On the density of rectangular unimodular matrices over the ring of algebraic integers

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The density of a set \( S \subset \mathbb{Z}^d \) is defined to be

\[
\rho(S) = \lim_{B \to \infty} \frac{|S \cap [-B, B]^d|}{(2B)^d}
\]

if the limit exists. Then one defines the upper density \( \bar{\rho} \) and the lower density \( \underline{\rho} \) equivalently with the \( \lim \sup \) and the \( \lim \inf \) respectively.
Theorem (Mertens, 1874 and Cesáro, 1883)

The density of the set of coprime pairs of $\mathbb{Z}$ is equal to

$$\frac{1}{\zeta(2)},$$

where $\zeta$ denotes the Riemann zeta function.
This can be generalized to

**Theorem (Nymann, 1972)**

*The density of the set of coprime m-tuples of \( \mathbb{Z} \) is equal to*

\[
\frac{1}{\zeta(m)},
\]

*where \( \zeta \) denotes the Riemann zeta function.*
This can be generalized to

**Theorem (Nymann, 1972)**

The density of the set of coprime \( m \)-tuples of \( \mathbb{Z} \) is equal to

\[
\frac{1}{\zeta(m)},
\]

where \( \zeta \) denotes the Riemann zeta function.

And this can be further generalized to

**Theorem (Ferraguti, Micheli, 2016)**

The density of the set of coprime \( m \)-tuples of \( \mathcal{O}_K \) is equal to

\[
\frac{1}{\zeta_K(m)},
\]

where \( \zeta_K \) denotes the Dedekind zeta function over \( K \).
Definition

Let $\mathcal{R}$ be a domain and $n < m \in \mathbb{N}$. Let $M \in \text{Mat}_{n \times m}(\mathcal{R})$. $M$ is said to be rectangular unimodular, if there exist $m - n$ rows in $\mathcal{R}^m$, such that when adjoining these rows to $M$ the resulting $m \times m$ matrix $\tilde{M}$ is invertible, i.e. $\det(\tilde{M})$ is a unit in $\mathcal{R}$. 
Over Dedekind domains, there are the following characterization of rectangular unimodular matrices:

**Proposition (Gustafson, Moore, Reiner, 1981)**

Let $\mathcal{D}$ be a Dedekind domain and $n < m \in \mathbb{N}$. Let $M \in \text{Mat}_{n \times m}(\mathcal{D})$. $M$ is rectangular unimodular, if and only if the ideal generated by all the $n \times n$ minors of $M$ is $\mathcal{D}$. 
Over Dedekind domains, there are the following characterization of rectangular unimodular matrices:

**Proposition (Gustafson, Moore, Reiner, 1981)**

Let $D$ be a Dedekind domain and $n < m \in \mathbb{N}$. Let $M \in \text{Mat}_{n \times m}(D)$. $M$ is rectangular unimodular, if and only if the ideal generated by all the $n \times n$ minors of $M$ is $D$.

**Proposition (Gustafson, Moore, Reiner, 1981)**

Let $D$ be a Dedekind domain and $n < m \in \mathbb{N}$. $M \in \text{Mat}_{n \times m}(D)$ is rectangular unimodular, if and only if $M \mod \mathfrak{p}$ has full rank for any $\mathfrak{p}$ non-zero prime ideal of $D$. 
Theorem (Micheli, W., 2018)

Let \( n \) and \( m \) be positive integers such that \( n < m \) and \( K \) be an algebraic number field. The density of the set of \( n \times m \) rectangular unimodular matrices over \( \mathcal{O}_K \) is

\[
\frac{1}{\prod_{i=0}^{n-1} \zeta_K(m-i)},
\]

where \( \zeta_K \) denotes the Dedekind zeta function of \( K \).
If $S$ is a set, then we denote by $2^S$ its powerset.
Let $M_Q = \{\infty\} \cup \{p \mid p \text{ prime}\}$ be the set of all places of $\mathbb{Q}$.
We denote by $\mathbb{Z}_p$ the $p$-adic integers.
Let $\mu_\infty$ denote the Lebesgue measure on $\mathbb{R}^d$ and $\mu_p$ the normalized Haar measure on $\mathbb{Z}_p^d$.
For $T$ a subset of a metric space, let us denote by $\partial T$ its boundary.
Theorem (Poonen, Stoll, 1999)

Let $U_\infty \subset \mathbb{R}^d$, such that $\mathbb{R}_{\geq 0} \cdot U_\infty = U_\infty$ and $\mu_\infty(\partial(U_\infty)) = 0$. Let $s_\infty = \frac{1}{2^d} \mu_\infty(U_\infty \cap [-1, 1]^d)$.

For each prime $p$, let $U_p \subset \mathbb{Z}_p^d$, such that $\mu_p(\partial(U_p)) = 0$ and define $s_p = \mu_p(U_p)$. Define

$$P : \mathbb{Z}^d \rightarrow 2^{M_\mathbb{Q}}$$

$$a \mapsto \{ \nu \in M_\mathbb{Q} \mid a \in U_\nu \}.$$

If the following is satisfied:

$$\lim_{M \rightarrow \infty} \bar{\rho} \left( \left\{ a \in \mathbb{Z}^d \mid a \in U_p \text{ for some prime } p > M \right\} \right) = 0,$$  \hspace{1cm} (1)
Theorem (continued)

then:

i) \( \sum_{\nu \in M_Q} s_{\nu} \) converges.

ii) For \( S \subset 2^M_Q \), \( \rho(P^{-1}(S)) \) exists, and defines a measure on \( 2^M_Q \).

iii) For each finite set \( S \in 2^M_Q \), we have that

\[
\rho(P^{-1}([S])) = \prod_{\nu \in S} s_{\nu} \prod_{\nu \notin S} (1 - s_{\nu}),
\]

and if \( S \) consists of infinite subsets of \( 2^M_Q \), then \( \rho(P^{-1}(S)) = 0 \).
Lemma (Poonen, Stoll, 1999)

Let $f, g \in \mathbb{Z}[x_1, \ldots, x_d]$ be relatively prime. Define

$$S_M(f, g) = \left\{ a \in \mathbb{Z}^d \mid p \mid f(a), p \mid g(a) \text{ for some prime } p > M \right\},$$

then

$$\lim_{M \to \infty} \bar{\rho}(S_M(f, g)) = 0.$$
Proof of Main Result

Let

$$\pi_p : \mathbb{Z}_p \to \mathbb{F}_p,$$

be the reduction modulo a rational prime $p$ and

$$E_p : \mathbb{F}_p^k \to \mathcal{O}_K/(p).$$

and the natural projection

$$\psi_p : \mathcal{O}_K/(p) \to \prod_{p | p} (\mathcal{O}_K/p).$$
Proof of Main Result

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and the natural projection
\[ \psi_p : \mathcal{O}_K/(p) \to \prod_{p \mid p} (\mathcal{O}_K/p). \]

Then the composition of their extension is \( F_p = \overline{\psi}_p \circ \overline{E}_p \circ \overline{\pi}_p : \)
\[
\mathbb{Z}^{knm}_p \xrightarrow{\overline{\pi}_p} \mathbb{F}^{knm}_p \xrightarrow{\overline{E}_p} (\mathcal{O}_K/(p))^{n \times m} \xrightarrow{\overline{\psi}_p} \prod_{p \mid p} (\mathcal{O}_K/p)^{n \times m} = T_p.
\]
Define

$$\mathcal{L}_p = \left\{ \left( a_{p_1}, \ldots, a_{p_{\ell_p}} \right) \in T_p \mid a_{p_i} \in \mathbb{F}^{n \times m}_{p^\text{deg}(p_i)} \text{ has full rank} \right\} .$$
Define

\[ \mathcal{L}_p = \left\{ \left( a_{p_1}, \ldots, a_{p_{\ell_p}} \right) \in T_p \mid a_{p_i} \in \mathbb{F}_{p^{\deg(p_i)}}^{n \times m} \right. \text{ has full rank} \right\}. \]

Consider now the following set

\[ A_p = \left\{ A \in \mathbb{Z}_{p}^{knm} \mid F_p(A) \in \mathcal{L}_p \right\}. \]
\[
\mu_p(A_p) = \mu_p \left( \bigsqcup_{f \in \mathcal{L}_p} F_p^{-1}(f) \right)
\]
\[
= \sum_{f \in \mathcal{L}_p} \mu_p(F_p^{-1}(f))
\]
\[
= \sum_{f \in \mathcal{L}_p} \mu_p(\pi_p^{-1}\overline{E}_p^{-1}\overline{\psi}_p^{-1}(f))
\]
\[
= \sum_{f \in \mathcal{L}_p} \mu_p(\pi_p^{-1}(\overline{E}_p^{-1}(f + \text{Ker}(\overline{\psi}_p))))
\]
\[
= \sum_{f \in \mathcal{L}_p} \mu_p \left( \bigsqcup_{i=1}^{\text{Ker}(\overline{\psi}_p)} \left( f_i + p\mathbb{Z}_p^{knm} \right) \right)
\]
\[
= |\mathcal{L}_p| \cdot |\text{Ker}(\overline{\psi}_p)| \cdot p^{-knm}.
\]
For

$$\overline{\psi}_p : (\mathcal{O}_K/(p))^{n \times m} \rightarrow \prod_{p | p} (\mathcal{O}_K/\mathfrak{p})^{n \times m}$$

we have

$$\dim_{\mathbb{F}_p} (\text{Ker}(\overline{\psi}_p)) = knm - \sum_{p | p} \deg(p) nm.$$
Proof of Main Result

For

$$\overline{\psi}_p : (\mathcal{O}_K/(p))^{n \times m} \rightarrow \prod_{p|p} (\mathcal{O}_K/p)^{n \times m}$$

we have

$$\dim_{\mathbb{F}_p} (\text{Ker}(\overline{\psi}_p)) = knm - \sum_{p|p} \text{deg}(p) nm.$$ 

Therefore

$$|\text{Ker}(\overline{\psi}_p)| = p^{knm - \sum_{p|p} \text{deg}(p) nm}.$$
Proof of Main Result

Since for a prime power $q$ the number of full rank matrices over $\mathbb{F}_q^{n \times m}$ is

$$
\prod_{i=0}^{n-1} (q^m - q^i),
$$

we have that

$$
|\mathcal{L}_p| = \prod_{p|p} \prod_{i=0}^{n-1} \left( p^{\deg(p)m} - p^{\deg(p)i} \right).
$$
Proof of Main Result

\[ \mu_p(A_p) = p^{-knm} \cdot \text{Ker}(\overline{\psi}_p) \cdot |\mathcal{L}_p| \]

\[ = \frac{1}{p^{knm}} \left( p^{knm - \sum_{p|p} \deg(p) nm} \right) \prod_{p|p} \prod_{i=0}^{n-1} \left( p^{\deg(p)m} - p^{\deg(p)i} \right) \]

\[ = p^{-\sum_{p|p} \deg(p) nm} \prod_{p|p} \prod_{i=0}^{n-1} \left( p^{\deg(p)m} - p^{\deg(p)i} \right) \]

\[ = \prod_{p|p} \prod_{i=0}^{n-1} p^{-\deg(p)m} \left( p^{\deg(p)m} - p^{\deg(p)i} \right) \]

\[ = \prod_{p|p} \prod_{i=0}^{n-1} \left( 1 - p^{\deg(p)(i-m)} \right). \]
Proof of Main Result

Let $U$ be the set of rectangular unimodular $n \times m$ matrices over $\mathcal{O}_K$, hence we can write

$$U = \{ M \in \text{Mat}_{n \times m}(\mathcal{O}_K) \mid M \mod \mathfrak{p} \text{ has full rank for any prime ideal } \mathfrak{p} \subseteq \mathcal{O}_K\}.$$ 

We choose $U_\infty = \emptyset$, then clearly $s_\infty = 0$.

We want to choose $U_p$ such that $P^{-1}(\{\emptyset\}) = U$. 

Violetta Weger Rectangular Unimodular Matrices
Proof of Main Result

Let $U$ be the set of rectangular unimodular $n \times m$ matrices over $\mathcal{O}_K$, hence we can write

$$U = \{ M \in \text{Mat}_{n \times m}(\mathcal{O}_K) \mid M \text{ mod } \mathfrak{p} \text{ has full rank for any prime ideal } \mathfrak{p} \subset \mathcal{O}_K \}.$$

We choose $U_\infty = \emptyset$, then clearly $s_\infty = 0$.

We want to choose $U_p$ such that $P^{-1}(\{\emptyset\}) = U$.

We choose $U_p = \mathbb{Z}_p^{knm} \setminus A_p$. Hence

$$s_p = \mu_p(U_p) = 1 - \mu_p(A_p) = 1 - \prod_{\mathfrak{p} | p} \prod_{i=0}^{n-1} \left( 1 - p^{\deg(\mathfrak{p})(i-m)} \right).$$
Proof of Main Result

To show:

$$\lim_{M \to \infty} \bar{\rho} \left( \left\{ A \in \mathbb{Z}^{knm} \mid A \in U_p \text{ for some prime } p > M \right\} \right) = 0.$$
Proof of Main Result

To show:

$$\lim_{M \to \infty} \bar{\rho} \left( \left\{ A \in \mathbb{Z}^{km} \mid A \in U_p \text{ for some prime } p > M \right\} \right) = 0.$$ 

Let $\bar{E} : \mathbb{Z}^{km} \to \mathcal{O}_K^{n \times m}$. 

For $A \in \mathbb{Z}^{km}$, let us denote the $n \times n$ minors of $\bar{E}(A)$ by $A_i$ for $i \in \{1, \ldots, (m \choose n)\}$. 

Hence $A \in U_p$ is equivalent to $\langle A_1, \ldots, A_{(m \choose n)} \rangle \subseteq \mathfrak{p}$ for some $\mathfrak{p} \mid p$. 

Thus for all $i \in \{1, \ldots, (m \choose n)\}$ we have that $\langle A_i \rangle \subseteq \mathfrak{p}$ and hence that $N_{K/\mathbb{Q}}(A_i) \equiv 0 \mod p$. Hence it is a subset of 

$$B_M = \left\{ A \in \mathbb{Z}^{km} \mid p \mid N_{K/\mathbb{Q}}(A_1), p \mid N_{K/\mathbb{Q}}(A_2) \right. \text{ for some prime } p > M \}.$$
Let $\mathcal{R}$ be an integral domain and $n \in \mathbb{N}$. Recall that, if $X$ is an $n \times n$ polynomial matrix over $\mathcal{R}[x_{1,1}, \ldots, x_{n,n}]$ having as $(i,j)$ entry the variable $x_{i,j}$, then $\det(X) \in \mathcal{R}[x_{1,1}, \ldots, x_{n,n}]$ is irreducible.
**Remark**

Let $\mathcal{R}$ be an integral domain and $n \in \mathbb{N}$. Recall that, if $X$ is an $n \times n$ polynomial matrix over $\mathcal{R}[x_{1,1}, \ldots, x_{n,n}]$ having as $(i,j)$ entry the variable $x_{i,j}$, then $\det(X) \in \mathcal{R}[x_{1,1}, \ldots, x_{n,n}]$ is irreducible.

Let $\ell, k \in \mathbb{N}, f \in \mathbb{C}[x_1, \ldots, x_\ell]$ and $e = (e_1, \ldots, e_k) \in (\mathbb{C} \setminus \{0\})^k$. In the new polynomial ring $\mathbb{C}[x_1^{(1)}, x_1^{(2)}, \ldots, x_\ell^{(k)}]$ let us denote by $f_e$

$$f \left( \sum_{u=1}^{k} x_1^{(u)} e_u, \ldots, \sum_{u=1}^{k} x_\ell^{(u)} e_u \right).$$

**Lemma**

Let $\ell, k, f$ and $e$ be as above. If $f \in \mathbb{C}[x_1, \ldots, x_\ell]$ is irreducible, then $f_e$ is irreducible in $\mathbb{C}[x_1^{(1)}, x_1^{(2)}, \ldots, x_\ell^{(k)}]$. 

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Rectangular Unimodular Matrices
Lemma

Let $N$ be the norm map for the extension field $K(x_1, \ldots x_M)/\mathbb{Q}(x_1, \ldots x_M)$. Let $F, G \in \mathcal{O}_K[x_1, \ldots, x_M]$ be irreducible and such that there are variables appearing in $F$ but not in $G$, then $N(F)$ and $N(G)$ are coprime.
Proof of Main Result

Using the local to global principle, we get

\[ \rho(U) = \rho(P^{-1}({\emptyset})) = \prod_{\nu \in \emptyset} s_{\nu} \prod_{\nu \not\in \emptyset} (1 - s_{\nu}) \]

\[ = (1 - s_\infty) \prod_{p \text{ prime}} (1 - s_p) \]

\[ = \prod_{p \text{ prime}} \prod_{p|p} \prod_{i=0}^{n-1} \left(1 - p^{\deg(p)(i-m)}\right) \]

\[ = \prod_{i=0}^{n-1} \prod_{p \text{ prime}} \prod_{p|p} \left(1 - \frac{1}{p^{\deg(p)(m-i)}}\right) \]

\[ = \prod_{i=0}^{n-1} \frac{1}{\zeta_K(m-i)}. \]
Thank you