Bounds and optimal codes in the Lee metric

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joint work with Eimear Byrne

Motivation

- From an information theoretic perspective:
  The Lee metric is best suited for channels, where the error $+x, -x$ are equally likely and the magnitude matters.

  C. Lee "Some properties of nonbinary error-correcting codes", IIRE Transactions on Information Theory, 1958.

- From an algebraic perspective:
  Some excellent but non-linear binary codes can be represented as linear codes over $\mathbb{Z}/4\mathbb{Z}$ endowed with the Lee metric.

  A. Roger Hammons, P. Vijay Kumar, A. Robert Calderbank, Neil J.A. Sloane and Patrick Solé "The $\mathbb{Z}_4$-linearity of Kerdock, Preparata, Goethals, and related codes", IEEE Transactions on Information Theory, 1994.

- From a cryptographic perspective:
  The Lee metric promises lower key sizes/signature sizes, since one can insert more errors.

Outline

1. Preliminaries
   - Ring-Linear Coding Theory
   - Lee Metric
2. Singleton Bound in the Lee Metric
   - Maximum Lee-Distance Codes
3. Plotkin Bound in the Lee Metric
   - Constant Lee-Weight Codes
4. Open Problems

Ring-Linear Coding Theory

	Classical	$\mathbb{Z}/p^s\mathbb{Z}$-Linear
Ambient space	Finite field $\mathbb{F}_q$	
Linear code	$\mathcal{C} \subseteq \mathbb{F}_q^n$ linear subspace	
Parameters	length n dimension k	

Ring-Linear Coding Theory

	Classical	$\mathbb{Z}/p^s\mathbb{Z}$-Linear
Ambient space	Finite field $\mathbb{F}_q$	Integer residue ring $\mathbb{Z}/p^s\mathbb{Z}$
Linear code	$\mathcal{C} \subseteq \mathbb{F}_q^n$ linear subspace	$\mathcal{C} \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n$ $\mathbb{Z}/p^s\mathbb{Z}$-submodule
Parameters	length n dimension k	length n ?

Let $\mathcal{C} \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n$ be a code, then

$$\mathcal{C} \cong (\mathbb{Z}/p^s\mathbb{Z})^{k_1} \times \left(\mathbb{Z}/p^{s-1}\mathbb{Z}\right)^{k_2} \times \cdots \times (\mathbb{Z}/p\mathbb{Z})^{k_s} .$$

Then we say $\mathcal{C}$ has

- **subtype** $(k_1, \ldots, k_s)$,
- **type** $k = \sum_{i=1}^{s} \frac{s-i+1}{s} k_i = \log_{p^s}(| \mathcal{C} |)$,
- **rate** $R = k/n$,
- **rank** $K = \sum_{i=1}^{s} k_i$,
- **free rank** k_1.

$$0 \leq k_1 \leq k \leq K \leq n.$$

If $k_1 = k = K$, we say that $\mathcal{C}$ is a **free code**.

Systematic Form

If $\mathcal{C}$ has subtype $(k_1, \ldots, k_s)$ and rank K then

$$G = \begin{pmatrix} \mathrm{Id}_{k_1} & * & \cdots & * & * \\ 0 & p\mathrm{Id}_{k_2} & \cdots & p* & p* \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & p^{s-1}\mathrm{Id}_{k_s} & p^{s-1}* \end{pmatrix} \in (\mathbb{Z}/p^s\mathbb{Z})^{K \times n}.$$

If $\mathcal{C}$ is a free code, then

$$G = \begin{pmatrix} \mathrm{Id}_k & A \end{pmatrix} \in (\mathbb{Z}/p^s\mathbb{Z})^{k \times n}.$$

Lee Metric

Definition (Lee Metric)

$$\begin{array}{rcrcl} x \in \mathbb{Z}/p^s\mathbb{Z} & : & \mathrm{wt}_L(x) & = & \min\{x, \mid p^s - x \mid\}, \\ x \in (\mathbb{Z}/p^s\mathbb{Z})^n & : & \mathrm{wt}_L(x) & = & \sum_{i=1}^n \mathrm{wt}_L(x_i), \\ x, y \in (\mathbb{Z}/p^s\mathbb{Z})^n & : & d_L(x, y) & = & \mathrm{wt}_L(x - y). \end{array}$$

Maximal Lee weight: $M = \left\lfloor \frac{p^s}{2} \right\rfloor$.

Connection to Hamming metric:

$$0 \leq \mathrm{wt}_H(x) \leq \mathrm{wt}_L(x) \leq M\mathrm{wt}_H(x) \leq Mn.$$

For a linear code $\mathcal{C} \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n$ its **minimum Lee distance** is given by

$$d_L(\mathcal{C}) = \min\{\mathrm{wt}_L(x) \mid x \in \mathcal{C}, x \neq 0\}.$$

Techniques

Let $\mathcal{C} \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n$ be a code of subtype $(k_1, \ldots, k_s)$ and rank K.

- Define the subcodes $\mathcal{C}_i = \mathcal{C} \cap \langle p^{s-1-i} \rangle$ for $i \in \{0, \ldots, s-1\}$.
- We have a sequence of subcodes $\mathcal{C}_0 \subseteq \mathcal{C}_1 \subseteq \cdots \subseteq \mathcal{C}_{s-1} = \mathcal{C}$.
- The socle $\mathcal{C}_0 = \mathcal{C} \cap \langle p^{s-1} \rangle$ can be seen as

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$$\{xG \mid x \in p^{s-1} (\mathbb{Z}/p^s\mathbb{Z})^{k_1} \times p^{s-2} (\mathbb{Z}/p^s\mathbb{Z})^{k_2} \times \cdots \times (\mathbb{Z}/p^s\mathbb{Z})^{k_s}\}$$

$$\begin{pmatrix} p^{s-1}\star \\ p^{s-2}\star \\ \vdots \\ \star \end{pmatrix}^\top \begin{pmatrix} \mathrm{Id}_{k_1} & & & & \star \\ 0 & p\mathrm{Id}_{k_2} & & & p\star \\ \vdots & & \ddots & & \\ 0 & \cdots & & p^{s-1}\mathrm{Id}_{k_s} & p^{s-1}\star \end{pmatrix}$$

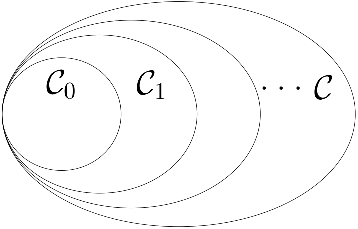
$$| \mathcal{C}_0 | = p^{k_1+k_2+\cdots+k_s} = p^K.$$

Techniques

1. Use the socle:
   $\mathcal{C}_0 = \mathcal{C} \cap \langle p^{s-1} \rangle$ can be identified with a $[n, K]$ linear code over $\mathbb{F}_p$.
2. Use the Hamming metric:

$$d_H(\mathcal{C}) \leq d_L(\mathcal{C}) \leq M d_H(\mathcal{C}).$$

We always find minimum Hamming weight codewords in the socle.



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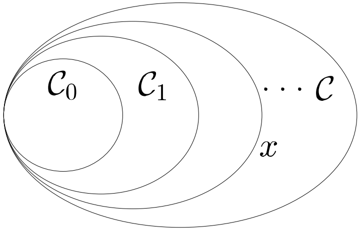

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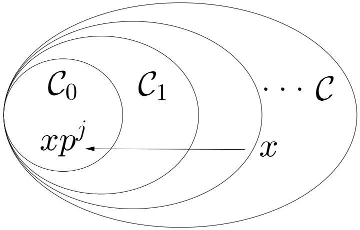

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Where do the minimum Lee weight codewords live?

Example

$\langle (1,4,5),(0,3,6) \rangle \subset \mathbb{Z}/9\mathbb{Z}^3$ has all minimum Lee weight codewords outside the socle.

Example

$\langle (1,2,3),(0,3,0) \rangle \subset \mathbb{Z}/9\mathbb{Z}^3$ has all minimum Lee weight codewords in the socle.

General Singleton Bound

For any finite ring R of size r and additive weight wt with maximum weight $M = \max\{\mathrm{wt}(x) \mid x \in R\}$.

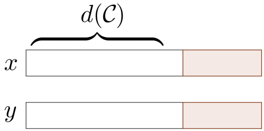
Theorem (Singleton Bound)

A code $\mathcal{C} \subseteq R^n$ of size r^k has minimum distance

$$\left\lfloor \frac{d(\mathcal{C}) - 1}{M} \right\rfloor \leq n - k.$$

Proof Idea

- Puncture $\mathcal{C}$ in $\left\lfloor \frac{d(\mathcal{C})-1}{M} \right\rfloor$ positions to get $\mathcal{C}'$.
- Any two codewords of $\mathcal{C}'$ are still distinct: $\mid \mathcal{C}' \mid = r^k$.
- Since $\mathcal{C}' \subseteq R^{n-\left\lfloor \frac{d(\mathcal{C})-1}{M} \right\rfloor}$, we have $k \leq n - \left\lfloor \frac{d(\mathcal{C})-1}{M} \right\rfloor$.



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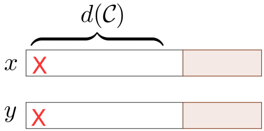

MDS Codes

For $(\mathbb{F}_q, d_H)$: set $M = 1$, we get the classical Singleton bound

$$d_H(\mathcal{C}) \leq n - k + 1.$$

Codes that achieve the Singleton bound are called

maximum distance separable (MDS) codes.

- For $n \leq q + 1$ we have a construction of MDS codes: (extended) RS codes.
- For $q \to \infty$ MDS codes have density 1.
- For $n \to \infty$ MDS codes have density 0 (assuming the MDS conjecture).
- Dual of MDS codes are also MDS codes.
- Binary MDS codes are trivial, that is $k \in \{1, n, n-1\}$.

Singleton Bound in the Lee Metric

For $(\mathbb{Z}/p^s\mathbb{Z}, d_L)$: set $M = \lfloor p^s/2 \rfloor$, we get:

Theorem (Shiromoto)

For any code $\mathcal{C} \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n$ of type k, we have that

$$\left\lfloor \frac{d_L(\mathcal{C}) - 1}{M} \right\rfloor \leq n - k.$$

Keisuke Shiromoto "Singleton bounds over finite rings.", Journal of Algebraic Combinatorics, 2000.

Codes attaining this bound are called

maximum Lee distance (MLD) codes.

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Example

$\mathcal{C} = \langle (1,2) \rangle \subset \mathbb{Z}/5\mathbb{Z}^2$ with $d_L = 3$ is MLD:

$$\left\lfloor \frac{3-1}{2} \right\rfloor = 2 - 1.$$

Singleton Bound in the Lee Metric

Theorem (Byrne, W.)

The only non-trivial linear codes that attain the Lee-metric Singleton bound are equivalent to $\mathcal{C} = \langle (1,2) \rangle \subseteq (\mathbb{Z}/5\mathbb{Z})^2$.

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- MLD codes have density 0 for $p \to \infty$.
- MLD codes have density 0 for $n \to \infty$.

Is the dual of an MLD code also an MLD code?

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- MLD codes have density 0 for $p \to \infty$.
- MLD codes have density 0 for $n \to \infty$.

Is the dual of an MLD code also an MLD code?

Yes

Since $\mathcal{C} = \langle (1,2) \rangle \subseteq (\mathbb{Z}/5\mathbb{Z})^2$ is self-dual.

Singleton Bound in the Lee Metric

Theorem (Alderson-Huntemann)

Let $\mathcal{C} \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n$ be a linear code of type $1 < k < n$ a positive integer, then

$$d_L(\mathcal{C}) \leq M(n-k).$$

Tim L. Alderson and Svenja Huntemann "On maximum Lee distance codes.", Journal of Discrete Mathematics, 2013.

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Characterization

- for $p = 2$: only for $s = 2$, or $s = 3$ and $k \in \{n-2, n-1\}$.
- for p odd: only for $p^s \in \{5, 7, 9\}$ and $k + 1 \leq n \leq k + 3$.

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Density

- For $p \to \infty$ MLD codes have density 0.
- For $n \to \infty$ MLD codes have density 0.

General Plotkin Bound

For any finite ring R and additive weight wt.

Theorem

Let $\mathcal{C} \subseteq R^n$, then

$$d(\mathcal{C}) \leq \frac{|\mathcal{C}|}{|\mathcal{C}|-1} n\overline{wt}(R).$$

Proof

- $$d(\mathcal{C})(|\mathcal{C}|-1) \leq \sum_{c \in \mathcal{C}} \mathrm{wt}(c).$$
- Define the average weight of a code
$$\overline{\mathrm{wt}}(\mathcal{C}) = \frac{1}{|\mathcal{C}|} \sum_{c \in \mathcal{C}} \mathrm{wt}(c).$$
- Note that
$$\overline{\mathrm{wt}}(\mathcal{C}) \leq n\overline{\mathrm{wt}}(R).$$

Classical Plotkin Bound

For $(\mathbb{F}_q, d_H)$: set $\overline{\mathrm{wt}}_H(\mathbb{F}_q) = \frac{q-1}{q} \Rightarrow$ classical Plotkin bound

$$d_H(\mathcal{C}) \leq \frac{\mid \mathcal{C} \mid}{\mid \mathcal{C} \mid -1} \frac{q-1}{q} n = \frac{q^{k-1}}{q^k - 1}(q-1)n.$$

Linear codes that achieve the Plotkin bound are called

constant Hamming-weight codes.

- The simplex code of length $n = \frac{q^m - 1}{q-1}$, dimension m is defined through a generator matrix G, which has one representative of each 1-dimensional subspace $\langle x \rangle \subseteq \mathbb{F}_q^m$ as column.

Example

Let $q = 3, m = 2$ and thus $n = 4$. $G = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}$.

- Any constant Hamming-weight code is an ℓ-fold duplicate of simplex codes.

Plotkin Bound in the Lee Metric

For $(\mathbb{Z}/p^s\mathbb{Z}, d_L)$: set

$$\overline{\mathrm{wt}}_L(\mathbb{Z}/p^s\mathbb{Z}) = \begin{cases} \frac{p^{2s}-1}{4p^s} & \text{if } p \text{ is odd,} \\ 2^{s-2} & \text{if } p = 2. \end{cases}$$

Theorem (Wyner and Graham)

For any code $\mathcal{C} \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n$ of type k we have that

$$d_L(\mathcal{C}) \leq \frac{n\overline{wt}_L(\mathbb{Z}/p^s\mathbb{Z})}{1 - 1/p^{sk}}.$$

Since

$$\frac{1}{1-1/p^{sk}} = \frac{\mid \mathcal{C} \mid}{\mid \mathcal{C} \mid -1}.$$

Aaron D. Wyner and Ronald L. Graham "An upper bound on minimum distance for a k-ary code.", Inf. Control., 1968.

Plotkin Bound in the Lee Metric

Theorem (Chiang and Wolf (adapted))

For a linear code $\mathcal{C} \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n$ of free rank $k_1 > 0$ we have that

$$d_L(\mathcal{C}) \leq \frac{(n-k_1+1)\overline{wt}_L(\mathbb{Z}/p^s\mathbb{Z})}{1-1/p^s}.$$

J. Chung-Yaw Chiang and Jack K. Wolf "On channels and codes for the Lee metric", Information and Control, 1971.

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$$d_L(\mathcal{C}) \leq \frac{(n-k_1+1)\overline{\mathrm{wt}}_L(\mathbb{Z}/p^s\mathbb{Z})}{1-1/p^s}.$$

Proof

For any subcode $\mathcal{C}' \subseteq \mathcal{C}$

$$d_L(\mathcal{C}) \leq d_L(\mathcal{C}') \leq \frac{\mid \mathcal{C}' \mid}{\mid \mathcal{C}' \mid -1}\overline{\mathrm{wt}}_L(\mathcal{C}').$$

$$d_L(\mathcal{C}) \leq \frac{(n-k_1+1)\overline{\mathrm{wt}}_L(\mathbb{Z}/p^s\mathbb{Z})}{1-1/p^s}.$$

Proof

For any $c \in \mathcal{C}$

$$d_L(\mathcal{C}) \leq d_L(\langle c \rangle) \leq \frac{|\ \langle c \rangle\ |}{|\ \langle c \rangle\ | -1}\overline{\mathrm{wt}}_L(\langle c \rangle).$$

Plotkin Bound in the Lee Metric

$$d_L(\mathcal{C}) \leq \frac{(n-k_1+1)\overline{\mathrm{wt}}_L(\mathbb{Z}/p^s\mathbb{Z})}{1-1/p^s}.$$

Proof

For any $c \in \mathcal{C}$ in the free part

$$d_L(\mathcal{C}) \leq d_L(\langle c \rangle) \leq \frac{1}{1-1/p^s}\mathrm{wt}_H(c)\overline{\mathrm{wt}}_L(\mathbb{Z}/p^s\mathbb{Z}).$$

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- Let G be a $K \times n$ generator matrix for the code $\mathcal{C}$.

$$G = \begin{pmatrix} \mathrm{Id}_{k_1} & A \\ 0 & pB \end{pmatrix}.$$

Plotkin Bound in the Lee Metric

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- Let G' be the $k_1 \times n$ generator matrix for the free part $\mathcal{C}' \subseteq \mathcal{C}$.

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- Let G' be the $k_1 \times n$ generator matrix for the free part $\mathcal{C}' \subseteq \mathcal{C}$.
- $c \in \mathcal{C}'$ with $\mathrm{wt}_H(c) \leq n-k_1+1$.

$$G' = \begin{pmatrix} \mathrm{Id}_{k_1} & A \end{pmatrix}.$$

Plotkin Bound in the Lee Metric

$$d_L(\mathcal{C}) \leq \frac{|\langle c \rangle|}{|\langle c \rangle| - 1} \overline{\mathrm{wt}}_L(\langle c \rangle),$$

for a minimum Hamming weight codeword c.

- If we can take c in the free part: we get the Chiang and Wolf bound with k_1.
- If $c \in \langle p^{s-\ell} \rangle$: how do we bound $\overline{\mathrm{wt}}_L(\langle c \rangle)$?

We introduce the support subtype

- For $j \in \{1, \ldots, n\}$ let π_j be the j-th coordinate map.
- Define

$$n_i(\mathcal{C}) := |\{j \in \{1, \ldots, n\} \;\mid\; \langle \pi_j(\mathcal{C}) \rangle = \langle p^i \rangle\}|.$$

- Linear code $\mathcal{C} \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n$ has **support subtype** $(n_0, \ldots, n_s)$.

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Example

Let $\mathcal{C}$ be the code over $\mathbb{Z}/8\mathbb{Z}$ generated by

$$G = \begin{pmatrix} 1 & 3 & 5 & 0 & 2 \\ 0 & 2 & 4 & 2 & 6 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 4 \end{pmatrix}$$

then $\mathcal{C}$ has subtype $(1, 1, 2)$ and support subtype $(3, 2, 0, 0)$.

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Lemma (Byrne, W.)

Let $\mathcal{C} \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n$ be a linear code of support subtype $(n_0, \ldots, n_s)$. Then

$$\overline{wt}_L(\mathcal{C}) = \begin{cases} \frac{1}{4p^s}\left(p^{2s}|n - n_s| - \sum_{i=0}^{s-1} p^{2i} n_i\right) & \text{if } p \text{ is odd,} \\ 2^{s-2}|n - n_s| & \text{if } p = 2. \end{cases}$$

Plotkin Bound in the Lee Metric

Theorem (Byrne, W.)

Let $\mathcal{C} \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n$ be linear code. Let $\ell \in \{1, \ldots, s\}$ be maximal such that there exists $y \in \mathcal{C}$ satisfying $wt_H(y) = d_H(\mathcal{C})$ and $y \in \langle p^{s-\ell} \rangle$. Then

$$d_L(\mathcal{C}) \leq \begin{cases} \dfrac{p^{s-\ell}(p^\ell + 1)}{4} d_H(\mathcal{C}) & \text{if } p \text{ is odd,} \\[2ex] \dfrac{2^{s-2+\ell}}{2^\ell - 1} d_H(\mathcal{C}) & \text{if } p = 2. \end{cases}$$

We can always choose $\ell = 1$ (there is always a minimal Hamming weight codeword in the socle)

Corollary (Byrne, W.)

Let $\mathcal{C} \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n$ be a linear code of rank K. Then

$$\left\lfloor \frac{d_L(\mathcal{C}) - 1}{A} \right\rfloor \leq n - K,$$

for

$$A := \begin{cases} \frac{p^{s-1}(p+1)}{4} & \text{if } p \text{ is odd}, \\ 2^{s-1} & \text{if } p = 2. \end{cases}$$

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Example

We consider the code $\mathcal{C} = \langle (0,1,1), (2,0,0), (0,0,2) \rangle \subset (\mathbb{Z}/4\mathbb{Z})^3$. This code attains the new bound for $\ell = 1$ since

$$d_L = 2 = 2(n - K + 1).$$

It does not attain the bound of Chiang and Wolf with k_1, as

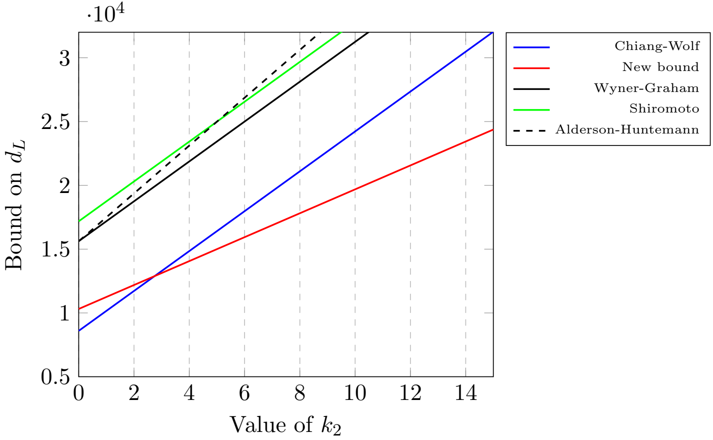
$$d_L \leq \frac{4}{3}(3 - 1 + 1) = 4.$$

We also note that we cannot choose $\ell = 2$, since the only codewords that have minimal Hamming weight are divisible by 2. In fact:

$$d_L = 2 \not\leq \frac{4}{3} = \frac{4}{3}(3 - 3 + 1).$$

Comparison of Bound

Comparison of bounds for codes over $\mathbb{Z}/5^5\mathbb{Z}$ of type $(10, k_2, 0, 0, 0)$ and length $2K, K = 10 + k_2$.



Density

Note that in order to meet the new bound with $\ell = 1$, we need:

1. The socle $\mathcal{C}_0 = \mathcal{C} \cap \langle p^{s-1} \rangle$ is an MDS code, we can identify it with a $[n, K]$ linear code over $\mathbb{F}_p$.
2. There is an $x \in \mathcal{C}_0$ which generates a constant Lee-weight code.

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$$\Downarrow$$

1. Due to the MDS conjecture: density is 0 if $n \to \infty$.
2. Due to the characterization of constant Lee-weight codes of Wood: x consists of repetitions of $(\pm 1, \ldots, \pm \frac{p-1}{2})$, hence the density is 0 if $p \to \infty$.

Jay Wood "The structure of linear codes of constant weight", Transactions of the American Mathematical Society, 2002.

Assume $K = 1$.

Which cyclic modules are constant Hamming-weight over $\mathbb{Z}/p^s\mathbb{Z}$?

- If $s = 1$: any cyclic module is constant Hamming-weight.
- If $s > 1$: any cyclic module with support subtype $(0, \ldots, 0, n_i, 0, \ldots, 0, n_s)$ is constant Hamming-weight.

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Example

- $\langle (1,4) \rangle \subseteq \mathbb{Z}/5\mathbb{Z}^2$ is a constant Hamming-weight code.
- $\langle (1,0,3) \rangle \subseteq \mathbb{Z}/4\mathbb{Z}^3$ is a constant Hamming-weight code.

Constant Lee-Weight Codes

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- $\langle (1,0,3) \rangle \subseteq \mathbb{Z}/4\mathbb{Z}^3$ is a constant Hamming-weight code. Not constant Lee-weight $\mathrm{wt}_L(2,0,2) = 4, \mathrm{wt}_L(1,0,3) = 2$.

Constant Lee-Weight Codes

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Theorem

Any constant Lee-weight code is equivalent to an ℓ-fold duplicate of shortest length constant Lee-weight codes.

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Let U be the collection of orbits of $(\mathbb{Z}/p^s\mathbb{Z})^K$ under the action of $\{1, -1\}$.

1. If $s = 1$: a representative of each member of U appears as a column of a generator matrix with the same multiplicity.

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Example

$\mathcal{C} = \langle(1, 2)\rangle \subseteq \mathbb{Z}/5\mathbb{Z}^2$ is a constant Lee-weight code.

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2. If $p = 2$: every non-zero element of $(\mathbb{Z}/2^s\mathbb{Z})^K$ appears as a column of the generator matrix with the same multiplicity.

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$\mathcal{C} = \langle (1, 2, 3) \rangle \subseteq \mathbb{Z}/4\mathbb{Z}^3$ is a constant Lee-weight code.

3. We have $K \leq 2$.

Constant Lee-Weight Codes

Theorem (Byrne, W.)

Let $\mathcal{C}$ be a shortest-length constant Lee-weight code over $\mathbb{Z}/p^s\mathbb{Z}$ of rank $K = 1$ and weight w. Let i be such that $k_i = 1$. Then $\mathcal{C}$ has support subtype $(0, \dots, 0, n_{i-1}, n_i, \dots, n_{s-1}, 0)$ with

$$w = \frac{p+1}{4} p^{s-1} n_{i-1},$$
$$n_{i-1}(p-1) = p^{j-i+2} n_j \ \forall j \in \{1, \dots, s\}.$$

Proof idea:

Use the exact average weight, i.e.,

$$(\mid \mathcal{C} \mid -1)\overline{\mathrm{wt}}_L(\mathcal{C}) = \frac{\mid \mathcal{C} \mid}{4p^s} \sum_{i=0}^{s-1} n_i \left(p^{2s} - p^{2i}\right)$$

inductively on the subcodes $\mathcal{C}_{j-i+1} = \mathcal{C} \cap \langle p^{j-i+2} \rangle$ of size p^{j-i+2} and support subtype $(0, \dots, 0, n_{i-1}, \dots, n_j, z)$.

Constant Lee-Weight Codes

Theorem (Byrne, W.)

Let $g \in \langle p^{i-1} \rangle$ consist of

- *p repetitions of all elements in $\langle p^{i-1} \rangle \setminus \langle p^i \rangle$ up to ± 1 and*
- *$p-1$ repetitions of all elements in $\langle p^j \rangle \setminus \langle p^{j+1} \rangle$ up to ± 1 for all $j \in \{i, \dots, s-1\}$,*

then $\langle g \rangle$ is a shortest constant Lee-weight code over $\mathbb{Z}/p^s\mathbb{Z}$ with $k_i = 1$.

Example

Over $\mathbb{Z}/9\mathbb{Z}$ for $k_1 = 1$ we have

$$g = (1, 2, 4, 1, 2, 4, 1, 2, 4, 3, 3).$$

Constant Lee-Weight Codes

Theorem (Byrne, W.)

A constant Lee-weight code over $\mathbb{Z}/p^s\mathbb{Z}$ of rank 2 with $k_s = 0$ cannot exist.

Example

Over $\mathbb{Z}/27\mathbb{Z}$ for $k_2 = k_3 = 1$ we have

$$G = \begin{bmatrix} 3 & 3 & 3 & 6 & 6 & 6 & 12 & 12 & 12 & 9 & 9 & 0 \\ 0 & 9 & 18 & 0 & 9 & 18 & 0 & 9 & 18 & 9 & 18 & 9 \end{bmatrix}.$$

Summary

- The density of MLD codes is 0 for $n \to \infty$.
- The density of MLD codes is 0 for $p \to \infty$.
- Plotkin-optimal linear codes in the Lee metric are sparse.

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Open Questions

- Is there a 'better' Lee-metric Singleton bound?
- How close do codes get to the Lee-metric Singleton bound, i.e., are there almost-MLD codes?
- What about other ambient spaces, other metrics, other bounds?

Eimear Byrne and Violetta Weger "Bounds in the Lee metric and optimal codes", 2021.

New Singleton Bound

Define for $i, j \in \{1, \dots, s-1\}$
$M_i = \frac{p^{s-i}-1}{2}p^i, \quad A_j = \sum_{i=1}^{j} n_{s-i}M_{s-i}, \quad B_j = \sum_{i=1}^{j} n_{s-i}.$

Theorem

Let $\mathcal{C} \subset (\mathbb{Z}/p^s\mathbb{Z})^n$ be a linear code of support subtype $(n_0, \dots, n_{s-1}, 0)$ and rank K. Let $j \in \{1, \dots, s-1\}$ be the largest integer such that $A_j < d_L(\mathcal{C})$, then

$$K \leq n - B_j.$$

- The maximal Lee weight of a position belonging to n_i (living in $\langle p^i \rangle$) is given by M_i.
- We puncture in positions of lowest possible Lee weights, i.e., n_{s-1}, then n_{s-2}, and so on. We possibly killed positions of Lee weight up to A_j.
- In order to still have rank K, we need $A_j < d_L(\mathcal{C})$. The length of the punctured code is then $n - B_j$.

Thank you!