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Open Problems in the Lee Metric

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The history of the Lee metric/ why do we care about it

- Introduced in 1958 by Lee for non-binary codes
- Some good non-linear binary codes can be represented as linear codes in the Lee metric over $\mathsf{Z}/4\mathsf{Z}$
- Introduced to code-based cryptography
- First Lee-metric signature scheme


C. Lee. "Some properties of nonbinary error-correcting codes.", IRE Transactions on Information Theory, 1958.


A.R. Hammons, P.V. Kumar, A.R. Calderbank, N.J. Sloane, P. Solé. "The Z_4-linearity of Kerdock, Preparata, Goethals, and related codes."IEEE Transactions on Information Theory, 1994.


A.-L. Horlemann, **V.W.** "Information set decoding in the Lee metric with applications to cryptography."Advances in Mathematics of Communications, 2019.


S. Ritterho□, G. Maringer, S. Bitzer, **V.W.**, P. Karl, T. Schamberger, J. Schupp, A. Wachter-Zeh. "FuLeeca: A Lee-based Signature Scheme.", NIST Submission 2023.

What is a ring-linear code?

C $\quad$ F_q^n is a **code** if C is a linear subspace

Generator matrix in systematic form

$$\begin{pmatrix} \mathrm{Id}_k & A \end{pmatrix}$$

dimension $k = \log_q(|\,\mathsf{C}\,|)$ number of generators

What is a ring-linear code?

$\mathtt{C}$ $(\mathbb{Z}/p^s\mathbb{Z})^n$ is a **code** if $\mathtt{C}$ is a $\mathbb{Z}/p^s\mathbb{Z}$-submodule

$$\mathtt{C} = (\mathbb{Z}/p^s\mathbb{Z})^{k_0} \times (\mathbb{Z}/p^{s-1}\mathbb{Z})^{k_1} \times \cdots \times (\mathbb{Z}/p\mathbb{Z})^{k_{s-1}}$$

Generator matrix in systematic form

$$\begin{pmatrix} \mathrm{Id}_{k_0} & A_{1,2} & \cdots & A_{1,s} & A_{1,s+1} \\ 0 & p\mathrm{Id}_{k_1} & \cdots & pA_{2,s} & pA_{2,s+1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & p^{s-1}\mathrm{Id}_{k_{s-1}} & p^{s-1}A_{s,s+1} \end{pmatrix},$$

- **subtype** $(k_0, \ldots, k_{s-1})$,
- **rank** $K = \sum_{i=0}^{s-1} k_i$,
- **type** $k = \sum_{i=0}^{s-1} \frac{s-i}{s} k_i = \log_{p^s}(|\,\mathtt{C}\,|)$,
- $0 \leq k \leq K \leq n$ and if $k = K$ **free code**

The Lee metric

Hamming metric

- x $(\mathbb{Z}/p^s\mathbb{Z})^n$: $\mathrm{wt}_H(x) = |\{i \quad \{1,\dots,n\} \mid x_i = 0\}|$
- x, y $(\mathbb{Z}/p^s\mathbb{Z})^n$: $d_H(x,y) = |\{i \quad \{1,\dots,n\} \mid x_i = y_i\}| = \mathrm{wt}_H(x-y)$
- C $(\mathbb{Z}/p^s\mathbb{Z})^n$: $d_H(C) = \min\{\mathrm{wt}_H(x) \mid 0 = x \quad C\}$

e 

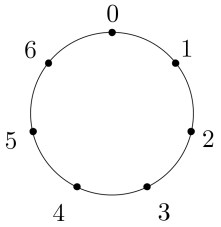
Example

- $x = (1,2,3,0,0,2)$ $(\mathbb{Z}/4\mathbb{Z})^6$ $\mathrm{wt}_H(x) = 4$
- $C =$ $(1,2,3),(2,0,0)$ $(\mathbb{Z}/4\mathbb{Z})^3$ $d_H(C) = 1$

The Lee metric

Lee metric

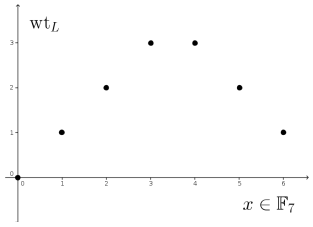
- $x \quad \mathbb{Z}/p^s\mathbb{Z} = \{0, \ldots, p^s - 1\}$ $\rightarrow$ $\mathrm{wt}_L(x) = \min\{x, |p^s - x|\}$



The Lee metric

Lee metric

- $x \quad \{-\frac{p^s}{2}, \ldots, \frac{p^s}{2}\}$ $\rightarrow$ $\mathrm{wt}_L(x) = |x|$



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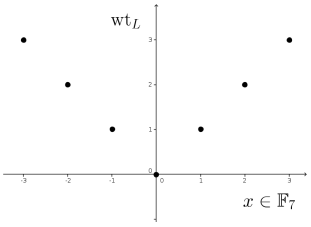

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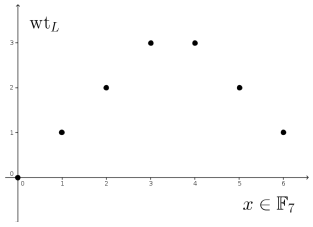

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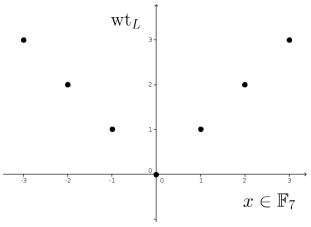

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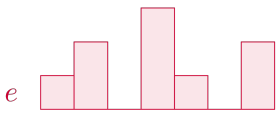

The Lee metric

Lee metric

- $x \quad \{-\frac{p^s}{2}, \ldots, \frac{p^s}{2}\}$
- $x \quad (\mathbb{Z}/p^s\mathbb{Z})^n$
- $x, y \quad (\mathbb{Z}/p^s\mathbb{Z})^n$
- $C \quad (\mathbb{Z}/p^s\mathbb{Z})^n$ linear code

→ $\mathrm{wt}_L(x) = |x|$

→ $\mathrm{wt}_L(x) = \sum_{i=1}^{n} \mathrm{wt}_L(x_i)$

→ $d_L(x, y) = \mathrm{wt}_L(x - y)$

→ $d_L(C) = \min\{\mathrm{wt}_L(x) \mid x \quad C, x = 0\}$

Example

- $x = (1, 2, 3, 0, 0, 2) \quad (\mathbb{Z}/4\mathbb{Z})^6$
- $C = (1, 2, 3), (2, 0, 0) \quad (\mathbb{Z}/4\mathbb{Z})^3$

$\mathrm{wt}_H(x) = 4$ $\quad$ $\mathrm{wt}_L(x) = 6$

$d_H(C) = 1$ $\quad$ $d_L(C) = 2$

→ Maximal Lee weight $M = \left\lfloor \frac{p^s}{2} \right\rfloor$

→ $d_H(C) \leq d_L(C) \leq M d_H(C)$

What is a ring-linear code?

Filtration

For $C \quad (\mathbb{Z}/p^s\mathbb{Z})^n$, define for all $i \quad \{0, \dots, s-1\}$:

$$C_i = C \cap \quad p^i \qquad p^i\,(\mathbb{Z}/p^s\mathbb{Z})^n = \left(\mathbb{Z}/p^{s-i}\mathbb{Z}\right)^n$$

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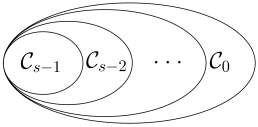

- new maximal Lee weight in C_i is $M_i = \frac{p^{s-i}}{2} p^i$
- $C_{s-1} \subseteq C_{s-2} \subseteq \cdots \subseteq C_1 \subseteq C_0 = C$

→ $d_L(C) \leq d_L(C_1) \leq \cdots \leq d_L(C_{s-2}) \leq d_L(C_{s-1})$

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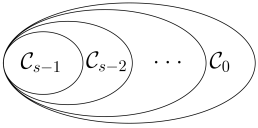

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→ $d_L(C) \leq d_L(C_1) \leq \cdots \leq d_L(C_{s-2}) \leq d_L(C_{s-1})$

$$C_{s-1} = \{xG \mid x \in p^{s-1}\mathbb{Z}/p^s\mathbb{Z}^{k_0} \times p^{s-2}\mathbb{Z}/p^s\mathbb{Z}^{k_1} \times \cdots \times \mathbb{Z}/p^s\mathbb{Z}^{k_{s-1}}\}$$

- C_{s-1} **socle** of C
- $|C_{s-1}| = p^{k_0+k_1+\cdots+k_{s-1}} = p^K$

→ $C_{s-1} \cong \mathbb{F}_p^n$ of dimension K

$$\begin{pmatrix} p^{s-1}\star \\ p^{s-2}\star \\ \vdots \\ \star \end{pmatrix} \begin{pmatrix} \mathrm{Id}_{k_0} & & & \star \\ 0 & p\mathrm{Id}_{k_1} & & p\star \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & p^{s-1}\mathrm{Id}_{k_{s-1}} & p^{s-1}\star \end{pmatrix}$$

What do we know?

Quite an old metric, but how much do we know?

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F_q and Hamming metric:

- If $n \to \infty$ random codes attain the Gilbert-Varshamov bound w.h.p.
- If $q \to \infty$ random codes attain the Singleton bound w.h.p.
- Many bounds: Hamming, Plotkin, LP, Elias-Bassalygo, Griesmer, Johnson
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$\mathsf{Z}/p^s\mathsf{Z}$ and Lee metric:

- If $n \to \infty$ do we know the d_L of a random code?
- If $p \to \infty$ do we know the d_L of a random code?
- Which bounds are known?
- Do we have characterizations and constructions for optimal codes?

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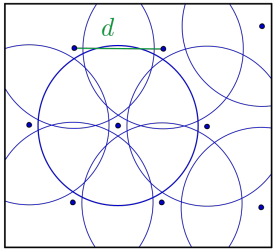
Sphere covering and packing bounds

$A_H(q, n, d)$: largest size of code C F_q^n of minimum distance $d = 2t + 1$

Sphere covering/ Gilbert-Varshamov bound

$$A_H(q, n, d) \geq \frac{q^n}{|B_H(q, n, d-1)|}$$



optimal codes are dense for $n \to \infty$

Sphere packing/ Hamming bound

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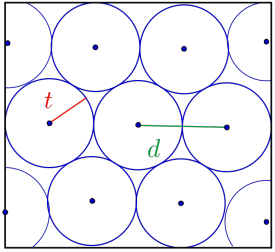


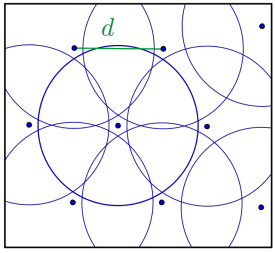
optimal codes: perfect codes

Lee-metric sphere covering and packing bounds

$A_L(p^s, n, d)$: largest size of code C $(\mathbb{Z}/p^s\mathbb{Z})^n$ of minimum Lee distance $d = 2t + 1$

Sphere covering/ Gilbert-Varshamov bound

$$A_L(p^s, n, d) \geq \frac{p^{sn}}{|B_L(p^s, n, d-1)|}$$



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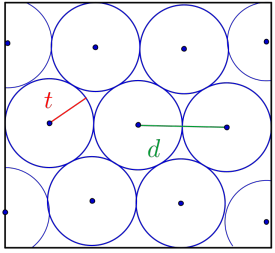

optimal codes: perfect code

E. Byrne, A.-L. Horlemann, K. Khathuria, **V.W.** "Density of free modules over finite chain rings." Linear Algebra and its Applications, 2022

Golomb-Welch conjecture

C $(\mathbb{Z}/p^s\mathbb{Z})^n$ with minimum Lee distance $d = 2t + 1$ is perfect

→ for every x $(\mathbb{Z}/p^s\mathbb{Z})^n$ there exists a unique c C with $d_L(x, c) \leq t$

Golomb-Welch conjecture

- weak version
  There exists no perfect code C $(\mathbb{Z}/p^s\mathbb{Z})^n$ for $n \geq 3$ with minimum Lee distance $5 \leq d \leq p^s$.
- strong version
  There exists no perfect code C $\mathbb{Z}^n$ for $n \geq 3$ with minimum L1 distance $5 \leq d$.

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50 years later - How much do we know?

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50 years later - How much do we know?

- true for $n < (t + 2)^2/2.1, t \geq 285$
- true for $n \geq 6, t \geq \frac{\sqrt{2}}{2} - \frac{3}{4}\sqrt{2} - \frac{1}{2}$

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The Singleton bound

- Hamming metric: Singleton 1964
- → Optimal codes: Maximum Distance Separable (MDS)
- MDS dense for $q \to \infty$
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- Lee metric: Shiromoto 2000
- → Optimal codes and their densities?


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For $C \subseteq \mathbb{F}_q^n$ linear code of minimum Hamming distance $d_H(C)$ has dimension:

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For C F_q^n linear code of minimum Hamming distance $d_H(C)$ has dimension:

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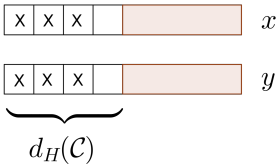
- Puncture in $d_H(C) - 1$ positions

→ new code $|C| = |C|$

→ C $\mathsf{F}_q^{n-(d_H(C)-1)}$



The Singleton bound

Lee-metric Singleton bound

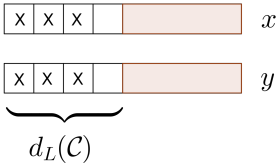
For $C \quad (\mathbb{Z}/p^s\mathbb{Z})^n$ linear code of minimum Lee distance $d_L(C)$ has type:

$$k \leq n - \left\lfloor \frac{d_L(C) - 1}{M} \right\rfloor$$

- Puncture in $\left\lfloor \frac{d_L(C)-1}{M} \right\rfloor$ positions

→ new code $|C| = |C|$

→ $C \quad (\mathbb{Z}/p^s\mathbb{Z})^{n - \left\lfloor \frac{d_L(C)-1}{M} \right\rfloor}$



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Optimal codes

Lee-metric Singleton bound

$\mathcal{C}$ length n, type k, $M = \frac{p^s}{2}$:

$$k \leq n - \left\lfloor \frac{d_L(\mathcal{C}) - 1}{M} \right\rfloor$$

Example

$\mathcal{C} = (1, 2) \quad (\mathbb{Z}/5\mathbb{Z})^2$

$$1 = 2 - \left\lfloor \frac{3 - 1}{2} \right\rfloor$$

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→ Need new techniques!

Other Singleton bounds

Maximum Distance with respect to Rank (MDR)
For $C \quad (\mathbb{Z}/p^s\mathbb{Z})^n$ linear code with rank K:

$$d_H(C) \leq n - K + 1 \qquad (\leq n - k + 1)$$

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$$d_H(C) \leq d_H(C_{s-1}) \leq n - K + 1 \qquad C_{s-1} \quad \mathbb{F}_p^n \text{ dimension } K$$

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optimal codes: dense for $p \to \infty$, sparse for $n \to \infty$

- → can do the same to get $d_L(C) \leq M(n - K + 1)$
  - for $1 < k < n$ integer Alderson-Huntemann: $d_L(C) \leq M(n - k)$
- → full characterization and only few optimal codes exist
- → always sparse for n or $p \to \infty$

T. Alderson, S. Huntemann. "On maximum Lee distance codes.", Journal of Discrete Mathematics, 2013

Generalized Hamming weights

J. Bari□, V.W. "Better bounds on the minimal Lee distance.", 2023

Support and weight of code

x F_q^n $\quad \mathrm{supp}_H(x) = \{i \quad \{1,\dots,n\} \mid x_i = 0\} \quad \rightarrow \mathrm{wt}_H(x) = |\mathrm{supp}_H(x)|$

C F_q^n $\quad \mathrm{supp}_H(C) = \{i \quad \{1,\dots,n\} \mid \quad x \quad C : x_i = 0\} \quad \rightarrow \mathrm{wt}_H(C) = |\mathrm{supp}_H(C)|$

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Generalized weights

C F_q^n of dimension k. For all r $\{1,\dots,k\}$:

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Example

C F_2^4 generated by $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

$d_H^1(C) = 1$

$d_H^2(C) = 3$

$d_H^3(C) = 4$

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Properties

- $d_H(C) = d_H^1(C)$
- $d_H^r(C) < d_H^{r+1}(C)$ for $r < k$
- $d_H^k(C) = \mathrm{wt}_H(C)$

$\rightarrow d_H(C) = \underbrace{d_H^1(C) < d_H^2(C) < \cdots < d_H^k(C)}_{k-1} = \mathrm{wt}_H(C) \rightarrow$ Singleton Bound: $d_H(C) \leq n-k+1$

Generalization to Lee metric

- $\mathbb{Z}/4\mathbb{Z}$  S. Dougherty, M. Gupta, K. Shiromoto. “On Generalized weights for codes over finite rings.”, 2002

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- **Lee support** $\mathrm{supp}_L(x) = (\mathrm{wt}_L(x_1),\dots,\mathrm{wt}_L(x_n)) = s,\ |\ s\ | = \sum s_i$
- **join Lee support**

$$\mathrm{wt}_L(C) = |\bigvee_{c\ C} \mathrm{supp}_L(c)\ |$$
$$= \sum_{i=1}^{n} \max\{\mathrm{wt}_L(c_i) \mid c \quad C\}$$

→ Resulting Bound $d_L(C) \leq \frac{p}{2}\ p^{s-1}(n - K + 1)$

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Very few optimal codes

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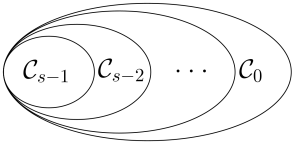
Filtration bound

Filtration

For $C \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n$, define for all $i \in \{0, \dots, s-1\}$: $C_i = C \cap p^i$

maximal Lee weight in C_i is $M_i = \frac{p^{s-i}}{2} p^i$



$$C_{s-1} \quad C_{s-2} \quad \cdots \quad C_1 \quad C_0 = C$$

$$\rightarrow d_L(C) \leq d_L(C_1) \leq \cdots \leq d_L(C_{s-2}) \leq d_L(C_{s-1})$$

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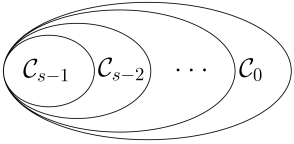

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New Lee-metric Singleton bound

$C \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n$, subtype $(k_0, \dots, k_\sigma)$, ℓ: max prime power $\ell = \sigma, s$ in G, appears n times:

$$d_L(C) \leq p^{s-\ell+\sigma} + (n - K - n) \frac{p^{\ell-\sigma}}{2} p^{s-\ell+\sigma}$$

Examples

$$G_1 = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

$\mathbb{Z}/9\mathbb{Z}$, $d_L(G_1) = 3$

- Shiromoto: $d_L \leq 5$
- Join: $d_L \leq 6$
- Filtration: $d_L \leq 3$

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Are the optimal codes dense?

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Are the optimal codes dense? NO

Open Problem

Find a tighter Lee-metric Singleton bound, for which optimal codes are dense

What do we know?

Quite an old metric, but how much do we know?

F_q and Hamming metric:

- If $n \to \infty$ random codes attain the Gilbert-Varshamov bound w.h.p.
- If $q \to \infty$ random codes attain the Singleton bound w.h.p.
- Many bounds: Hamming, Plotkin, LP, Elias-Bassalygo, Griesmer, Johnson
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Plotkin bound

Plotkin bound

$C \subseteq \mathbb{F}_q^n$ has minimum Hamming distance

$$d_H(C) \leq \frac{|C|}{|C|-1} n \frac{q-1}{q}.$$

- Weight of non-zero codeword is at least $d_H(C)$:

$$(|C|-1) d_H(C) \leq \sum_{c \in C} \mathrm{wt}_H(c)$$

- average weight of code

$$\overline{\mathrm{wt}}_H(C) = \frac{1}{|C|} \sum_{c \in C} \mathrm{wt}_H(c) \leq \overline{\mathrm{wt}}_H(\mathbb{F}_q^n)$$

- additive weight

$$\overline{\mathrm{wt}}_H(\mathbb{F}_q^n) = n \overline{\mathrm{wt}}_H(\mathbb{F}_q) = n \frac{q-1}{q}$$

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optimal codes: constant Hamming weight codes $\rightarrow$ ℓ-fold duplicates of simplex code

Lee-metric Plotkin bound

Plotkin bound

C $(\mathbb{Z}/p^s\mathbb{Z})^n$ has minimum Lee distance

$$d_L(C) \leq \frac{|C|}{|C|-1} n D_L = n D_L \frac{1}{1-p^{-sk}}.$$

- average Lee weight in $\mathbb{Z}/p^s\mathbb{Z}$:

$$D_L = \begin{cases} \frac{p^{2s}-1}{4p^s} & \text{if } p = 2 \\ 2^{s-2} & \text{if } p = 2 \end{cases}$$

R. Graham, A. Wyner. "An upper bound on the minimum distance for a q-ary code.", Information and Control, 1968

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using $d_L(C) \leq d_L(c)$

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Improvements

Support subtype

C has **support subtype** $(n_0, \ldots, n_s)$:

$$n_i(C) = |\{j \in \{1, \ldots, n\} \mid \pi_j(C) = p^i\}|$$

for π_j projection on jth coordinate

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Example

C $\mathbb{Z}/8\mathbb{Z}^5$ generated by

$$G = \begin{pmatrix} 1 & 3 & 5 & 0 & 2 \\ 0 & 2 & 4 & 2 & 6 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 4 \end{pmatrix}$$

has subtype $(1, 1, 2)$ and support subtype $(3, 2, 0, 0)$

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$$\overline{\mathrm{wt}}_L(C) \le nD_L = n \begin{cases} \frac{p^{2s}-1}{4p^s} & \text{if } p = 2 \\ 2^{s-2} & \text{if } p = 2 \end{cases} \quad \rightarrow \quad \overline{\mathrm{wt}}_L(C) = \begin{cases} \left(p^{2s}n - \sum_{i=0}^{s-1} p^{2i} n_i\right)/4p^s & \text{if } p = 2 \\ 2^{s-2}n & \text{if } p = 2 \end{cases}$$

Plotkin bound

C $(\mathbb{Z}/p^s\mathbb{Z})^n$ has minimum Lee distance

$$\left\lfloor \frac{d_L - 1}{A} \right\rfloor \le n - K \quad \text{for} \quad A = \begin{cases} p^{s-1}(p+1)/4 & \text{if } p = 2 \\ 2^{s-1} & \text{if } p = 2 \end{cases}$$

E. Byrne, **V.W.** "Bounds in the Lee metric and optimal codes.", Finite Fields and Their Applications, 2022

optimal codes: ℓ-folds of smallest constant Lee weight codes

- characterization and construction by Wood for $s = 1$ or $p = 2$
- else $K \le 2$: finished construction

J. Wood. "The structure of linear codes of constant weight.", Transactions of the American Mathematical Society, 2002

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MacWilliams identities and LP bound

Weight enumerator $W_C(i) = |\{c \quad C \mid \mathrm{wt}_H(c) = i\}|$

Dual code $C \quad = \{x \quad \mathbb{F}_q^n \mid \quad x, c \ = 0 \text{ for all } c \quad C\} = \quad H$

MacWilliams identity

$$W_C \quad (\ell) = \frac{1}{|C|} \sum_{i=0}^{n} K_\ell(i) W_C(i)$$

F. J. MacWilliams. "A theorem on the distribution of weights in a systematic code.", Bell System Technical Journal, 1963.

LP bound

$$W_C\ (\ell) = \frac{1}{|C|} \sum_{i=0}^{n} K_\ell(i) W_C(i)$$

Linear Programming (LP) bound

Maximize $\sum_{i=0}^{n} A_i$ under the linear constraints

- $A_0 = 1$
- $A_i = 0$ for $1 < i < d$
- $A_i \geq 0$
- $\sum_{i=0}^{n} K_\ell(i) A_i \geq 0$ for all ℓ

For $A_i = W_C(i)$ → upper bound on max. $|C| = \sum_{i=0}^{n} A_i$

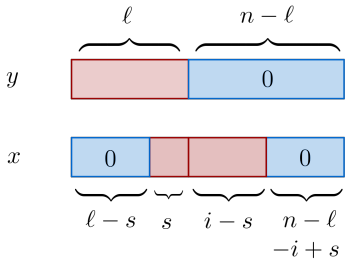
Krawtchouk Coefficient

Krawtchouk coefficient: arbitrary y with $\mathrm{wt}_H(y) = \ell$

$$K_\ell(i) = \sum_{x:\mathrm{wt}_H(x)=i} \chi(x,y)$$

Character: ζ pth root of unity

$$\chi(x,y) = \zeta^{\mathrm{Tr}(x,y)}$$



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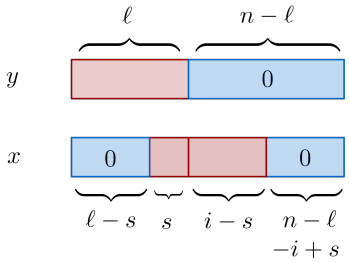

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$$= \sum_{s=0}^{i} \binom{\ell}{s}\binom{n-\ell}{i-s} \prod_i \sum_{x_i} \zeta^{\mathrm{Tr}(x_i y_i)}$$

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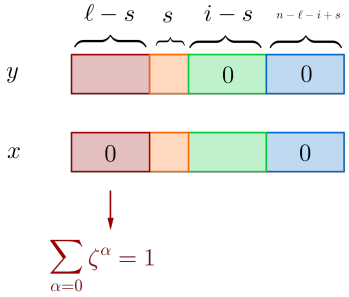

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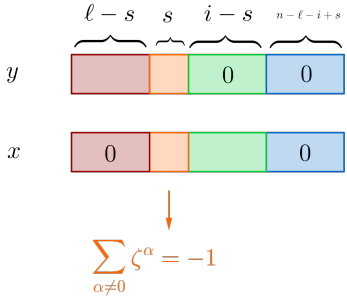

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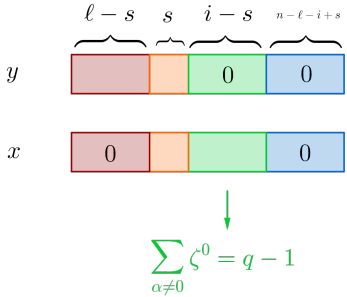

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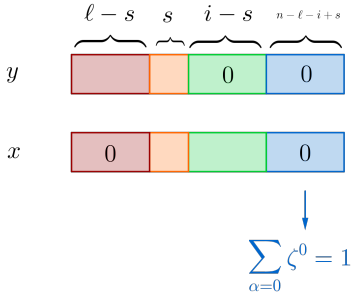

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Lee-Metric MacWilliams identity

- Lee-metric MacWilliams identity does not exist

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- Hamming metric — Partition F_q^n into $P_i = \{x \mid \mathrm{wt}_H(x) = i\}$
- Lee metric — Partition $\mathsf{Z}/p^s\mathsf{Z}^n$ into $P_i = \{x \mid \mathrm{wt}_L(x) = i\}$

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- Lee metric — Partition $\mathbb{Z}/p^s\mathbb{Z}^n$ into $P_\rho = \{x \mid x \text{ has Lee type } \rho\}$

Lee Type

$x \in \mathbb{Z}/p^s\mathbb{Z}^n$ has **Lee type** $\rho = (i_0, \ldots, i_M)$ if $i_\ell = |\{j \mid x_j = \pm\ell\}|$

$$\sum_{i=0}^{M} i\rho_i = \mathrm{wt}_L(x)$$

Example

$x = (0,1,4,8) \quad \mathbb{Z}/9\mathbb{Z}^4$ has Lee type $(1,2,0,0,1)$ and $1 \cdot 0 + 2 \cdot 1 + 1 \cdot 4 = 6 = \mathrm{wt}_L(x)$

Lee-Metric MacWilliams identity

Type enumerator $W_C(\rho) = |\{c \quad C \mid c \text{ of Lee type } \rho\}|$

Krawtchouk coe□cient: arbitrary $y \quad P_\rho$ $K_\rho(\pi) = \sum_{x \quad P_\pi} \chi(x, y)$

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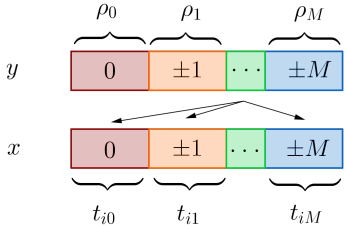

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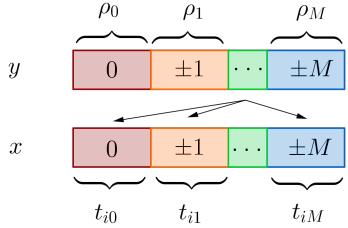

$$K_\rho(\pi) = \prod_{i=0}^{M} \prod_{(t_{i0},\ldots,t_{iM}):\sum_i t_{ij}=\rho_j} \binom{\pi_i}{t_{i0},\ldots,t_{iM}} \prod_{j=0}^{M} (\zeta^{-ij} + \zeta^{ij})^{t_{ij}}$$

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Set of all types $D = \{\rho \quad \{0, \ldots, n\}^M \mid \sum_{i=0}^{M} \rho_i = n\}$

Lee-Metric MacWilliams identity

$$W_C \ (\rho) = \frac{1}{|C|} \sum_{\pi \ D} K_\rho(\pi) W_C(\pi)$$

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too many variables for the program to run → no e□ciently computable LP bound

Other bounds

- Elias-Bassalygo bound
- → and improvements

- Griesmer bound
- → only over $\mathbb{Z}/4\mathbb{Z}$

- Johnson bound


T. Lepistö. "A modification of the Elias-bound and nonexistence theorems for perfect codes in the Lee-metric.", Information and Control, 1981


J. Astola. "An Elias-type bound for Lee codes over large alphabets and its application to perfect codes.", IEEE TIT, 1982.


A. Ashikhmin. "On generalized Hamming weights for Galois ring linear codes.", Designs, Codes and Cryptography, 1998.


I. Tal. "List Decoding of Lee Metric Codes.", PhD thesis, 2003.

Questions?

Summary

- Many Lee-metric bounds only exist over $\mathbb{Z}/4\mathbb{Z}$
- Many bounds obtained by classic arguments

→ Need new techniques to generalize and get tighter bounds

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- Many Lee-metric bounds only exist over $\mathbb{Z}/4\mathbb{Z}$
- Many bounds obtained by classic arguments

→ Need new techniques to generalize and get tighter bounds

Thank you!

Generalized Filtration Weight

$$\mathsf{C} = G_{sys}$$

$$\max \sigma : \quad k_\sigma = 0$$

$$G_{sys} = \begin{pmatrix} \mathsf{Id}_{k_1} & & & & \star \\ 0 & p\mathsf{Id}_{k_2} & & & p\star \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & p^\sigma \mathsf{Id}_{k_\sigma} & p^\sigma \star \end{pmatrix}$$

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$$\downarrow$$

$$C_\sigma$$

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$$\downarrow$$

$$G_\sigma = \begin{pmatrix} p^\sigma \mathrm{Id}_K & p^\sigma A_\sigma \end{pmatrix}$$

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1. If $\ell = \sigma$ $\rightarrow$ $d_L(\mathsf{C}_\sigma) \leq p^\sigma + (n - K)M_\sigma$

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Generalized Filtration Weight

$$\mathsf{C} = G_{sys}$$

$$\max \sigma : \quad k_\sigma = 0$$

$$\downarrow$$

$$\mathsf{C}_\sigma$$

$$G_{sys} = \begin{pmatrix} \mathsf{Id}_{k_1} & & & & \star \\ 0 & p\mathsf{Id}_{k_2} & & & p\star \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & p^\sigma \mathsf{Id}_{k_\sigma} & p^\sigma \star \end{pmatrix}$$

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n : max number of p^ℓ in one row

1. If $\ell = \sigma$ → $d_L(\mathsf{C}_\sigma) \leq p^\sigma + (n - K)M_\sigma$
2. If $\ell = s$ → $d_L(\mathsf{C}_\sigma) \leq p^\sigma + (n - K - n)M_\sigma$
3. If $\ell = \sigma, \ell = s$ → go to $\mathsf{C}_{s-\ell+\sigma}$: multiply with $p^{s-\ell}$

Generalized Filtration Weight

$$\begin{array}{ccc} \mathsf{C}_{\sigma} & & G_{\sigma} = \begin{pmatrix} p^{\sigma}\mathsf{Id}_K & p^{\sigma}A_{\sigma} \end{pmatrix} \\ \downarrow & & \downarrow \\ \mathsf{C}_{s-\ell+\sigma} & & G_{s-\ell+\sigma} = \begin{pmatrix} p^{s-\ell+\sigma}\mathsf{Id}_K & p^{s-\ell+\sigma}A_{s-\ell+\sigma} \end{pmatrix} \end{array}$$

Generalized Filtration Weight

C_σ $\qquad$ $G_\sigma = \begin{pmatrix} p^\sigma \mathrm{Id}_K & p^\sigma A_\sigma \end{pmatrix}$ $\qquad$ $(\underbrace{0 p^\sigma 0}_{K}\ \underbrace{p^\ell \cdots p^\ell}_{n}\ \underbrace{\star \cdots \star}_{n-K-n})$

$\downarrow$ $\qquad$ $\downarrow$ $\qquad$ $\downarrow$

$C_{s-\ell+\sigma}$ $\qquad$ $G_{s-\ell+\sigma} = \begin{pmatrix} p^{s-\ell+\sigma} \mathrm{Id}_K & p^{s-\ell+\sigma} A_{s-\ell+\sigma} \end{pmatrix}$ $\qquad$ $(\underbrace{0 p^{s-\ell+\sigma} 0}_{K}\ \underbrace{0 \cdots 0}_{n}\ \underbrace{\star \cdots \star}_{n-K-n})$

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New Lee-metric Singleton bound:

C $(\mathbb{Z}/p^s\mathbb{Z})^n$, subtype $(k_0, \ldots, k_\sigma)$, max prime power $\ell = \sigma, s$, appears n times:

$$d_L(C) \leq p^{s-\ell+\sigma} + (n - K - n) \frac{p^{\ell-\sigma}}{2} p^{s-\ell+\sigma}$$

Support and Weights of Codes: Lee Metric

Support and weight of code:

$x \quad (\mathbb{Z}/p^s\mathbb{Z})^n : \quad \mathrm{supp}_H(x) = \{i \quad \{1,\dots,n\} \mid x_i = 0\} \quad \rightarrow \mathrm{wt}_H(x) = |\mathrm{supp}_H(x)|$

$C \quad (\mathbb{Z}/p^s\mathbb{Z})^n : \quad \mathrm{supp}_H(C) = \{i \quad \{1,\dots,n\} \mid \quad x \quad C : x_i = 0\} \quad \rightarrow \mathrm{wt}_H(C) = |\mathrm{supp}_H(C)|$

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$C \quad (\mathbb{Z}/p^s\mathbb{Z})^n$: $\quad \mathrm{supp}_H(C) = \bigvee_{c \ C} \mathrm{supp}_H(c) \quad \rightarrow \mathrm{wt}_H(C) = |\mathrm{supp}_H(C)|$

$s, t \quad \mathbb{N}^n$:

- **size** $|s| = \sum_{i=1}^{n} s_i$
- **join** $s \quad t = (\max\{s_1, t_1\}, \ldots, \max\{s_n, t_n\})$

Support and Weights of Codes: Lee Metric

Support and weight of code:

$x \quad (\mathbb{Z}/p^s\mathbb{Z})^n$: $\quad \mathrm{supp}_L(x) = (\mathrm{wt}_L(x_1), \ldots, \mathrm{wt}_L(x_n)) \quad \rightarrow \quad \mathrm{wt}_L(x) = |\mathrm{supp}_L(x)|$

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Generalized Lee weights:

$C \quad (\mathbb{Z}/p^s\mathbb{Z})^n$ of rank K. For all $r \quad \{1, \ldots, K\}$:

$$d_L^r(C) = \min\{\mathrm{wt}_L(D) \mid D \quad C \text{ of rank } r\}$$

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Example:

C $(\mathbb{Z}/9\mathbb{Z})^4$ generated by $\begin{pmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 3 & 3 \end{pmatrix}$

$d_L(C) = 2$
$d_L^1(C) = 6$
$d_L^2(C) = 9$
$d_L^3(C) = 12$
$\mathrm{wt}_L(C) = 16$

Generalized Lee Weights

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$$d_L^r(C) = \min\{\mathrm{wt}_L(D) \mid D \quad C \text{ of rank } r\}$$

Properties:

- $d_L(C) \leq d_L^1(C)$
- $d_L^r(C) \leq d_L^{r+1}(C)$ for $r < K$
- $d_L^K(C) \leq \mathrm{wt}_L(C)$

Generalized Lee Weights

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socle: $C_{s-1} = C \cap \ p^{s-1}$ of maximal Lee weight $M_{s-1} = \ \frac{p}{2}\ p^{s-1}$

Properties:
All subcodes attaining the rth generalized Lee weights are in the socle: $d_L^r(C) = d_H^r(C) M_{s-1}$

Generalized Lee Weights

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$$d_L(C) \leq M_{s-1}(n - K + 1)$$

Better than previous $d_L(C) \leq M(n - K + 1)$

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New Lee-metric Singleton bound:

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Better than previous $d_L(C) \leq M(n - K + 1)$

only codes with $p = 3$ can attain it

Need different approach

Lee Column Weight

Example:

C $\mathbb{F}_2^4$ generated by

$$G = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$G_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \rightarrow d_H^1(C) = 1$$

$$G_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \rightarrow d_H^2(C) = 3$$

$$G \rightarrow d_H^3(C) = 4$$

Lee Column Weight

Example:

C $\mathbb{F}_2^4$ generated by

$$G = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$G_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \rightarrow d_H^1(C) = 1 = \text{colwt}(G_1)$$

$$G_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \rightarrow d_H^2(C) = 3 = \text{colwt}(G_2)$$

$$G \rightarrow d_H^3(C) = 4 = \text{colwt}(G)$$

Lee Column Weight

Example:

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Lee column weight:

$$A = \begin{pmatrix} \vdots & & \vdots \\ a_1 & \cdots & a_n \\ \vdots & & \vdots \end{pmatrix} \quad \to \quad \text{colwt}_L(A) = |(\max \text{supp}_L(a_1), \ldots, \max \text{supp}_L(a_n))|$$

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Example: $G = \begin{pmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 3 & 3 \end{pmatrix} \quad \rightarrow \quad \mathrm{colwt}_L(G) = |(1, 1, 3, 3)| = 8$

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Lee column weight:

$C \quad (\mathbb{Z}/p^s\mathbb{Z})^n$: $\mathrm{colwt}_L(C) = \min\{\mathrm{colwt}(G) \mid \quad G \quad = C\}$

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Highly depends on the choice of generator matrix

Lee Column Weight

Generalized Lee column weights:
C $(\mathbb{Z}/p^s\mathbb{Z})^n$ of rank K. For all r $\{1, \ldots, K\}$:

$$d_L^r(C) = \min\{\text{colwt}_L(D) \mid D \quad C \text{ of rank } r\}$$

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Properties:

- $d_L(C) = d_L^1(C)$
- $d_L^r(C) < d_L^{r+1}(C)$ for $r < K$
- $d_L^K(C) = \mathrm{colwt}_L(C)$

Lee Column Weight

support subtype of a code is $(n_0, \ldots, n_{s-1})$, where

$$n_i = |\{j \quad \{1, \ldots, n\} \mid \ c_j \ = \ p^i \ \}|$$

→ Remainder support subtype $(\mu_0, \ldots, \mu_{s-1})$ is support subtype in $C_{n-K,\ldots,n}$

$$\text{colwt}_L(C)m \leq \sum_{i=0}^{s-1} p^i k_i + \sum_{i=0}^{s-1} \mu_i M_i,$$

where $M_i = \ \frac{p^{s-1}}{2} \ p^i$

Singleton bound:
$C \quad (\mathbb{Z}/p^s\mathbb{Z})^n$ with subtype $(k_0, \ldots, k_{s-1})$, σ largest with $k_\sigma = 0$, support subtype in redundant part $(\mu_{n-K}, \ldots, \mu_n)$,

$$d_L(C) \leq \sum_{i=0}^{s-1} p^i k_i + \sum_{i=n-K}^{n} \mu_i M_i - \left(\sum_{i=0}^{\sigma-1} \left(\sum_{j=0}^{i} k_j \right) \ \frac{p}{2} \ p^i + (k_\sigma - 1) p^\sigma \right)$$

Much better than previous bound $d_L(C) \leq M(n - K + 1)$

Torsion Codes

Torsion codes:

$C \quad (\mathbb{Z}/p^s\mathbb{Z})^n$: for $i \quad \{1, \ldots, s\}$: $\quad \tilde{C}_i = C \mod p^i \quad \left(\mathbb{Z}/p^i\mathbb{Z}\right)^n$

Torsion Codes

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$C \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n$: for $i \in \{1, \ldots, s\}$: $\tilde{C}_i = C \mod p^i \subseteq \left(\mathbb{Z}/p^i\mathbb{Z}\right)^n$

$p^{s-i}\tilde{C}_i \quad C_{s-i} = C \cap p^{s-i}$ with $\mathrm{rank}(p^{s-i}\tilde{C}_i) = k_0 + \cdots + k_{i-1} < \mathrm{rank}(C_{s-i}) = K$

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$$C : G = \begin{pmatrix} \mathrm{Id}_{k_0} & & & \star \\ 0 & p\mathrm{Id}_{k_1} & & p\star \\ \vdots & & & \vdots \\ 0 & & p^{i-1}k_{i-1} & p^{i-1}\star \\ \vdots & & & \vdots \\ 0 & & p^{s-1}\mathrm{Id}_{k_s-1} & p^{s-1}\star \end{pmatrix}$$

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$$C_{s-i} : G_{s-i} = \begin{pmatrix} p^{s-i}\mathrm{Id}_{k_0} & & & & p^{s-i}\star \\ 0 & p^{s-i}\mathrm{Id}_{k_1} & & & p^{s-i}\star \\ \vdots & & & & \vdots \\ 0 & & p^{s-i}k_{i-1} & & p^{s-i}\star \\ \vdots & & & & \vdots \\ 0 & & & p^{s-1}\mathrm{Id}_{k_s-1} & p^{s-1}\star \end{pmatrix}$$

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$$d_L(C) \leq d_L(C_{s-i}) \leq d_L(p^{s-i}\tilde{C}_i) \leq \text{upper bound}$$

Fixing the subtype

Generalized Lee weights:

C $(\mathbb{Z}/p^s\mathbb{Z})^n$ of subtype $(k_0, \ldots, k_{s-1})$. For all $(\tilde{k_0}, \ldots, \tilde{k}_{s-1})$ with $\tilde{k}_i \leq k_i$

$$d_L^{(\tilde{k_0}, \ldots, \tilde{k}_{s-1})}(C) = \min\{\mathrm{wt}_L(D) \mid D \quad C \text{ of subtype } (\tilde{k_0}, \ldots, \tilde{k}_{s-1})\}$$

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$(k_0, \ldots, k_{s-1})$	$(k_0, \ldots, k_{s-1} - 1)$	$\cdots$	$(k_0, \cdots, 0)$
$(k_0 - 1, \ldots, k_{s-1})$	$(k_0 - 1, \ldots, k_{s-1} - 1)$	$\cdots$	$(k_0 - 1, \ldots, 0)$
$\vdots$			-
$(k_0 - i, \ldots, k_{s-1})$	$\cdots$	$(k_0 - i, \ldots, k_{s-1} - i)$	-
$\vdots$			-
$(0, \ldots, k_{s-1})$	$(0, \ldots, k_{s-1} - 1)$	-	-

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all our bounds go to the socle or the subcode of subtype $(0, \dots, 0, k_i, 0, \dots, 0)$ → already considered

Alderson-Huntemann

Alderson-Hunteman:
$C \quad (\mathbb{Z}/p^s\mathbb{Z})^n$ of integer type $1 < k < n$:

$$d_L(C) \leq (n - K)M$$

T. Alderson, S. Huntemann. "On maximum Lee distance codes.", Discrete Math, 2013

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only optimal codes:

- p odd: $p^s = 5, k + 1 \leq n \leq k + 3$ or $p^s \quad \{7, 9\}, n = k + 1$
- $p = 2$: free, $s = 2, k + 1 \leq n \leq k + 2$ or $s = 3$, $n = k + 1$ or $k + 1 = K \quad \{n, n + 1\}$

→ sparse

E. Byrne, V.W. "Bounds in the Lee metric and optimal codes.", Finite Fields and Their Applications, 2022