_0(\mathcal{B}^{n+1}_{-R_n}).$$

Then $I^{n+1}_{-R_n}$ and I^{n+1}_0 are disjoint intervals in I^n_0. Moreover, we have $|I^{n+1}_0| = O(1)$ and

$$|I^{n+1}_{-R_n}|, |I^{n+1}_{R_n}| \to 0 \quad \text{as} \quad n \to \infty.$$

However,

$$G^n(I^{n+1}_{-R_n}) = I^{n+1}_0 \quad \text{and} \quad G^n(I^{n+1}_0) = I^{n+1}_{R_n}.$$

This is a contradiction. The result follows. □

5. Return Times of Bounded Type

For $N \in \mathbb{N} \cup \{\infty\}$, let F be the N-times regular Hénon-like renormalizable diffeomorphism considered in Subsection 3.1. Suppose that the return times are of $\mathbf{b}$-bounded type for some integer $\mathbf{b} \geq 2$. Moreover, assume that ε is sufficiently small so that (1.5) holds with $\varepsilon_0 \geq \bar{\varepsilon}$. By only considering every other returns if necessary, we may also assume without loss of generality that $r_n \geq 3$.

Lemma 5.1. *For $s \in \{1, 2\}$ and $1 \leq n \leq N - s$, let Γ_0 be a $\lambda^{-\bar{\varepsilon}R_n}$-horizontal curve in $\mathcal{B}^{n+s}_0$. Then for $1 \leq k \leq R_{n+s}/R_n$, the following statements hold:*

i) $\Gamma_{(k-1)R_n}$ *is* $\lambda^{-\bar{\varepsilon}R_n}$*-horizontal in* $\mathcal{B}_0^n$*; and*
ii) Γ_{kR_n-1} *is* $\lambda^{(1-\bar{\varepsilon})R_n}$*-horizontal in* $\mathcal{B}_{-1}$*.*

Proof. The result is an immediate consequence of Lemmas 3.9 iii) and 4.1, and Proposition 4.6. □

Proposition 5.2. *For* $1 \leq n < N$*, denote*

$$u_k^n := \Psi^n(v_{kR_n}) \in B_0^n \quad \text{and} \quad a_k^n := \pi_h(u_k^n) \quad \text{for} \quad k \geq -1.$$

If v_{kR_n} *does not converge to a sink as* $k \to \infty$*, then the following holds.*
i) We have

$$a_1^n < a_2^n < a_0^n = 0 \quad \text{and} \quad |a_i^n - a_2^n| > \lambda^{\bar{\varepsilon}R_n} \quad \text{for} \quad i \in \{0, 1\}.$$

ii) Define

$$\tilde{\mathcal{B}}_0^n := V_{[u_1^n, u_0^n]}(\lambda^{\bar{\varepsilon}R_n}) \quad \text{and} \quad \tilde{\mathcal{B}}_0^n := (\Psi^n)^{-1}(\tilde{B}_0^n).$$

Then $F^{R_n}(\tilde{\mathcal{B}}_0^n) \Subset \tilde{\mathcal{B}}_0^n$*.*

Proof. By Proposition 4.6, we have

$$|a_1^n - a_{-1}^n| > \lambda^{\bar{\varepsilon}R_n}.$$

Thus, by Theorem 3.6, we have

$$|a_2^n - a_0^n| > (\lambda^{\bar{\varepsilon}R_n})^2 - \lambda^{(1-\bar{\varepsilon})R_n} > \lambda^{\bar{\varepsilon}R_n}.$$

Suppose towards a contradiction that

$$|a_1^n - a_2^n| < \lambda^{\bar{\varepsilon}R_n}.$$

Proceeding by induction, suppose that

$$|a_{i-1}^n - a_i^n| < \lambda^{\bar{\varepsilon}R_n} \quad \text{for} \quad 1 < i < r_n.$$

Iterating $v_{(i-1)R_n}$ and v_{iR_n}, and applying Propositions A.4 and A.5, and Theorem 3.6, we see that

$$|a_i^n - a_{i+1}^n| < \lambda^{-\bar{\varepsilon}R_n}|a_{i-1}^n - a_i^n| + \lambda^{(1-\bar{\varepsilon})R_n} < \lambda^{\bar{\varepsilon}R_n}.$$

Consequently,

$$|a_1^n - a_{r_n}^n| < r_n\lambda^{\bar{\varepsilon}R_n} < \lambda^{\bar{\varepsilon}R_n}.$$

By Propositions 3.2 and 4.6, we have $v_{-R_n} \in \mathcal{B}_0^{n+1}$. This is a contradiction.

Suppose towards a contradiction that

$$a_2^n < a_1^n - \lambda^{\bar{\varepsilon}R_n} < a_1^n. \tag{5.1}$$

Consider

$$J_0^n := [a_1^n - \lambda^{\bar{\varepsilon}R_n}, a_{-1}^n - \lambda^{\varepsilon R_n}] \quad \text{and} \quad \mathcal{J}_0^n := (\Psi^n)^{-1}(J_0^n \times \{0\}).$$

By Lemma 4.1, we see that $\mathcal{J}_{R_n}^n$ is $\lambda^{-\bar{\varepsilon}R_n}$-horizontal in $\mathcal{B}_0^n$. Let $F_n := p\mathcal{R}^n(F)$ and $f_n := \Pi_{1\text{D}}(F_n)$. It follows that f_n maps J_0^n onto its image $f_n(J_0^n)$ as an orientation preserving diffeomorphism. Observe that by (5.1), $f_n(J_0^n)$ must contain a $\lambda^{\bar{\varepsilon}R_n}$-neighborhood of J_0^n.

For $y \in I^v_0$, let

$$\mathcal{J}^{n,y}_0 := (\Psi^n)^{-1}(J^n_0 \times \{y\}).$$

By Lemma 3.9, we conclude that

$$\|\mathcal{J}^{n,y}_{R_n} - \mathcal{J}^n_{R_n}\|_{C^r} < \lambda^{(1-\bar{\varepsilon})R_n}.$$

Let

$$D^n_0 := J^n_0 \times I^v_0 \quad \text{and} \quad \mathcal{D}^n_0 := (\Psi^n)^{-1}(D^n_0).$$

Consider the quadrilateral

$$\hat{\mathcal{D}}^n_{R_n} := \mathcal{D}^n_{R_n} \cap \mathcal{B}^n_0$$

as horizontally foliated by $\{\mathcal{J}^{n,y}_{R_n}\}$ and vertically foliated by the vertical leaves in $\mathcal{B}^n_0$. Define

$$\mathcal{K}_0 := (\Psi^n)^{-1}(\{(a^n_1, t) \mid t \in I^v_0\})$$

and

$$\mathcal{K}_{-i} := F^{-R_n}(\mathcal{K}_{-i+1} \cap \hat{\mathcal{D}}^n_{R_n}) \quad \text{for} \quad i \in \mathbb{N}.$$

It follows from Lemma 4.2 and Lemma 3.9 iv) that $\{\mathcal{K}_{-i}\}_{i=0}^{\infty}$ is a sequence of vertically proper and $\lambda^{(1-\bar{\varepsilon})R_n}$-vertical curves in $\mathcal{D}^n_0$. Moreover, by Lemma 4.3, we see that any point $p \in \mathcal{K}_{-i}$ is iR_n-times forward $(\bar{L}, \bar{\varepsilon}, \lambda)$-regular along the tangent direction to $\mathcal{K}_{-i}$ at p. It follows that $\mathcal{K}_{-i}$ converges as $i \to \infty$ to a subarc in the stable manifold of some R_n-periodic saddle $q \in \mathcal{D}^n_0$ of non-flip type.

Let $\mathcal{B}^{n,r}_0$ and $\mathcal{B}^{n,l}_0$ be the connected components of $\mathcal{B}^n_0 \setminus W^{ss}(q)$ containing v_0 and v_{R_n} respectively. It follows that $\mathcal{B}^{n,r/l}_{R_n} \subset \mathcal{B}^{n,r/l}_0$. This is a contradiction.

Property ii) now follows immediately. □

By Proposition 5.2 ii), we may henceforth assume that

$$B^n_0 := V_{[v_{R_n}, v_0]}(\lambda^{\bar{\varepsilon}R_n}) \quad \text{and} \quad \mathcal{B}^n_0 := (\Psi^n)^{-1}(B^n_0) \quad \text{for} \quad 1 \leq n \leq N.$$

Proposition 5.3. *Let $s \in \{1, 2\}$ and $1 \leq n \leq N - s$. For $0 \leq k < R_{n+s}/R_n$, Denote*

$$u^n_k := \Psi^n(v_{kR_n}), \quad w^n_k := \Psi^n(v_{R_{n+s}+kR_n}), \quad a^n_k := \pi_h(u^n_k) \quad \text{and} \quad b^n_k := \pi_h(w^n_k).$$

Define

$$\hat{B}^{n,s}_{kR_n} := V_{[u^n_k, w^n_k]}(\lambda^{\bar{\varepsilon}R_n}) \subset B^n_0 \quad \text{and} \quad \hat{\mathcal{B}}^{n,s}_{kR_n} := (\Psi^n)^{-1}(\hat{B}^{n,s}_{kR_n}).$$

If v_{kR_n} does not converge to a sink as $k \to \infty$, then the following properties hold.

i) For integers $2 \leq k < R_{n+s}/R_n$, we have

$$a^n_1 < b^n_1 < a^n_k\ ,\ b^n_k < b^n_0 < a^n_0 = 0.$$

ii) For integers $0 \leq k, l \leq R_{n+s}/R_n$ with $k \neq l$, we have

$$|a^n_k - a^n_l|\ ,\ |b^n_k - b^n_l|\ ,\ |a^n_k - b^n_l|\ ,\ |a^n_k - b^n_k| > \lambda^{\bar{\varepsilon}R_n}.$$

iii) For $0 \leq k < R_{n+s}/R_n$, we have

$$\hat{\mathcal{B}}^{n,s}_{kR_n} \supset \mathcal{B}^{n+s}_{kR_n} \quad \text{and} \quad F^{R_{n+s}-kR_n}(\hat{\mathcal{B}}^{n,s}_{kR_n}) \Subset \mathcal{B}^{n+s}_0.$$

Proof. By Propositions 4.6 and 5.2, we have

$$|a_0^n - b_0^n| > \lambda^{\bar{\varepsilon} R_n} \quad \text{and} \quad F^{R_{n+s}}(\hat{\mathcal{B}}_0^{n,s}) \Subset \hat{\mathcal{B}}_0^{n,s}$$

respectively. Applying Lemma 3.8 $(R_{n+s}/R_n - 1)$-times starting from u_0^n and w_0^n, we obtain

$$|a_k^n - b_k^n| > \lambda^{\bar{\varepsilon} R_n} \quad \text{for} \quad 0 \le k < R_{n+s}/R_n.$$

By (3.7) and (3.4), we see that

$$F^{R_n}(\hat{\mathcal{B}}_{kR_n}^{n,s}) \Subset \hat{\mathcal{B}}_{(k+1)R_n}^{n,s}.$$

Hence, by Proposition 5.2 ii), we also have

$$F^{R_{n+s} - kR_n}(\hat{\mathcal{B}}_{kR_n}^{n,s}) \Subset \mathcal{B}_0^{n+s}.$$

It follows that for $0 \le k, l < R_{n+s}/R_n$ with $k \ne l$, we have

$$\hat{\mathcal{B}}_{kR_n}^{n,s} \cap \hat{\mathcal{B}}_{lR_n}^{n,s} = \varnothing.$$

This implies the result. □

Theorem 5.4. *Suppose F_N is topologically renormalizable with return time $2 \le r_N \le$ $\mathbf{b}$, and that not every r_N-periodic Jordan domain of F_N contains a sink. Then F is $(N+1)$-times $(\bar{L}, \bar{\varepsilon}, \lambda)$-regular Hénon-like renormalizable.*

Proof. Let $\mathcal{D}_0^{N+1} \Subset \mathcal{B}_0^n$ be an $\hat{R}_{N+1}$-periodic Jordan domain with

$$\hat{r}_N := \hat{R}_{N+1}/R_N \le \mathbf{b}.$$

Define

$$\mathcal{A}_0 := \bigcap_{i=1}^{\infty} \mathcal{D}_{i\hat{R}_{N+1}}^{N+1}.$$

By Proposition 4.5, we see that

$$\mathcal{V}_{v_0}^N(\lambda^{\bar{\varepsilon} R_N}) \cap \mathcal{A} \ne \varnothing, \quad \text{where} \quad \mathcal{A} := \bigcup_{i=0}^{\hat{r}_N - 1} \mathcal{A}_{iR_N}.$$

Without loss of generality, assume that

$$\mathcal{V}_{v_0}^N(\lambda^{\bar{\varepsilon} R_N}) \cap \mathcal{A}_0 \ne \varnothing.$$

By (3.5) and Proposition A.4, it follows that

$$\operatorname{dist}(v_{\hat{R}_{N+1}}, \mathcal{A}_0) < \lambda^{\bar{\varepsilon} R_N}.$$

For $m \ge -1$, let

$$a_m^N := \pi_h \circ \Psi^N(v_{mR_N}).$$

Define

$$\check{I}_0 := (a_{\hat{r}_N}^N + \lambda^{\bar{\varepsilon} R_N}, -\lambda^{\bar{\varepsilon} R_N}) \quad \text{and} \quad \check{\mathcal{V}}_0 := (\Psi^N)^{-1}(\check{I}_0 \times I_0^v).$$

We claim that for some $r_N \le \hat{r}_N$, we have

$$a_{-1}^N \in \pi_h \circ \Psi^N(\check{\mathcal{V}}_{(r_N - 1)R_N}).$$

Suppose not. For $y \in I_0^v$, let

$$\check{I}_0^y := \check{I}_0 \times \{y\} \quad \text{and} \quad \check{\mathcal{I}}_0^y := (\Psi^N)^{-1}(\check{I}_0^y).$$

Arguing inductively using Lemmas 3.9 and 4.1, and Propositions 4.6, 5.3 ii), A.4 and A.5, we see that for $l \geq 1$ such that

$$a_{-1}^N \not\in \pi_h \circ \Psi^N(\check{\mathcal{I}}^y_{(m-1)R_N}) \quad \text{for} \quad 0 \leq m \leq l, \tag{5.2}$$

the arc $\hat{\mathcal{I}}^y_{lR_N-1}$ is $\lambda^{(1-\bar{\varepsilon})lR_N}$-horizontal in $\mathcal{B}_{-1}$, and

$$\check{\mathcal{I}}^y_{lR_N} \cap \left(\check{\mathcal{V}}_{mR_N} \cup \mathcal{V}^N_{v_0}(\lambda^{\bar{\varepsilon}R_N})\right) = \varnothing \quad \text{for} \quad 0 \leq m < l.$$

If (5.2) holds for all $l \in \mathbb{N}$, then it is easy to see that the sequence $\check{\mathcal{V}}_{lR_N}$ converges to a sink. Otherwise, let $l > \hat{r}_N$ be the smallest integer such that

$$a_{-1}^N \in \pi_h \circ \Psi^N(\check{\mathcal{V}}_{(l-1)R_N}).$$

Denote

$$\check{I}_{iR_N} := \pi_h \circ \Psi^N(\check{\mathcal{I}}^0_{iR_N}) \quad \text{for} \quad 0 \leq i \leq l.$$

Note that for $s \in \check{I}_{iR_N}$ and $t \in \check{I}_{jR_N}$ with $i < j$, we have

$$t < s < -\lambda^{\bar{\varepsilon}R_N}.$$

For $0 \leq m \leq l$, let $\hat{I}_m$ be the convex hull of the union

$$\bigcup_{i=0}^{m-1} \check{I}_{iR_N} \subset I_0^N.$$

Proposition 7.7 implies that $f_N^l|_{\hat{I}_l}$ is a unimodal map that maps $\hat{I}_{l-1}$ as an orientation preserving diffeomorphism to the interval $f_N(\hat{I}_{l-1})$ disjoint from $\check{I}_0$, and maps the turning point $c^N \in \hat{I}_l \setminus \hat{I}_{l-1}$ of f_N to $f_N(c^N)$ that is $\lambda^{(1-\bar{\varepsilon})R_N}$-close to 0. This is clearly impossible.

Denote $R_{N+1} := r_N R_N$. Define

$$I_0^{N+1} := (a^N_{R_{N+1}} - \lambda^{\bar{\varepsilon}R_N}, \lambda^{\bar{\varepsilon}R_N}) \Supset \check{I}_0,$$

and let

$$B_0^{N+1} := I_0^{N+1} \times I_0^v \quad \text{and} \quad \mathcal{B}_0^{N+1} := (\Psi^N)^{-1}(B_0^{N+1}).$$

We showed that $\mathcal{B}^{N+1}_{R_{N+1}-1} \ni v_{-1}$, and that for any $y \in I_0^v$, the following holds:

- $\check{\mathcal{I}}^y_{mR_N} \cap \mathcal{V}^N_{v_0}(\lambda^{\bar{\varepsilon}R_N}) = \varnothing$ for $1 \leq m < \hat{r}_N$;
- $\check{\mathcal{I}}^y_{\hat{R}_N-1}$ is $\lambda^{(1-\bar{\varepsilon})\hat{R}_{N+1}}$-horizontal in $\mathcal{B}_{-1}$; and
- $\check{\mathcal{I}}^y_{\hat{R}_{N+1}}$ is vertical quadratic in $\mathcal{B}_0^n$.

Arguing as in Proposition 5.2, we see that $F^{R_{N+1}}(\mathcal{B}_0^{N+1}) \Subset \mathcal{B}_0^{N+1}$.

Adjust the left and right boundaries of $\mathcal{B}^{N+1}_{\hat{R}_{N+1}-1} \subset \mathcal{B}_{-1}$ so that they map to genuine vertical leaves under Φ_{-1}. Consider the genuine vertical foliation over $\Phi_{-1}\left(\mathcal{B}^{N+1}_{\hat{R}_{N+1}-1}\right)$. By Lemma 4.2, we see that the pull back of this foliation under $\Phi_{-1} \circ F^{R_{N+1}-1}$ is a

$\lambda^{(1-\bar{\varepsilon})R_{N+1}}$-vertical and vertically proper foliation over $\mathcal{B}_0^{N+1}$. Let Ψ^{N+1} be the genuine horizontal chart that rectifies this foliation. We conclude that $(F^{R_{N+1}}, \Psi^{N+1})$ is a Hénon-like return.

It remains to prove that this Hénon-like return is $(\bar{L}, \bar{\varepsilon}, \lambda)$-regular. The forward regularity follows immediately from Proposition 4.3.

For $s \in \{0, 1\}$ and $p_0 \in \mathcal{B}^{N+s}_{R_{N+s}}$, let

$$E^{v,N+s}_{p_0} := D\Phi_0^{-1}(E^{gh}_{\Phi_0(p_0)}).$$

Let $s = 1$. By the regularity of the Nth Hénon-like return, p_0 is R_N-times backward $(L, \varepsilon, \lambda)$-regular along

$$E^{v,N+1}_{p_0} = E^{v,N}_{p_0}.$$

Proceeding by induction, suppose that for some $1 \leq l < r_{N+1}$, the point p_0 is lR_N-times backward $(\bar{L}, \bar{\varepsilon}, \lambda)$-regular along $E^{v,N+1}_{p_0}$.

By Proposition A.8, $E^{v,N+1}_{p_{-lR_N}}$ is $\lambda^{(1-\bar{\varepsilon})R_N}$-vertical in $\mathcal{B}_0^N$. By (4.2) and (4.3), we see that

$$\lambda^{\bar{\varepsilon}R_N} < \frac{\|DF^{-i}|_{E^{v,N+1}_{p_{-lR_N}}}\|}{\|DF^{-i}|_{E^{v,N}_{p_{-lR_n}}}\|} < \lambda^{-\bar{\varepsilon}R_N} \quad \text{for} \quad 1 \leq i \leq R_N.$$

Concatenating with the lR_N-times backward $(\bar{L}, \bar{\varepsilon}, \lambda)$-regularity of p_0, we conclude that p_0 is actually $(l+1)R_N$-times backward $(\bar{L}, \bar{\varepsilon}, \lambda)$-regular along $E^{v,N+1}_{p_0}$ (with $\bar{L}$ and $\bar{\varepsilon}$ increased some uniform amount from the lth step). □

6. A Priori Bounds

For $N \in \mathbb{N} \cup \{\infty\}$, let F be the N-times regularly Hénon-like diffeomorphism considered in Section 5.

For $1 \leq n \leq N$, we define a sequence of maps $\{H_i^n\}_{i=0}^\infty$ as follows. First, let $H_i^0 := F^i$. Proceeding inductively, suppose H_i^{n-1} is defined. Write $i = j + kR_n$ with $k \geq 0$ and $0 \leq j < R_n$. Define

$$H_i^n := H_j^{n-1} \circ \mathcal{P}_0^n \circ F^{kR_n}.$$

Observe that H_i^n is well-defined on $F^{-kR_n}(\mathcal{B}_0^n)$.

Recall that

$$\mathcal{I}_0^n := (\Psi^n)^{-1}(I_0^n \times \{0\}) = \Phi_0^{-1}(I_0^n \times \{0\}) = \mathcal{I}_0^h \cap \mathcal{B}_0^n \ni v_0.$$

Lemma 6.1. *Let $s \in \{1, 2\}$ and $1 \leq n \leq N - s$. Then $H_i^n|_{\mathcal{I}_1^{n+s}}$ is a diffemorphism for $0 \leq i < R_{n+s}$.*

Proof. The statement is clearly true for $n = 0$. Suppose the statement is true for $n - 1$. If $i < R_n$, then

$$H_i^n|_{\mathcal{I}_1^{n+s}} = H_i^{n-1}|_{\mathcal{I}_1^{n+s}}$$

is a diffeomorphism. Suppose the same is true for $i < (k-1)R_n$ with $2 \leq k < R_{n+s}/R_n$. Observe that

$$H^n_{kR_n} = \mathcal{P}^n_0 \circ F^{kR_n}.$$

By Lemma 5.1 i), the map $\mathcal{P}^n_0|_{\mathcal{I}^{n+s}_{kR_n}}$ is a diffeomorphism. For $i = j + kR_n$ with $j < R_n$, we have

$$H^n_i := H^{n-1}_j \circ \mathcal{P}^n_0 \circ F^{kR_n}.$$

Since

$$\mathcal{P}^n_0(\mathcal{I}^{n+s}_{kR_n}) \subset \mathcal{I}^n_0,$$

the result follows. □

Lemma 6.2. *For $s \in \{1, 2\}$ and $1 \leq n \leq N - s$, let Γ_0 be a C^r-curve which is $\lambda^{-\bar{\varepsilon}R_n}$-horizontal in $\mathcal{B}^{n+s}_0$. Then for $1 \leq k \leq R_{n+s}/R_n$, we have*

$$F^{kR_n-1}|_{\Gamma_0} = \left(\mathcal{P}^1_{-1}|_{\Gamma_{kR_n-1}}\right)^{-1} \circ H^n_{kR_n-1}|_{\Gamma_0}.$$

Proof. If $n = k = 1$, then the result follows immediately from Lemma 3.10. Suppose the result is true for some $1 \leq n < N - s$ and $1 \leq k < R_{n+s}/R_n$. By definition, we have

$$H^n_{(k+1)R_n-1} = H^n_{kR_n-1} \circ F^{R_n}.$$

If Γ_0 is a C^r-curve which is $\lambda^{-\bar{\varepsilon}R_n}$-horizontal in $\mathcal{B}^{n+s}_0$, then by Lemma 5.1 i), we see that $\Gamma_{R_n} := F^{R_n}(\Gamma_0)$ is a C^r-curve which is $\lambda^{-\bar{\varepsilon}R_n}$-horizontal in $\mathcal{B}^n_0$. Thus, by induction, we have

$$F^{kR_n-1}|_{\Gamma_{R_n}} = \left(\mathcal{P}^1_{-1}|_{\Gamma_{(k+1)R_n-1}}\right)^{-1} \circ H^n_{kR_n-1}|_{\Gamma_{R_n}}.$$

Composing on the right by $F^{R_n}|_{\Gamma_0}$, the result is true in this case.

Finally, suppose that the result is true for some $1 \leq n < N - s$ and $k = R_{n+1}/R_n$. Let $\gamma_0 := \mathcal{P}^{n+1}_0(\Gamma_0)$. By the induction hypothesis, we have:

$$F^{R_{n+1}-1}|_{\gamma_0} = \left(\mathcal{P}^1_{-1}|_{\gamma_{R_{n+1}-1}}\right)^{-1} \circ H^n_{R_{n+1}-1}|_{\gamma_0}.$$

Applying Lemma 3.10:

$$\begin{aligned} F^{R_{n+1}-1}|_{\Gamma_0} &= \left(\mathcal{P}^{n+1}_{-1}|_{\Gamma_{R_{n+1}-1}}\right)^{-1} \circ \left(\mathcal{P}^1_{-1}|_{\gamma_{R_{n+1}-1}}\right)^{-1} \circ H^n_{R_{n+1}-1} \circ \mathcal{P}^{n+1}_0|_{\Gamma_0} \\ &= \left(\mathcal{P}^1_{-1}|_{\Gamma_{R_{n+1}-1}}\right)^{-1} \circ H^{n+1}_{R_{n+1}-1}|_{\Gamma_0}. \end{aligned}$$

□

We also define another sequence of maps $\{\hat{H}_i\}_{i=0}^{R_N-1}$ as follows (if $N = \infty$, then $R_N = \infty$). If $i < 2R_1$, let $\hat{H}_i := F^i$. Otherwise, let $1 \leq n < N$ be the largest number such that $i \geq 2R_n$, and define $\hat{H}_i := H^n_i$. Observe that by Lemma 5.1, we have

$$\hat{H}_{R_n-1}|_{\mathcal{I}^n_0} = H^{n-1}_{R_n-1}|_{\mathcal{I}^n_0} = \mathcal{P}^1_{-1}|_{\mathcal{I}^n_{R_n-1}} \circ F^{R_n-1}|_{\mathcal{I}^n_0}. \tag{6.1}$$

Theorem 6.3. *There exists a uniform constant* $\mathbf{K} = \mathbf{K}(\|F\|_{C^2}, R_1) > 1$ *such that for all* $1 \leq n \leq N$*, we have*

$$\text{Dis}(\hat{H}_i, \mathcal{I}_0^n) < \mathbf{K} \quad \textit{for} \quad 0 \leq i < R_n.$$

Corollary 6.4. *For* $1 \leq n \leq N$*, let* $h_n : I_0^n \to h_n(I_0^n)$ *be the diffeomorphism given in Theorem 3.6 iv). Then* $\text{Dis}(h_n, I_0^n) < \mathbf{K}$*, where* $\mathbf{K} > 1$ *is the uniform constant given in Theorem 6.3.*

Observe that any number $2R_1 \leq i < R_N$ can be uniquely expressed as

$$i = j + a_1 R_1 + a_2 R_2 + \ldots + a_n R_n$$

for some $1 \leq n < N$, where

i) $0 \leq j < R_1$;
ii) $0 \leq a_m < r_m$ for $1 \leq m < n$; and
iii) $2 \leq a_n < 2r_n$.

In this case, we denote

$$i := j + [a_1, a_2, \ldots, a_n].$$

We extend this notation to $i < 2R_1$ by writing

$$i = j + [a_1] \quad \text{for some} \quad a_1 \in \{0, 1\}$$

We record the following easy observation.

Lemma 6.5. *Let* $2R_1 \leq i < R_N$ *be given by*

$$i = j + [a_1, \ldots, a_n].$$

Then we have

$$\hat{H}_i = H_i^n = F^j \circ \left(\mathcal{P}_0^1 \circ F^{a_1 R_1}\right) \circ \ldots \circ \left(\mathcal{P}_0^n \circ F^{a_n R_n}\right).$$

For $1 \leq n \leq N$, we define a collection of arcs $\{\mathcal{J}_i^n\}_{i=0}^{R_n - 1}$ by

$$\mathcal{J}_i^n := \hat{H}_i(\mathcal{I}_0^n) \quad \text{for} \quad 0 \leq i < R_n. \tag{6.2}$$

Lemma 6.6. *Let* $1 \leq n \leq N$ *and* $0 \leq i < R_n$*. If*

$$i = [0, \ldots, 0, a_m, a_{m+1}, \ldots, a_k]$$

for some $1 \leq m \leq k < n$*, then we have* $\mathcal{J}_i^n \subset \mathcal{I}_0^m$*. Moreover, we have*

$$\mathcal{J}_{i+l}^n = H_l^{m-1}(\mathcal{J}_i^n) \quad \textit{for} \quad 0 \leq l < R_m.$$

Proof. Observe that

$$\mathcal{P}_1^k \circ F^{a_k R_k}(\mathcal{I}_1^{k+1}) \subset \mathcal{I}_1^k.$$

By Lemma 6.5, the result follows from induction. □

Lemma 6.7. *For* $1 \leq n \leq N$ *and* $0 \leq i < R_n$*, we have* $\mathcal{J}_i^n \subset \mathcal{I}^1_{i \, (\text{mod } R_1)}$.

Proof. The result follows immediately from Lemma 6.6. □

Let $\Gamma : [0,1] \to \mathbb{R}^2$ be a parameterized Jordan arc. For

$$0 \leq a < b < c < d \leq 1,$$

Let

$$\Gamma_1 := \Gamma(a,b) \quad \text{and} \quad \Gamma_2 := \Gamma(c,d).$$

Then we denote $\Gamma_1 <_\Gamma \Gamma_2$. Let Γ_3 be a subarc of Γ. We denote $\Gamma_1 \leq_\Gamma \Gamma_3$ if either $\Gamma_1 <_\Gamma \Gamma_3$ or $\Gamma_1 = \Gamma_3$.

Henceforth, we consider $\mathcal{I}_0^1$ with parameterization given by

$$\mathcal{I}_0^1(t) := (\Psi^1)^{-1}(t,0) \quad \text{for} \quad t \in I_0^1.$$

Note that $\mathcal{I}_0^1 \circ P_0^1 = \mathcal{P}_0^1$. Moreover,

$$P_0^1(v_{R_1}) < 0 = P_0^1(v_0).$$

Lemma 6.8. *For $s \in \{1,2\}$; $1 \leq n \leq N-s$ and $1 < k < R_{n+s}/R_n$, we have*

$$\mathcal{J}^{n+s}_{R_n} <_{\mathcal{I}_0^1} \mathcal{J}^{n+s}_{kR_n} <_{\mathcal{I}_0^1} \mathcal{J}^{n+s}_0.$$

Proof. Observe that

- For $s \in \{1,2\}$:
$$\mathcal{J}^{n+s}_{R_n} = H^{n-1}_{R_n}(\mathcal{I}_0^{n+s}) = \mathcal{P}_0^{n-1} \circ F^{R_n}(\mathcal{I}_0^{n+s}).$$
- For $1 < k < r_n$:
$$\mathcal{J}^{n+1}_{kR_n} = H^n_{kR_n}(\mathcal{I}_0^{n+1}) = \mathcal{P}_0^n \circ F^{kR_n}(\mathcal{I}_0^{n+1}).$$
- For $1 < k < 2r_n$:
$$\mathcal{J}^{n+2}_{kR_n} = H^n_{kR_n}(\mathcal{I}_0^{n+2}) = \mathcal{P}_0^n \circ F^{kR_n}(\mathcal{I}^{n+2}_{kR_n}).$$

In the case $s = 1$, and the case $s = 2$ and $1 < k < 2r_n$ follow immediately from Proposition 5.3.

Replacing n by $n+1$ and applying the above conclusion, we see that for $1 < l < r_{n+1}$:

$$\mathcal{J}^{n+2}_{R_{n+1}} <_{\mathcal{I}_0^1} \mathcal{J}^{n+2}_{lR_{n+1}} <_{\mathcal{I}_0^1} \mathcal{J}^{n+2}_0.$$

Note that for $2 < k < r_n$:

$$\mathcal{J}^{n+2}_{lR_{n+1}+kR_n} = H^n_{kR_n}|_{\mathcal{I}_0^{n+1}}(\mathcal{J}^{n+2}_{lR_{n+1}}).$$

The result now follows from Lemma 6.1. □

Let $\Gamma_0 : [0, |\Gamma_0|] \to \mathbb{R}^2$ be a C^1-curve parameterized by its arclength, and let $\Gamma_1 = \Gamma_0(a,b)$ with $(a,b) \subset [0,|\Gamma_0|]$ be a subarc of Γ_0. If for some $0 < l < |\Gamma_0|/2$, we have $a < l$ and $b > |\Gamma_0| - l$ then we denote

$$\Gamma_1 = \Gamma_0\{-l\} \quad \text{and} \quad \Gamma_0 = \Gamma_1\{+l\}.$$

Let $\Gamma_2 := \Gamma_0(l, |\Gamma_0| - l)$. Then we denote

$$\Gamma_2 = \Gamma_0[-l] \quad \text{and} \quad \Gamma_0 = \Gamma_2[+l].$$

If Γ_3 and Γ_4 are C^1-curves in $\mathbb{R}^2$ and we have $\Gamma_3[-l] \subset \Gamma_4 \subset \Gamma_3[+l]$, then we denote

$$\Gamma_4 = \Gamma_3\{\sim l\}.$$

These notations can be extended to intervals in $\mathbb{R}$ in the obvious way.

Let $2 \le n \le N$, and consider the collection of arcs $\{\mathcal{J}_i^n\}_{i=0}^{R_n-1}$. By Lemma 6.7 and Lemma 6.8, for $2R_1 \le i < R_n$, there exist unique numbers $0 \le \iota_-^n(i), \iota_+^n(i) < R_n$ such that

$$\iota_\pm^n(i) = i \ (\mathrm{mod}\ \ R_1),$$

and the arcs $\mathcal{J}^n_{\iota_-^n(i)}$ and $\mathcal{J}^n_{\iota_+^n(i)}$ are the two nearest neighbors of $\mathcal{J}_i^n$ (one on each side) in $\mathcal{I}^1_{i\,(\mathrm{mod}\ R_1)}$. Define $\hat{\mathcal{J}}_i^n$ as the convex hull of $\mathcal{J}^n_{\iota_-^n(i)} \cup \mathcal{J}_i^n \cup \mathcal{J}^n_{\iota_+^n(i)}$ in $\mathcal{I}^1_{i\,(\mathrm{mod}\ R_1)}$.

We also define a subarc $\tilde{\mathcal{J}}_i^n$ of $\mathcal{I}^1_{i\,(\mathrm{mod}\ R_1)}$ containing $\mathcal{J}_i^n$ as follows. Write

$$i = j + [a_1, a_2, \dots, a_m]$$

for some $1 \le m < n$. If $m < n-1$, define

$$\tilde{\mathcal{J}}_i^n := \hat{\mathcal{J}}_i^n[+\lambda^{\bar{\varepsilon} R_m}].$$

Otherwise, define

$$\tilde{\mathcal{J}}_i^n := \hat{\mathcal{J}}_i^n[-\lambda^{\bar{\varepsilon} R_{n-1}}].$$

Proposition 6.9. *There exists a uniform constant $K > 0$ such that for $1 \le n \le N$, we have*

$$\sum_{i=2R_1}^{R_n-1} |\tilde{\mathcal{J}}_i^n| < K.$$

Proof. Observe that

$$\sum_{i=2R_1}^{R_n-1} |\tilde{\mathcal{J}}_i^n| < \sum_{i=2R_1}^{R_n-1} |\hat{\mathcal{J}}_i^n| + \sum_{m=1}^{n-1} 2R_{m+1}\lambda^{\bar{\varepsilon} R_m}.$$

By Lemma 6.8, the maximum number of overlaps among arcs in $\{\hat{\mathcal{J}}_i^n\}_{2R_1}^{R_n-1}$ is three. Hence, the above sum has a uniform upper bound. ☐

Lemma 6.10. *For $1 \le n \le N$, let $\Gamma_0 \subset \mathcal{I}_0^n$ be an arc. Then we have*

$$\bar{L}^{-1}\lambda^{\bar{\varepsilon} i} < \frac{|H_i^n(\Gamma_0)|}{|\Gamma_0|} < \bar{L}\lambda^{-\bar{\varepsilon} i} \quad \textit{for} \quad 0 \le i < R_n.$$

Proof. For $p_0 \in \Gamma_0$, let $E_{p_0} \in \mathbb{P}^2_{p_0}$ be the direction tangent to Γ_0 at p_0. Note that p_0 is R_n-times forward $(L, \varepsilon, \lambda)$-regular along $E^v_{p_0}$. Thus, by Theorem A.2 and Proposition A.5, we have

$$\bar{L}^{-1}\lambda^{\bar{\varepsilon} l} < \|DF^l|_{E_{p_0}}\| < \bar{L}\lambda^{-\bar{\varepsilon} l} \quad \text{for} \quad 0 \le l < R_n.$$

By Proposition 5.3 and Lemma 5.1 i), the curve $\Gamma_{kR_m} := F^{kR_m}(\Gamma_0)$ for $0 \le k < r_m$ is $\lambda^{-\bar{\varepsilon} R_m}$ horizontal in $\mathcal{B}_0^m$. Hence, by Theorem 3.6, we see that

$$\bar{L}^{-1}\lambda^{\bar{\varepsilon} R_m} < \|D\mathcal{P}_0^m|_{E_{p_{kR_m}}}\| < \bar{L}.$$

Write

$$i = j + [a_1, \dots, a_m]$$

for some $1 \leq m < n$. Then by Lemma 6.5 we have

$$H^n_i = F^j \circ \mathcal{P}^1_1 \circ F^{a_1 R_1} \circ \dots \circ \mathcal{P}^m_1 \circ F^{a_m R_m}.$$

Concatenating the previous estimates, we obtain the desired result. □

Lemma 6.11. *For $s \in \{1, 2\}$; $1 \leq n \leq N - s$ and $2 \leq k < 2r_n$, let $X_{-1} \subset \mathcal{B}^n_{R_n - 1}$ be a set such that*

$$\mathcal{P}^1_{-1}(X_{-1}) = \mathcal{J}^{n+s}_{kR_n - 1}.$$

Then

$$\mathcal{P}^n_0 \circ F(X_{-1}) = \mathcal{J}^{n+s}_{kR_n}\{\sim \lambda^{(1-\bar{\varepsilon})R_n}\}.$$

Proof. By Lemma 6.2, we have

$$\mathcal{I}^{n+s}_{kR_n - 1} = \left(\mathcal{P}^1_{-1}|_{\mathcal{I}^{n+s}_{kR_n-1}}\right)^{-1} (\mathcal{J}^{n+s}_{kR_n - 1}) = \left(\mathcal{P}^1_{-1}|_{\mathcal{I}^{n+s}_{kR_n-1}}\right)^{-1} \circ \mathcal{P}^1_{-1}(X_{-1}).$$

Since

$$\mathcal{J}^{n+s}_{kR_n} = \mathcal{P}^n_0 \circ F(\mathcal{I}^{n+s}_{kR_n - 1}),$$

the claim follows from (3.4) and (3.7). □

Proposition 6.12. *For $1 \leq n \leq N - 2$ and $2R_n \leq i < 2R_{n+1}$, there exists an arc $\mathcal{K}_{0,i}$ containing $\mathcal{I}^{n+2}_0$ such that the following properties are satisfied.*

i) We have $\mathcal{K}_{0,i} \supset \mathcal{K}_{0,i+1}$.

ii) The map $\hat{H}_i|_{\mathcal{K}_{0,i}}$ is a diffeomorphism.

iii) We have $\hat{H}_i(\mathcal{K}_{0,i}) \supset \tilde{\mathcal{J}}^{n+1}_i$.

iv) Denote $\mathcal{K}_i := F^i(\mathcal{K}_{0,i})$. Then for $2 < k \leq 2r_n$, the arc $\mathcal{K}_{kR_n - 1}$ is $\lambda^{(1-\bar{\varepsilon})R_n}$-horizontal in $\mathcal{B}_{-1}$, and

$$\mathcal{K}_{kR_n} \subset \mathcal{B}^n_{R_n} \setminus \mathcal{V}_{v_0}(\lambda^{\bar{\varepsilon}R_n}).$$

Proof. We first extend $\mathcal{I}^2_{2R_1 - 1}$ to an arc $\mathcal{K}_{2R_1 - 1} \subset \mathcal{B}_{-1}$ such that $\mathcal{K}_{2R_1 - 1}$ is $\lambda^{(1-\bar{\varepsilon})R_1}$-horizontal in $\mathcal{B}_{-1}$, and the curve $\mathcal{K}_{2R_1} := F(\mathcal{K}_{2R_1 - 1})$ maps diffeomorphically onto $\mathcal{I}^1_0 \setminus \mathcal{V}_{v_0}(\lambda^{\bar{\varepsilon}R_1})$ under $\mathcal{P}^1_0|_{\mathcal{K}_{2R_1}}$. We define

$$\mathcal{K}_{0,2R_1} := F^{-2R_1}(\mathcal{K}_{2R_1}).$$

Proceeding by induction, suppose the result holds for $i \leq (k-1)R_n$ with $2 < k \leq 2r_n$. For $0 \leq l < R_n$, define

$$\mathcal{K}_{0,(k-1)R_n + l} := \mathcal{K}_{0,(k-1)R_n}.$$

Observe that

$$\hat{H}_{(k-1)R_n + l} = H^n_l \circ F^{(k-1)R_n}.$$

Thus, property ii) follows from Lemma 6.1; property iii) follows from Lemmas 6.6 and 6.10; and property iv) for $\mathcal{K}_{kR_n - 1}$ follows from Lemma 5.1 ii).

If $k < 2r_n$, then define $\mathcal{K}_{kR_n}$ to be the component of $F(\mathcal{K}_{kR_n-1}) \setminus \mathcal{V}_{v_0}(\lambda^{\bar{\varepsilon}R_n})$ containing $\mathcal{I}^{n+2}_{kR_n}$. By Lemma 5.1 i), $\mathcal{K}_{kR_n}$ maps injectively under $\mathcal{P}^n_0$. Lastly, property iii) follows from Lemma 6.11.

If $k = 2r_n$, then define $\mathcal{K}_{2R_{n+1}}$ to be the component of

$$F(\mathcal{K}_{2R_{n+1}-1}) \cap \left(\mathcal{B}^{n+1}_0 \setminus \mathcal{V}_{v_0}(\lambda^{\bar{\varepsilon}R_{n+1}})\right)$$

containing $\mathcal{I}^{n+3}_{2R_{n+1}}$. Properties ii) and iii) for $\mathcal{K}_{2R_{n+1}}$ can be checked similarly as above. □

By Lemma 6.8, for $1 \leq n \leq N-2$, there exists a unique number $2 \leq \kappa_n < r_n$ such that

$$\mathcal{J}^{n+1}_{kR_n} <_{\mathcal{I}^1_0} \mathcal{J}^{n+1}_{\kappa_n R_n} \leq_{\mathcal{I}^1_0} \mathcal{J}^{n+1}_0 \quad \text{for all} \quad 1 \leq k < r_n.$$

After relabelling $\iota^n_\pm$ if necessary, the following results hold.

Lemma 6.13. *Let $1 \leq n \leq N-2$. Then*

$$\iota^{n+1}_+(i) = i + \kappa_n R_n \quad \textit{for} \quad 2R_1 \leq i < R_n.$$

Proof. The claim follows immediately from Lemmas 6.1 and 6.6. □

Lemma 6.14. *Let $3 \leq n \leq N$. For $1 \leq m \leq n-2$ and $2 \leq k < 2r_m$, we have*

$$\iota^n_-(kR_m) = \iota^{m+2}_-(kR_m) = iR_m \quad \textit{for some} \quad 1 \leq i < 2r_m.$$

Proof. By Lemmas 6.8, 6.1 and 6.6, we see that the extremal intervals in $\mathcal{J}^{m+1}_{lR_m}$ for $0 \leq l < r_m$ are $\mathcal{J}^n_{lR_m}$ and $\mathcal{J}^n_{lR_m+R_{m+1}}$. Moreover, by Lemma 6.13, we have

$$\mathcal{J}^n_{\iota^n_+(lR_m+jR_{m+1})} \subset \mathcal{J}^{m+1}_{lR_m} \quad \text{for} \quad j \in \{0,1\}.$$

The claim follows. □

Proposition 6.15. *For $3 \leq n \leq N$ and $2R_1 \leq i < R_n$, there exists an arc $\tilde{\mathcal{I}}^n_{0,i}$ such that the following conditions hold for all $2R_1 \leq j \leq i$.*

i) We have $\mathcal{I}^n_0 \subset \tilde{\mathcal{I}}^n_{0,i} \subset \mathcal{K}_{0,i}$.

ii) Denote

$$\tilde{\mathcal{J}}^n_{j,i-j} := \hat{H}_j(\tilde{\mathcal{I}}^n_{0,i}).$$

Then we have

$$\tilde{\mathcal{J}}^n_{j,i-j} \subset \tilde{\mathcal{J}}^n_j \quad \textit{and} \quad \tilde{\mathcal{J}}^n_{i,0} \supset \tilde{\mathcal{J}}^n_i.$$

Proof. First consider the case when $i < 2R_{n-1}$. Proceeding by induction, suppose that the result is true for $j \leq kR_m$ with $1 \leq m \leq n-2$ and $2 \leq k < 2r_m$. Then the result holds for $kR_m < j < (k+1)R_m$ by Lemmas 6.1 and 6.6.

Note that we have,

$$\mathcal{P}^m_0(\mathcal{K}_{kR_m}) \supset \tilde{\mathcal{J}}^{m+2}_{kR_m} \supset \mathcal{J}^{m+2}_{\iota^{m+2}_-(kR_m)} \cup \mathcal{J}^{m+2}_{kR_m} \cup \mathcal{J}^{m+2}_{\iota^{m+2}_+(kR_m)},$$

where by Lemmas 6.13 and 6.14, we have

$$\mathcal{J}^{m+2}_{\iota^{m+2}_-(kR_m)} = \mathcal{J}^{m+2}_{\iota^n_-(kR_m)} \supset \mathcal{J}^n_{\iota^n_-(kR_m)} \quad \text{and} \quad \mathcal{J}^{m+2}_{kR_m} \supset \mathcal{J}^n_{kR_m} \cup \mathcal{J}^n_{\iota^n_+(kR_m)}.$$

Hence, there exists an arc $\mathcal{I}'_{kR_m} \subset \mathcal{K}_{kR_m}$ such that

$$\mathcal{P}^m_0(\mathcal{I}'_{kR_m}) = \tilde{\mathcal{J}}^{m+2}_{kR_m}.$$

By Lemmas 6.10 and 6.2, we have

$$\mathcal{P}^1_{-1} \circ F^{R_m-1}(\mathcal{I}'_{kR_m}) = \hat{\mathcal{J}}^{m+2}_{(k+1)R_m-1}[+\lambda^{\bar{\varepsilon}R_m}].$$

Thus, by Lemmas 6.11 and 6.13, we see that

$$\mathcal{P}^m_0 \circ F^{R_m}(\mathcal{I}'_{kR_m}) \supset \hat{\mathcal{J}}^{m+2}_{(k+1)R_m},$$

and hence, the result holds for $j = (k+1)R_m$.

Next, consider the case when $i \geq 2R_{n-1}$. For $j < 2R_{n-1}$, the result follows by the same argument as in the previous case. Proceeding by induction, suppose that the result is true for $j \leq kR_{n-1}$ with $2 \leq k < r_{n-1}$. Then the result holds for $kR_{n-1} < j < (k+1)R_{n-1}$ by Lemmas 6.1, 6.6 and Lemma 6.10.

Similar to the previous case, there exists an arc $\mathcal{I}'_{kR_{n-1}} \subset \mathcal{K}_{kR_{n-1}}$ such that

$$\mathcal{P}^{n-1}_0(\mathcal{I}'_{kR_{n-1}}) \supset \tilde{\mathcal{J}}^n_{kR_{n-1}}$$

and

$$\mathcal{P}^1_{-1} \circ F^{R_{n-1}-1}(\mathcal{I}'_{kR_{n-1}}) = \hat{\mathcal{J}}^{m+2}_{(k+1)R_{n-1}-1}[-\lambda^{\bar{\varepsilon}R_n}].$$

Let $\mathcal{I}''_{(k+1)R_{n-1}}$ be the connected component of

$$F(\mathcal{I}'_{(k+1)R_{n-1}}) \setminus \mathcal{V}_{v_0}(\lambda^{\bar{\varepsilon}R_n})$$

containing $\mathcal{I}^n_{(k+1)R_{n-1}}$. By Lemma 6.11, we have

$$\mathcal{P}^{n-1}_0(\mathcal{I}''_{(k+1)R_{n-1}}) \supset \hat{\mathcal{J}}^n_{(k+1)R_{n-1}}[-\lambda^{\bar{\varepsilon}R_n}].$$

Thus, the result holds for $j = (k+1)R_{n-1}$. □

Let $i \geq 2R_1$ be a number given by

$$i = [0, \dots, 0, a_m, a_{m+1}, \dots, a_k]$$

for some $1 \leq m \leq k$ so that $a_m > 0$. Denote

$$\hat{m}(i) := m, \quad \hat{k}(i) := k \quad \text{and} \quad \hat{a}(i) := a_m.$$

We extend this notation to the case when $i = a_1R_1$ with $a_1 \in \{0, 1\}$ by letting

$$\hat{m}(i) := 1, \quad \hat{k}(i) := 1 \quad \text{and} \quad \hat{a}(i) := a_1.$$

Proposition 6.16. *Let $1 \leq n \leq N$ and $i = j + sR_1$ with $0 \leq j < R_1$ and $0 \leq s < R_n/R_1$. For $0 \leq l \leq s$, denote*

$$\hat{m}_l := \hat{m}(lR_1), \quad \hat{k}_l := \hat{k}(lR_1) \quad \textit{and} \quad \hat{a}_l := \hat{a}(lR_1).$$

If $\hat{m}_l = \hat{k}_l$, let

$$\check{\mathcal{I}}^n_l := F^{lR_1-1}(\tilde{\mathcal{I}}^n_{0,i}).$$

Otherwise, let

$$\check{\mathcal{I}}^n_l := \mathcal{I}^{\hat{m}_l+1}_{\hat{a}_lR_{\hat{m}_l}-1}.$$

Then $\check{\mathcal{I}}^n_l$ *is* $\lambda^{(1-\bar{\varepsilon})R_{\hat{m}_l}}$*-horizontal. Moreover, define*

$$\check{H}_l := \mathcal{P}_0^{\hat{m}_l} \circ F \circ \left(\mathcal{P}^1_{-1}|_{\check{\mathcal{I}}^n_l}\right)^{-1} \circ F^{R_1-1}|_{\mathcal{I}^1_0}.$$

Then we have

$$\hat{H}_i|_{\tilde{\mathcal{I}}^n_{0,i}} = F^j|_{\mathcal{I}^1_0} \circ \check{H}_s \circ \ldots \circ \check{H}_4 \circ \check{H}_3 \circ \mathcal{P}^1_0 \circ F^{2R_1}|_{\tilde{\mathcal{I}}^n_{0,i}}.$$

Proof. We proceed by induction. Clearly, the result is true for $i < 2R_1$. Suppose that the result is true for all $i' < i$.

First, suppose $i = 2R_{k+1}$ for some $1 \le k+1 < n$. Denote

$$\Gamma_d := F^d(\tilde{\mathcal{I}}^n_{0,i}) \quad \text{for} \quad 0 \le d \le i.$$

By Lemma 6.5:

$$\hat{H}_{2R_{k+1}}|_{\Gamma_0} = \mathcal{P}^{k+1}_0 \circ F^{2R_{k+1}} = \mathcal{P}^{k+1}_0 \circ F \circ F^{R_k-1} \circ F^{(2r_k-1)R_k}|_{\Gamma_0}. \tag{6.3}$$

By Proposition 6.12 iv), $\Gamma_{(2r_k-1)R_k}$ is $\lambda^{-\bar{\varepsilon}R_k}$-horizontal in $\mathcal{B}^k_0$. So it follows from Lemma 3.10 that

$$F^{R_k-1}|_{\Gamma_{(2r_k-1)R_k}} = \left(\mathcal{P}^1_{-1}|_{\Gamma_{2R_{k+1}-1}}\right)^{-1} \circ F^{R_k-1} \circ \mathcal{P}^k_0|_{\Gamma_{(2r_k-1)R_k}}.$$

Note that

$$\hat{H}_{(2r_k-1)R_k} = H^k_{(2r_k-1)R_k} = \mathcal{P}^k_0 \circ F^{(2r_k-1)R_k}.$$

Substituting into (6.3), we obtain

$$\hat{H}_{2R_{k+1}}|_{\Gamma_0} = \mathcal{P}^{k+1}_0 \circ F \circ \left(\mathcal{P}^1_{-1}|_{\Gamma_{2R_{k+1}-1}}\right)^{-1} \circ F^{R_k-1} \circ \hat{H}_{(2r_k-1)R_k}|_{\Gamma_0}.$$

By Lemma 6.2, we have

$$F^{R_k-1}|_{\mathcal{I}^k_0} = \left(\mathcal{P}^1_{-1}|_{\mathcal{I}^k_{R_k-1}}\right)^{-1} \circ H^k_{R_k-1}|_{\mathcal{I}^k_0}.$$

Thus, we conclude:

$$\hat{H}_{2R_{k+1}}|_{\Gamma_0} = \mathcal{P}^{k+1}_0 \circ F \circ \left(\mathcal{P}^1_{-1}|_{\Gamma_{2R_{k+1}-1}}\right)^{-1} \circ H^k_{R_k-1}|_{\mathcal{I}^k_0} \circ \hat{H}_{(2r_k-1)R_k}|_{\Gamma_0}.$$

We can apply the induction hypothesis to decompose $\hat{H}_{(2r_k-1)R_k}$ into factors of the form $\check{H}_l$. Observe that for

$$e_0 := (2r_k-1)R_k < e < 2R_{k+1},$$

we have

$$\hat{m}(e) = \hat{m}(e-e_0) < \hat{k}(e) \le k \quad \text{and} \quad \hat{a}(e) = \hat{a}(e-e_0).$$

Hence, we can also apply the induction hypothesis to $H^k_{R_k-1}|_{\mathcal{I}^k_1}$ to decompose them into factors of the form $\check{H}_l$. The claim follows.

Next, suppose that $i = a_k R_k$ for some $1 \le k < n$ and $a_k \ge 3$. Proceeding in the same way as in the previous case, we obtain (in place of (6.3)):

$$\hat{H}_i|_{\Gamma_0} = \mathcal{P}^k_0 \circ F^{a_k R_k} = \mathcal{P}^k_0 \circ F \circ F^{R_k-1} \circ F^{(a_k-1)R_k}|_{\Gamma_0}.$$

The rest of the argument is identical *mutatis mutandis*.

Lastly, suppose that

$$i = j + [a_1, \dots, a_k]$$

for some $1 < k < n$ such that

$$\hat{m}(i) < k = \hat{k}(i) < n.$$

Then

$$\hat{H}_i = H^{k-1}_{i-a_kR_k} \circ \mathcal{P}^k_0 \circ F^{a_kR_k} = H^{k-1}_{i-a_kR_k}|_{\mathcal{I}^k_0} \circ \hat{H}_{a_kR_k}.$$

Applying the induction hypothesis to $\hat{H}_{a_kR_k}$ and $H^{k-1}_{i-a_kR_k}|_{\mathcal{I}^k_0}$ and arguing as above, we obtain the desired result. □

Let $G : U \to G(U)$ be a C^1-diffeomorphism defined on a domain $U \subset \mathbb{R}^2$. For a C^1-curve $\Gamma \subset U$, we define the *cross-ratio distortion* $\mathrm{CrD}(G, \Gamma)$ *of* G *on* Γ as the cross-ratio distortion of

$$G_\Gamma := \phi^{-1}_{G(\Gamma)} \circ G \circ \phi_\Gamma,$$

where ϕ_Γ and $\phi_{G(\Gamma)}$ are parameterizations of Γ and $G(\Gamma)$ by their respective arclengths (see Section B).

Proposition 6.17. *Let $1 \leq n \leq N$ and $1 \leq i < R_n$. Then there exists a uniform constant $\nu > 0$ such that the maps $\hat{H}_i$ and $\hat{H}_{R_n-1} \circ \hat{H}_i^{-1}$ have ν-bounded cross-ratio distortion on $\tilde{\mathcal{I}}^n_{0,i}$ and $\hat{H}_i(\tilde{\mathcal{I}}^n_{0,R_n-1})$ respectively.*

Proof. Consider the decomposition of $\hat{H}_i$ given in Proposition 6.16:

$$\hat{H}_i|_{\tilde{\mathcal{I}}^n_{0,i}} = F^j|_{\mathcal{I}^1_0} \circ \check{H}_s \circ \dots \circ \check{H}_3 \circ \mathcal{P}^1_0 \circ F^{2R_1}|_{\tilde{\mathcal{I}}^n_{0,i}}.$$

Denote

$$\mathcal{J} := \mathcal{P}^1_0 \circ F^{2R_1}(\tilde{\mathcal{I}}^n_{0,i}) \quad \text{and} \quad \check{H} := \check{H}_s \circ \dots \circ \check{H}_3.$$

To prove the cross-ratio distortion bound for $\hat{H}_i$, it suffices to prove it for $\check{H}$ on $\mathcal{J}$.

The maps

$$\phi_0 := (P^1_0|_{\mathcal{I}^1_0})^{-1} : I^1_0 \to \mathcal{I}^1_0 \quad \text{and} \quad \phi_{-1} := (P_{-1}|_{\mathcal{I}^1_{R_1-1}})^{-1} : I^1_{R_1-1} \to \mathcal{I}^1_{R_1-1}$$

give parameterizations of $\mathcal{I}^1_0$ and $\mathcal{I}^1_{R_1-1}$ by their respective arclengths. Denote

$$J_2 := \phi_0^{-1}(\mathcal{J}) \quad \text{and} \quad h_1 := \phi^{-1}_{-1} \circ F^{R_1-1}|_{\mathcal{I}^1_0} \circ \phi_0.$$

For $3 \leq l \leq s$, let

$$H_l := \phi_0^{-1} \circ \check{H}_l \circ \dots \circ \check{H}_3 \circ \phi_0;$$

and

$$J'_l := h_1(J_{l-1}) \quad \text{and} \quad J_l := H_l(J_2).$$

By Propositions 6.16 and 3.11, there exist a diffeomorphism $\psi_l : J'_l \to \psi_l(J'_l)$ and a constant $a_l \in \mathbb{R}$ such that

$$H_l(x) = a_l - (\psi_l \circ h_1 \circ H_{l-1}(x))^2.$$

By (B.2) and Lemma B.2, we see that

$$\mathrm{CrD}(\check{H}, \mathcal{J}) := \mathrm{CrD}(H_s, J_2) > \left(\prod_{l=2}^{s-1} \mathrm{CrD}(h_1, J_l)\right) \cdot \left(\prod_{l=3}^{s} \mathrm{CrD}(\psi_l, J_l')\right).$$

Note that the diffeomorphisms h_1 and $\{\psi_l\}_{l=3}^s$ have uniformly bounded second derivatives. Moreover, Propositions 6.9 and 6.15 implies that the total length of $\{J_l, J_l'\}_{l=3}^s$ is uniformly bounded. The bound on the cross ratio distortion of $\hat{H}_i$ now follows from Lemma B.3.

Now, consider the decomposition of $\hat{H}_{R_n-1}$ on $\tilde{\mathcal{I}}^n_{0,R_n-1}$:

$$\hat{H}_{R_n-1}|_{\tilde{\mathcal{I}}^n_{0,R_n-1}} = F^{R_1-1}|_{\mathcal{I}^1_0} \circ \check{H}_S \circ \ldots \circ \check{H}_3 \circ \mathcal{P}^1_0 \circ F^{2R_1}|_{\tilde{\mathcal{I}}^n_{0,R_n-1}},$$

where $S := R_n/R_1 - 1$. The same argument as above implies the bound on the cross ratio distortion of

$$\hat{H}_{R_n-1} \circ \hat{H}_i^{-1}|_{\mathcal{I}} = F^{R_1-1}|_{\mathcal{I}^1_0} \circ \check{H}_S \circ \ldots \circ \check{H}_{S-s} \circ F^{R_1-1-j}|_{\mathcal{I}}$$

on $\mathcal{I} := \hat{H}_i(\tilde{\mathcal{I}}^n_{0,R_n-1})$. □

Proof of Theorem 6.3. Consider the arcs $\{\mathcal{J}^n_i\}_{i=0}^{R_n-1}$. There exists $2R_1 \leq i_1 < R_n$ such that

$$|\mathcal{J}^n_{\iota^n_+(i_1)}| \ , \ |\mathcal{J}^n_{\iota^n_-(i_1)}| > k|\mathcal{J}^n_{i_1}|$$

for some uniform constant $k > 0$. By Proposition 6.15, there exists an arc $\tilde{\mathcal{I}}^n_{0,i_1} \supset \mathcal{I}^n_0$ which is mapped diffeomorphically onto $\tilde{\mathcal{J}}^n_{i_1}$ by $\hat{H}_{i_1}$.

Recall that the nearest neighbor of $\mathcal{I}^n_0$ in $\mathcal{I}^1_0$ is given by $\mathcal{J}^n_{\kappa_{n-1}R_{n-1}}$. Let $\hat{\mathcal{I}}^n_0$ be the convex hull of $\mathcal{I}^n_0 \cup \mathcal{J}^n_{\kappa_{n-1}R_{n-1}}$. Then

$$(\tilde{\mathcal{I}}^n_{0,i_1} \cap \mathcal{I}^1_0) \setminus \mathcal{I}^n_0 \subset \hat{\mathcal{I}}^n_0 \setminus \mathcal{I}^n_0.$$

Hence, Proposition 6.17 and Theorem B.4 imply

$$\left|\hat{\mathcal{I}}^n_0 \setminus \mathcal{I}^n_0\right| > k\,|\mathcal{I}^n_0|\,.$$

By Lemma 6.11, we conclude that the two components of $\tilde{\mathcal{J}}^n_{R_n-1} \setminus \mathcal{J}^n_{R_n-1}$ have lengths greater than $k\left|\mathcal{J}^n_{R_n-1}\right|$. By Proposition 6.15, $\hat{H}_{R_n-1}$ maps $\tilde{\mathcal{I}}^n_{0,R_n-1} \supset \mathcal{I}^n_0$ diffeomorphically onto $\tilde{\mathcal{J}}^n_{R_n-1}$. The result now follows from Proposition 6.17 and Theorem B.4. □

7. Uniform C^1-Bounds

7.1. **For unimodal maps.** Define

$$\mathrm{sign}(x) := \begin{cases} +1 & : \text{ if } x \geq 0 \\ -1 & : \text{ otherwise.} \end{cases}$$

Lemma 7.1. *Let $f : I \to I$ be a C^r-unimodal map with the critical point at $c \in I$. Then there exists a unique orientation-preserving C^r-diffeomorphism $h_f : I \to h_f(I)$ such that $h_f(c) = 0$ and*

$$f(x) = f(c) + \operatorname{sign}(f''(c))(h_f(x))^2.$$

Consider a C^2-unimodal map $f : I \to I$, and let $h := h_f$ be the diffeomorphism given in Lemma 7.1. Suppose that for some $K \geq 1$, we have

$$\sup_{x,y \in I} \frac{h'(x)}{h'(y)} \leq K. \tag{7.1}$$

Proposition 7.2. *There exists a constant $C \geq 1$ independent of f such that $\|f\|_{C^1} < CK$.*

Proof. Let $\hat{f} : \hat{I} \to \hat{I}$ be the normalization of f_n, so that $|\hat{I}| \asymp 1$. Let $\hat{h} := h_{\hat{f}}$ given in Lemma 7.1. Note that $\hat{h}$ is h composed with some affine transformation, which does not affect its distortion. Hence:

$$\sup_{x,y \in \hat{I}} \frac{\hat{h}'(x)}{\hat{h}'(y)} < K.$$

Since $|\hat{h}(\hat{I})| = O(1)$, it follows that there exists a uniform constant $\tilde{C} \geq 1$ independent of f such that $\|\hat{h}\|_{C^1} < \tilde{C}K$. Since $\|\hat{f}'\| = \|f'\|$, the result follows. □

Proposition 7.3. *Suppose that the critical orbit of f does not converge to a sink. Then for any $N \in \mathbb{N}$, there exists a uniform constant $\tau = \tau(K, N) > 0$ such that*

$$|f^n(c) - c| > \tau |I| \quad \text{for} \quad n \leq N.$$

Proof. By conjugating with an affine map, we may assume that $c = 0$ and $f(c) = 1$. Since $f(I) \Subset I$, we see that there exists a uniform constant $C = C(K) > 0$ such that $|I| < C$.

There exists a uniform constant $C' = C'(K, N) > 1$ such that for any interval $J \subset I$, we have $|f^n(J)| < C'|J|$. Let $J := (-t, t)$ for some $t \ll 1/C'$. Observe that $|f^n(J)| < C't^2 \ll t$. Hence, if $f^n(0) \in (-t/2, t/2)$, then the orbit of 0 converges to sink. □

Proposition 7.4. *Suppose that $|I| = O(1)$. Then there exists a uniform constant $c > 0$ independent of f such that*

$$\inf_{x \in I} |h_f'(x)| > cK^{-1}.$$

Proof. Observe that $|h_f(I)|^2 \asymp |I|$. It follows that $|h_f(I)| > C|I|$ for some uniform constant $C > 0$ independent of f. Thus, there exists $x \in I$ such that $h_f'(x)$ is uniformly bounded below. The result follows. □

Proposition 7.5. *Suppose that f is valuably renormalizable: there exist $I^1 \subset I$ and $R \geq 2$ such that $v \in f^R(I^1) \subset I^1$. If the critical orbit of f does not converge to a sink, then*

$$|f^i(I^1)| > \rho|I| \quad \text{for} \quad 0 \leq i \leq R,$$

where $\rho = \rho(K, R) \in (0,1)$ is a uniform constant.

Proof. The result is an immediate consequence of Proposition 7.3. □

Proposition 7.6. *Suppose that f is twice valuably renormalizable: there exist $I^2 \subset I^1 \subset I$ and $R_2 > R_1 \geq 2$ such that $v \in f^{R_n}(I^n) \subset I^n$ for $n \in \{1, 2\}$. Let J be a connected component of*

$$I \setminus \bigcup_{i=0}^{R_1-1} f^i(I^1).$$

If the critical orbit of f does not converge to a sink, then we have $|J| > \rho|I|$, where $\rho = \rho(K, R_2) \in (0,1)$ is a uniform constant.

Proof. Denote $I^1_i := f^i(I^1)$ for $0 \leq i < R_1$. By Lemma 13.1, we may choose $I^1_i := [f^i(v), f^{i+R_1}(v)]$.

For $t > 0$, suppose that the gap J_0 between I^1_k and I^1_l with $0 \leq k < l < R_1$ is smaller than t. If $J_m := f^m(J_0)$ with $m = O(R_2)$ maps onto an interval I^1_i for some $0 \leq i < R_1$, then by Proposition 7.2, we have $t \asymp |I^1_i|$.

By this previous observation, we may assume, after replacing J_0 with J_{R_1} if necessary, that $\partial J_0 \ni f^{k+R_1}(v)$. Under $f^{R_2-k+R_1}$, the point $f^{k+R_1}(v)$ maps to the endpoint $f^{R_2}(v)$ of I^2. Since

$$I^1_{l+R_2-k+R_1} \cap I^1_0 = \varnothing,$$

the image $J_{R_2-k+R_1}$ of the gap must contain $I^1_0 \setminus I^2_0$. Again, by Proposition 7.2, we have $t \asymp |I^2_0|$. The result now follows from Proposition 7.5. □

7.2. **For Hénon-like maps.** For $N \in \mathbb{N} \cup \{\infty\}$, let F be the N-times regularly Hénon-like diffeomorphism considered in Section 5. For $1 \leq n \leq N$, recall that the nth pre-renormalization of F is given by

$$F_n := p\mathcal{R}^n(F) := \Psi^n \circ F^{R_n} \circ (\Psi^n)^{-1},$$

and its 1D profile is given by

$$f_n := \Pi_{1\text{D}} \circ p\mathcal{R}^n(F).$$

Additionally, let $h_n := h_{f_n}$ be the diffeomorphism given by Lemma 7.1.

Proposition 7.7. *Let $\mathbf{K}$ be the constant given in Theorem 6.3. Then there exists a uniform constant $C \geq 1$ independent of F such that for all $1 \leq n \leq N$, we have*

$$\|f_n\|_{C^1}\ ,\ \|F_n\|_{C^1} < C\mathbf{K} \quad \text{and} \quad \inf_{x \in I^n_0} |h'_n(x)| > (C\mathbf{K})^{-1}.$$

Proof. The estimate on $\|f_n\|_{C^1}$ is an immediate consequence of Theorem 6.3 and Proposition 7.2. The estimate on $\|F_n\|_{C^1}$ then follows from the fact that F_n is a $\lambda^{(1-\bar{\varepsilon})R_n}$-thin Hénon-like map. Lastly, the estimate on $|h'_n|$ is implied by Theorem 6.3 and Proposition 7.4. □

8. Compositions of Nearby Maps

We first record the following general estimate.

Lemma 8.1. *Let $d \in \mathbb{N}$. Consider C^{r-1}-maps $H_0, \tilde{H}_0 : U \to U'$ and C^r-maps $H_1, \tilde{H}_1 : V \to V'$ defined on domains $U, V \subset \mathbb{R}^d$ with $H_0(U) \Subset V$. Suppose*

$$\|\tilde{H}_i - H_i\|_{C^{r-1}} < \delta \quad \text{for} \quad i \in \{0, 1\}.$$

Then we have

$$\|H_1 \circ H_0 - \tilde{H}_1 \circ \tilde{H}_0\|_{C^{r-1}} < \delta P(\|H_1\|_{C^r}, \|\tilde{H}_0\|_{C^{r-1}}),$$

where P is a two-variable polynomial of degree r independent of the maps $H_i, \tilde{H}_i$ for $i \in \{0, 1\}$.

Proof. Let $d_i := H_i - \tilde{H}_i$. A straightforward computation shows that

$$\begin{aligned} H_1 \circ H_0 &= H_1 \circ (\tilde{H}_0 - d_0) \\ &= H_1 \circ \tilde{H}_0 + O(\|DH_1 \circ \tilde{H}_0\| \|d_0\|) \\ &= \tilde{H}_1 \circ \tilde{H}_0 + d_1 \circ \tilde{H}_0 + O(\|DH_1 \circ \tilde{H}_0\| \|d_0\|). \end{aligned}$$

The result follows. □

For $N \in \mathbb{N} \cup \{\infty\}$, let F be the N-times regularly Hénon-like diffeomorphism considered in Section 5. Denote

$$F_n := \Psi^n \circ F^{R_n} \circ (\Psi^n)^{-1} \quad \text{and} \quad f_n := \Pi_{1\mathrm{D}}(F_n).$$

Define

$$\Pi_h(x, y) := (x, 0) \quad \text{and} \quad \Pi_v(x, y) := (0, y).$$

Proposition 8.2. *Let $1 \leq n \leq N$. Then for $1 \leq k < r_n$, we have*

$$\|f_n^k - \Pi_{1\mathrm{D}} \circ F_n^k\|_{C^{r-1}} < \|F_n^k - F_n^k \circ \Pi_h\|_{C^{r-1}} < K\lambda^{(1-\bar{\varepsilon})R_n},$$

where $K \geq 1$ is a constant depending only on $\|f_n\|_{C^r}$ and $\mathbf{b}$.

Proof. By Theorem 3.6 and Proposition 7.7, $\|\pi_h \circ \Psi^n\|_{C^r}$ and $\|F_n\|_{C^1}$ are uniformly bounded. Moreover, by Theorem 3.6 iv), we have

$$\|F_n - F_n \circ \Pi_h\|_{C^r} < \lambda^{(1-\bar{\varepsilon})R_n},$$

where $\Pi_h(x, y) := (x, 0)$. The result now follows from Lemma 8.1. □

9. Robustness of Regularity

For $N \in \mathbb{N} \cup \{\infty\}$, let F be the N-times regularly Hénon-like diffeomorphism considered in Section 5.

Proposition 9.1. *There exists a uniform constant $\mathbf{K} \geq 1$ depending only on $\|F\|_{C^2}$, R_1 and $\mathbf{b}$ such that the following condition holds. For $1 \leq n < N$ and $0 \leq k < r_n$, let*

$$p_0 \in \mathcal{B}^{n+1}_{kR_n} \subset \mathcal{B}^n_0 \quad \textit{and} \quad z_0 = (x_0, y_0) := \Psi^n(p_0)$$

Then

$$\frac{1}{\mathbf{K}} < \|D(\pi_h \circ F^i_n)|_{E^{gh}_{z_0}}\| \leq \|DF^i_n|_{E^{gh}_{z_0}}\| < \mathbf{K} \quad \textit{for} \quad 0 \leq i < r_n - k.$$

Proof. The upper bound is given in Proposition 7.7. For the lower bound, by Proposition 8.2, it suffices to show that

$$|f'_n(x_0)| > 1/\mathbf{K} \quad \text{for} \quad x_0 = \pi_h \circ \Psi^n(p_0) \quad \text{with} \quad p_0 \in \mathcal{B}^{n+1}_{kR_n}.$$

Denote the critical point and the critical value of f_n by c^n and v^n respectively. Normalize $f_n : I^n_0 \to I^n_0$ to $\hat{f}_n : \hat{I}^n_0 \to \hat{I}^n_0$ by conjugating it with an affine map $S : I^n_0 \to \hat{I}^n_0$ so that the critical point and the critical value of $\hat{f}_n$ are 0 and 1 respectively. Let $\hat{h}_n := h_{\hat{f}_n}$ be the diffeomorphism given in Proposition 7.1. By Corollary 6.4, we have

$$\inf_{x \in \hat{I}^n_0} |\hat{h}'_n(x)| > 1/\mathbf{K}.$$

By Proposition 5.3 and Proposition 7.7, we see that $\hat{x}_0 := S(x_0)$ is contained in a $\lambda^{\bar{\varepsilon}R_n}$-neighborhood of the interval $(\hat{f}^k_n(1), \hat{f}^{k+r_n}_n(1))$. Then Proposition 7.3 implies that $|\hat{x}_0| > \tau$, where τ only depends on $\mathbf{K}$ and $\mathbf{b}$. The result follows. □

Proposition 9.2. *There exists a constant $\mathbf{L} \geq 1$ depending only on $\|\Phi_0\|_{C^1}$ such that the following holds. Let $\mathbf{K} \geq 1$ be the constant given in Proposition 9.1. For $1 \leq n \leq N$, let $p_0 \in \mathcal{B}^n_0$. Then*

$$(\mathbf{L}\mathbf{K}^n)^{-1} \lambda^{(1+\varepsilon)i} < \operatorname{Jac}_{p_0} F^i < \mathbf{L}\mathbf{K}^n \lambda^{(1-\varepsilon)i} \quad \textit{for} \quad 0 \leq i < R_n.$$

Proof. Let $z_0 := \Psi^n(p_0)$, and define

$$E^{v/h,n}_{p_0} := (D\Psi^n)^{-1}(E^{gv/gh}_{z_0}).$$

By Theorem 3.6, we have

$$\|(\Psi^n)^{-1} \circ \Phi_0 - \operatorname{Id}\|_{C^r} < \lambda^{(1-\bar{\varepsilon})R_n}.$$

Consequently,

$$\mathbf{L}^{-1} < \frac{\operatorname{Jac}_{p_0} F^i}{\|DF^i|_{E^{h,n}_{p_0}}\| \|DF^i|_{E^{v,n}_{p_0}}\|} < \mathbf{L}.$$

Plugging in the above inequality and the estimates in Proposition 9.1 into the forward regularity condition for p_0 along $E^{v,n}_{p_0}$, the result follows. □

Theorem 9.3. *Fix $\delta \in (\bar{\varepsilon}, 1)$ such that $\mathbf{b}\bar{\delta} < 1$. Suppose that*

$$\mathbf{LK}^N \lambda^{\delta R_N} < 1, \tag{9.1}$$

where $\mathbf{K}$ and $\mathbf{L}$ are constants given in Propositions 9.1 and 9.2 respectively. Let

$$\mathbf{C} := \overline{\mathbf{LK}^N}.$$

Then the following holds.

For $m \in \mathbb{N} \cup \{\infty\}$, suppose that F_N is $(m+1)$-times topologically renormalizable with return times of $\mathbf{b}$-bounded type. Then F has $N + m$ nested $(\mathbf{C}, \delta, \lambda)$-regular Hénon-like returns.

Proof. Proceeding by induction, suppose that for $N \leq M < N + m$, the map F has M nested $(\mathbf{C}, \delta, \lambda)$-regular Hénon-like returns

$$\{(F^{R_n}, \Psi^n : \mathcal{B}_0^n \to B_0^n)\}_{n=1}^M.$$

By Theorem 5.4, F has a $(\overline{\mathbf{C}}, \bar{\delta}, \lambda)$-regular Hénon-like return

$$(F^{R_{M+1}}, \Psi^{M+1} : \mathcal{B}_0^{M+1} \to B_0^{M+1}).$$

Let $p_0 \in \mathcal{B}_0^{M+1}$ and

$$E_{p_0}^{v/h} := (D\Psi^{M+1})^{-1}(E^{gv/gh}_{\Psi^{M+1}(p_0)}).$$

By Propositions 9.1 and 9.2, p_0 is R_{M+1}-times forward $(\mathbf{LK}^N, \bar{\varepsilon}, \lambda)$-regular horizontally along $E^h_{p_0}$, and $p_{R_{M+1}}$ is R_{M+1}-times backward $(\mathbf{LK}^N, \bar{\varepsilon}, \lambda)$-regular horizontally along $E^h_{p_{R_{M+1}}}$. By Propositions A.13 and A.14, it follows that p_0 is R_{M+1}-times forward $(\mathbf{C}, \delta, \lambda)$-regular (vertically) along $E^v_{p_0}$, and $p_{R_{M+1}}$ is R_{M+1}-times backward $(\mathbf{C}, \delta, \lambda)$-regular (vertically) along $E^v_{p_{R_{M+1}}}$. □

10. Uniform C^r-Bounds

Let F be the diffeomorphism considered in Section 5. Suppose that $N = \infty$, so that F is infinitely regular Hénon-like renormalizable. For $n \in \mathbb{N}$, denote the nth pre-renormalization F and its 1D profile by

$$F_n = p\mathcal{R}^n(F) := \Psi^n \circ F^{R_n} \circ (\Psi^n)^{-1} \quad \text{and} \quad f_n := \Pi_{1\mathrm{D}}(F_n)$$

respectively.

Consider the arcs

$$\mathcal{I}_0^n := (\Psi^n)^{-1}(I_0^n \times \{0\}) = \mathcal{I}_0^h \cap \mathcal{B}_0^n \ni v_0$$

and $\mathcal{I}_i^n := F^i(\mathcal{I}_0^n)$ for $i \in \mathbb{N}$. Let $\{\mathcal{J}_i^n\}_{i=0}^{R_n-1}$ be the collection of arcs given in (6.2). Recall that for $1 \leq m \leq n$; $0 \leq k < R_n/R_m$ and $0 \leq i < R_m$, we have

$$\mathcal{J}_0^n := \mathcal{I}_0^n, \quad \mathcal{J}_{kR_m}^n \subset \mathcal{J}_0^m \quad \text{and} \quad \mathcal{J}_{i+kR_m}^n = \hat{H}_i(\mathcal{J}_{kR_m}^n). \tag{10.1}$$

Moreover, $\{\mathcal{J}_i^n\}_{i=0}^{R_n-1}$ is pairwise disjoint by Lemma 6.8.

The map

$$\phi_0 := P_0|_{\mathcal{I}_0^h} : \mathcal{I}_0^h \to I_0^h$$

gives a parameterization of $\mathcal{I}^h_0$ by its arclength. For $n \in \mathbb{N}$ and $0 \le l < R_n/R_1$, let

$$J^n_{lR_1} := \phi_0(\mathcal{J}^n_{lR_1}).$$

Observe that $\{J^n_{lR_1}\}_{l=0}^{R_n/R_1-1}$ is a pairwise disjoint set of intervals contained in $\mathbb{R}$. Moreover,

$$J^{n+1}_{kR_n} = \Pi_{1\mathrm{D}} \circ F^k_n(J^{n+1}_0) \quad \text{for} \quad 0 \le k < r_n. \tag{10.2}$$

Let $\gamma \subset \Gamma$ be C^1-curves in $\mathbb{R}^2$. We say that γ is *commensurable with* Γ if $|\gamma| \asymp |\Gamma|$.

Proposition 10.1. *Let $n \in \mathbb{N}$ and $0 \le i < R_n$. Then any arc $\mathcal{J}^{n+1}_{i+kR_n}$ for some $0 \le k < r_n$, or any component of*

$$\mathcal{J}^n_i \setminus \bigcup_{k=0}^{r_n-1} \mathcal{J}^{n+1}_{i+kR_n}$$

is commensurable with $\mathcal{J}^n_i$. Consequently, there exists a uniform constant $\rho \in (0,1)$ such that

$$\sum_{i=0}^{R_n-1} |\mathcal{J}^n_i| < O(\rho^n).$$

Proof. By Lemma 7.7 and Proposition 8.2, it follows that

$$\|f^k_n - \Pi_{1\mathrm{D}}(F^k_n)\|_{C^0} = O(\lambda^{(1-\bar{\varepsilon})R_n}). \tag{10.3}$$

Denote the critical value of f_n by v^n. Then by Corollary 6.4 and Proposition 7.3, we see that each component of

$$J^n_0 \setminus \bigcup_{k=0}^{2r_n-1} f^k_n(v^n)$$

is commensurate with J^n_0. Thus, by (10.2) and (10.3), this implies the result in the case $i = 0$. The case $0 < i < R_n$ then follows immediately from Theorem 6.3 and (10.1). □

The map

$$\phi_{-1} := P_{-1}|_{\mathcal{I}^1_{R_1-1}} : \mathcal{I}^1_{R_1-1} \to I^1_{R_1-1}$$

gives a parameterization of $\mathcal{I}^1_{R_1-1}$ by its arclength. Denote

$$J^n_{lR_1-1} := \phi_{-1}(\mathcal{J}^n_{lR_1-1}) \quad \text{for} \quad 1 \le l \le R_n/R_1.$$

Observe that $\{J^n_{lR_1-1}\}_{l=1}^{R_n/R_1}$ is a pairwise disjoint set of intervals contained in $\mathbb{R}$. Define

$$\gamma^n_{-1} := \bigcup_{l=3}^{R_n/R_1-1} J^n_{lR_1-1} \subset I^h_{-1} \quad \text{and} \quad \gamma^n_0 := \bigcup_{l=3}^{R_n/R_1-1} J^n_{lR_1} \subset I^h_0. \tag{10.4}$$

Proposition 6.16 gives the following decomposition of $\hat{H}_{R_n-1}$:

$$\hat{H}_{R_n-1}|_{\mathcal{I}^n_0} = F^{R_1-1}|_{\mathcal{I}^1_0} \circ \check{H}_{\frac{R_n}{R_1}-1} \circ \ldots \circ \check{H}_3 \circ \mathcal{P}^1_0 \circ F^{2R_1}|_{\mathcal{I}^n_0}.$$

where for $3 \leq l < R_n/R_1$, we have

$$\check{H}_l := \mathcal{P}_0^{\hat{m}_l} \circ F \circ \left(\mathcal{P}_{-1}^1|_{\check{\mathcal{I}}_l^n}\right)^{-1} \circ F^{R_1-1}|_{\mathcal{I}_0^1}.$$

Define

$$\Gamma_{-1}^n := \bigcup_{l=3}^{R_n/R_1-1} \check{\mathcal{I}}_l^n \subset \mathcal{U}_{-1} \subset \mathbb{R}^2.$$

Lemma 10.2. *For $n \in \mathbb{N}$ and $3 \leq l < R_n/R_1$, the map P_{-1} restricts to a diffeomorphism from $\check{\mathcal{I}}_l^n$ to $J_{lR_1-1}^n$ (and hence, also from Γ_{-1}^n to γ_{-1}^n). Define*

$$g_{-1}^n := \pi_v \circ \Phi_{-1} \circ (P_{-1}|_{\Gamma_{-1}^n})^{-1}.$$

Then

$$\|g_{-1}^n|_{(-t,t)}\|_{C^r} = O(t^{1/\varepsilon}).$$

Proof. The first claim follows immediately from Proposition 6.16.

Observe that $\hat{m}_l$ is the largest integer such that

$$\{0\} \cup J_{lR_1-1}^n \subset J_{R_{\hat{m}_l}-1}^{\hat{m}_l}.$$

Moreover,

$$J_{lR_1-1}^n \subset J_{\hat{a}_l R_1-1}^{\hat{m}_l+1} \quad \text{and} \quad 0 \notin J_{\hat{a}_l R_1-1}^{\hat{m}_l+1}.$$

By Proposition 6.16, $\check{\mathcal{I}}_l^n$ is $\lambda^{(1-\bar{\varepsilon})R_{\hat{m}_l}}$-horizontal. Additionally, by Proposition 10.1, we have

$$\operatorname{dist}(0, \check{I}_l^n) \asymp \rho^{\hat{m}_l}$$

for some uniform constant $\rho \in (0,1)$. The estimate on G_{-1}^n follows. □

Let $G : \mathcal{I} \to \mathcal{J}$ be a C^1-diffeomorphism between two C^1-curves $\mathcal{I}, \mathcal{J} \subset \mathbb{R}^2$. Define the *zoom-in operator* $\mathbf{Z}$ by

$$\mathbf{Z}(G)(t) := |\mathcal{J}|^{-1} \cdot \phi_{\mathcal{J}}^{-1} \circ G \circ \phi_{\mathcal{I}}(|\mathcal{I}|t),$$

where $\phi_{\mathcal{I}} : [0, |\mathcal{I}|] \to \mathcal{I}$ is the parameterization of $\mathcal{I}$ by its arclength (and $\phi_{\mathcal{J}}$ similarly defined). Note that $\mathbf{Z}(G) : [0,1] \to [0,1]$.

This rest of this section is devoted to proving the following theorem.

Theorem 10.3. *There exists a universal constant $K > 0$ such that for all $n \in \mathbb{N}$ sufficiently large and $1 \leq i < R_n$, we have*

$$\|\mathbf{Z}(\hat{H}_i|_{\mathcal{I}_0^n})\|_{C^r} < K.$$

Define

$$\mathbf{q}(x) := \operatorname{sign}(x)x^2.$$

Denote $\check{I}_0^h := \mathbf{q}^{-1}(I_0^h)$. For $n \in \mathbb{N}$ and $0 \leq l < R_n/R_1$, let $\check{J}_{lR_1}^n := \mathbf{q}^{-1}(J_{lR_1}^n)$. The proof of Theorem 10.3 relies on the following key result.

Proposition 10.4. *Let $n \in \mathbb{N}$. There exists a C^r-diffeomorphism $\check{h}^n : I^h_0 \to \check{I}^h_0$ with*

$$\|(\check{h}^n)^{\pm 1}\|_{C^r} = O(1)$$

such that for $1 \leq l \leq R_n/R_1$, we have

$$\phi_0 \circ \hat{H}_{lR_1} \circ \phi_0^{-1}|_{I^n_0} = (\mathbf{q}^n_l \circ \check{h}^n_l) \circ \ldots \circ (\mathbf{q}^n_2 \circ \check{h}^n_2) \circ (\mathbf{q}^n_1 \circ \check{h}^n_1),$$

where $\check{h}^n_l : J^n_{(l-1)R_1} \to \check{J}^n_{lR_1}$ and $\mathbf{q}^n_l : \check{J}^n_{lR_1} \to J^n_{lR_1}$ are diffeomorphisms given by

$$\check{h}^n_l := \check{h}^n|_{J^n_{(l-1)R_1}} \quad \text{and} \quad \mathbf{q}^n_l := \mathbf{q}|_{\check{J}^n_{lR_1}}. \tag{10.5}$$

Lemma 10.5. *For $n \in \mathbb{N}$ and $3 \leq l < R_n/R_1$, we have*

$$P^{\hat{m}_l}_0 \circ F \circ (\mathcal{P}^1_{-1}|_{\check{\mathcal{I}}^n_l})^{-1} \circ F^{R_1-1} \circ \phi_0^{-1}|_{J^n_{(l-1)R_1}} = \mathbf{q}^n_l \circ \check{h}^n_l(x),$$

where $\check{h}^n_l$ and $\mathbf{q}^n_l$ are as defined in (10.5).

Proof. Define $\check{\gamma}^n_0 := \mathbf{q}^{-1}(\gamma^n_0)$, where γ^n_0 is given in (10.4). By Lemmas 3.11 and 10.2, there exists a C^r-diffeomorphism $\psi^n_{-1,0} : \gamma^n_{-1} \to \check{\gamma}^n_0$ with

$$\|(\psi^n_{-1,0})^{\pm 1}\|_{C^r} = O(1)$$

such that

$$P^{\hat{m}_l}_0 \circ F \circ \Phi^{-1}_{-1} \circ G^n_{-1}|_{\check{I}^n_l} = \mathbf{q} \circ \psi^n_{-1,0}|_{\check{I}^n_l},$$

where $G^n_{-1}(x) := (x, g^n_{-1}(x))$. Precomposing with $P_{-1} \circ F^{R_1-1} \circ \phi_0^{-1}|_{J^n_{(l-1)R_1}}$ gives the desired result. □

Lemma 10.6. *Let $\phi : U \to \phi(U)$ be a C^r-diffeomorphism defined on a domain $U \subset \mathbb{R}$. Then there exists a uniform constant*

$$K = K(\|\phi\|_{C^r}, \|\phi''/\phi'\|_{C^0}) \geq 1$$

such that for any interval $I \subset U$, we have

$$\|\mathbf{Z}(\phi|_I) - \mathrm{Id}\,\|_{C^r} \leq K|I|.$$

Lemma 10.7. *For $1 \leq i \leq n$, let $\phi_i : [0,1] \to [0,1]$ be a C^r-diffeomorphism such that*

$$\sum_{i=1}^{n} \|\phi_i - \mathrm{Id}\,\|_{C^r} = O(1).$$

Then

$$\|\phi_n \circ \ldots \circ \phi_1\|_{C^r} = O(1).$$

Proof of Theorem 10.3. For $1 \leq l < R_n/R_1$, let $1 \leq \hat{m}_l \leq n$ be the largest integer such that

$$\{0\} \cup \check{J}^n_{lR_1} \subset \check{J}^{\hat{m}_l}_{R_{\hat{m}_l}}.$$

Denote $\mathbb{L}^n_m := \{1 \leq l < R_n/R_1 \mid \hat{m}_l = m\}$. Then $l \in \mathbb{L}^n_m$ if and only if

$$\check{J}^n_{lR_1} \subset \check{J}^m_{R_m} \quad \text{and} \quad \check{J}^n_{lR_1 - 1} \cap \check{J}^{m+1}_{R_{m+1}} = \varnothing.$$

Note that

$$\bigcup_{m=1}^{n} \mathbb{L}_m^n = \{1 \leq l < R_n/R_1\}.$$

Let $U_{R_m}^m$ be the component of $\check{J}_{R_m}^m \setminus \check{J}_{R_{m+1}}^{m+1}$ contained in $\mathbb{R}^-$. Applying Proposition 10.1 and Lemma 10.6 to $\mathbf{Z}\left(\mathbf{q}|_{U_{R_m}^m}\right)$, we see that

$$\sum_{l \in \mathbb{L}_m^n} \|\mathbf{Z}(\mathbf{q}_l^n) - \mathrm{Id}\,\|_{C^r} = O(\rho^m)$$

for some uniform constant $\rho \in (0,1)$. The result now follows from Proposition 10.1, Proposition 10.4, and Lemmas 10.6 and 10.7. □

Theorem 10.8. *For all $n \in \mathbb{N}$ sufficiently large, we have*

$$\|\mathcal{R}^n(F)\|_{C^r} = O(1).$$

Proof. By Theorem 10.3 and (6.1), we see that

$$\|\Pi_{\mathrm{1D}} \circ \mathcal{R}^n(F)\|_{C^r} = O(1).$$

Since $\mathcal{R}^n(F)$ is a $\lambda^{(1-\bar{\varepsilon})R_n}$-thin Hénon-like map, the result follows. □

11. Exponentially Small Pieces

Let F be the infinitely regular Hénon-like renormalizable diffeomorphism considered in Section 10.

Recall that for $a \geq 0$, we have

$$H_{aR_n}^n = \mathcal{P}_0^n \circ F^{aR_n},$$

where $\mathcal{P}_0^n : \mathcal{B}_0^n \to \mathcal{I}_0^n$ is the projection map onto $\mathcal{I}_0^n$. Any integer $i \geq 2R_1$ can be uniquely expressed as

$$i = a_1 R_{n_1} + \ldots + a_l R_{n_l}, \tag{11.1}$$

where $1 \leq a_k < R_{n_k}$ for $1 \leq k < l$, and $2 \leq a_l < 2r_{n_l}$. Define

$$\hat{\mathcal{H}}_i := F^{a_1 R_{n_1}} \circ H_{a_2 R_{n_2}}^{n_2} \circ \ldots \circ H_{a_l R_{n_l}}^{n_l} \circ \mathcal{P}_0^{n_l}.$$

Denote $\hat{m}(i) := n_1$ and $\hat{k}(i) := n_l$. Then

$$\mathcal{P}_0^{\hat{m}(i)} \circ \hat{\mathcal{H}}_i = \hat{H}_i \circ \mathcal{P}_0^{\hat{k}(i)}. \tag{11.2}$$

For convenience, we let $\hat{\mathcal{H}}_0 := \mathrm{Id}$.

Lemma 11.1. *Let $2R_1 \leq i < R_n$. Then*

$$\|\hat{\mathcal{H}}_i \circ \mathcal{P}_0^n - F^i|_{\mathcal{B}_0^n}\|_{C^0} < K^n \lambda^{(1-\bar{\varepsilon})R_{\hat{m}(i)}}$$

for some uniform constant $K \geq 1$.

Proof. By Theorem 3.6 and Proposition 7.7, $\|(\Psi^m)^{\pm 1}\|_{C^r}$ and $\|F_m\|_{C^1}$ are uniformly bounded. Moreover, by Theorem 3.6 iv), we have

$$\|F_m - F_m \circ \Pi_h\|_{C^r} < \lambda^{(1-\bar{\varepsilon})R_m}, \tag{11.3}$$

where $\Pi_h(x,y) := (x,0)$.

Let i be given by (11.1) with $n_l < n$. Note that

$$F^{R_{n_l}} = (\Psi^{n_l})^{-1} \circ F_{n_l} \circ \Psi^{n_l}$$

and

$$\hat{\mathcal{H}}_{R_{n_l}} \circ \mathcal{P}^n_0 = F^{R_{n_l}} \circ \mathcal{P}^n_0 = (\Psi^{n_l})^{-1} \circ (F_{n_l} \circ \Pi_h) \circ \Psi^n.$$

Moreover,

$$\hat{\mathcal{H}}_{a_l R_{n_l}} = \left((\Psi^{n_l})^{-1} \circ F^{a_l-1}_{n_l} \circ \Psi^{n_l}\right) \circ \hat{\mathcal{H}}_{R_{n_l}}$$

and

$$F^{a_l R_{n_l}} = \left((\Psi^{n_l})^{-1} \circ F^{a_l-1}_{n_l} \circ \Psi^{n_l}\right) \circ F^{R_{n_l}}$$

By Theorem 3.6, (11.3) and Lemma 8.1, we obtain

$$\|\hat{\mathcal{H}}_{a_l R_{n_l}} \circ \mathcal{P}^n_0 - F^{a_l R_{n_l}}|_{\mathcal{B}^n_0}\|_{C^0} < K \lambda^{(1-\bar{\varepsilon})R_{n_l}}$$

for some uniform constant $K \geq 1$.

Proceeding by induction, suppose that

$$\|\hat{\mathcal{H}}_{i_{j+1}} \circ \mathcal{P}^n_0 - F^{i_{j+1}}|_{\mathcal{B}^n_0}\|_{C^0} < K^{l-j} \lambda^{(1-\bar{\varepsilon})R_{n_{j+1}}}.$$

where $1 \leq j < l$ and

$$i_{j+1} := a_{n_{j+1}} R_{n_{j+1}} + \ldots + a_{n_l} R_{n_l}.$$

Write

$$\hat{\mathcal{H}}_{i_j} = (\Psi^{n_j})^{-1} \circ F^{a_{n_j}-1}_{n_j} \circ \left(F_{n_j} \circ \Pi_h\right) \circ \Psi^{n_j} \circ \hat{\mathcal{H}}_{i_{j+1}}$$

and

$$F^{i_j}|_{\mathcal{B}^n_0} = (\Psi^{n_j})^{-1} \circ F^{a_{n_j}-1}_{n_j} \circ F_{n_j} \circ \Psi^{n_j} \circ F^{i_{j+1}}|_{\mathcal{B}^n_0}.$$

Applying Lemma 8.1, the result follows. □

Lemma 11.2. *There exists a uniform constant $\rho \in (0,1)$ such that*

$$\sum_{i=0}^{R_n-1} \operatorname{diam}(\hat{\mathcal{H}}_i(\mathcal{I}^n_0)) = O(\rho^n).$$

Proof. For $3 \leq l \leq R_n/R_1$, consider the curve $\check{\mathcal{I}}^n_l \subset \mathcal{U}_{-1}$ given in Proposition 6.16. By (11.2), we have

$$\hat{\mathcal{H}}_{lR_1}(\mathcal{I}^n_0) = F(\check{\mathcal{I}}^n_l) = F \circ \left(\mathcal{P}^1_{-1}|_{\check{\mathcal{I}}^n_l}\right)^{-1} \circ F^{R_1-1}(\mathcal{J}^n_{(l-1)R_1}).$$

Thus, $\{\hat{\mathcal{H}}_{lR_1}(\mathcal{I}^n_0)\}_{l=3}^{R_n/R_1}$ is the image of $\{\mathcal{J}^n_{lR_1}\}_{l=2}^{R_n/R_1-1}$ under

$$G_n := F \circ \left(\mathcal{P}^1_{-1}|_{\Gamma^n_{-1}}\right)^{-1} \circ F^{R_1-1},$$

where

$$\Gamma^n_{-1} := \bigcup_{l=3}^{R_n/R_1-1} \check{\mathcal{I}}^n_l.$$

Since Γ^n_{-1} is uniformly horizontal, $\|G_n\|_{C^r} = O(1)$. The result now follows from Proposition 10.1. □

Theorem 11.3. *There exists a uniform constant $\tilde{\rho} \in (0,1)$ such that for $n \in \mathbb{N}$, we have*

$$\sum_{i=0}^{R_n-1} \operatorname{diam}(F^i(\mathcal{B}^n_{R_n})) = O(\tilde{\rho}^n).$$

Proof. Choose $1 \leq m < n$ to be determined later. By Lemma 11.1, we see that for $1 \leq l < R_n/R_m$, we have

$$\operatorname{diam}(F^{lR_m}(\mathcal{B}^n_{R_n})) < \operatorname{diam}(\hat{\mathcal{H}}_{lR_m}(\mathcal{I}^n_0)) + K^n \lambda^{(1-\bar{\varepsilon})R_{\hat{m}(i)}}.$$

Thus, by Lemma 11.2, we have

$$\sum_{l=0}^{R_n/R_m-1} \operatorname{diam}(F^{lR_m}(\mathcal{B}^n_{R_n})) = O(\rho^n) + \frac{R_n}{R_m} K^n \lambda^{(1-\bar{\varepsilon})R_m}.$$

For m sufficiently large, the expression on the right is bounded by $O(\rho_1^n)$ for some uniform constant $\rho_1 \in (\rho, 1)$.

Let $i = a_0 + a_1 R_1 + \ldots + a_{m-1}R_{m-1} + lR_m$ with $0 \leq a_j < r_j$ for $0 \leq j < m$ and $1 \leq l < R_n/R_m$. We can write

$$F^{i-lR_m} = F^{a_0} \circ (\Psi^1)^{-1} \circ F_1^{a_1} \circ \Psi^1 \circ \ldots \circ (\Psi^{m-1})^{-1} \circ F_{m-1}^{a_{m-1}} \circ \Psi^{m-1}.$$

By Theorem 3.6 and Proposition 7.7, we see that

$$\|F^{i-lR_m}\|_{C^1} < K^m$$

for some uniform constant $K \geq 1$. Hence,

$$\sum_{i=0}^{R_n-1} \operatorname{diam}(F^i(\mathcal{B}^n_{R_n})) = R_m K^m \sum_{l=0}^{R_n/R_m-1} \operatorname{diam}(F^{lR_m}(\mathcal{B}^n_{R_n})) = O(R_m K^m \rho_1^n).$$

For n/m sufficiently large, the expression on the right is bounded by $O(\tilde{\rho}^n)$ for some uniform constant $\tilde{\rho} \in (\rho_1, 1)$. □

12. Regular Unicriticality

Let F be the infinitely regular Hénon-like renormalizable diffeomorphism considered in Section 10. Recall that the renormalization limit set of F is given by

$$\Lambda_F := \bigcap_{n=1}^{\infty} \bigcup_{i=0}^{R_n-1} \mathcal{B}^n_{R_n+i}.$$

By Theorem B, Λ_F supports a unique invariant probability measure μ given by the counting measure:

$$\mu(\mathcal{B}_i^n) = 1/R_n \quad \text{for} \quad n, i \in \mathbb{N}.$$

Proposition 12.1. *With respect to μ, the Lyapunov exponents of F on Λ_F are 0 and $\log \lambda_\mu < 0$ for some $\lambda_\mu \in (0,1)$.*

Proposition 12.2. *For any $\eta > 0$, there exist uniform constants $N_\eta \in \mathbb{N}$ and $C_\eta \geq 1$ such that for $p \in \mathcal{B}_k^n$ and $E_p \in \mathbb{P}_p^2$ with $n \geq N_\eta$ and $k \geq 0$, we have for all $i \in \mathbb{N}$:*

$$C_\eta^{-1} \lambda_\mu^{(1+\eta)i} < \|DF^i|_{E_p}\| < C_\eta \lambda_\mu^{-\eta i} \tag{12.1}$$

and

$$C_\eta^{-1} \lambda_\mu^{(1+\eta)i} < \operatorname{Jac}_p(F^i) < C_\eta \lambda_\mu^{(1-\eta)i}. \tag{12.2}$$

For $p \in \mathcal{B}_0^n$, define

$$E_p^{v,n} := D(\Psi^n)^{-1}(E_{\Psi^n(p)}^{gv})$$

and

$$E_p^h := D(\Psi^n)^{-1}(E_{\Psi^n(p)}^{gh}) = D(\Phi_0)^{-1}(E_{\Phi_0(p)}^{gh}).$$

Theorem 12.3. *For any $\varepsilon > 0$, there exists $L_\varepsilon \geq 1$ such that for all $n \in \mathbb{N}$, the nth Hénon-like return (F^{R_n}, Ψ^n) is $(L_\varepsilon, \varepsilon, \lambda_\mu)$-regular.*

Proof. Choose $\eta \in (0, \underline{\varepsilon})$. It suffices to show the result for $n \geq N_\eta$ given Proposition 12.2. Let $p_0 \in \mathcal{B}_0^n$. By Proposition 9.1 and (12.2), we see that p_0 is R_n-times forward $(O(1), \bar{\eta}, \lambda_\mu)$-regular horizontally along $E_{p_0}^h$; and p_{R_n} is R_n-times backward $(O(1), \bar{\eta}, \lambda_\mu)$-regular horizontally along $E_{p_{R_n}}^h$. The result now follows from Propositions A.13 and A.14. □

Recall that by Theorem 4.7, we have

$$\bigcap_{n=1}^{\infty} \mathcal{B}_{R_n}^n = \{v_0\}.$$

Theorem 12.4. *The orbit $\{v_m\}_{m \in \mathbb{Z}}$ is a regular quadratic critical orbit.*

Proof. By Theorem 12.3, v_0 is infinitely forward and backward $(L_\varepsilon, \varepsilon, \lambda_\mu)$-regular along $E_{v_0}^* = E_{v_0}^{ss} = E_{v_0}^c$ for all $\varepsilon > 0$. Thus, $\{v_m\}_{m \in \mathbb{Z}}$ is a regular critical orbit. The quadratic tangency of $W^{ss}(v_0)$ and $W^c(v_0)$ at v_0 is given in Proposition 3.4 iii). □

12.1. **Critical cover.** Let $\delta = \bar{\varepsilon}$ for some $\varepsilon \in (0,1)$. Choose $\eta \in (0, \underline{\varepsilon})$. Proposition 12.2 and Theorem 12.3 imply that by replacing F on Ω with $F^{R_{n_1}}$ on $\mathcal{B}_0^{n_1}$ for some $n_1 \in \mathbb{N}$ sufficiently large, we may henceforth assume the following.

- The map F is *η-homogeneous*: for all $p \in \Omega$ and $E_p \in \mathbb{P}_p^2$, we have

$$\lambda_\mu^{1+\eta} < \|DF|_{E_p}\| < \lambda_\mu^{-\eta} \quad \text{and} \quad \lambda_\mu^{1+\eta} < \operatorname{Jac}_p F < \lambda_\mu^{1-\eta}.$$

- For $n \in \mathbb{N}$, the nth Hénon-like return (F^{R_n}, Ψ^n) is $(1, \eta, \lambda_\mu)$-regular.

Denote $\varepsilon' := (1+\bar{\varepsilon})\varepsilon > \varepsilon$. For $z = (a,b) \in B_0^n$ and $t \geq 0$, let

$$V_z(t) := [a-t, a+t] \times I_0^v.$$

If $V_{\Psi^n(p)}(t) \subset B_0^n$ for some $p \in \mathcal{B}_0^n$; $t \geq 0$ and $1 \leq n \leq N$, then we denote

$$\mathcal{V}_p^n(t) := (\Psi^n)^{-1}(V_{\Psi^n(p)}(t)).$$

We now show that F is (δ, ε)-regularly unicritical on Λ_F. First, we need to define a suitable cover of the iterated preimages of critical value v_0. For $n \geq 0$ and $1 \leq i < r_n$, let $\mathcal{C}^n$ be the connected component of

$$\mathcal{B}_{R_n}^n \cap \mathcal{V}_{v_{-R_n}}^n(\lambda_\mu^{\varepsilon' R_n})$$

containing v_{-R_n}. Define

$$\mathcal{C}_i^n := F^i(\mathcal{C}^n) \quad \text{for} \quad 0 \leq j < R_n,$$

and

$$\mathbf{C}^N := \bigcup_{n=0}^{N} \bigcup_{i=0}^{R_{n+1}-1} \mathcal{C}_i^{n+1}.$$

Note that $\{v_{-i}\}_{i=1}^{R_{N+1}} \subset \mathbf{C}^N$.

Proposition 12.5. *We have*

$$\operatorname{diam}(\mathcal{C}_i^n) < \lambda_\mu^{\varepsilon R_n}.$$

Consequently,

$$\mathbf{C}^N \subset \bigcup_{i=1}^{R_{N+1}} \mathbb{D}_{v_{-i}}(\lambda_\mu^{\varepsilon i}).$$

Proof. By Theorem 3.6 iv), $\mathcal{B}_{R_n}^n$ is a $\lambda_\mu^{(1-\bar{\varepsilon})R_n}$-thick strip around the curve $F^{R_n}(\mathcal{I}_0^n)$, which is vertical quadratic in $\mathcal{B}_0^n$ with the vertical tangency $\lambda_\mu^{(1-\bar{\eta})R_n}$-close to v_0. By Proposition 4.6, we have

$$\mathcal{V}_{v_{-R_n}}(\lambda_\mu^{\bar{\eta}R_n}) \cap \mathcal{V}_{v_0}(\lambda_\mu^{\bar{\eta}R_n}) = \varnothing.$$

By Lemma 4.1, the connected component Γ^n of the curve

$$\mathcal{I}_{R_n}^n \cap \mathcal{V}_{v_{-R_n}}(\lambda_\mu^{\bar{\eta}R_n})$$

is $\lambda^{\bar{\eta}R_n}$-horizontal in $\mathcal{B}_0^n$. Consequently,

$$\operatorname{diam}(\mathcal{C}^n) \asymp |\Gamma^n| < \lambda^{-\bar{\eta}R_n} \lambda^{\varepsilon' R_n}.$$

Then by η-homogeneity of F, we have

$$\operatorname{diam}(\mathcal{C}_i^n) < \lambda^{-\bar{\eta}i} \operatorname{diam}(\mathcal{C}^n)$$

for $0 \leq i < R_n$. The result follows. □

12.2. **Forward regularity away from the critical cover.** For all $p \in \Lambda_F \setminus \{v_0\}$, there exists a unique number $d_p \geq 0$ such that $p \in \mathcal{B}_0^{d_p} \setminus \mathcal{B}_0^{d_p+1}$. Define $\operatorname{depth}(p) := d_p$. If $p = v_0$, define $\operatorname{depth}(p) = \infty$. Let $p_0 \in \Lambda_F$. For $N \in \mathbb{N}$, let $0 \leq S \leq N$ be the largest number satisfying

$$d = \operatorname{depth}(p_S) \geq \operatorname{depth}(p_i) \quad \text{for} \quad 0 \leq i \leq N.$$

Define the *valuable moment* and the *valuable depth of the N-times forward orbit of* p_0 as

$$\operatorname{vm}(p_0, N) := S \quad \text{and} \quad \operatorname{vd}(p_0, N) := d$$

respectively.

Lemma 12.6. *Let $p_0 \in \Lambda_F$ and $N \in \mathbb{N}$. Denote $S := \operatorname{vm}(p_0, N)$ and $d := \operatorname{vd}(p_0, N)$. Write*

$$S = s_0 + s_1 R_1 + \ldots + s_d R_d,$$

where $0 \leq s_i < r_i$ for $0 \leq i \leq d$. If $p_0 \setminus \mathbf{C}^d$, then for $0 \leq n \leq d$ and $0 \leq s \leq s_n$, we have

$$p_{S_{n-1}+sR_n} \notin \mathcal{V}^n_{v_0}(\lambda_\mu^{\bar{\varepsilon}R_n}) \quad \textit{where} \quad S_{n-1} := s_0 + s_1 R_1 + \ldots + s_{n-1} R_{n-1}.$$

Proof. If $q_0 \in \Lambda_F \cap \mathcal{V}^n_{v_0}(\lambda^{\bar{\varepsilon}R_n})$, then it follows from Theorem 3.6 iv) and η-homogeneity that $q_{-R_{n+1}} \in \mathcal{C}^{n+1}$. Thus, if $p_{S'} \in \mathcal{V}^n_{v_0}(\lambda_\mu^{\bar{\varepsilon}R_n})$, where $S' := S_{n-1} + sR_n$, then $p_{-R_{n+1}+S'} \in \mathcal{C}^{n+1}$. Therefore,

$$p_0 \in \mathcal{C}^{n+1}_{R_{n+1}-S'} \subset \mathbf{C}^n \subset \mathbf{C}^d.$$

This is a contradiction. $\square$

Lemma 12.7. *Denote*

$$\varepsilon_i = (1+\bar{\varepsilon})^i \bar{\varepsilon} \quad \textit{for} \quad i \geq 0.$$

Let $q_0 \in \mathcal{B}_0^n$ and $E_{q_0} \in \mathbb{P}^2_{q_0}$. If

$$\measuredangle(E_{q_0}, E^{v,n}_{q_0}) > \lambda_\mu^{\varepsilon_1 R_n},$$

then

$$\|DF^{R_n}|_{E_{q_0}}\| > \lambda_\mu^{\varepsilon_2 R_n}.$$

Moreover, if $q_{R_n} \notin \mathcal{V}^n_{v_0}(\lambda_\mu^{\varepsilon_0 R_n})$, then

$$\measuredangle(E_{q_{R_n}}, E^{v,n}_{q_{R_n}}) > \lambda_\mu^{\varepsilon_1 R_n}.$$

Proof. The estimate on $\|DF^{R_n}|_{E_{q_0}}\|$ follows immediately from the $(1, \eta, \lambda_\mu)$-regularity of the Hénon-like return (F^{R_n}, Ψ^n). The estimate on $\measuredangle(E_{q_{R_n}}, E^{v,n}_{q_{R_n}})$ follows immediately from Lemma 4.1. $\square$

Lemma 12.8. *For $n, k \in \mathbb{N}$, let $q_0 \in \mathcal{B}_0^{n+k}$ and $E_{q_0} \in \mathbb{P}^2_{q_0}$. If*

$$R_n \geq \bar{\varepsilon} R_{n+k} \quad \textit{and} \quad \measuredangle(E_{q_0}, E^{v,n+k}_{q_0}) > \lambda_\mu^{\bar{\varepsilon}R_{n+k}},$$

then

$$\|DF^{R_n}|_{E_{q_0}}\| > \lambda_\mu^{\bar{\varepsilon}R_n} \quad \textit{and} \quad \measuredangle(E_{q_{R_n}}, E^{v,n}_{q_{R_n}}) > \lambda_\mu^{\bar{\eta}R_n}.$$

Proof. Observe that

$$\bar{\eta}R_n > \bar{\eta}\bar{\varepsilon}R_{n+k} = \bar{\varepsilon}R_{n+k}.$$

So

$$\lambda_\mu^{\bar{\eta}R_n} < \lambda_\mu^{\bar{\varepsilon}R_{n+k}}.$$

By Theorem 3.6 iii), we have

$$\measuredangle(E^{v,n+k}_{q_0}, E^{v,n}_{q_0}) < \lambda_\mu^{(1-\bar{\eta})R_n}.$$

Hence,

$$\measuredangle(E_{q_0}, E^{v,n}_{q_0}) > \lambda_\mu^{\bar{\varepsilon}R_{n+k}} - \lambda_\mu^{(1-\bar{\eta})R_n} > \lambda_\mu^{\bar{\eta}R_n} - \lambda_\mu^{(1-\bar{\eta})R_n} = \lambda_\mu^{\bar{\eta}R_n}.$$

Since $\operatorname{depth}(q_{R_n}) < n$, we have $q_{R_n} \notin \mathcal{V}^n_{v_0}(\lambda_\mu^{\bar{\eta}R_n})$ by Proposition 4.6. The result then follows from Lemma 4.1. □

Theorem 12.9. *Let $p_0 \in \Lambda_F$ and $N \in \mathbb{N}$. Define*

$$\hat{E}_{p_i} := D(F^i \circ \Phi_0^{-1})(E^{gh}_{p_0}) \quad \textit{for} \quad i \geq 0.$$

If $p_0 \notin \mathbf{C}^d$ with $d := \operatorname{vd}(p_0, N)$, then

$$\|DF^N|_{\hat{E}_{p_0}}\| > \lambda_\mu^{\bar{\varepsilon}N}.$$

Proof. Write

$$S := \operatorname{vm}(p_0, N) = s_0R_0 + \ldots + s_{d_{\text{in}}}R_{d_{\text{in}}}$$

with $0 \leq s_n < r_n$ for $0 \leq n \leq d_{\text{in}} \leq d$. Using Lemmas 12.6 and 12.7, and arguing inductively, we see that

$$\|DF^S|_{\hat{E}_{p_0}}\| > \lambda_\mu^{\bar{\varepsilon}S}, \quad p_S \notin \mathcal{V}^{d_{\text{in}}}_{v_0}(\lambda_\mu^{\bar{\eta}R_{d_{\text{in}}}}) \quad \text{and} \quad \measuredangle(\hat{E}_{p_S}, E^{v,d_{\text{in}}}_{p_S})) > \lambda_\mu^{\bar{\eta}R_{d_{\text{in}}}}.$$

Let

$$T := N - S = t_0R_0 + \ldots + t_{d_{\text{out}}}R_{d_{\text{out}}}$$

with $0 \leq t_n < r_n$ for $0 \leq n \leq d_{\text{out}} < d$. If $d_{\text{out}} \geq d_{\text{in}}$, then

$$p_S \notin \mathcal{V}^{d_{\text{out}}}_{v_0}(\lambda_\mu^{\bar{\eta}R_{d_{\text{out}}}}) \subset \mathcal{V}^{d_{\text{in}}}_{v_0}(\lambda_\mu^{\varepsilon R_{d_{\text{in}}}}) \quad \text{and} \quad \measuredangle(\hat{E}_{p_S}, E^{v,d_{\text{out}}}_{p_S})) > \lambda_\mu^{\bar{\eta}R_{d_{\text{out}}}}.$$

Thus, by Lemma 12.6, we have

$$\|DF^{t_{d_{\text{out}}}R_{d_{\text{out}}}}|_{\hat{E}_{p_S}}\| > \lambda_\mu^{\bar{\varepsilon}t_{d_{\text{out}}}R_{d_{\text{out}}}}.$$

Denote

$$T_n := t_0R_0 + \ldots + t_nR_n \quad \text{and} \quad 0 \leq n \leq d_{\text{out}}.$$

Note that $T_n < R_{n+1} \leq \mathbf{b}R_n$.

If $d_{\text{out}} < d_{\text{in}}$, let $\check{d} := d_{\text{out}}$, and denote $t_{d_{\text{in}}} := s_{d_{\text{in}}}$. Otherwise, let $\check{d} < d_{\text{out}}$ be the largest integer such that $t_{\check{d}} > 0$. Proceeding by induction, suppose for some $n \leq \check{d}$ with $t_n > 0$, we have

$$\|DF^{N-T_n}|_{\hat{E}_{p_0}}\| > \lambda_\mu^{\bar{\varepsilon}(N-T_n)} \quad \text{and} \quad \measuredangle(\hat{E}_{p_{N-T_n}}, E^{v,n+k}_{p_{N-T_n}})) > \lambda_\mu^{\bar{\eta}R_{n+k}},$$

where $k > 0$ is the smallest number such that $t_{n+k} > 0$.

If $R_n \geq \bar{\varepsilon} R_{n+k}$, then Lemma 12.8 implies that

$$\|DF^{t_n R_n}|_{\hat{E}_{p_{N-T_n}}}\| > \lambda_\mu^{\bar{\varepsilon} t_n R_n} \quad \text{and} \quad \measuredangle(\hat{E}_{p_{N-T_{n-1}}}, E^{v,n}_{p_{N-T_{n-1}}})) > \lambda_\mu^{\bar{\eta} R_n}.$$

If $R_n < \bar{\varepsilon} R_{n+k}$, then by η-homogeneity, we have

$$\|DF^N|_{\hat{E}_{p_0}}\| > \lambda_\mu^{(1+\eta)T_n} \|DF^{N-T_{n+k}}|_{\hat{E}_{p_0}}\| > \lambda_\mu^{\bar{\varepsilon} R_{n+k}} \lambda_\mu^{\bar{\varepsilon}(N-T_{n+k})} > \lambda_\mu^{\bar{\varepsilon} N}.$$

□

13. Renormalization Convergence

13.1. **For unimodal maps.** Let $r \geq 2$ be an integer. Consider a C^r-unimodal map $f : I \to I$ with the critical value $v \in I$. For an integer $0 \leq s \leq r$ and a number $t > 0$, the t-neighborhood of f with respect to the C^s-topology is denoted $\mathfrak{N}^s(f, t)$. For $K \geq 1$, we say that f has K*-bounded non-linearity* if (7.1) holds for the diffeomorphism $h := h_f$ given by Lemma 7.1. Let $\mathfrak{U}^r$ be the space of all normalized C^r-unimodal maps, and let $\mathfrak{U}^r(K)$ the set of maps in $\mathfrak{U}^r$ with K-bounded non-linearity.

Suppose f is valuably renormalizable: there exists an R-periodic interval $I^1 \subset I$ for some integer $R \geq 2$ such that $f^R(I^1) \ni v$. Then the corresponding *renormalization type* $\tau(f)$ is given by the order of points in $\{f^i(v)\}_{i=0}^{R_n-1} \subset I$. Note that there is only one renormalization type for the period-doubling case $R = 2$. If f is N-times renormalizable, then its N*-renormalization type* is given by

$$\tau_N(f) := [\tau(f), \ldots, \tau(\mathcal{R}^{N-1}_{1\mathrm{D}}(f))].$$

Lemma 13.1. *Let $f : I \to I$ be a C^2-unimodal map with the critical value v. If f is topologically renormalizable with return time $R \geq 2$, and not every R-periodic subinterval $I^1 \subset I$ of f contains a sink, then f is valuably renormalizable. In this case, the minimal R-periodic interval containing v is given by $I^1 = [f^R(v), v]$.*

Lemma 13.2. *For an integer $\mathbf{b} \geq 2$ and a constant $K \geq 1$, there exists a uniform constant $t_0 = t_0(\mathbf{b}, K) > 0$ such that the following holds. Let $f \in \mathfrak{U}^r(K)$ be twice valuably renormalizable with return times of $\mathbf{b}$-bounded type, and suppose the critical orbit of f does not converge to sink. If $\tilde{f} \in \mathfrak{N}^s(f, t) \cap \mathfrak{U}^2$ with $0 \leq s < r$ and $t \in [0, t_0]$, then $\tilde{f}$ is valuably renormalizable with $\tau(\tilde{f}) = \tau(f)$. Moreover,*

$$\|\mathcal{R}_{1\mathrm{D}}(f) - \mathcal{R}_{1\mathrm{D}}(\tilde{f})\|_{C^s} < Ct,$$

where $C \geq 1$ is a uniform constant depending only on $\mathbf{b}$ and $\|f\|_{C^{s+1}}$.

Proof. Let R_i for $i \in \{1, 2\}$ be the return times of the renormalizations of f. By Lemma 13.1, we have

$$f(1) < f^{R_1+1}(1) < f^{R_1}(1) < f^{R_2+R_1}(1) \leq f^{2R_1}(1) \leq f^{R_2}(1) < 1.$$

Moreover, by Propositions 7.3, 7.5 and 7.6, there exists a uniform constant $\eta = \eta(\mathbf{b}, K) \in (0, 1)$ such that the components of

$$I \setminus \bigcup_{i=-1}^{2R_1} f^i(1)$$

have length greater than η. The renormalizability of $\tilde{f}$ now follows immediately from Lemma 13.1. Then Proposition 7.6 implies the claim $\tau(\tilde{f}) = \tau(f)$.

By Lemma 8.1, we see that

$$\|f^R - \tilde{f}^R\|_{C^s} < Ct.$$

Proposition 7.5 implies that $\mathcal{R}_{1D}(f)$ is a rescaling of f^R by a uniform factor $\rho = \rho(\mathbf{b}, K) \in (0,1)$. The result now follows. □

Consider the full renormalization attractor $\mathfrak{A}$ contained in the space $\mathfrak{U}^\omega$ of analytic unimodal maps. For an integer $\mathbf{b} \geq 2$, the compact invariant subset of $\mathfrak{A}$ consisting of all infinitely renormalizable unimodal maps with return times of $\mathbf{b}$-bounded type is denoted $\mathfrak{A}_{\mathbf{b}}$.

The following is a consequence of the fact that $\mathfrak{A}_{\mathbf{b}}$ is a hyperbolic attractor for the renormalization operator $\mathcal{R}_{1D}$ acting on $\mathfrak{U}^3$.

Lemma 13.3. *Let $r \geq 3$ and $N \in \mathbb{N}$ be integers, and let $K \geq 1$ be a number. Suppose $f \in \mathfrak{U}^r(K)$ is N-times valuably renormalizable. Then for any $f^* \in \mathfrak{A}_{\mathbf{b}}$ with $\tau_N(f) = \tau_N(f^*)$, we have:*

$$\|\mathcal{R}^n_{1D}(f) - \mathcal{R}^n_{1D}(f^*)\|_{C^r} = C\rho^n \|f - f^*\|_{C^r} \quad \text{for} \quad 1 \leq n < N/2,$$

where $\rho = \rho(\mathbf{b}) \in (0,1)$ is a universal constant and $C \geq 1$ is a uniform constant depending only on $\mathbf{b}, K$ and $\|f\|_{C^r}$.

13.2. **For Hénon-like maps.** Consider a C^r-Hénon-like map $F : B \to B$. For $K \geq 1$, we say that F has *K-bounded non-linearity* if $\Pi_{1D}(F) \in \mathfrak{U}^r(K)$. For $\beta \in (0,1]$, let $\mathfrak{H}\mathfrak{L}^r_\beta$ be the space of normalized β-thin C^r-Hénon maps, and let $\mathfrak{H}\mathfrak{L}^r_\beta(K)$ be the set of all maps in $\mathfrak{H}\mathfrak{L}^r_\beta$ with K-bounded non-linearity.

Proposition 13.4. *For an integer $\mathbf{b} \geq 2$, let $\varepsilon \in (0,1)$ be a sufficiently small constant such that $\mathbf{b}\bar{\varepsilon} < 1$. Then for $K \geq 1$, there exists a uniform constant $\beta_0 = \beta_0(\varepsilon, K, \|F\|_{C^r}) \in (0,1)$ such that the following holds. Let $F \in \mathfrak{H}\mathfrak{L}^r_\beta(K)$ with $\beta \leq \beta_0$, and let $f := \Pi_{1D}(F)$. If F is twice Hénon-like renormalizable with return times of $\mathbf{b}$-bounded type, and the orbit of the critical value of F does not converge to a sink, then f is valuably renormalizable. Conversely, if f is twice valuably renormalizable with return times of $\mathbf{b}$-bounded type, and the critical orbit of f does not converge to a sink, then F is $(1, \varepsilon, \beta)$-regular Hénon-like renormalizable. In either case, we have*

$$\|\Pi_{1D} \circ \mathcal{R}(F) - \mathcal{R}_{1D}(f)\|_{C^{r-1}} < \beta^{1-\varepsilon}.$$

Proof. Choose β_0 sufficiently small such that we have $C\beta_0^{\underline{\varepsilon}} < \rho$, where $C \geq 1$ (depending only on K and $\|F\|_{C^r}$) and $\rho \in (0,1)$ (independent of F) are suitable uniform constants. By Lemma 8.1, we have

$$\|f^k - \Pi_{1D}(F^k)\|_{C^{r-1}} \leq \|F^k - F^k \circ \Pi_h\|_{C^{r-1}} < \beta^{1-\underline{\varepsilon}} \quad \text{for} \quad 0 \leq k < \mathbf{b}^2, \tag{13.1}$$

where $\Pi_h(x,y) := (x,0)$.

Suppose that F is twice Hénon-like renormalizable. Let

$$\{(F^{R_n}, \Psi^n : \mathcal{B}_0^n \to B_0^n)\}_{n=1}^2$$

be the Hénon-like returns of F. Then by Theorem 5.4, we see that $\{(F^{R_n}, \Psi^n)\}_{n=1}^2$ is $(1, \underline{\varepsilon}, \beta)$-regular. Note that the critical value of f is given by 1. Let $v_0 \in \mathcal{B}_0^2$ be the critical value of $\{(F^{R_n}, \Psi^n)\}_{n=1}^2$ as defined in Section 3. Then by Theorem 3.6 iv), we see that

$$|\pi_h(v_0) - 1| < \beta^{1-\underline{\varepsilon}}.$$

We conclude from Proposition 5.2 and (13.1) that f is valuably renormalizable.

Conversely, suppose that f is twice valuably renormalizable: for $i \in \{1, 2\}$, there exist R_i-periodic subinterval $I^i \ni 1$ of f. Arguing as in the proof of Lemma 13.2, we have $f^{2R_1}(1) \in I^1$ and the components of

$$I^1 \setminus \bigcup_{i=-1}^{2R_1} f^i(1)$$

have lengths bounded below by some uniform constant $\eta = \eta(\mathbf{b}, K) \in (0, 1)$.

For $0 \leq i < R_1$, let $\tilde{I}_i^1$ be an interval that compactly contains $f^i(I^1)$, and the components of $\tilde{I}_i^1 \setminus f^i(I^1)$ have lengths commensurate to $\beta^{\bar{\varepsilon}}$. Define

$$V_i := \tilde{I}_i^1 \times \pi_v(B).$$

By (13.1) and the previous observation, it follows that we have $F(V_i) \Subset V_{i+1}$, and $F(V_{R_1-1}) \Subset V_0$.

For $p_0 \in V_0$, let

$$E_{p_0}^{v,1} := DF^{-R_1}(E_{p_{R_1}}^{gh}).$$

By Lemma 4.2, we see that $DF^i(E_{p_0}^{v,1})$ is $\beta^{1-\underline{\varepsilon}}$-vertical for $0 \leq i < R_1$. It follows that there is a genuine chart $\Psi : V_0 \to \Psi(V_0)$ that rectifies $E_p^{v,1}$ for $p \in V_0$ to genuine vertical directions such that

$$\|\Psi^{\pm 1} - \mathrm{Id}\|_{C^r} < \beta^{1-\underline{\varepsilon}}.$$

It follows immediately that (F^{R_1}, Ψ) is a $(1, \varepsilon, \beta)$-regular Hénon-like return.

Finally, by Proposition 7.3, $\mathcal{R}_{1\mathrm{D}}(f)$ is a rescaling of f^{R_1} by a uniform constant $\rho \in (0, 1)$ depending only on $\mathbf{b}$ and K. The last inequality now follows from (13.1). $\square$

Let F be the infinitely regular Hénon-like renormalizable diffeomorphism considered in Section 10. For $n \in \mathbb{N}$, denote

$$\hat{F}_n := \mathcal{R}^n(F) \quad \text{and} \quad \hat{f}_n := \Pi_{1\mathrm{D}}(\hat{F}_n).$$

By Theorem 3.6 iv) and Corollary 6.4, there exists a uniform constant $\mathbf{K} \geq 1$ such that $\hat{F}_n \in \mathfrak{H}\mathfrak{L}_{\beta_n}^r(\mathbf{K})$ with $\beta_n = \lambda^{(1-\bar{\varepsilon})R_n}$. By replacing F with $F^{R_{n_0}}|_{\mathcal{B}_0^{n_0}}$ for some sufficiently large $n_0 \in \mathbb{N}$, we may assume that β_n is less than the value β_0 given in

Proposition 13.4. Then $\hat{f}_n$ is valuably renormalizable for $n \geq 0$. For $k \in \mathbb{N} \cup \{\infty\}$, define the *k-renormalization type of* $\hat{F}_n$ as

$$\tau_k(\hat{F}_n) := [\tau(\hat{f}_n), \tau(\hat{f}_{n+1}), \dots, \tau(\hat{f}_{n+k-1})].$$

Proposition 13.5 (Shadowing Lemma). *For $N \in \mathbb{N}$, there exists $n_1 = n_1(N) \in \mathbb{N}$ such that for all $n \geq n_1$, the map $\hat{f}_n$ is N-times valuably renormalizable with $\tau_N(\hat{f}_n) = \tau_N(\hat{F}_n)$. Moreover, we have*

$$\|f_{n+k} - \mathcal{R}^k_{1\mathrm{D}}(f_n)\|_{C^{r-1}} < C^k \lambda^{(1-\bar{\varepsilon})R_n} \quad \text{for} \quad 1 \leq k \leq N$$

for some uniform constant $C \geq 1$.

Proof. The case $N = 1$ follows from Proposition 13.4. Proceeding inductively, suppose that the result is true for all $1 \leq k < N$. In particular, we have

$$\|f_{n+N-1} - \mathcal{R}^{N-1}_{1\mathrm{D}}(f_n)\|_{C^{r-1}} < C^{N-1}\lambda^{(1-\bar{\varepsilon})R_n}.$$

Choosing $n_1 \leq n$ sufficiently large, it follows from Lemma 13.2 and Proposition 13.4 that f_{n+N-1} and $\mathcal{R}^{N-1}_{1\mathrm{D}}(f_n)$ are both valuably renormalizable, and

$$\tau(f_{n+N-1}) = \tau(\mathcal{R}^{N-1}_{1\mathrm{D}}(f_n)).$$

Hence, f_n is N-times valuably renormalizable, and

$$\tau_N(f_n) = \tau_N(\hat{F}_n).$$

For $m \in \mathbb{N}$, Proposition 13.4 implies that

$$\|f_{n+m} - \mathcal{R}_{1\mathrm{D}}(f_{n+m-1})\|_{C^{r-1}} < \lambda^{(1-\bar{\varepsilon})R_{n+m}}.$$

Applying Lemma 13.2 $0 \leq k < N$ times, we obtain

$$\|\mathcal{R}^k_{1\mathrm{D}}(f_{n+m}) - \mathcal{R}^{k+1}_{1\mathrm{D}}(f_{n+m-1})\|_{C^{r-1}} < C^k\lambda^{(1-\bar{\varepsilon})R_{n+m}}.$$

Thus,

$$\begin{aligned}\|f_{n+N} - \mathcal{R}^N_{1\mathrm{D}}(f_n)\|_{C^{r-1}} &\leq \sum_{k=0}^{N-1} \|\mathcal{R}^k_{1\mathrm{D}}(f_{n+N-k}) - \mathcal{R}^{k+1}_{1\mathrm{D}}(f_{n+N-(k+1)})\|_{C^{r-1}} \\ &< \sum_{k=0}^{N-1} C^k \lambda^{(1-\bar{\varepsilon})R_{n+N-k}} \\ &< O(C^N\lambda^{(1-\bar{\varepsilon})R_n}).\end{aligned}$$

□

Proof of Theorem D. Statements i) and ii) are given by Theorem 3.6. Statement iii) is given by Theorem 10.8.

Suppose $r \geq 4$. Let $f^* \in \mathfrak{A}_{\mathbf{b}}$ so that

$$\mathcal{T}_\infty(f^*) = \tau_\infty(F) := [\tau(\hat{f}_0), \tau(\hat{f}_1), \dots].$$

Denote $f^*_n := \mathcal{R}^n_{1\mathrm{D}}(f^*)$ for $n \geq 0$.

Consider the constants $C \geq 1$ and $\rho \in (0,1)$ given in Lemma 13.3. Choose $N \in \mathbb{N}$ sufficiently large so that $C\rho^N < \tilde{\rho} < 1$. Let $n_1 = n_1(2N) \in \mathbb{N}$ be the number given in Proposition 13.5. Then for all $n \geq n_1$, we have

$$\begin{aligned}\|f_{n+N} - f^*_{n+N}\|_{C^{r-1}} &\leq \|f_{n+N} - \mathcal{R}^N_{1\mathrm{D}}(f_n)\|_{C^{r-1}} + \|\mathcal{R}^N_{1\mathrm{D}}(f_n) - \mathcal{R}^N_{1\mathrm{D}}(f^*_n)\|_{C^{r-1}} \\ &\leq O(\lambda^{(1-\bar{\varepsilon})R_n}) + \tilde{\rho}\|f_n - f^*_n\|_{C^{r-1}} \\ &< \tilde{\rho}'\|f_n - f^*_n\|_{C^{r-1}},\end{aligned}$$

for some uniform constant $\tilde{\rho}' \in (0,1)$. □

Appendix A. Quantitative Pesin Theory

Consider an orientation preserving C^r-diffeomorphism $F : \Omega \to F(\Omega) \Subset \Omega$ satisfying $\|F\|_{C^r} = O(1)$. Let $\lambda, \varepsilon \in (0,1)$. Assume $\bar{\varepsilon} < 1$.

Let $p_0 \in \Omega$ and $E^v_{p_0} \in \mathbb{P}^2_{p_0}$. For $m \in \mathbb{Z}$, decompose the tangent space at p_m as

$$\mathbb{P}^2_{p_m} = (E^v_{p_m})^\perp \oplus E^v_{p_m}.$$

In this decomposition, we have

$$D_{p_m}F =: \begin{bmatrix}\alpha_m & 0 \\ \zeta_m & \beta_m\end{bmatrix},$$

where $\alpha_m, \beta_m > 0$ and $\zeta_m \in \mathbb{R}$.

For some $M, N \in \mathbb{N} \cup \{0, \infty\}$ and $L \geq 1$, suppose for $s \in \{r-1, -r\}$, we have

$$L\lambda^{(1+\varepsilon)n} \leq (\alpha_0 \dots \alpha_{n-1})^s \beta_0 \dots \beta_{n-1} \leq L\lambda^{(1-\varepsilon)n} \quad \text{for} \quad 1 \leq n \leq N,$$

and

$$L\lambda^{(1+\varepsilon)n} \leq (\alpha_{-n} \dots \alpha_{-1})^s \beta_{-n} \dots \beta_{-1} \leq L\lambda^{(1-\varepsilon)n} \quad \text{for} \quad 1 \leq n \leq M.$$

Then we say that p_0 is *(M,N)-times $(L, \varepsilon, \lambda)$-regular along $E^v_{p_0}$*.

Proposition A.1. *For $-M \leq m \leq N$, let $L_{p_m} \geq 1$ be the minimum value such that p_m is $(M+m, N-m)$-times $(L_{p_m}, \varepsilon, \lambda)$-regular along $E^v_{p_m}$. Then*

$$L_{p_m} < \bar{L}\lambda^{-\bar{\varepsilon}|m|}.$$

Theorem A.2. *For $-M \leq m \leq N$, let*

$$l_{p_m} := \bar{L}^{-1}\lambda^{\bar{\varepsilon}|m|} > 0 \quad \textit{and} \quad U_{p_m} := [-l_{p_m}, l_{p_m}] \times [-l_{p_m}, l_{p_m}] \subset \mathbb{R}^2$$

Then there exists a chart

$$\Phi_{p_m} : (\mathcal{U}_{p_m}, p_m) \to (U_{p_m}, 0)$$

such that

$$\|\Phi^{\pm 1}_{p_m}\|_{C^r} = O(\bar{L}\lambda^{-\bar{\varepsilon}|m|}), \quad D\Phi_{p_m}(E^v_{p_m}) = E^{gv}_0,$$

and $\Phi_{p_{n+1}} \circ F|_{\mathcal{U}_{p_m}} \circ \Phi^{-1}_{p_m}$ extends to a globally defined C^r-diffeomorphism

$$F_{p_m} : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$$

satisfying the following properties.

i) We have $\|F_{p_m}^{\pm 1}\|_{C^r} = O(1)$.

ii) The map F_{p_m} *is uniformly* C^1*-close to*

$$D_0 F_{p_m} = A_m = \begin{bmatrix} a_m & 0 \\ 0 & b_m \end{bmatrix},$$

with

$$b_m < \lambda^{1-\bar{\varepsilon}} \quad \textit{and} \quad a_m > \lambda^{\bar{\varepsilon}}.$$

iii) We have

$$F_{p_m}(x, y) = (f_{p_m}(x), e_{p_m}(x, y)) \quad \textit{for} \quad (x, y) \in \mathbb{R}^2,$$

where $f_{p_m} : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ *is a* C^r*-diffeomorphism, and* $e_{p_m} : \mathbb{R}^2 \to \mathbb{R}$ *is a* C^r*-map with* $e_{p_m}(\cdot, 0) \equiv 0$.

The construction in Theorem A.2 is referred to as *a linearization of* F *along the* (M, N)*-orbit of* p_0 *with vertical direction* $E^v_{p_0}$. For $0 \leq n \leq N$, we refer to $\mathcal{U}_{p_m}$, Φ_{p_m} and F_{p_m} as a *regular neighborhood*, a *regular chart* and a *linearized map at* p_m respectively.

Proposition A.3. *For* $-M \leq m \leq N$*, we have*

$$\operatorname{diam}(\mathcal{U}_{p_m}) \asymp \bar{L}^{-1} \lambda^{\bar{\varepsilon}|m|}.$$

Lemma A.4. *Consider the coefficients* $\{a_m, b_m\}_{m=-M}^{N}$ *given in Theorem A.2 ii). Then for all* $0 \leq n \leq N$*:*

$$b_0 \cdot \ldots \cdot b_{n-1} > \bar{L}^{-1} \lambda^{(1+\bar{\varepsilon})n} \quad \textit{and} \quad a_0 \cdot \ldots \cdot a_{n-1} < \bar{L} \lambda^{-\bar{\varepsilon} n}$$

and for all $0 \leq m \leq M$*:*

$$b_{-m} \cdot \ldots \cdot b_{-1} > \bar{L}^{-1} \lambda^{(1+\bar{\varepsilon})n} \quad \textit{and} \quad a_{-m} \cdot \ldots \cdot a_{-1} < \bar{L} \lambda^{-\bar{\varepsilon} n}.$$

For $1 \leq n \leq N - m$, we denote

$$F_{p_m}^n := F_{p_{m+n-1}} \circ \ldots \circ F_{p_{m+1}} \circ F_{p_m}.$$

The following result states that restricted to the regular neighborhoods, iterates of F are nearly linear.

Proposition A.5. *For any constant* $k > 0$*, the values* $\{l_{p_m}\}_{m=-M}^{N}$ *in Theorem A.2 can be chosen sufficiently small so that the following holds. Let* $-M \leq m \leq N$ *and* $-M - m \leq l \leq N - m$*. Suppose that* $q_{m+i} \in \mathcal{U}_{p_{m+i}}$ *for* $i \in [m, m+l] \cap \mathbb{Z}$*. Write* $z_m := \Phi_{p_m}(q_m) \in U_{p_m}$*. Then for all* $v \in \mathbb{R}^2$*, we have*

$$\|D_{z_m} F_{p_m}^l(v) - D_0 F_{p_m}^l(v)\| < k \|D_0 F_{p_m}^l(v)\|$$

and

$$\|D_{q_m} F^l(v) - D_{p_m} F^l(v)\| < k \|D_{p_m} F^l(v)\|.$$

Moreover,

$$1 - k < \frac{\operatorname{Jac}_{z_m} F_{p_m}^l}{\operatorname{Jac}_0 F_{p_m}^l}, \frac{\operatorname{Jac}_{q_m} F^l}{\operatorname{Jac}_{p_m} F^l} < 1 + k.$$

Let $-M \leq m \leq N$. For $q \in \mathcal{U}_{p_m}$, write $z := \Phi_{p_m}(q)$. Denote

$$E_q^{v/h} := D\Phi_{p_m}^{-1}(E_z^{gv/gh}).$$

By the construction of regular charts in Theorem A.2, vertical directions are invariant under F:

$$\text{i.e.} \quad DF(E_q^v) = E_{F(q)}^v \quad \text{for} \quad q \in \mathcal{U}_{p_m}.$$

Note that the same is not true for horizontal directions. However, the following result states that they are still nearly invariant under F.

Proposition A.6. *Let $-M \leq m \leq N$ and $-M - m \leq l \leq N - m$. Suppose that*

$$q_{m+i} \in \mathcal{U}_{p_{m+i}} \quad \textit{for} \quad i \in [m, m+l] \cap \mathbb{Z}.$$

Let

$$\tilde{E}_{q_{m+l}}^h := DF^l(E_{q_m}^h).$$

Write

$$z_m = (x_m, y_m) := \Phi_{p_m}(q_m) \quad \textit{and} \quad \tilde{E}_{z_{m+l}}^h := DF_{p_m}^l(E_{z_m}^{gh}) = D\Phi_{p_{m+l}}(\tilde{E}_{q_{m+l}}^h).$$

Then we have

$$\measuredangle(\tilde{E}_{z_{m+l}}^h, E_{z_{m+l}}^{gh}), \ \ \measuredangle(\tilde{E}_{q_{m+l}}^h, E_{q_{m+l}}^h) < K|y_{m+l}|^{1-\bar{\varepsilon}}$$

for some uniform constant $K > 1$.

For $n \in \mathbb{N}$, denote

$$U_{p_0}^{\bar{\varepsilon}n} := [-\lambda^{\bar{\varepsilon}n} l_{p_0}, \lambda^{\bar{\varepsilon}n} l_{p_0}] \times [-l_{p_0}, l_{p_0}].$$

The *n-times truncated regular neighborhood of* p_0 is defined as

$$\mathcal{U}_{p_0}^{\bar{\varepsilon}n} := \Phi_{p_0}^{-1}\left(U_{p_0}^{\bar{\varepsilon}n}\right) \subset \mathcal{U}_{p_0}. \tag{A.1}$$

Lemma A.7. *For $1 \leq m \leq M$, we have*

$$F^i(\mathcal{U}_{p_{-m}}) \subset \mathcal{U}_{p_{-m+i}} \quad \textit{for} \quad 0 \leq i \leq m.$$

Moreover, for $1 \leq n \leq N$, we have

$$F^i(\mathcal{U}_{p_0}^{\bar{\varepsilon}n}) \subset \mathcal{U}_{p_i} \quad \textit{for} \quad 0 \leq i \leq n.$$

Proposition A.8. *Let $q_0 \in \mathcal{U}_{p_0}$ and $\tilde{E}_{q_0}^v \in \mathbb{P}_{q_0}^2$. Suppose for some $0 < n \leq N$, we have $q_i \in \mathcal{U}_{p_i}$ for $0 \leq i \leq n$. If*

$$\nu := \|DF^n|_{\tilde{E}_{q_0}^v}\| < \bar{L}^{-1}\lambda^{\bar{\varepsilon}n},$$

then

$$\measuredangle(\tilde{E}_{q_0}^v, E_{q_0}^v) < \bar{L}\lambda^{-\bar{\varepsilon}n}\nu + \bar{L}\lambda^{(1-\bar{\varepsilon})n}.$$

Proposition A.9. *Let $q_0 \in \mathcal{U}_{p_0}$ and $\tilde{E}^h_{q_0} \in \mathbb{P}^2_{q_0}$. Suppose for some $0 < m \leq M$, we have $q_{-i} \in \mathcal{U}_{p_{-i}}$ for $0 \leq i \leq m$. If*

$$\mu := \|DF^{-m}|_{\tilde{E}^h_{q_0}}\| < \bar{L}^{-1}\lambda^{-(1-\bar{\varepsilon})m},$$

then

$$\measuredangle(\tilde{E}^h_{q_0}, E^h_{q_0}) < \bar{L}\lambda^{(1-\bar{\varepsilon})m}(1+\mu).$$

Let

$$\mathcal{E} : \mathcal{D} \to T^1\mathcal{D}$$

be a unit vector field on $\mathcal{D} \subset \Omega$. Define

$$DF_*(\mathcal{E})(p) := \frac{DF(\mathcal{E}(p))}{\|DF(\mathcal{E}(p))\|} \in T^1_{F(p)}F(\mathcal{D}) \quad \text{for} \quad p \in \mathcal{D}.$$

Let

$$\Psi : \mathcal{B} \to B$$

be a chart with $\mathcal{D} \subset \mathcal{B}$. For $t \geq 0$, we say that $\mathcal{E}$ is *t-vertical in $\mathcal{B}$* if

$$\frac{\measuredangle(D\Psi(\mathcal{E}(p)), E^{gv}_{\Psi(p)})}{\measuredangle(D\Psi(\mathcal{E}(p)), E^{gh}_{\Psi(p)})} \leq t \quad \text{for} \quad p \in \mathcal{D}.$$

For $-N \leq m \leq N$, define $\mathcal{E}^v_{p_m} : \mathcal{U}_{p_m} \to T^1(\mathcal{U}_{p_m})$ to be a C^{r-1}-unit vector field given by

$$\mathcal{E}^v_{p_m}(q) \in E^v_q \quad \text{for} \quad q \in \mathcal{U}_{p_m}.$$

Proposition A.10. *Let $\mathcal{D}_0 \subset \mathcal{U}_{p_0}$ and $0 \leq n \leq N$. Suppose*

$$\mathcal{D}_i := F^i(\mathcal{D}_0) \subset \mathcal{U}_{p_i} \quad \textit{for} \quad 0 \leq i \leq n.$$

Let $\mathcal{E} : \mathcal{D}_n \to T^1(\mathcal{D}_n)$ be a C^{r-1}-unit vector field. If $\mathcal{E}$ is t-vertical in $\mathcal{U}_{p_n}$ for some $t \geq 0$, then we have

$$\|DF^{-n}_*(\mathcal{E}) - \mathcal{E}^v_{p_0}|_{\mathcal{D}_0}\|_{C^{r-1}} \leq (1+t^2)\|\mathcal{E}\|_{C^{r-1}}\bar{L}\lambda^{(1-\bar{\varepsilon})n}.$$

Proposition A.11. *There exists a uniform constant $\delta_0 > 0$ depending only on $\|F\|_{C^r}$ such that the following holds. Let $\tilde{F} : \tilde{\Omega} \to \tilde{F}(\tilde{\Omega})$ be a C^r-diffeomorphism such that*

$$\|\tilde{F} - F\|_{C^r} = \delta \leq \delta_0.$$

Moreover, suppose that p_0 is also N-times forward $(L, \varepsilon, \lambda)$-regular along $E^v_{p_0}$ under $\tilde{F}$. Let $\mathcal{E} : \mathcal{D}_n \to T^1(\mathcal{D}_n)$ be a t-vertical unit vector field considered in Proposition A.10 with $t \leq \bar{L}\lambda^{-\bar{\varepsilon}n}$. Then we have

$$\|DF^{-n}_*(\mathcal{E}) - D\tilde{F}^{-n}_*(\mathcal{E})\|_{C^{r-1}} \leq \|\mathcal{E}\|_{C^{r-1}}\bar{L}\lambda^{(1-\bar{\varepsilon})}\delta.$$

If $N = \infty$, then Proposition A.8 implies that $E^v_{p_0}$ is the unique direction along which p_0 is infinitely forward $(L, \varepsilon, \lambda)$-regular. In this case, we denote $E^{ss}_{p_0} := E^v_{p_0}$, and refer to this direction as the *strong stable direction at p_0*. Moreover, we define the *local strong stable manifold at p_0* as

$$W^{ss}_{\text{loc}}(p_0) := \Phi^{-1}_{p_0}(\{(0, y) \in U_{p_0}\}),$$

and the *strong stable manifold at* p_0 as

$$W^{ss}(p_0) := \{q \in \Omega \mid F^n(q) \in W^{ss}_{\mathrm{loc}}(p_m) \text{ for some } n \geq 0\}.$$

If $M = \infty$, we denote $E^c_{p_0} := E^h_{p_0}$, and refer to this direction as the *center direction at* p_0. Moreover, we define the *(local) center manifold at* p_0 as

$$W^c(p_0) := \Phi^{-1}_{p_0}(\{(x, 0) \in U_{p_0}\}).$$

Unlike stable manifolds, the center manifold at an infinitely backward regular point is not unique. However, the following result states that it still has a canonical jet.

Proposition A.12. *Suppose $M = \infty$. Let*

$$\Gamma_0 : (-l, l) \to \mathcal{U}_{p_0}$$

be a C^r-curve parameterized by its arclength such that $\Gamma_0(0) = p_0$, and for all $n \in \mathbb{N}$, we have

$$\|DF^{-n}|_{\Gamma_0'(t)}\| < \lambda^{-(1-\bar{\varepsilon})n} \quad \textit{for} \quad |t| < \lambda^{\varepsilon n}.$$

Then Γ_0 has a degree r tangency with $W^c(p_0)$ at p_0.

We say that p is *N-times forward horizontally (L, ε)-regular along $E^{h,+}_p \in \mathbb{P}^2_p$* if for $s \in \{-r+1, r\}$, we have

$$L^{-1}\lambda^{(1+\varepsilon)n} \leq \frac{\operatorname{Jac}_p F^n}{\|D_pF^n|_{E^{h,+}_p}\|^{s+1}} \leq L\lambda^{(1-\varepsilon)n} \quad \text{for} \quad 1 \leq n \leq N. \tag{A.2}$$

Similarly, we say that p is *M-times backward horizontally (L, ε)-regular along $E^{h,-}_p \in \mathbb{P}^2_p$* if for $s \in \{-r+1, r\}$, we have

$$L^{-1}\lambda^{-(1-\varepsilon)n} \leq \frac{\operatorname{Jac}_p F^{-n}}{\|D_pF^{-n}|_{E^{h,-}_p}\|^{s+1}} \leq L\lambda^{-(1+\varepsilon)n} \quad \text{for} \quad 1 \leq n \leq M. \tag{A.3}$$

If both (A.2) and (A.3) hold with $E^h_p := E^{h,+}_p = E^{h,-}_p$, then p is *(M, N)-times horizontally (L, ε)-regular along E^h_p.*

Proposition A.13 (Vertical forward regularity = horizontal forward regularity)**.** *If p is N-times forward horizontally (L, ε)-regular along $E^h_p \in \mathbb{P}^2_p$, then there exists $E^v_p \in \mathbb{P}^2_p$ such that p is N-times forward $(\bar{L}, \bar{\varepsilon})$-regular along E^v_p.*

Proposition A.14 (Horizontal backward regularity = vertical backward regularity)**.** *Suppose p is M-times backward horizontally (L, ε)-regular along $E^h_p \in \mathbb{P}^2_p$. Let $E^v_p \in \mathbb{P}^2_p \setminus \{E^h_p\}$. If $\measuredangle(E^h_p, E^v_p) > \theta$, then the point p is M-times backward $(\bar{L}/\theta^2, \varepsilon)$-regular along E^v_p.*

Appendix B. Distortion Theorems for 1D Maps

Let $f : I \to f(I)$ be a C^1-diffeomorphism on an interval $I \subset \mathbb{R}$. For $J \subset I$, the *distortion of f on J* is defined as

$$\mathrm{Dis}(f, J) := \sup_{x,y \in J} \frac{|f'(x)|}{|f'(y)|}.$$

We denote $\mathrm{Dis}(f) := \mathrm{Dis}(f, I)$. For $K \geq 1$, we say that f has *K-bounded distortion on J* if

$$\mathrm{Dis}(f, J) \leq K.$$

Clearly, if $g : I' \to g(I')$ is another C^1-diffeomorphism on an interval $I' \supset f(J)$, then we have

$$\mathrm{Dis}(g \circ f, J) \leq \mathrm{Dis}(g, f(J)) \cdot \mathrm{Dis}(f, J). \tag{B.1}$$

Theorem B.1 (Denjoy Lemma)**.** *Let $f : I \to I$ be a C^r-map on an interval $I \subset \mathbb{R}$. Then there exists a uniform constant $K > 0$ such that if $f^n|_J$ is a diffeomorphism on a subinterval $J \subset I$ for some $n \in \mathbb{N}$, then*

$$\log(\mathrm{Dis}(f^n, J)) \leq K \sum_{i=0}^{n-1} |f(J)|.$$

B.1. **Cross Ratios.** Let $J \Subset I \subset \mathbb{R}$ be bounded open intervals. The complement $I \setminus \overline{J}$ consists of two intervals L and R. The *cross-ratio of J in I* is given by

$$\mathrm{Cr}(I, J) := \frac{|I||J|}{|L||R|}.$$

For $\tau > 0$, we say that I *contains a τ-scaled neighborhood of J* if

$$|L|, |R| > \tau|J|.$$

Let $f : I \to f(I)$ be a homeomorphism. The *cross-ratio distortion under f of J in I* is given by

$$\mathrm{CrD}(f, I, J) := \frac{\mathrm{Cr}(f(I), f(J))}{\mathrm{Cr}(I, J)}.$$

Clearly, if $g : f(I) \to g \circ f(I)$ is another homeomorphism, then

$$\mathrm{CrD}(g \circ f, I, J) = \mathrm{CrD}(g, f(I), f(J)) \cdot \mathrm{CrD}(f, I, J). \tag{B.2}$$

For $\nu > 0$, we say that f has *ν-bounded cross-ratio distortion on I* if

$$\mathrm{CrD}(f, I', J) > \nu$$

for all bounded open intervals $J \Subset I' \subset I$.

Lemma B.2. *For $\alpha > 1$, let $P_\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ be an α-power map such that*

$$P_\alpha(x) = x^\alpha \quad \textit{for} \quad x \in \mathbb{R}^+.$$

Then $P_\alpha|_{\mathbb{R}^+}$ has negative Schwarzian derivative. Consequently, $P_\alpha|_{\mathbb{R}^+}$ has 1-bounded cross-ratio distortion on $\mathbb{R}_+$.

Lemma B.3. *Let $I \subset \mathbb{R}$ be a bounded open interval, and let $f : I \to f(I)$ be a C^1-diffeomorphism with K-bounded distortion on I for some $K > 0$. Then there exists a uniform constant $\nu = \nu(K) > 0$ such that f has ν-bounded cross-ratio distortion on I.*

Theorem B.4 (Koebe distortion theorem)**.** *Let $J \Subset I \subset \mathbb{R}$ be bounded open intervals, and let $f : I \to f(I)$ be a C^1-diffeomorphism with ν-bounded cross-ratio distortion on I for some $\nu > 0$. If $f(I)$ contains a τ-scaled neighborhood of $f(J)$, then there exists a uniform constant $K = K(\nu, \tau) > 0$ depending only on ν and τ such that f has K-bounded distortion on J.*

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