The Mysterious Case of Code Equivalence

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VT-Swiss Summer School

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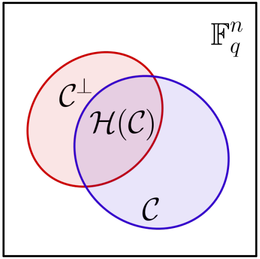
Basics



- Linear code: $\mathcal{C} \subseteq \mathbb{F}_q^n$ linear subspace of dimension k
- Generator matrix: $G \in \mathbb{F}_q^{k \times n}$ with $\langle G \rangle = \mathcal{C}$
- Dual code: $\mathcal{C}^\perp = \{x \in \mathbb{F}_q^n \mid \langle x, c \rangle = 0 \ \forall \ c \in \mathcal{C}\}$
- Parity-check matrix: $H \in \mathbb{F}_q^{n-k \times n}$ with $\langle H \rangle = \mathcal{C}^\perp$
- Hull: $\mathcal{H}(\mathcal{C}) = \mathcal{C} \cap \mathcal{C}^\perp$

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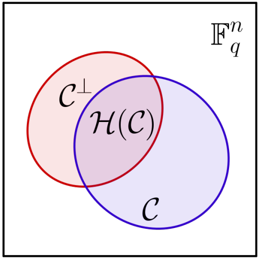

$$\begin{aligned} cG^\top &= mGG^\top = 0 \\ \dim(\ker(GG^\top)) &= k - \mathrm{rk}(GG^\top) \\ &= 0 \text{ w.h.p} \end{aligned}$$

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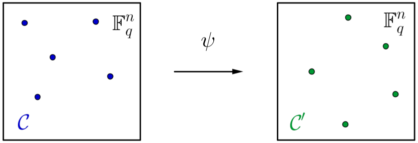
Basics

- Hamming weight: $\mathrm{wt}(c) = |\{i \in \{1, \ldots, n\} \mid c_i \neq 0\}|$
- Linear isometry: linear map $\psi : \mathbb{F}_q^n \to \mathbb{F}_q^n$ with $\mathrm{wt}(c) = \mathrm{wt}(\psi(c))\ \forall c \in \mathbb{F}_q^n$
- Hamming isometries $\mathcal{L} = (\mathbb{F}_q^\star)^n \rtimes (\mathrm{Aut}(\mathbb{F}_q) \times \mathcal{S}_n)$
- Code equivalence $\mathcal{C}$ is equivalent to $\mathcal{C}'$ if exists $\psi \in \mathcal{L} : \psi(\mathcal{C}) = \mathcal{C}'$



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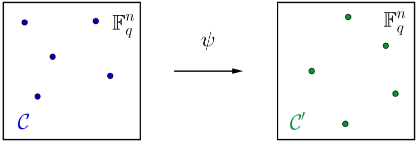


Coding Theory: Distinguish if codes belong to new class or not

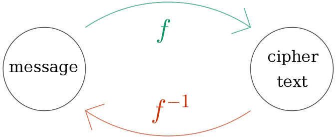
E. M. Gabidulin, "New Rank Codes with Efficient Decoding", EnT, 2017.

A. Neri, S. Puchinger, A.-L. Horlemann, "Invariants and Inequivalence of Linear Rank-Metric Codes.", ISIT, 2019.

Motivation

Public-key cryptography

Encryption



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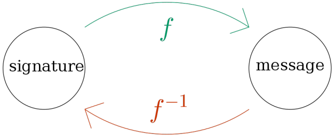

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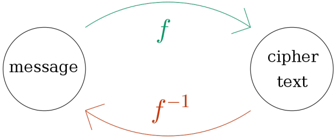

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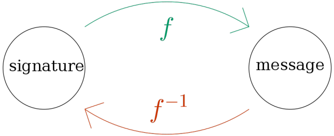

- f easy to compute with
- f^{-1} hard to compute with
- f^{-1} easy with secret

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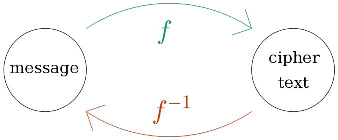

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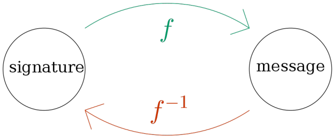

computing f^{-1}: hard mathematical problem

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- Linear equivalence: $\mathcal{C}$ is linear equivalent to $\mathcal{C}'$ if $\exists \varphi \in (\mathbb{F}_q^\star)^n \rtimes \mathcal{S}_n : \varphi(\mathcal{C}) = \mathcal{C}'$
- Permutation equivalence: $\mathcal{C}$ is permutation equivalent to $\mathcal{C}'$ if $\exists \sigma \in \mathcal{S}_n$: $\sigma(\mathcal{C}) = \mathcal{C}'$

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Linear Equivalence Problem (LEP):
Given $\mathcal{C}, \mathcal{C}'$ find $\varphi \in (\mathbb{F}_q^\star)^n \rtimes \mathcal{S}_n$: $\varphi(\mathcal{C}) = \mathcal{C}'$

Permutation Equivalence Problem (PEP):
Given $\mathcal{C}, \mathcal{C}'$ find $\sigma \in \mathcal{S}_n$: $\sigma(\mathcal{C}) = \mathcal{C}'$

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Linear Equivalence Problem (LEP):
Given G, G' find $S \in \mathrm{GL}_k(q), P \in S_n, D = \mathrm{diag}(v) : SGPD = G'$

Permutation Equivalence Problem (PEP):
Given G, G' find $S \in \mathrm{GL}_k(q), P \in S_n : SGP = G'$

Motivation

Can build Zero-Knowledge (ZK) protocol from group action

Prover

secret

Verifier

public

✓

Motivation

Can build Zero-Knowledge (ZK) protocol from group action

Prover		Verifier
	$\xrightarrow{\text{commitment}}$
	$\xleftarrow{\text{challenge}}$
secret	$\xrightarrow{\text{response}}$	public
		✓

Motivation

Can build Zero-Knowledge (ZK) protocol from group action

Prover	Interaction	Verifier
	$\xrightarrow{\text{commitment}}$	
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Motivation

Can build Zero-Knowledge (ZK) protocol from group action

Prover		Verifier
secret	$\xrightarrow{\text{Fiat-Shamir}}$	public
		✓

$\rightarrow$ Signature scheme

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Prover

secret

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$$\xleftarrow{\text{challenge}}$$
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Verifier

public

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$$G \xrightarrow{\varphi} G'$$

secret key: $\varphi \in (\mathbb{F}_q^\star)^n \rtimes \mathcal{S}_n$

public key: G, G' with $\varphi(G) = G'$

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$$\begin{tikzcd} G \arrow[rr, "\varphi"] \arrow[dr, "\widetilde{\varphi}"'] & & G' \\ & \widetilde{G} & \end{tikzcd}$$

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$$\begin{tikzcd} G \arrow[rr, "\varphi"] \arrow[dr, "\widetilde{\varphi}"'] & & G' \arrow[dl, "\bar{\varphi} = \varphi^{-1} \circ \widetilde{\varphi}"] \\ & \widetilde{G} & \end{tikzcd}$$

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Challenges: 1. show $\widetilde{\varphi}$, 2. show $\bar{\varphi}$

ZK: $\widetilde{\varphi}$, $\bar{\varphi}$ do not reveal anything on φ

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Soundness error: t rounds 2^{-t}

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ongoing NIST standardization process for post-quantum signature schemes

- LESS linear equivalence
- MEDS matrix code equivalence
- PERK subcode equivalence

Main question: How hard is code equivalence?

- complexity class
- solvers

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- complexity class
- can reduce PEP to GI

- solvers
- can reduce LEP to PEP if $q < 5$

Is Code Equivalence NP-hard?

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...no

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...no

Arthur-Merlin Protocol

Merlin has a ”no” instance: $\mathcal{C}_1, \mathcal{C}_2$, wants to convince Arthur $\nexists\varphi$

- Arthur chooses G_i and ψ
- Arthur sends $G' = \psi(G_i)$
- Merlin replies with i

- t rounds $\rightarrow 2^{-t}$
- not NP-hard, else AM = PH $\rightarrow$ complexity hierarchy collapses

Sneak Peek

Reduction from PEP to GI

$\mathcal{G} = (V, E)$ weighted graph weight on edge $\{u, v\}$ is $w(u, v)$ $V = [1, n]$

Graph Isomorphism (GI)

Given $\mathcal{G} = (V, E), \mathcal{G}' = (V, E')$, find $\sigma \in \mathcal{S}_n$, s.t.

1. $\{u, v\} \in E \leftrightarrow \{\sigma(u), \sigma(v)\} \in E'$ 2. $w(u, v) = w(\sigma(u), \sigma(v))$

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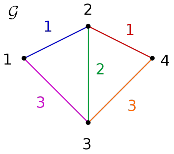

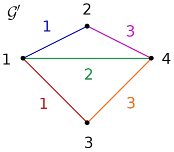

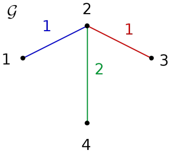

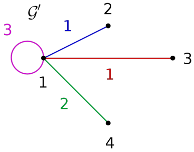

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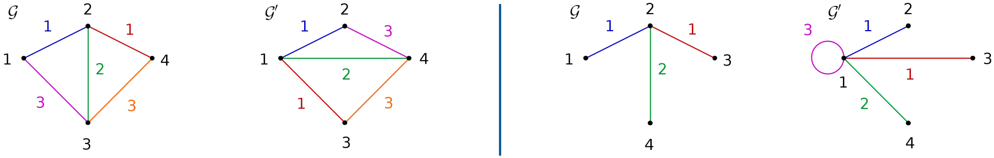

L. Babai. "Graph isomorphism in quasipolynomial time", ACM, 2016.

Sneak Peek

Adjacency matrix A: $A_{i,j} = w(i,j)$ if $\{i,j\} \in E$ and 0 else. $\rightarrow$ symmetric

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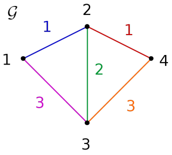

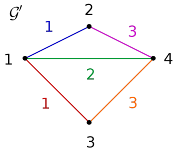

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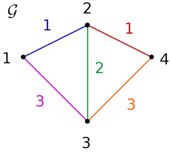

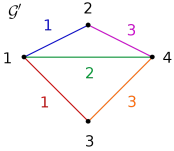

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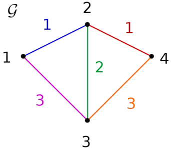

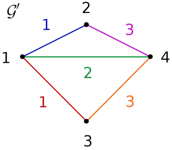

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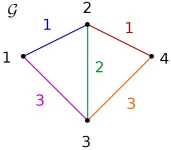

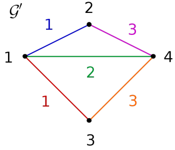

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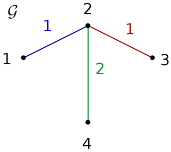

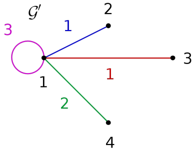

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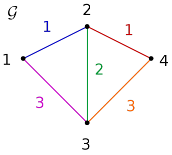

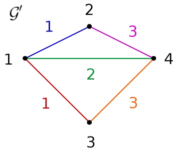

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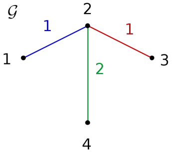

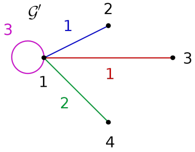

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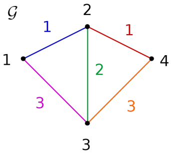
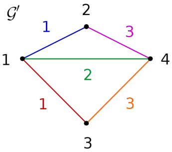
Sneak Peek

Adjacency matrix A: $A_{i,j} = w(i,j)$ if $\{i,j\} \in E$ and 0 else. $\rightarrow$ symmetric





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$\rightarrow$ PEP easier than GI (quasi-polynomial)

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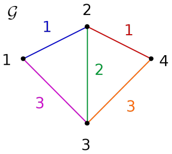

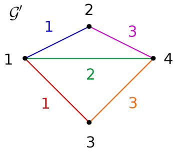

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How to choose A for code $\mathcal{C}$ to form graph $\mathcal{G}$?

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M. Bardet, A. Otmani, and M. Saeed-Taha. "Permutation code equivalence is not harder than graph isomorphism when hulls are trivial", ISIT, 2019.

For $\mathcal{C} = \langle G \rangle$ with trivial hull:

$$A = G^\top (GG^\top)^{-1} G \in \mathbb{F}_q^{n \times n}$$

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$$(SG)^\top (SG(SG)^\top)^{-1} SG = G^\top (GG^\top)^{-1} G$$

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take $\mathcal{G}, \mathcal{G}'$ having adjacency matrices A, A'

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$\leftarrow$: If $\sigma(\mathcal{G}) = \mathcal{G}'$ then $P^\top AP = A'$ $\quad \rightarrow \sigma(\mathcal{C}) = \mathcal{C}'$

Sneak Peek

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For $\mathcal{C} = \langle G \rangle$ with trivial hull: only works for instance $(\mathcal{C}, \mathcal{C}')$ w.h.p.

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$$\sigma(\mathcal{C}) = \mathcal{C}' \leftrightarrow \sigma(\mathcal{G}) = \mathcal{G}'$$

$\rightarrow$: If $\sigma(\mathcal{C}) = \mathcal{C}'$ then for any gen. matrices G, G' ex. S: $SGP = G'$

$$A' = (GP)^\top (GP(GP)^\top)^{-1} GP = P^\top G^\top (GG^\top)^{-1} GP = P^\top AP$$

Sneak Peek

Reduction from LEP to PEP and why only for $q < 5$

Closure of code: For $\mathcal{C} \subset \mathbb{F}_q^n$ its closure is

$$\widetilde{\mathcal{C}} = \{(\alpha c_i)_{(i,\alpha)\in[1,n]\times\mathbb{F}_q^\star} \mid (c_i)_{i\in[1,n]} \in \mathcal{C}\} \subset \mathbb{F}_q^{n(q-1)}$$

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$$G = \begin{pmatrix} | & & | \\ g_1 & \cdots & g_n \\ | & & | \end{pmatrix} \to \widetilde{G} = \begin{pmatrix} | & | & & | & & | & | & & | \\ g_1 & \alpha g_1 & \cdots & \alpha^{q-2} g_1 & \cdots & g_n & \alpha g_n & \cdots & \alpha^{q-2} g_n \\ | & | & & | & & | & | & & | \end{pmatrix}$$

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$$G P \mathrm{diag}(v) = G' \quad \leftrightarrow \quad \widetilde{G}\widetilde{P} = \widetilde{G}'$$

$$\widetilde{P} = \begin{pmatrix} P_1 & & \\ & \ddots & \\ & & P_n \end{pmatrix} Q, \; P_i \in S_{q-1}, \; Q \text{ block permutation}$$

Sneak Peek

.. and why only for $q < 5$?

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For $q \geq 5$ $\quad \widetilde{\mathcal{C}} \subset \widetilde{\mathcal{C}}^{\perp}$ weakly self dual and $\mathcal{H}(\widetilde{\mathcal{C}}) = \widetilde{\mathcal{C}}$ $\lightning$

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For $q \geq 5$ $\quad \widetilde{\mathcal{C}} \subset \widetilde{\mathcal{C}}^{\perp}$ weakly self dual and $\mathcal{H}(\widetilde{\mathcal{C}}) = \widetilde{\mathcal{C}}$ ⚡

$$X = \widetilde{G}\widetilde{G}^{\top} = \begin{pmatrix} | & | & & | \\ g_1 & \alpha g_1 & \cdots & \alpha^{q-2} g_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} - & g_1 & - \\ - & \alpha g_1 & - \\ & \vdots & \\ - & \alpha^{q-2} g_n & - \end{pmatrix}$$

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$\alpha^2 \neq 1$ $\quad$ needs $q \geq 4$

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What is known/Spoilers

Complexity Class

- Code equivalence is not NP-hard
- Easy instance of PEP:
  Random codes!

- rand. reduction from PEP to GI
- reduction from LEP to PEP
  if $q \geq 5$: weakly self dual

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Solvers

$$\begin{tikzcd} \mathcal{C} \arrow[d, "\varphi"'] \\ \mathcal{C}' \end{tikzcd}$$

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Solvers

- all solvers have exponential cost
- use Information Set Decoding (ISD)

$$
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Methods/Buzzwords

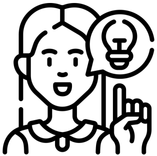
Tools we use/ what to expect in the project

- behavior of different hulls
- automorphism groups
- weight enumerators
- supports of subcodes

- code-based crypto
- algorithmics
- complexity theory
- ISD

Questions

- Other reductions (not randomized)?
- Other easy instances?
- Other solvers: other invariants/subcodes?
- Other metrics?



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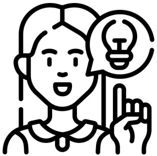

Notes on Code Equivalence

Thank you!

References

- M. Bardet, A. Otmani, and M. Saeed-Taha. "Permutation code equivalence is not harder than graph isomorphism when hulls are trivial", ISIT, 2019.
- N. Sendrier, D. E. Simos. "How easy is code equivalence over Fq?", WCC, 2013.
- E. Petrank, R.M. Roth. "Is code equivalence easy to decide?", IEEE TIT, 1997.
- N. Sendrier. "On the dimension of the hull.", SIAM Discr. Math., 1997.
- N. Sendrier. "Finding the permutation between equivalent linear codes: The support splitting algorithm.", IEEE TIT, 2000.
- I. Bouyukliev. "About the code equivalence.", Adv. in Coding and Crypto., 2007.
- L. Babai, P. Codenotti, J. Grochow, Y. Qiao. "Code equivalence and group isomorphism.", SIAM Discr. Math., 2011.
- A. Barenghi, J.F. Biasse, E. Persichetti, P. Santini. "On the computational hardness of the code equivalence problem in cryptography.", AMC, 2023.
- W. Beullens "Not enough LESS: an improved algorithm for solving code equivalence problems over F q.", Crypto, 2020.
- N. Sendrier, D. E. Simos. "The hardness of code equivalence over and its application to code-based cryptography.", PQCrypto, 2013.
- J. Leon. "Computing automorphism groups of error-correcting codes.", IEEE TIT, 1982.
- A. Barenghi, J.F. Biasse, E. Persichetti, P. Santini. "LESS-FM: fine-tuning signatures from the code equivalence problem.", PQCrypto, 2021.