The Local-to-Global Principle for Densities

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Fix a subset $T \subseteq \mathbb{Z}$. How likely is it for a randomly chosen $x \in \mathbb{Z}$ to be in $T$?
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No uniform probability distribution over $\mathbb{Z}$.

$$1 = \mathbb{P}(\mathbb{Z}) = \sum_{x \in \mathbb{Z}} \mathbb{P}\{x\} = \sum_{x \in \mathbb{Z}} p \neq 1$$
**Definition (Natural Density over \( \mathbb{Z}^d \))**

Let \( d \in \mathbb{N} \) and \( T \subseteq \mathbb{Z}^d \). The density of \( T \) is given by

\[
\rho(T) = \lim_{H \to \infty} \frac{|T \cap [-H, H]^d|}{(2H)^d},
\]

if the limit exists.

We define the upper density \( \bar{\rho} \) and the lower density \( \underline{\rho} \) with the limsup, respectively with the liminf.
1 Introduction
   - Properties
   - First Examples

2 Mertens-Cesàro Theorem

3 Local-to-Global Principle
   - Rectangular Unimodular Matrices
   - Eisenstein Polynomials

4 Mean
   - Strategy

5 Local-to-Global Principle for Mean
   - Eisenstein Polynomials
Properties

Proposition

1. $\rho(\emptyset) = 0$ and $\rho(\mathbb{Z}^d) = 1$,

2. if $A \subseteq B \subseteq \mathbb{Z}^d$, then $\rho(A) \leq \rho(B)$,

3. if $T \subseteq \mathbb{Z}^d$, then $\rho(T) \in [0, 1]$,

4. if $F \subseteq \mathbb{Z}^d$ is finite, then $\rho(F) = 0$,

5. if $A, B \subseteq \mathbb{Z}^d$ with $A \cap B = \emptyset$, then $\rho(A \cup B) = \rho(A) + \rho(B)$,

6. if $B \subseteq A$, then $\rho(A \setminus B) = \rho(A) - \rho(B)$,

assuming that $\rho(A), \rho(B)$ and $\rho(T)$ exist.
Important difference to probability:

$$\rho \left( \bigcup_{i \in I} A_i \right) \not\leq \sum_{i \in I} \rho(A_i)$$

for a countable set $I$ and $A_i \subseteq \mathbb{Z}$. 

Counterexample: $I = \mathbb{Z}, A_i = \{i\}$. 

$1 = \rho(\mathbb{Z}) = \rho \left( \bigcup_{i \in \mathbb{Z}} \{i\} \right) \not\leq \sum_{i \in \mathbb{Z}} \rho(\{i\}) = 0$. 

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for a countable set \( I \) and \( A_i \subseteq \mathbb{Z} \).

Counterexample: \( I = \mathbb{Z}, A_i = \{i\} \).

\[ 1 = \rho(\mathbb{Z}) = \rho \left( \bigcup_{i \in \mathbb{Z}} \{i\} \right) \leq \sum_{i \in \mathbb{Z}} \rho(\{i\}) = 0. \]
Example (Primes)

The density of primes $\mathcal{P}$ is

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By the prime number theorem we have that

$$\pi(x) = \left| \{ p \in \mathcal{P} \mid p \leq x \} \right| \sim \frac{x}{\ln(x)}.$$ 

Hence

$$\rho(\mathcal{P}) = \lim_{H \to \infty} \frac{\left| \mathcal{P} \cap [-H, H] \right|}{2H} = \lim_{H \to \infty} \frac{\pi(H)}{2H} = 0.$$
Example (Invertible Matrices)

For \( n \) a positive integer, the density of invertible matrices \( GL_n(\mathbb{Z}) \) is

\[
\rho(GL_n(\mathbb{Z})) = 0.
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Example (Invertible Matrices)

For $n$ a positive integer, the density of invertible matrices $GL_n(\mathbb{Z})$ is

$$\rho( GL_n(\mathbb{Z}) ) = 0.$$

Let us fix all entries of $A \in [-H, H]^{n \times n}$ except for $a_{n,n}$. Since

$$\pm 1 = \det(A) = \sum_{j=1}^{n-1} (-1)^{n+j} a_{n,j} \det(A_{n,j}) + a_{n,n} \det(A_{n,n}),$$

we have at most two choices for $a_{n,n}$. Hence

$$\lim_{H \to \infty} \frac{|GL_n(\mathbb{Z}) \cap [-H, H]^{n^2}|}{(2H)^{n^2}} \leq \lim_{H \to \infty} \frac{2(2H)^{n^2}-1}{(2H)^{n^2}} = 0.$$
Example (Divisibility by $n$)

For $n$ a positive integer, the density of $n\mathbb{Z}$ is

$$\rho(n\mathbb{Z}) = \frac{1}{n}.$$
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Since $|n\mathbb{Z} \cap [-H, H]| = \lceil \frac{2H}{n} \rceil$, we have

$$\lim_{H \to \infty} \frac{|n\mathbb{Z} \cap [-H, H]|}{2H} = \lim_{H \to \infty} \frac{\lceil \frac{2H}{n} \rceil}{2H} = \frac{1}{n}.$$
How likely is it that two randomly chosen integers are coprime?
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**Theorem (F. Mertens, 1874 and E. Cesàro, 1884)**

Let the set of coprime pairs be denoted by $C$. Then

$$\rho(C) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}.$$ 

$\zeta$ denotes the Riemann-zeta function

$$\zeta(s) = \sum_{s \geq 1} \frac{1}{n^s} = \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$ 

**References**

- E. Cesàro, “Probabilité de certains faits arithmétiques”, Mathesis, 1884.
Theorem (J. Nymann, 1972)

Let $m \geq 2$ be a positive integer and the set of coprime $m$-tuples be denoted by $C_m$. Then

$$\rho(C_m) = \frac{1}{\zeta(m)}.$$
Definition (Rectangular Unimodular Matrix)

Let $n < m \in \mathbb{N}$. $M \in \text{Mat}_{n \times m}(\mathbb{Z})$ is rectangular unimodular, if we can extend it to an $m \times m$ matrix in $GL_m(\mathbb{Z})$. 

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Definition (Rectangular Unimodular Matrix)

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Let $n < m$ then we define the rectangular unimodular matrices as

$$U_{n,m} = \{ A \in \mathbb{Z}^{n \times m} \mid \forall p \in \mathcal{P} \; \text{rk}(A \mod p) = n \}.$$
**p-adic Integers**

- **p-adic valuation:**
  \[ \text{ord}_p(x) = \max \{ v : p^v \mid x \} \quad x \in \mathbb{Z} \setminus \{0\}, \]
  \[ \text{ord}_p(0) = \infty, \]
  \[ \text{ord}_p(y) = \text{ord}_p(a) - \text{ord}_p(b) \quad y = a/b \in \mathbb{Q} \setminus \{0\}. \]

- **p-adic norm:**
  \[ |x|_p = \begin{cases} 
    \frac{1}{p^{\text{ord}_p(x)}} & x \neq 0, \\
    0 & x = 0. 
  \end{cases} \]

- The \( p \)-adic number \( \mathbb{Q}_p \) are the completion of \( \mathbb{Q} \) with respect to \( |\cdot|_p \)

- The \( p \)-adic integers \( \mathbb{Z}_p = \{ \alpha \in \mathbb{Q}_p : |\alpha|_p \leq 1 \} \)

- Haar measure \( \mu_p \) on \( \mathbb{Z}_p \): 
  \[ \mu_p(a + p^k \mathbb{Z}_p) = \frac{1}{p^k} \]
\textbf{\textit{p}-adic Integers}

- \textit{p}-adic valuation:
  \[ \text{ord}_p(x) = \max\{v : p^v \mid x\} \quad x \in \mathbb{Z} \setminus \{0\}, \]
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The \textit{p}-adic number \( \mathbb{Q}_p \) are the completion of \( \mathbb{Q} \) with respect to \( |\cdot|_p \)
\[ \alpha \in \mathbb{Q}_p : \alpha = \sum_{i=k}^{\infty} a_i p^i \]

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- Haar measure \( \mu_p \) on \( \mathbb{Z}_p \):
  \[ \mu_p(a + p^k \mathbb{Z}_p) = \frac{1}{p^k} \]
Local-to-Global Principle

Idea

\[
T = \bigcap_{p \in \mathcal{P}} (U_p^C \cap \mathbb{Z})
\]

\[
T_M = \bigcap_{p < M} (U_p^C \cap \mathbb{Z})
\]

\[
T_M \setminus T \subseteq \bigcup_{p > M} (U_p \cap \mathbb{Z})
\]

\[
\rho(T) = \rho(T_M) - \rho(T_M \setminus T)
\]
Theorem (B. Poonen and M. Stoll, 1999)

Let $U_\infty \subset \mathbb{R}^d$, such that $\mathbb{R}_{\geq 0} \cdot U_\infty = U_\infty$ and $\mu_\infty(\partial(U_\infty)) = 0$. Let $s_\infty = \frac{1}{2^d} \mu_\infty(U_\infty \cap [-1, 1]^d)$.

For each $p \in \mathcal{P}$, let $U_p \subset \mathbb{Z}_p^d$, such that $\mu_p(\partial(U_p)) = 0$ and define $s_p = \mu_p(U_p)$. Define

$$P : \mathbb{Z}^d \rightarrow 2^{M_\mathbb{Q}},$$
$$a \mapsto \{ \nu \in M_\mathbb{Q} \mid a \in U_\nu \}.$$

If

$$\lim_{M \to \infty} \bar{\rho} \left( \left\{ a \in \mathbb{Z}^d \mid a \in U_p \text{ for some prime } p > M \right\} \right) = 0,$$ (1)
Theorem (B. Poonen and M. Stoll, 1999)

then:

i) \( \sum_{\nu \in M_\mathbb{Q}} s_\nu \) converges.

ii) For \( S \subset 2^{M_\mathbb{Q}} \), \( \rho(P^{-1}(S)) \) exists, and defines a measure on \( 2^{M_\mathbb{Q}} \).

iii) For each finite set \( S \in 2^{M_\mathbb{Q}} \), we have that

\[
\rho(P^{-1}(\{S\})) = \prod_{\nu \in S} s_\nu \prod_{\nu \not\in S} (1 - s_\nu),
\]

and if \( S \) consists of infinite subsets of \( 2^{M_\mathbb{Q}} \), then \( \rho(P^{-1}(S)) = 0 \).


Local-to-Global Principle

Strategy

\[ T = \{ a \in \mathbb{Z}^d \mid \forall p \in \mathcal{P} \text{ } a \mod p \text{ satisfies } C_p \} \]

1. Choose system \((U_\nu)\):
   - Choose \(S = \emptyset\).
   - Choose \(U_\infty = \emptyset\).
   - Choose \(U_p\) such that \(P^{-1}(\{\emptyset\}) = T\), that is
     \[ U_p = \{ a \in \mathbb{Z}_p^d \mid a \mod p \text{ does not satisfy } C_p \} \].

2. Compute \(s_p = \mu_p(U_p)\).

3. Show that Condition (1) is satisfied.

4. Compute density.
Lemma

Let $d$ and $M$ be positive integers. Let $f, g \in \mathbb{Z}[x_1, \ldots, x_d]$ be relatively prime. Define $S_M(f, g)$ as

$$\left\{ a \in \mathbb{Z}^d \mid f(a) \equiv g(a) \equiv 0 \mod p \text{ for some prime } p > M \right\},$$

then

$$\lim_{M \to \infty} \bar{\rho}(S_M(f, g)) = 0.$$ 

Strategy

Choose coprime $f, g$ such that

$$\left\{ a \in \mathbb{Z}^d \mid a \in U_p \text{ for some prime } p > M \right\} \subset S_M(f, g).$$
Theorem (G. Micheli and V. W., 2019)

Let $n < m$ be a positive integers and the set of rectangular unimodular $n \times m$ matrices be denoted by $U_{n,m}$. Then

$$
\rho(U_{n,m}) = \prod_{i=0}^{n-1} \frac{1}{\zeta(m-i)}.
$$

Proof

1. Choose system \((U_\nu)\):
   - \(T = U_{n,m} = \{ A \in \mathbb{Z}^{n \times m} \mid \forall p \in \mathcal{P} \ \text{rk}(A \mod p) = n \}\)
     thus \(C_p\) is \(A \mod p\) has full rank.
   - Define \(\pi_p : \mathbb{Z}_p^{n \times m} \to \mathbb{F}_p^{n \times m}\) as \(A \mapsto A \mod p\).
   - \(\mathcal{L}_p = \{ A \in \mathbb{F}_p^{n \times m} \mid \text{rk}(A) = n \}\).
   - \(|\mathcal{L}_p| = \prod_{i=0}^{n-1} (p^m - p^i)\).
   - If \(X \in \mathbb{Z}_p^{n \times m}\) is such that \(\pi_p(X) \in \mathcal{L}_p\), then also \(X + p\mathbb{Z}_p^{nm}\).
   - Define \(A_p = \{ X \in \mathbb{Z}_p^{n \times m} \mid \pi_p(X) \in \mathcal{L}_p \} + p\mathbb{Z}_p^{nm}\).
   - \(U_p\) should be the elements that do not satisfy \(C_p\):
     Choose \(U_p = A_p^C\).
Proof

2. Compute $s_p = \mu_p(U_p)$:
   - $A_p = \bigcup_{A \in \mathcal{L}_p} (\pi_p^{-1}(A) + p\mathbb{Z}_p^{nm})$.
   - Compute the Haar measure of $A_p$ as

\[
\mu_p(A_p) = \sum_{A \in \mathcal{L}_p} \mu_p(\pi_p^{-1}(A) + p\mathbb{Z}_p^{nm}) \\
= \mu_p(p\mathbb{Z}_p^{nm}) \mid \mathcal{L}_p \mid \\
= \frac{1}{p^{nm}} \prod_{i=0}^{n-1} (p^m - p^i) \\
= \prod_{i=0}^{n-1} \left(1 - \frac{1}{p^{m-i}}\right).
\]

   - $s_p = \mu_p(U_p) = 1 - \mu_p(A_p)$. 

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Rectangular Unimodular Matrices

Proof

3. Show that Condition (1) is satisfied:
Use the helpful lemma with $f$ giving the first basic minor and $g$ the second.

$$S_M(f, g) = \{A \in \mathbb{Z}^{nm} \mid f(A) \equiv g(A) \equiv 0 \mod p$$
for some $p > M$$$
\supset \{A \in \mathbb{Z}^{nm} \mid A \in U_p \text{ for some } p > M\}.$$

4. Compute density:

$$\rho(P^{-1}(\emptyset)) = \prod_{\nu \in \emptyset} s_{\nu} \prod_{\nu \notin \emptyset} (1 - s_{\nu}) = \prod_{p \in \mathcal{P}} (1 - s_p)$$

$$= \prod_{p \in \mathcal{P}} \prod_{i=0}^{n-1} \left(1 - \frac{1}{p^{m-i}}\right) = \prod_{i=0}^{n-1} \frac{1}{\zeta(m-i)}.$$
Let \( f \in \mathbb{Z}[x] \) be a polynomial of degree \( d \), i.e.,

\[
f = \sum_{i=0}^{d} a_i x^i,
\]

then we identify \( f \) by \((a_0, \ldots, a_d) \in \mathbb{Z}^{d+1}\).

**Definition (Eisenstein Polynomials)**

Let \( f \in \mathbb{Z}[x] \) be of degree \( d \) and having associated tuple \((a_0, \ldots, a_d)\). Then we call \( f \) an Eisenstein polynomial, if \( p^2 \nmid a_0, p \nmid a_d \) and for all \( i < d \) we have \( p \mid a_i \), for some \( p \in \mathcal{P} \).

In addition, we say that \( f \) is \( p \)-Eisenstein if \( f \) satisfies the criterion of Eisenstein for this prime \( p \).
Theorem (R. Heymann and I. Shparlinski, 2013)

Let the set of Eisenstein polynomials be denoted by $E_d$. Then

$$\rho(E_d) = 1 - \prod_{p \in \mathcal{P}} \left(1 - \frac{(p - 1)^2}{p^{d+2}}\right).$$
Theorem (R. Heymann and I. Shparlinski, 2013)

Let the set of Eisenstein polynomials be denoted by $E_d$. Then

$$\rho(E_d) = 1 - \prod_{p \in \mathcal{P}} \left(1 - \frac{(p - 1)^2}{p^{d+2}}\right).$$

Using the Local-to-Global Principle we can set

$$U_p = (p\mathbb{Z}_p \setminus p^2\mathbb{Z}_p) \times (p\mathbb{Z}_p)^{d-1} \times (\mathbb{Z}_p \setminus p\mathbb{Z}_p),$$

with $s_p = \frac{(p-1)^2}{p^{d+2}}$.
How many primes on average are such that an Eisenstein polynomial satisfies the criterion of Eisenstein?

Notation:

- Target set $T = \{a \in \mathbb{Z}^d \mid \exists p \in \mathcal{P} \ a \text{ satisfies } C_p\}$.
- Counting function $\psi$:

$$\psi : \mathbb{Z}^d \to \mathbb{N},$$

$$a \mapsto |\{p \in \mathcal{P} \mid a \text{ satisfies } C_p\}|.$$

- For $H \in \mathbb{N}$, define $T(H) = T \cap [-H, H]^d$.

**Definition (Mean)**

Let $T \subset \mathbb{Z}^d$, such that $\rho(T) \neq 0$ exists. Then we define the mean of $\psi$ as

$$\mu = \lim_{H \to \infty} \frac{\sum_{a \in T(H)} \psi(a)}{|T(H)|},$$

if it exists.
Strategy

For $s \geq 2$ a positive integer, let

$$\mathcal{H}(s, H) = \{a \in \mathbb{Z}^d \cap [-H, H^d] \mid a \text{ satisfies } C_s\}.$$ 

1. Compute the size of $\mathcal{H}(s, H)$.
2. Compute
   $$\sum_{a \in T(H)} \psi(a) = \sum_{p \in \mathcal{P}, p < H} |\mathcal{H}(p, H)|.$$
3. Compute $|T(H)|$ as
   $$|T(H)| = -\sum_{s=2}^{H} \mu(s) |\mathcal{H}(s, H)|.$$
4. Compute $\mu$ as
   $$\mu = \lim_{H \to \infty} \frac{\sum_{a \in T(H)} \psi(a)}{|T(H)|}.$$
How many primes on average are such that an Eisenstein polynomial satisfies the criterion of Eisenstein?

Theorem (M. Shillin, K. McGown, D. Rhodes and M. Wanner, 2018)

Let $\psi(a)$ be the number of $p \in \mathcal{P}$ such that $f$ associated to $a$ is $p$-Eisenstein. The mean of $\psi(a)$, is given by

$$
\mu(E_d) = \lim_{H \to \infty} \frac{\sum_{a \in E_d(H)} \psi(a)}{|E_d(H)|} = \frac{\alpha}{\rho(E_d)},
$$

where

$$
\alpha = \sum_{p \in \mathcal{P}} \frac{(p - 1)^2}{p^{d+2}}.
$$

Local-to-Global Principle for Mean Pattern

\[ \mu = \lim_{H \to \infty} \frac{\sum_{a \in T(H)} \psi(a)}{|T(H)|} = \frac{\alpha}{\rho(T)}, \]

where

\[ \alpha = \sum_{\nu \in M_\mathbb{Q}} s_\nu. \]
Definition (Mean of a System)

Let $H$ and $d$ be positive integers, then we define the expected value of the system $(U_\nu)_{\nu \in M_\mathbb{Q}}$ to be

$$\mu = \lim_{H \to \infty} \frac{\sum_{A \in [-H,H]^d} | \{ \nu \in M_\mathbb{Q} | A \in U_\nu \} |}{(2H)^d},$$

if it exists.

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We define the expected value of the system \((U_\nu)_{\nu \in M_Q}\) restricted to \(T\) to be

\[
\mu_T = \lim_{H \to \infty} \frac{\sum_{A \in [-H, H]^d \cap T} | \{ \nu \in M_Q \mid A \in U_\nu \} |}{| [-H, H]^d \cap T |},
\]

if it exists.

**Lemma**

*If the density of \(T\) exists and is nonzero and \(T\) is such that \(T^C \subseteq P^{-1}(\{\emptyset\})\), then \(\mu_T\) exists and is given by*

\[
\mu_T = \frac{\mu}{\rho(T)}.
\]
Theorem (G. Micheli, S. Schraven and V. W., 2020)

Let $H, d \in \mathbb{N}$. Let $U_\infty \subset \mathbb{R}^d$, such that $\mathbb{R}_{\geq 0} \cdot U_\infty = U_\infty$ and $\mu_\infty(\partial(U_\infty)) = 0$. Let $s_\infty = \frac{1}{2d} \mu_\infty(U_\infty \cap [-1,1]^d)$. For each prime $p$, let $U_p \subset \mathbb{Z}_p^d$, such that $\mu_p(\partial(U_p)) = 0$ and define $s_p = \mu_p(U_p)$. Define the following map

$$P : \mathbb{Z}^d \rightarrow 2^{M_\mathbb{Q}},$$

$$a \mapsto \{ \nu \in M_\mathbb{Q} \mid a \in U_\nu \}.$$

If

1. $\lim_{M \rightarrow \infty} \bar{\rho} \left( \left\{ a \in \mathbb{Z}^d \mid a \in U_p \text{ for some prime } p > M \right\} \right) = 0,$
Local-to-Global Principle for Mean

**Theorem (G. Micheli, S. Schraven and V. W., 2020)**

2. there exists an absolute constant $c \in \mathbb{Z}$, and some $m \in \mathbb{N}$ such that for all $H \geq 1$ and for all $A \in \mathbb{Z}^d$

$$\ell_{A,H} = | \{p \in \mathcal{P} \mid p > H, A \in U_p \cap [-H, H]^d \} | < c$$

3. there exists a sequence $(v_p)_{p \in \mathcal{P}}$, such that for all $p < H$

$$| U_p \cap [-H, H]^d | \leq v_p (2H)^d \quad \text{and} \quad \sum_{p \in \mathcal{P}} v_p \text{ converges.}$$

Then the mean of the system $(U_\nu)_{\nu \in M_0}$ exists and is given by

$$\mu = \sum_{\nu \in M_0} s_\nu.$$
Mean of Eisenstein Polynomials

Choose

\[ U_p = (p\mathbb{Z}_p \setminus p^2\mathbb{Z}_p) \times (p\mathbb{Z}_p)^{d-1} \times (\mathbb{Z}_p \setminus p\mathbb{Z}_p). \]

1. Choose \( f(x_1, \ldots, x_{d+1}) = x_1, g(x_1, \ldots, x_{d+1}) = x_2. \)
2. Clear:

\[ | \{ p \in \mathcal{P} \mid p > H, A \in U_p \cap [-H, H^d] \} | = 0. \]

3. Clear:

\[ | U_p \cap [-H, H^{d+1}]| \leq \lceil (2H)/p \rceil^d H \]

and

\[ \sum_{p \in \mathcal{P}} \frac{1}{p^d} \text{ converges}. \]

Then

\[ \mu(E_d) = \frac{\sum_{p \in \mathcal{P}} \frac{(p-1)^2}{p^{d+2}}}{\rho(E_d)}. \]
Thank you!