Weight two Masking in the McEliece system

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Outline

Preliminaries

McEliece System

Choose $n = 2^m$, $t < \frac{n}{m}$ and Γ a binary Goppa code of length n, dimension $k \geq n - mt$, which can correct upto t errors. Γ has a generator matrix G of size $k \times n$. Choose a $k \times k$ invertible matrix S and a $n \times n$ permutation matrix P and compute $G' = SGP$.

$$\begin{aligned} \text{Public Key} &= (G', t) \\ \text{Private Key} &= (S, G, P) \end{aligned}$$

Encryption: Let $x \in \mathbb{F}_2^k$ be the message and $e \in \mathbb{F}_2^n$ the error vector, s.t. $\mathrm{wt}(e) \leq t$, then the cipher is computed as

$$y = xG' + e.$$

Decryption: Compute

$$yP^{-1} = xSG + eP^{-1},$$

then xSG is a code word of Γ and since $\mathrm{wt}(eP^{-1}) \leq t$, we can apply the decoding algorithm and get xS and by multiplication with the inverse of S we get the message x.

Niederreiter system

Let $\mathbb{F}_q$ be a finite field. Let $1 \leq k < n \leq q$ be integers. Construct a $[n, k]$-linear code C, that can correct upto t errors and has an efficient decoding algorithm. C has a parity check matrix H of size $r \times n$, where $r = n - k$. Choose a $r \times r$ invertible matrix S and a $n \times n$ permutation matrix P and compute $H' = SHP$.

$$\begin{aligned} \text{Public Key} &= (H', t) \\ \text{Private Key} &= (S, H, P) \end{aligned}$$

Niederreiter system

Encryption: Let $x \in \mathbb{F}_q^n$ be the message, s.t. $\mathrm{wt}(x) \leq t$, then the cipher is computed as

$$y^T = H' x^T.$$

Decryption: Compute

$$S^{-1} y^T = HPx^T = H(xP^T)^T.$$

Since $\mathrm{wt}(xP^T) \leq t$, we can apply syndrome decoding to get xP^T and by multiplication with the inverse of P^T we get the message x.

BBCRS Scheme

BBCRS Scheme

Baldi, Bianchi, Chiaraluce, Rosenthal and Schipani proposed a variant of the McEliece cryptosystem, in order to reconsider the use of GRS codes as secret code.

Instead of the permutation matrix they use as scrambling matrix the sum $T + R$, where T is a sparse matrix of row weight m and R is a matrix of rank z.

BBCRS Scheme for $m = 1, z = 1$

Let $\mathbb{F}_q$ be a finite field. Let $1 \leq k < n \leq q$ be integers. Let

$$\begin{aligned} G &= k \times n \text{ generator matrix of GRS code,} \\ T &= n \times n \text{ permutation matrix,} \\ R &= n \times n \text{ rank 1 matrix, } R = \alpha^T \beta, \\ Q &= n \times n \text{ invertible matrix, } Q = R + T, \\ S &= k \times k \text{ invertible matrix.} \end{aligned}$$

Compute: $G' = S^{-1} G Q^{-1}$ and $t_{\text{pub}} = t = \lfloor \frac{n-k}{2} \rfloor$.

$$\begin{aligned} \text{Public Key} &= (G', t) \\ \text{Private Key} &= (G, T, R, Q, S) \end{aligned}$$

BBCRS Scheme for $m = 1, z = 1$

Encryption: Let $x \in \mathbb{F}_q^k$ be the message and $e \in \mathbb{F}_q^n$, s.t. $\mathrm{wt}(e) \leq t$ be the error vector. Compute the cipher as

$$y = xG' + e.$$

Decryption: Guess the value of eR. Then compute

$$y' = yQ - eR = xS^{-1}G + eT.$$

Since $\mathrm{wt}(eT) \leq t$ by decoding algorithm we get xS^{-1} and by multiplication with S we get the message x.

Distinguisher Attack

Definition (Schur Product)

Let $x, y \in \mathbb{F}_q^n$. The Schur product of x and y is

$$x \star y = (x_1 y_1, \ldots, x_n y_n).$$

Square Code

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$$x \star y = (x_1 y_1, \ldots, x_n y_n).$$

Definition (Schur Product of Codes and Square Code)

Let A, B be two codes of length n. The Schur product of A and B is

$$\langle A \star B \rangle = \langle \{a \star b \mid a \in A, b \in B\} \rangle.$$

If $A = B$, then we call $\langle A \star A \rangle$ the square code of A and denote it by $\langle A^2 \rangle$.

Definition (Schur Matrix)

Let G be a $k \times n$ matrix, with rows g_i for $1 \leq i \leq k$. We denote by $S(G)$ the Schur matrix of G, which consists of the rows $g_i \star g_j$ for $1 \leq i \leq j \leq k$. Thus $S(G)$ is of the size $\frac{1}{2}(k^2 + k) \times n$.

Proposition

Let A be a code of length n and dimension k, then

$$dim(\langle A^2 \rangle) \leq min\left\{n, \binom{k+1}{2}\right\} \tag{1}$$

Square Code

Proposition (Márquez-Corbella, Pellikaan (2016))

Let A be an $[n, k]$ linear code chosen at random, then with high probability the square code of A has maximal dimension.

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Proposition

If $2k - 1 < n$

$$\langle GRS_{n,k}(\alpha, \beta)^2 \rangle = GRS_{n,2k-1}(\alpha, \beta \star \beta) \tag{2}$$

Distinguisher Attack

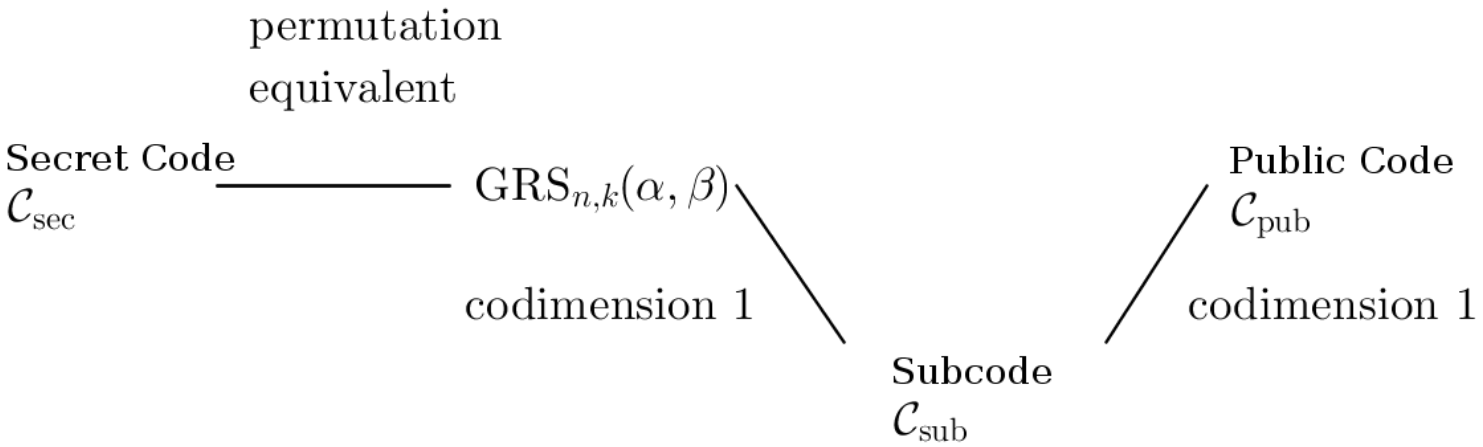
Couvreur, Gaborit, Gauthier-Umaña, Otmani and Tillich presented a distinguisher attack on the BBCRS scheme for the parameters $m < 2, z = 1$.

Proposition (Couvreur, Gaborit, Gauthier-Umaña, Otmani, Tillich (2015))

Let $\mathcal{C}_{pub}$ denote the public code of length n and dimension k of the BBCRS scheme. Then

$$dim(\langle \mathcal{C}_{pub}^2 \rangle) \leq 3k - 1.$$

Distinguisher Attack



Distinguisher Attack

1. Find subcode $\mathcal{C}_{\text{sub}}$

Take a basis $g_1, \ldots, g_k$ of $\mathcal{C}_{\text{pub}}$ and random other elements z_1, z_2, z_3 from $\mathcal{C}_{\text{pub}}$. Then define

$$\mathcal{B} = \langle \{ z_i \star g_j \mid 1 \leq i \leq 3, 1 \leq j \leq k \} \rangle.$$

Proposition (Couvreur, Gaborit, Gauthier-Umaña, Otmani, Tillich (2015))

If $dim(\mathcal{B}) \leq 2k + 2$, then z_i is in $\mathcal{C}_{sub}$ for $i \in \{1, 2, 3\}$.

2. Find $\text{GRS}_{n,k}(x, y)$

Remark (Márquez-Corbella, Martínez-Moro, Pellikaan (2013))

Let $\mathcal{A}$ be an ℓ dimensional subspace of $GRS_{n,k}(\alpha, \beta)$. If ℓ is large enough, then with high probability we have

$$\langle \mathcal{A}^2 \rangle = \langle GRS_{n,k}(\alpha, \beta)^2 \rangle.$$

Weight two Masking

Weight two Masking

The Weight two Masking in the McEliece system is a proposal of Bolkema, Gluessing-Luerrsen, Kelley, Lauter, Malmskog and Rosenthal.

They use as scrambling matrix instead of a permutation matrix, a matrix of constant row weight two.

Observe that this is a special case of the BBCRS scheme by choosing $m = 2, z = 0$.

McEliece Version

Let $\mathbb{F}_q$ be a finite field and $1 \leq k < n \leq q$ integers. Let G be a $k \times n$ generator matrix of $\mathrm{GRS}_{n,k}(\alpha, \beta)$ code over $\mathbb{F}_q^n$, which is able to correct upto $t = \lfloor \frac{n-k}{2} \rfloor$ errors. We choose a $k \times k$ invertible matrix S, and a $n \times n$ invertible matrix Q, which is of constant row and column weight 2, both over $\mathbb{F}_q$. We define $t_{\mathrm{pub}} = \lfloor \frac{t}{2} \rfloor$ and compute $G' = S^{-1}GQ^{-1}$.

$$\begin{aligned} \text{Public Key} &= (G', t_{\mathrm{pub}}) \\ \text{Private Key} &= (G, S, Q) \end{aligned}$$

Encryption: Let $x \in \mathbb{F}_q^k$ be the message and $e \in \mathbb{F}_q^n$ be the error vector, s.t. $\mathrm{wt}(e) \leq t_{\mathrm{pub}}$ and compute the cipher

$$y = xG' + e.$$

Decryption: Compute

$$y' = yQ = xS^{-1}G + eQ.$$

Since $\mathrm{wt}(eQ) \leq t$ we can decode and get xS^{-1} and by multiplication with S we get the message x.

Key Size

In order for the ISD attack to reach a work factor greater than 2^{80} the following key sizes are needed with the different systems.

	q	n	k	Key Size
Original McEliece		1632	1269	460647
BBCRS scheme	347	346	252	199899
Weight two Masking	457	450	225	447326

Monte Carlo test with 1000 tries

q	n	r	Success rate
512	500	250	1
256	255	100	1
151	100	50	1
128	100	50	1

Remark

Let R be a $n \times n$ matrix over $\mathbb{F}_q$ of constant row weight two, then there exist permutation matrices P, P', s.t.

$$PRP' = \begin{bmatrix} Q_{n_1} & & \\ & \ddots & \\ & & Q_{n_l} \end{bmatrix}, \tag{3}$$

where Q_{n_i} are $n_i \times n_i$ matrices of the form (4) for all $1 \leq i \leq l < n$.

Define Q_n to be

$$Q_n = \begin{bmatrix} x_1 & & & y_n \\ y_1 & x_2 & & \\ & \ddots & \ddots & \\ & & y_{n-1} & x_n \end{bmatrix}. \tag{4}$$

To avoid the distinguisher attack we want that the public code has maximal square code dimension.

We claim that for a random constant row weight two matrix R, this is satisfied with high probability, i.e. the Schur matrix of the public matrix $S(HR)$ has full rank.

Let $H_{n,r}(\alpha, \beta)$ denote a generator matrix of GRS code of length n and dimension r.

$$H_{n,r}(\alpha, \beta) = \begin{bmatrix} \beta_1 & \cdots & \beta_n \\ \beta_1\alpha_1 & \cdots & \beta_n\alpha_n \\ \vdots & & \vdots \\ \beta_1\alpha_1^{r-1} & \cdots & \beta_n\alpha_n^{r-1} \end{bmatrix}.$$

Let m denote the maximal square code dimension of an $[n, r]$ code, i.e.

$$m = \min\left\{n, \frac{1}{2}(r^2 + r)\right\}.$$

Define

$$\begin{aligned} A_n &= \{R_n \in \mathrm{GL}_n(\mathbb{F}_q) \mid R_n \text{ is of the form (3)}\}, \\ \mathcal{G}_{H_{n,r}} &= \left\{R_n^T \in A_n \mid S(H_{n,r}R_n^T) \text{ has rank } m\right\}. \end{aligned}$$

Lemma

Let $\mathbb{F}_q$ be a finite field and $1 \leq n \leq q$ integers. Let p be a nontrivial homogeneous polynomial in $\mathbb{F}_q[x_1, \ldots, x_n, y_1, \ldots, y_n]$, of total degree $2n$, in each variable of degree at most 2. Then there exist at least

$$((q-1)^2 - 2(q-1))^n$$

nonzero evaluations.

Under the assumption that there exists a nontrivial principal minor of $S(H_{n,r}R_n^T)$ we get that the probability for R_n to avoid the distinguisher attack is greater than or equal to

1. case: $n \leq \frac{1}{2}(r^2 + r)$

$$\frac{((q-1)^2 - 2(q-1))^n}{(q-1)^{2n}} = \left(1 - \frac{2}{q-1}\right)^n.$$

2. case: $n \geq \frac{1}{2}(r^2 + r)$

$$\frac{((q-1)^2 - 2(q-1))^m (q-1)^{2(n-m)}}{(q-1)^{2n}} = \left(1 - \frac{2}{q-1}\right)^m.$$

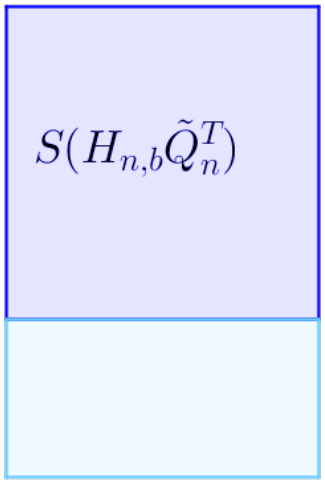
1. Assume constant row weight two matrix is of the form Q_n

$$Q_n = \begin{bmatrix} x_1 & & & y_n \\ y_1 & x_2 & & \\ & \ddots & \ddots & \\ & & y_{n-1} & x_n \end{bmatrix}.$$

2. Reduce to quadratic case: $n = \frac{1}{2}(r^2 + r)$

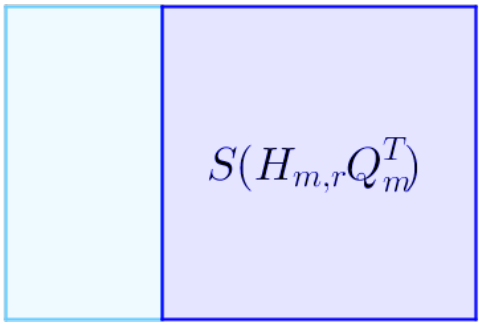
If $n \leq \frac{1}{2}(r^2 + r)$

$S(H_{n,r}Q_n^T) =$



If $n \geq \frac{1}{2}(r^2 + r)$

$S(H_{n,r}Q_n^T) =$



3. Transformations to get rid of β and y

Define

$$\begin{aligned} \widetilde{x}_i &= x_i\beta_i \ \forall\ 1 \leq i \leq n, \\ \widetilde{y}_i &= y_i\beta_{i+1} \ \forall\ 1 \leq i \leq n-1 \text{ and } \widetilde{y}_n = y_n\beta_1. \end{aligned}$$

Then

$$S(H_{n,r}(\alpha,\beta)Q_n^T(x,y)) = S(H_{n,r}(\alpha,\mathbf{1})Q_n^T(\widetilde{x},\widetilde{y})).$$

Now divide each column $j \in \{1,\ldots,n\}$ by y_j^2 and define

$$\widetilde{x}_i = \frac{x_i}{y_i} \ \forall\ 1 \leq i \leq n.$$

Then

$$\det\left(S(H_{n,r}(\alpha,\beta)Q_n^T(x,y))\right) = \det\left(S(H_{n,r}(\alpha,\beta)Q_n^T(\widetilde{x},\mathbf{1}))\right).$$

Remaining to show:

For all $\alpha \in \mathbb{F}_q^n$ distinct n-tuple, there exists a $x \in \left(\mathbb{F}_q^\times\right)^n$, such that

$$\det\left(S(H_{n,r}(\alpha, \mathbf{1})Q_n^T(x, \mathbf{1}))\right) \neq 0.$$