What is... the McEliece system?

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Outline

Toy Example: Repetition code

Repetition Code:

$$\begin{array}{ccc} \text{Me} & \to & \text{you} \\ 1 & \mapsto & 0 \end{array}$$

Toy Example: Repetition code

Repetition Code:

$$\begin{aligned} \text{Me} &\rightarrow \text{you} \\ 1 &\mapsto 0 \\ 111111 &\mapsto 111010 \end{aligned}$$

We can correct 2 errors and detect 3 errors.

Let $\mathbb{F}_q$ be a finite field.

Definition (Linear Code)

An $[n,k]$-linear code $\mathcal{C}$ over $\mathbb{F}_q$ is a k-dimensional linear subspace of $\mathbb{F}_q^n$. $c \in \mathcal{C}$ is called a codeword.

The toy example of the repetition code was a $[6,1]$-linear code over $\mathbb{F}_2$, with the codewords $\{000000, 111111\}$.

Coding Theory

Let $\mathcal{C}$ be an $[n, k]$-linear code over $\mathbb{F}_q$.

Definition (Generator Matrix)

There exists an $k \times n$ generator matrix G of $\mathcal{C}$ defined by:

$$\mathcal{C} = \left\{ uG \mid u \in \mathbb{F}_q^k \right\}.$$

Definition (Parity Check Matrix)

There exists an $(n-k) \times n$ parity check matrix H of $\mathcal{C}$ defined by:

$$\mathcal{C} = \left\{ x \in \mathbb{F}_q^n \mid Hx^T = \mathbf{0} \right\}.$$

Let $\mathcal{C}$ be an $[n, k]$-linear code over $\mathbb{F}_q$. Let G be its $k \times n$ generator matrix.

Definition (Information Set)

A set of k coordinates $I \subset \{1, \ldots, n\}$, for which the columns of G are linearly independent is called an information set.

Definition (Systematic Form)

If G is of the form

$$\left(Id_k \mid A\right),$$

we say G is of systematic form and then H is given by

$$\left(-A^T \mid Id_{n-k}\right).$$

Let $x, y \in \mathbb{F}_q^n$.

Definition (Hamming Distance)

The Hamming distance of x, y is defined as

$$d(x, y) = \mid \{i \in \{1, \dots, n\} \mid x_i \neq y_i\} \mid .$$

Definition (Hamming Weight)

The Hamming weight of x is defined as

$$wt(x) = \mid \{i \in \{1, \dots, n\} \mid x_i \neq 0\} \mid .$$

Let $\mathcal{C}$ be an $[n, k]$-linear code over $\mathbb{F}_q$.

Definition (Minimum Distance)

We define the minimum distance of $\mathcal{C}$ to be

$$\begin{aligned} d(\mathcal{C}) &= \min\{d(x,y) \mid x, y \in \mathcal{C}, x \neq y\} \\ &= \min\{wt(x) \mid x \in \mathcal{C}, x \neq \boldsymbol{0}\}. \end{aligned}$$

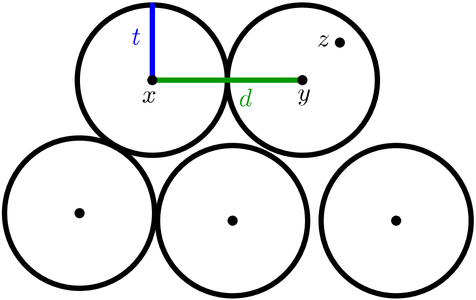
In our toy example of the $[6, 1]$-Repetition code we have $d(\mathcal{C}) = 6$.

Theorem (Singleton Bound)

Let $\mathcal{C}$ be an $[n, k]$-linear block code. Then $d(\mathcal{C}) \leq n - k + 1$.

Theorem

Let $\mathcal{C}$ be an $[n, k]$-linear code over $\mathbb{F}_q$ with minimum distance d. Then $\mathcal{C}$ can correct up to $t = \lfloor \frac{d-1}{2} \rfloor$ errors.



Let $\mathbb{F}_q$ be a finite field and $1 \leq k < n \leq q$ integers.

Definition (Generalized Reed-Solomon Code)

Let $\alpha \in \mathbb{F}_q^n$ be an n-tuple of distinct elements and $\beta \in \mathbb{F}_q^n$, be an n-tuple of nonzero elements.

$$GRS_{n,k}(\alpha, \beta) = \{(\beta_1 p(\alpha_1), \ldots, \beta_n p(\alpha_n)) \mid p \in \mathbb{F}_q[x], \; deg(p) < k\}.$$

We can write the generator matrix of $GRS_{n,k}(\alpha, \beta)$ as

$$G = \begin{pmatrix} \beta_1 & \cdots & \beta_n \\ \beta_1\alpha_1 & \cdots & \beta_n\alpha_n \\ \vdots & & \vdots \\ \beta_1\alpha_1^{k-1} & \cdots & \beta_n\alpha_n^{k-1} \end{pmatrix}.$$

Difference between Coding and Cryptography

Coding

$$m \xrightarrow{\textit{encoding}} c \xrightarrow{\textit{noisy channel}} c+e \xrightarrow{\textit{decoding}} m$$

$$m \xrightarrow[\textit{encryption}]{} c \xrightarrow[\textit{send}]{} c \xrightarrow[\textit{decryption}]{} m$$

Public Key Cryptography

Public-Key Cryptography

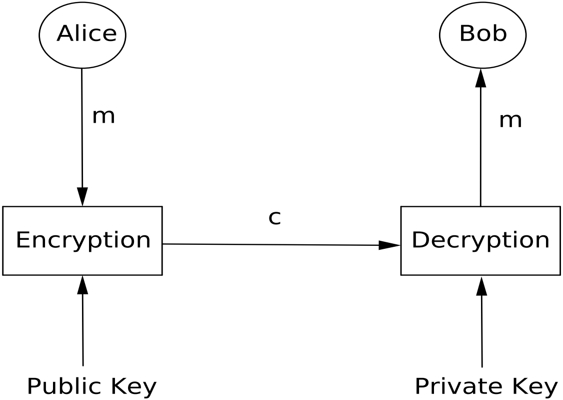
We consider two people: Bob and Alice.

Key generation:
Bob constructs a private key and a public key, which he publishes.

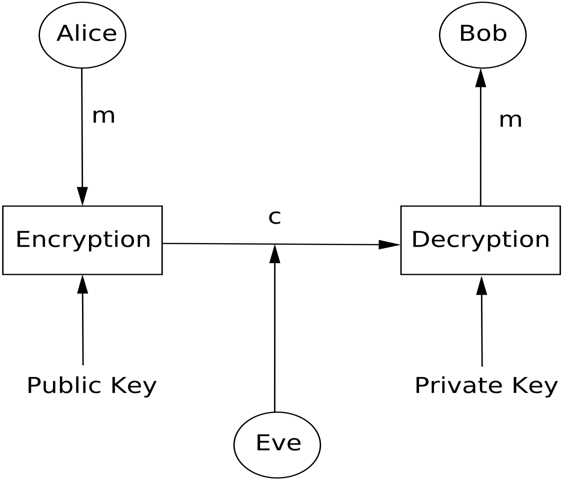
Encryption:
Alice uses the public key to encrypt the message m to get the cipher c and sends c to Bob.

Decryption:
Bob uses the private key to decrypt the cipher c and recover the message m.

Public-Key Cryptography



Public-Key Cryptography



Public-Key Cryptography

Example: RSA

Let p, q be primes. Compute $n = pq$ and the Euler-totient function $\phi(n) = (p-1)(q-1)$. Choose $e < \phi(n)$, s.t. $\gcd(e, \phi(n)) = 1$.

$$\begin{aligned} \text{Public Key} &= (n, e) \\ \text{Private Key} &= (p, q) \end{aligned}$$

Encryption: Let m be the message. The cipher is computed as

$$c = m^e \mod n.$$

Decryption: Compute d and b s.t.

$$de + b\phi(n) = 1.$$

Then by computing $c^d \mod n$ we recover the message, since

$$c^d = (m^e)^d = m^{1-b\phi(n)} = m(m^{\phi(n)})^{-b} \equiv m1^{-b} = m \mod n.$$

Post-quantum Cryptography

- The PKC systems, which we currently use are: RSA, DLP over elliptic curves or finite fields, ...
- NSA and NIST believe that a quantum computer will be available in 2030.
- Shor's Algorithm and Grover's Algorithm are quantum algorithms and will break those systems.
- Cryptosystems which will be resistant against attacks on a quantum computer are called post-quantum cryptosystems.
- Promising candidates for post-quantum cryptography are: lattice-based cryptosystems, multivariate cryptography and code-based cryptography.

Choose an $[n, k]$-linear code $\mathcal{C}$ over $\mathbb{F}_q$, which can correct upto t errors and has an efficient decoding algorithm. $\mathcal{C}$ has a generator matrix G of size $k \times n$. Choose a $k \times k$ invertible matrix S and a $n \times n$ permutation matrix P and compute $G' = SGP$.

$$\begin{aligned} \text{Public Key} &= (G', t) \\ \text{Private Key} &= (S, G, P) \end{aligned}$$

McEliece Cryptosystem

Encryption: Let $m \in \mathbb{F}_q^k$ be the message and $e \in \mathbb{F}_q^n$ the error vector, s.t. $\mathrm{wt}(e) \leq t$, then the cipher is computed as

$$c = mG' + e.$$

Decryption: Compute

$$cP^{-1} = mSG + eP^{-1},$$

then mSG is a code word of $\mathcal{C}$ and since $\mathrm{wt}(eP^{-1}) \leq t$, we can apply the decoding algorithm and get mS and by multiplication with the inverse of S we get the message m.

Niederreiter system

Choose an $[n, k]$-linear code $\mathcal{C}$, that can correct upto t errors and has an efficient decoding algorithm. $\mathcal{C}$ has a parity check matrix H of size $(n-k) \times n$. Choose a $(n-k) \times (n-k)$ invertible matrix S and a $n \times n$ permutation matrix P and compute $H' = SHP$.

$$\begin{aligned}\text{Public Key} &= (H', t)\\ \text{Private Key} &= (S, H, P)\end{aligned}$$

Niederreiter system

Encryption: Let $m \in \mathbb{F}_q^n$ be the message, s.t. $\mathrm{wt}(m) \leq t$, then the cipher is computed as

$$c^T = H' m^T.$$

Decryption: Compute

$$S^{-1} c^T = HPm^T = H(mP^T)^T.$$

Since $\mathrm{wt}(mP^T) \leq t$, we can apply syndrome decoding to get mP^T and by multiplication with the inverse of P^T we get the message m.

Security of McEliece Cryptosystem

The underlying problem of decoding a random linear code is an NP-complete problem, this makes it a quantum-secure cryptosystem.
Nevertheless, the codes we use are not random, hence there might exist structural attacks.
There also exists a nonstructural attack called Information Set Decoding (ISD), which has to be considered for the choice of secure parameters. The complexity of the best algorithms so far is $\mathcal{O}(2^{n/20})$.

ISD

The easiest version of the ISD algorithm is given by Lee-Brickell over the binary:
We denote by e_I, c_I, G_I its k columns indexed by the information set.

Input: $G \in \mathbb{F}_2^{k \times n}$, $c = mG + e$, where $e \in \mathbb{F}_2^n$ of weight $t \in \mathbb{N}$, $p < t$.
Output: $e \in \mathbb{F}_2^n$.

1. Choose an information set $I \subset \{1, \dots, n\}$ of size k.
2. Choose e_I with $wt(e_I) = p$.
3. If $wt(c + (c_I + e_I) G_I^{-1} G) = t$:
   Output $e = c + (c_I + e_I) G_I^{-1} G$.
4. Else: go back to 1.

To picture how this algorithm works, assume that G is given in systematic form and hence $I = \{1, \ldots, k\}$ and

$$G = \left(\mathrm{Id}_k \mid \quad A\right).$$

Hence if we have chosen e_I correctly, i.e. the correct error distribution in the first k bits, then $c_I + e_I = mG_I$ and hence $(c_I + e_I)G_I^{-1} = m$ and $c + (c_I + e_I)G_I^{-1}G = c + mG = e$.

Advantages and Disadvantages of McEliece Cryptosystem

Although the McEliece system is quantum secure, there is the major drawback of large key sizes:

Security Level	Key Size RSA	Key Size original McEliece
2^{80}	1248	520047
2^{128}	3248	1537536
2^{256}	15424	7667855

Research

The main idea to bring down the key sizes is to use another family of codes.

Proposal	Idea	Attack
Niederreiter	GRS codes	Sidelnikov-Shestakov
Berger, Loidreau	Subcodes of GRS codes	Wieschebrink
Gabidulin *et al.*	Gabidulin codes	Overbeck
Sidelnikov	Reed-Muller codes	Minder-Shokrollahi
Baldi *et al.*	LDPC codes	Couvreur *et al.*
Rosenthal *et al.*	GRS, new scrambling	Couvreur *et al.*

New proposals:

Proposal	Idea	Attack
Baldi *et al.*	QC-MDPC codes	
Baldi *et al.*	MDPC codes	
Khathuria, Rosenthal, W.	GRS, weight two matrix	
Horlemann-Trautmann, W.	Ring linear codes	

Thank you!