The Local-to-Global Principle for Densities

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Algebra and Number Theory Seminar

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Introduction

Fix a subset $T \subseteq \mathbb{Z}$. How likely is it for a randomly chosen $x \in \mathbb{Z}$ to be in T?

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No uniform probability distribution over $\mathbb{Z}$.

$$1 = \mathbb{P}(\mathbb{Z}) = \sum_{x \in \mathbb{Z}} \mathbb{P}(\{x\}) = \sum_{x \in \mathbb{Z}} p \neq 1$$

Definition (Natural Density over $\mathbb{Z}^d$)

Let $d \in \mathbb{N}$ and $T \subseteq \mathbb{Z}^d$. The density of T is given by

$$\rho(T) = \lim_{H \to \infty} \frac{| T \cap [-H, H[^d|}{(2H)^d},$$

if the limit exists.

We define the upper density $\overline{\rho}$ and the lower density $\underline{\rho}$ with the limsup, respectively with the the liminf.

Outline

Properties

Proposition

1. $\rho(\emptyset) = 0$ *and* $\rho(\mathbb{Z}^d) = 1$,
2. *if* $A \subseteq B \subseteq \mathbb{Z}^d$, *then* $\rho(A) \leq \rho(B)$,
3. *if* $T \subseteq \mathbb{Z}^d$, *then* $\rho(T) \in [0, 1]$,
4. *if* $F \subseteq \mathbb{Z}^d$ *is finite, then* $\rho(F) = 0$,
5. *if* $A, B \subseteq \mathbb{Z}^d$ *with* $A \cap B = \emptyset$, *then* $\rho(A \cup B) = \rho(A) + \rho(B)$,
6. *if* $B \subseteq A$, *then* $\rho(A \setminus B) = \rho(A) - \rho(B)$,

assuming that $\rho(A), \rho(B)$ *and* $\rho(T)$ *exist.*

Properties

Important difference to probability:

$$\rho\left(\bigcup_{i\in I} A_i\right) \not\leq \sum_{i\in I} \rho(A_i)$$

for a countable set I and $A_i \subseteq \mathbb{Z}$.

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for a countable set I and $A_i \subseteq \mathbb{Z}$.

Counterexample: $I = \mathbb{Z}$, $A_i = \{i\}$.

$$1 = \rho(\mathbb{Z}) = \rho\left(\bigcup_{i\in\mathbb{Z}} \{i\}\right) \not\leq \sum_{i\in\mathbb{Z}} \rho(\{i\}) = 0.$$

First Examples

Example (Primes)

The density of primes $\mathcal{P}$ *is*

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By the prime number theorem we have that

$$\pi(x) = \mid \{p \in \mathcal{P} \mid p \leq x\} \mid \sim \frac{x}{\ln(x)}.$$

Hence

$$\rho(\mathcal{P}) = \lim_{H \to \infty} \frac{\mid \mathcal{P} \cap [-H, H[\mid}{2H} = \lim_{H \to \infty} \frac{\pi(H)}{2H} = 0.$$

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Example (Invertible Matrices)

For n a positive integer, the density of invertible matrices $GL_n(\mathbb{Z})$ is

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Let us fix all entries of $A \in [-H, H[^{n\times n}$ except for $a_{n,n}$. Since

$$\pm 1 = \det(A) = \sum_{j=1}^{n-1} (-1)^{n+j} a_{n,j} \det(A_{n,j}) + a_{n,n} \det(A_{n,n}),$$

we have at most two choices for $a_{n,n}$. Hence

$$\lim_{H\to\infty} \frac{|\, GL_n(\mathbb{Z}) \cap [-H, H[^{n^2}|}{(2H)^{n^2}} \leq \lim_{H\to\infty} \frac{2(2H)^{n^2-1}}{(2H)^{n^2}} = 0.$$

Example (Divisibility by n)

For n a positive integer, the density of $n\mathbb{Z}$ is

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Since $\mid n\mathbb{Z} \cap [-H, H[\mid = \lceil \frac{2H}{n} \rceil$, we have

$$\lim_{H\to\infty} \frac{\mid n\mathbb{Z} \cap [-H, H[\mid}{2H} = \lim_{H\to\infty} \frac{\lceil \frac{2H}{n} \rceil}{2H} = 1/n.$$

Mertens-Cesàro Theorem

How likely is it that two randomly chosen integers are coprime?

Mertens-Cesàro Theorem

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Theorem (F. Mertens, 1874 and E. Cesàro, 1884)

Let the set of coprime pairs be denoted by C. Then

$$\rho(C) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}.$$

ζ denotes the Riemann-zeta function

$$\zeta(s) = \sum_{s \geq 1} \frac{1}{n^s} = \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

F. Mertens. "Über einige Asymptotische Gesetze der Zahlentheorie", Journal für die reine und angewandte Mathematik, 1874.

E. Cesàro, "Probabilité de certains faits arithméthiques", Mathesis, 1884

Generalization of Mertens-Cesàro

Theorem (J. Nymann, 1972)

Let $m \geq 2$ be a positive integer and the set of coprime m-tuples be denoted by C_m. Then

$$\rho(C_m) = \frac{1}{\zeta(m)}.$$

J. Nymann. "On the Probability that k Positive Integers are Relatively Prime", Journal of Number Theory, 1972.

Definition (Rectangular Unimodular Matrix)

Let $n < m \in \mathbb{N}$. $M \in Mat_{n\times m}(\mathbb{Z})$ is rectangular unimodular, if we can extend it to an $m \times m$ matrix in $GL_m(\mathbb{Z})$.

Definition (Rectangular Unimodular Matrix)

Let $n < m \in \mathbb{N}$. $M \in Mat_{n\times m}(\mathbb{Z})$ is rectangular unimodular, if we can extend it to an $m \times m$ matrix in $GL_m(\mathbb{Z})$.

Let $n < m$ then we define the rectangular unimodular matrices as

$$U_{n,m} = \{A \in \mathbb{Z}^{n\times m} \mid \forall p \in \mathcal{P} \ \mathrm{rk}(A \bmod p) = n\}.$$

p-adic Integers

- p-adic valuation:

$$\begin{aligned} \mathrm{ord}_p(x) &= \max\{v : p^v \mid x\} && x \in \mathbb{Z} \setminus \{0\}, \\ \mathrm{ord}_p(0) &= \infty, \\ \mathrm{ord}_p(y) &= \mathrm{ord}_p(a) - \mathrm{ord}_p(b) && y = a/b \in \mathbb{Q} \setminus \{0\}. \end{aligned}$$

- p-adic norm:

$$\mid x \mid_p = \begin{cases} \frac{1}{p^{\mathrm{ord}_p(x)}} & x \neq 0, \\ 0 & x = 0. \end{cases}$$

- The p-adic number $\mathbb{Q}_p$ are the completion of $\mathbb{Q}$ with respect to $\mid \cdot \mid_p$

- The p-adic integers $\mathbb{Z}_p = \{\alpha \in \mathbb{Q}_p : \mid \alpha \mid_p \leq 1\}$

- Haar measure μ_p on $\mathbb{Z}_p$: $\mu_p(a + p^k\mathbb{Z}_p) = \frac{1}{p^k}$

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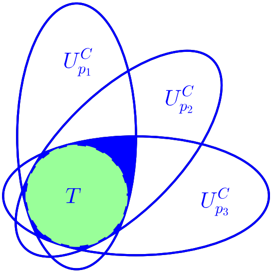
  $\beta \in \mathbb{Z}_p : \beta = \sum_{i=0}^{\infty} a_i p^i$

- Haar measure μ_p on $\mathbb{Z}_p$: $\mu_p(a + p^k\mathbb{Z}_p) = \frac{1}{p^k}$

Local-to-Global Principle

Idea



$$T = \bigcap_{p \in \mathcal{P}} \left(U_p^C \cap \mathbb{Z}\right)$$

$$T_M = \bigcap_{p < M} \left(U_p^C \cap \mathbb{Z}\right)$$

$$T_M \setminus T \subseteq \bigcup_{p > M} \left(U_p \cap \mathbb{Z}\right)$$

$$\rho(T) = \rho(T_M) - \rho(T_M \setminus T)$$

Local-to-Global Principle

Theorem (B. Poonen and M. Stoll, 1999)

Let $U_\infty \subset \mathbb{R}^d$, *such that* $\mathbb{R}_{\geq 0} \cdot U_\infty = U_\infty$ *and* $\mu_\infty(\partial(U_\infty)) = 0$.
Let $s_\infty = \frac{1}{2^d}\mu_\infty(U_\infty \cap [-1,1]^d)$.
For each $p \in \mathcal{P}$, *let* $U_p \subset \mathbb{Z}_p^d$, *such that* $\mu_p(\partial(U_p)) = 0$ *and define* $s_p = \mu_p(U_p)$. *Define*

$$\begin{aligned} P : \mathbb{Z}^d &\to 2^{M_\mathbb{Q}}, \\ a &\mapsto \{\nu \in M_\mathbb{Q} \mid a \in U_\nu\}. \end{aligned}$$

If

$$\lim_{M\to\infty} \bar{\rho}\left(\left\{a \in \mathbb{Z}^d \mid a \in U_p \text{ for some prime } p > M\right\}\right) = 0, \quad (1)$$

Theorem (B. Poonen and M. Stoll, 1999)

then:

i) $\sum_{\nu \in M_{\mathbb{Q}}} s_\nu$ *converges.*

ii) *For* $\mathcal{S} \subset 2^{M_{\mathbb{Q}}}$, $\rho(P^{-1}(\mathcal{S}))$ *exists, and defines a measure on* $2^{M_{\mathbb{Q}}}$.

iii) *For each finite set* $S \in 2^{M_{\mathbb{Q}}}$, *we have that*

$$\rho(P^{-1}(\{S\})) = \prod_{\nu \in S} s_\nu \prod_{\nu \notin S} (1 - s_\nu),$$

and if $\mathcal{S}$ *consists of infinite subsets of* $2^{M_{\mathbb{Q}}}$, *then* $\rho(P^{-1}(\mathcal{S})) = 0$.

B. Poonen and M. Stoll. "The Cassels-Tate Pairing on Polarized Abelian Varieties", Annals of Mathematics, 1999.

B. Poonen and M. Stoll. "A Local-Global Principle for Densities", Topics in Number Theory, 1999.

Strategy

$$T = \{a \in \mathbb{Z}^d \mid \forall p \in \mathcal{P} \ a \bmod p \text{ satisfies } C_p\}$$

1. Choose system (U_ν):
   - Choose $S = \emptyset$.
   - Choose $U_\infty = \emptyset$.
   - Choose U_p such that $P^{-1}(\{\emptyset\}) = T$, that is

     $$U_p = \{a \in \mathbb{Z}_p^d \mid a \bmod p \text{ does not satisfy } C_p\}.$$
2. Compute $s_p = \mu_p(U_p)$.
3. Show that Condition (1) is satisfied.
4. Compute density.

Local-to-Global Principle

Lemma

Let d and M be positive integers. Let $f, g \in \mathbb{Z}[x_1, \dots, x_d]$ be relatively prime. Define $S_M(f,g)$ as

$$\left\{ a \in \mathbb{Z}^d \mid f(a) \equiv g(a) \equiv 0 \bmod p \textit{ for some prime } p > M \right\},$$

then

$$\lim_{M \to \infty} \bar{\rho}(S_M(f,g)) = 0.$$

Strategy

Choose coprime f, g such that

$$\left\{ a \in \mathbb{Z}^d \mid a \in U_p \text{ for some prime } p > M \right\} \subset S_M(f,g).$$

Rectangular Unimodular Matrices

Theorem (G. Micheli and V. W., 2019)

Let $n < m$ be a positive integers and the set of rectangular unimodular $n \times m$ matrices be denoted by $U_{n,m}$. Then

$$\rho(U_{n,m}) = \prod_{i=0}^{n-1} \frac{1}{\zeta(m-i)}.$$

G. Micheli and V. Weger. "On rectangular unimodular matrices over the algebraic integers", SIAM Journal on Discrete Mathematics, 2019.

Proof

1. **Choose system (U_ν):**
   - $T = U_{n,m} = \{A \in \mathbb{Z}^{n\times m} \mid \forall p \in \mathcal{P} \ \mathrm{rk}(A \bmod p) = n\}$
     thus C_p is A mod p has full rank.
   - Define $\pi_p : \mathbb{Z}_p^{n\times m} \to \mathbb{F}_p^{n\times m}$ as $A \mapsto A \bmod p$.
   - $\mathcal{L}_p = \{A \in \mathbb{F}_p^{n\times m} \mid \mathrm{rk}(A) = n\}$.
   - $\mid \mathcal{L}_p \mid = \prod_{i=0}^{n-1} \left(p^m - p^i\right)$.
   - If $X \in \mathbb{Z}_p^{n\times m}$ is such that $\pi_p(X) \in \mathcal{L}_p$, then also $X + p\mathbb{Z}_p^{nm}$.
   - Define $A_p = \left\{X \in \mathbb{Z}_p^{n\times m} \mid \pi_p(X) \in \mathcal{L}_p\right\} + p\mathbb{Z}_p^{nm}$.
   - U_p should be the elements that do not satisfy C_p:
     Choose $U_p = A_p^C$.

Proof

2. Compute $s_p = \mu_p(U_p)$:

- $A_p = \bigcup_{A \in \mathcal{L}_p} (\pi_p^{-1}(A) + p\mathbb{Z}_p^{nm})$.
- Compute the Haar measure of A_p as

$$
\begin{aligned}
\mu_p(A_p) &= \sum_{A \in \mathcal{L}_p} \mu_p(\pi_p^{-1}(A) + p\mathbb{Z}_p^{nm}) \\
&= \mu_p(p\mathbb{Z}_p^{nm}) \mid \mathcal{L}_p \mid \\
&= \frac{1}{p^{nm}} \prod_{i=0}^{n-1} \left(p^m - p^i\right) \\
&= \prod_{i=0}^{n-1} \left(1 - \frac{1}{p^{m-i}}\right).
\end{aligned}
$$

- $s_p = \mu_p\left(U_p\right) = 1 - \mu_p(A_p)$.

Proof

3. Show that Condition (1) **is satisfied:**
Use the helpful lemma with f giving the first basic minor and g the second.

$$\begin{aligned} S_M(f,g) = \{A \in \mathbb{Z}^{nm} \mid f(A) \equiv g(A) \equiv 0 \bmod p \\ \text{for some } p > M\} \\ \supset \{A \in \mathbb{Z}^{nm} \mid A \in U_p \text{ for some } p > M\}. \end{aligned}$$

4. Compute density:

$$\begin{aligned} \rho(P^{-1}(\{\emptyset\})) &= \prod_{\nu \in \emptyset} s_\nu \prod_{\nu \notin \emptyset} (1 - s_\nu) = \prod_{p \in \mathcal{P}} (1 - s_p) \\ &= \prod_{p \in \mathcal{P}} \prod_{i=0}^{n-1} \left(1 - \frac{1}{p^{m-i}}\right) = \prod_{i=0}^{n-1} \frac{1}{\zeta(m-i)}. \end{aligned}$$

Eisenstein Polynomials

Let $f \in \mathbb{Z}[x]$ be a polynomial of degree d, i.e.,

$$f = \sum_{i=0}^{d} a_i x^i,$$

then we identify f by $(a_0, \dots, a_d) \in \mathbb{Z}^{d+1}$.

Definition (Eisenstein Polynomials)

Let $f \in \mathbb{Z}[x]^d$ be of degree d and having associated tuple $(a_0, \dots, a_d)$. Then we call f an Eisenstein polynomial, if $p^2 \nmid a_0, p \nmid a_d$ and for all $i < d$ we have $p \mid a_i$, for some $p \in \mathcal{P}$.

In addition, we say that f is p-Eisenstein if f satisfies the criterion of Eisenstein for this prime p.

Eisenstein Polynomials

Theorem (R. Heymann and I. Shparlinski, 2013)

Let the set of Eisenstein polynomials be denoted by E_d. Then

$$\rho(E_d) = 1 - \prod_{p \in \mathcal{P}} \left(1 - \frac{(p-1)^2}{p^{d+2}} \right).$$

Eisenstein Polynomials

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Let the set of Eisenstein polynomials be denoted by E_d. Then

$$\rho(E_d) = 1 - \prod_{p \in \mathcal{P}} \left(1 - \frac{(p-1)^2}{p^{d+2}}\right).$$

Using the Local-to-Global Principle we can set

$$U_p = (p\mathbb{Z}_p \setminus p^2\mathbb{Z}_p) \times (p\mathbb{Z}_p)^{d-1} \times (\mathbb{Z}_p \setminus p\mathbb{Z}_p),$$

with $s_p = \frac{(p-1)^2}{p^{d+2}}$.

R. Heymann and I. Shparlinski. "On the number of Eisenstein polynomials of bounded height", Applicable Algebra in Engineering, Communication and Computing, 2013.

How many primes on average are such that an Eisenstein polynomial satisfies the criterion of Eisenstein?

M. Shilin, K. McGown, D. Rhodes and M. Wanner. "On the number of primes for which a polynomial is Eisenstein", Integers, 2018.

Mean

Notation:

- Target set $T = \{a \in \mathbb{Z}^d \mid \exists p \in \mathcal{P} \ a \text{ satisfies } C_p\}$.
- Counting function ψ:

$$\begin{aligned}\psi : \mathbb{Z}^d &\to \mathbb{N}, \\ a &\mapsto \mid \{p \in \mathcal{P} \mid a \text{ satisfies } C_p\} \mid .\end{aligned}$$

- For $H \in \mathbb{N}$, define $T(H) = T \cap [-H, H[^d$.

Definition (Mean)

Let $T \subset \mathbb{Z}^d$, such that $\rho(T) \neq 0$ exists. Then we define the mean of ψ as

$$\mu = \lim_{H \to \infty} \frac{\sum\limits_{a \in T(H)} \psi(a)}{\mid T(H) \mid},$$

if it exists.

Strategy

For $s \geq 2$ a positive integer, let

$$\mathcal{H}(s,H) = \{a \in \mathbb{Z}^d \cap [-H,H[^d \mid a \text{ satisfies } C_s\}.$$

1. Compute the size of $\mathcal{H}(s,H)$.
2. Compute $\sum_{a \in T(H)} \psi(a) = \sum_{p \in \mathcal{P},\ p<H} \mid \mathcal{H}(p,H) \mid$.
3. Compute $\mid T(H) \mid$ as
$$\mid T(H) \mid = -\sum_{s=2}^{H} \mu(s) \mid \mathcal{H}(s,H) \mid .$$
4. Compute μ as
$$\mu = \lim_{H \to \infty} \frac{\sum_{a \in T(H)} \psi(a)}{\mid T(H) \mid}.$$

Mean

How many primes on average are such that an Eisenstein polynomial satisfies the criterion of Eisenstein?

Theorem (M. Shillin, K. McGown, D. Rhodes and M. Wanner, 2018)

Let $\psi(a)$ be the number of $p \in \mathcal{P}$ such that f associated to a is p-Eisenstein. The mean of $\psi(a)$, is given by

$$\mu(E_d) = \lim_{H \to \infty} \frac{\sum_{a \in E_d(H)} \psi(a)}{\mid E_d(H) \mid} = \frac{\alpha}{\rho(E_d)},$$

where

$$\alpha = \sum_{p \in \mathcal{P}} \frac{(p-1)^2}{p^{d+2}}.$$

M. Shilin, K. McGown, D. Rhodes and M. Wanner. "On the number of primes for which a polynomial is Eisenstein", Integers, 2018.

Pattern

$$\mu = \lim_{H\to\infty} \frac{\sum_{a\in T(H)} \psi(a)}{\mid T(H) \mid} = \frac{\alpha}{\rho(T)},$$

where

$$\alpha = \sum_{\nu\in M_{\mathbb{Q}}} s_\nu.$$

Local-to-Global Principle for Mean

Definition (Mean of a System)

Let H and d be positive integers, then we define the expected value of the system $(U_\nu)_{\nu \in M_{\mathbb{Q}}}$ to be

$$\mu = \lim_{H \to \infty} \frac{\sum_{A \in [-H,H[^d} |\{\nu \in M_{\mathbb{Q}} \mid A \in U_\nu\}|}{(2H)^d},$$

if it exists.

Local-to-Global Principle for Mean

We define the expected value of the system $(U_\nu)_{\nu \in M_{\mathbb{Q}}}$ restricted to T to be

$$\mu_T = \lim_{H \to \infty} \frac{\sum_{A \in [-H,H[^d \cap T} \mid \{\nu \in M_{\mathbb{Q}} \mid A \in U_\nu\} \mid}{\mid [-H, H[^d \cap T \mid},$$

if it exists.

Lemma

If the density of T exists and is nonzero and T is such that $T^C \subseteq P^{-1}(\{\emptyset\})$, then μ_T exists and is given by

$$\mu_T = \frac{\mu}{\rho(T)}.$$

Local-to-Global Principle for Mean

Theorem (G. Micheli, S. Schraven and V. W., 2020)

Let $H, d \in \mathbb{N}$. Let $U_\infty \subset \mathbb{R}^d$, such that $\mathbb{R}_{\geq 0} \cdot U_\infty = U_\infty$ and $\mu_\infty(\partial(U_\infty)) = 0$. Let $s_\infty = \frac{1}{2^d}\mu_\infty(U_\infty \cap [-1,1]^d)$.
For each prime p, let $U_p \subset \mathbb{Z}_p^d$, such that $\mu_p(\partial(U_p)) = 0$ and define $s_p = \mu_p(U_p)$. Define the following map

$$\begin{aligned} P : \mathbb{Z}^d &\to 2^{M_\mathbb{Q}}, \\ a &\mapsto \{\nu \in M_\mathbb{Q} \mid a \in U_\nu\}. \end{aligned}$$

If

1. $$\lim_{M\to\infty} \bar{\rho}\left(\left\{a \in \mathbb{Z}^d \mid a \in U_p \text{ for some prime } p > M\right\}\right) = 0,$$

Local-to-Global Principle for Mean

Theorem (G. Micheli, S. Schraven and V. W., 2020)

2. *there exists an absolute constant* $c \in \mathbb{Z}$, *and some* $m \in \mathbb{N}$ *such that for all* $H \geq 1$ *and for all* $A \in \mathbb{Z}^d$

$$\ell_{A,H} = \mid \{p \in \mathcal{P} \mid p > H, A \in U_p \cap [-H, H[^d\} \mid < c$$

3. *there exists a sequence* $(v_p)_{p \in \mathcal{P}}$, *such that for all* $p < H$

$$\mid U_p \cap [-H, H[^d \mid \leq v_p (2H)^d \text{ and } \sum_{p \in \mathcal{P}} v_p \text{ converges.}$$

Then the mean of the system $(U_\nu)_{\nu \in M_\mathbb{Q}}$ *exists and is given by*

$$\mu = \sum_{\nu \in M_\mathbb{Q}} s_\nu .$$

G. Micheli, S.Schraven and V. Weger. "Local to global principle for expected values", arXiv preprint arXiv:2008.06235, 2020.

Mean of Eisenstein Polynomials

Choose

$$U_p = (p\mathbb{Z}_p \setminus p^2\mathbb{Z}_p) \times (p\mathbb{Z}_p)^{d-1} \times (\mathbb{Z}_p \setminus p\mathbb{Z}_p).$$

1. Choose $f(x_1, \ldots, x_{d+1}) = x_1, g(x_1, \ldots, x_{d+1}) = x_2$.
2. Clear:
$$\mid \{p \in \mathcal{P} \mid p > H, A \in U_p \cap [-H, H[^d\} \mid = 0.$$
3. Clear:
$$\mid U_p \cap [-H, H[^{d+1} \mid \leq \lceil (2H)/p \rceil^d H$$
and
$$\sum_{p \in \mathcal{P}} \frac{1}{p^d} \text{ converges.}$$

Then

$$\mu(E_d) = \frac{\sum_{p \in \mathcal{P}} \frac{(p-1)^2}{p^{d+2}}}{\rho(E_d)}.$$

Thank you!