

**On the search for the right support:
Better bounds for the Lee metric**

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Joint work with Jessica Bariffi

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The history of the Lee metric/ why do we care about it

- Introduced in 1958 by Lee for non-binary codes
- Some good non-linear binary codes can be represented as linear codes in the Lee metric over $\mathbb{Z}/4\mathbb{Z}$
- Introduced to code-based cryptography
- First Lee-metric signature scheme



C. Lee. “Some properties of nonbinary error-correcting codes.”, IRE Transactions on Information Theory, 1958.



A.R. Hammons, P.V. Kumar, A.R. Calderbank, N.J. Sloane, P. Solé. “The \mathbb{Z}_4 -linearity of Kerdock, Preparata, Goethals, and related codes.”IEEE Transactions on Information Theory, 1994.



A.-L. Horlemann, V. Weger. “Information set decoding in the Lee metric with applications to cryptography.”Advances in Mathematics of Communications, 2019.



S. Ritterhoff, G. Maringer, S. Bitzer, V. Weger, P. Karl, T. Schamberger, J. Schupp, A. Wachter-Zeh. “FuLeecca: A Lee-based Signature Scheme.”, 2023.

What is a ring-linear code?

$\mathcal{C} \subseteq \mathbb{F}_q^n$ is a code if \mathcal{C} is a linear subspace

Generator matrix in systematic form

$$\left(\text{Id}_k \quad A \right)$$

dimension $k = \log_q(|\mathcal{C}|)$ number of generators

What is a ring-linear code?

$\mathcal{C} \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n$ is a **code** if \mathcal{C} is a $\mathbb{Z}/p^s\mathbb{Z}$ -submodule

$$\mathcal{C} \cong (\mathbb{Z}/p^s\mathbb{Z})^{k_0} \times (\mathbb{Z}/p^{s-1}\mathbb{Z})^{k_1} \times \cdots \times (\mathbb{Z}/p\mathbb{Z})^{k_{s-1}}$$

Generator matrix in systematic form

$$\begin{pmatrix} \text{Id}_{k_0} & A_{1,2} & \cdots & A_{1,s} & A_{1,s+1} \\ 0 & p\text{Id}_{k_1} & \cdots & pA_{2,s} & pA_{2,s+1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & p^{s-1}\text{Id}_{k_{s-1}} & p^{s-1}A_{s,s+1} \end{pmatrix},$$

- **subtype** (k_0, \dots, k_{s-1}) ,
- **rank** $K = \sum_{i=0}^{s-1} k_i$,
- **type** $k = \sum_{i=0}^{s-1} \frac{s-i}{s} k_i = \log_{p^s} (|\mathcal{C}|)$,
- $0 \leq k \leq K \leq n$ and if $k = K$ **free code**

The Lee metric

The Hamming metric

- $x \in (\mathbb{Z}/p^s\mathbb{Z})^n$: $\text{wt}_H(x) = |\{i \in \{1, \dots, n\} \mid x_i \neq 0\}|$
- $x, y \in (\mathbb{Z}/p^s\mathbb{Z})^n$: $d_H(x, y) = |\{i \in \{1, \dots, n\} \mid x_i \neq y_i\}| = \text{wt}_H(x - y)$
- $\mathcal{C} \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n$: $d_H(\mathcal{C}) = \min\{\text{wt}_H(x) \mid 0 \neq x \in \mathcal{C}\}$

Example

- $(1, 2, 3, 0, 0, 2) \in (\mathbb{Z}/4\mathbb{Z})^6$: $\text{wt}_H(x) = 4,$
- $\langle (1, 2, 3), (2, 0, 0) \rangle \subseteq (\mathbb{Z}/4\mathbb{Z})^3$: $d_H(\mathcal{C}) = 1,$

The Lee metric

The Lee metric

- $x \in \mathbb{Z}/p^s\mathbb{Z}$: $\text{wt}_L(x) = \min\{x, |p^s - x|\}$
- $x, y \in (\mathbb{Z}/p^s\mathbb{Z})^n$: $\text{wt}_L(x) = \sum_{i=1}^n \text{wt}_L(x_i)$, $d_L(x, y) = \text{wt}_L(x - y)$
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Example

- $(1, 2, 3, 0, 0, 2) \in (\mathbb{Z}/4\mathbb{Z})^6$: $\text{wt}_H(x) = 4$, $\text{wt}_L(x) = 6$
- $\langle (1, 2, 3), (2, 0, 0) \rangle \subseteq (\mathbb{Z}/4\mathbb{Z})^3$: $d_H(\mathcal{C}) = 1$, $d_L(\mathcal{C}) = 2$

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One of the main tasks: Bound minimum distance

The Singleton Bound

- Hamming metric: Singleton 1964 (Komamiya 1953)
- Optimal codes: MDS dense for $q \rightarrow \infty$
- Assuming MDS conjecture: sparse for $n \rightarrow \infty$



R. Singleton. “Maximum distance q -nary codes.”, *IEEE Transactions on Information Theory*, 1964.



B. Segre. “Curve razionali normali e k -archi negli spazi finiti.”, *Annali di Matematica Pura ed Applicata*, 1955.

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- Rank metric: Gabidulin 1985
 - \mathbb{F}_{q^m} -linear optimal codes: MRD dense for $m, q \rightarrow \infty$
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- Lee metric: Shiromoto 2000
 - Optimal codes and their densities?



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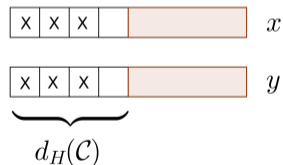
The Singleton Bound

Hamming-metric Singleton Bound

For $\mathcal{C} \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n$ linear code of type k :

$$k \leq n - d_H(\mathcal{C}) + 1$$

- Puncture in $d_H(\mathcal{C}) - 1$ positions
- new code $|\mathcal{C}'| = |\mathcal{C}|$
- $\mathcal{C}' \subseteq (\mathbb{Z}/p^s\mathbb{Z})^{n-(d_H(\mathcal{C})-1)}$



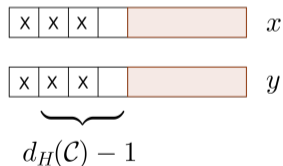
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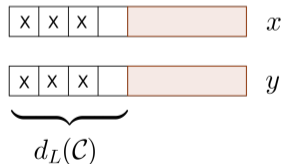
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Lee-metric Singleton Bound

For $\mathcal{C} \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n$ linear code of type k , for $M = \lfloor \frac{p^s}{2} \rfloor$:

$$k \leq n - \left\lfloor \frac{d_L(\mathcal{C}) - 1}{M} \right\rfloor$$

- Puncture in $\left\lfloor \frac{d_L(\mathcal{C}) - 1}{M} \right\rfloor$ positions
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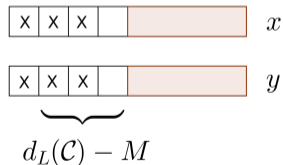
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Optimal Codes

Lee-metric Singleton Bound

\mathcal{C} length n , type k , $M = \lfloor \frac{p^s}{2} \rfloor$:

$$k \leq n - \lfloor \frac{d_L(\mathcal{C}) - 1}{M} \rfloor$$

Example

$$\mathcal{C} = \langle (1, 2) \rangle \subseteq (\mathbb{Z}/5\mathbb{Z})^2$$

$$1 = 2 - \lfloor \frac{3 - 1}{2} \rfloor$$

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This is the only linear non-trivial optimal code!



E. Byrne, V. Weger. "Bounds in the Lee metric and optimal codes.", Finite Fields and Their Applications, 2022

Need better Lee-metric Singleton Bound!

→ Need new technique

Generalized Hamming Weights

Support and Weight of Code

$$\begin{array}{lll} x \in \mathbb{F}_q^n : & \text{supp}_H(x) = \{i \in \{1, \dots, n\} \mid x_i \neq 0\} & \rightarrow \text{wt}_H(x) = |\text{supp}_H(x)| \\ \mathcal{C} \subseteq \mathbb{F}_q^n : & \text{supp}_H(\mathcal{C}) = \{i \in \{1, \dots, n\} \mid \exists x \in \mathcal{C} : x_i \neq 0\} & \rightarrow \text{wt}_H(\mathcal{C}) = |\text{supp}_H(\mathcal{C})| \end{array}$$

Generalized Hamming Weights

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Generalized Weights

$\mathcal{C} \subseteq \mathbb{F}_q^n$ of dimension k . For all $r \in \{1, \dots, k\}$:

$$d_H^r(\mathcal{C}) = \min\{\text{wt}_H(\mathcal{D}) \mid \mathcal{D} \subseteq \mathcal{C} \text{ of dimension } r\}$$

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Example: $\mathcal{C} \subseteq \mathbb{F}_2^4$ generated by $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

$$d_H^1(\mathcal{C}) = 1$$

$$d_H^2(\mathcal{C}) = 3$$

$$d_H^3(\mathcal{C}) = 4$$

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Properties

- $d_H(\mathcal{C}) = d_H^1(\mathcal{C})$
- $d_H^r(\mathcal{C}) < d_H^{r+1}(\mathcal{C})$ for $r < k$
- $d_H^k(\mathcal{C}) = \text{wt}_H(\mathcal{C})$

$$\rightarrow d_H(\mathcal{C}) = \underbrace{d_H^1(\mathcal{C}) < d_H^2(\mathcal{C}) < \dots < d_H^k(\mathcal{C})}_{k-1} = \text{wt}_H(\mathcal{C})$$

\rightarrow Singleton Bound: $d_H(\mathcal{C}) \leq \text{wt}_H(\mathcal{C}) - (k - 1) \leq n - k + 1$

Generalizations to Lee Metric

- over $\mathbb{Z}/4\mathbb{Z}$  S. Dougherty, M. Gupta, K. Shiromoto. “On Generalized weights for codes over finite rings.”, 2002

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- using the **join Lee support**

$$\text{supp}_L(x) = (\text{wt}_L(x_1), \dots, \text{wt}_L(x_n)) = s, \quad |s| = \sum s_i$$

$$\text{wt}_L(\mathcal{C}) = |\bigvee_{c \in \mathcal{C}} \text{supp}_L(c)| = \sum_{i=1}^n \max\{\text{wt}_L(c_i) \mid c \in \mathcal{C}\}$$

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→ Resulting Singleton Bound

$$d_L(\mathcal{C}) \leq \lfloor \frac{p}{2} \rfloor p^{s-1} (n - K + 1)$$

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$$d_L(\mathcal{C}) \leq \lfloor \frac{p}{2} \rfloor p^{s-1} (n - K + 1)$$

→ better than previous bound

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→ Problem

can only be attained for $p = 3$

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→ Resulting bound

$$d_L(\mathcal{C}) \leq \sum_{i=0}^{s-1} p^i k_i + \sum_{i=0}^{s-1} \mu_i M_i - \sum_{i=0}^{\sigma-1} \binom{i}{\sum_{j=0}^i k_j} \lfloor \frac{p}{2} \rfloor p^i - (k_\sigma - 1)p^\sigma$$

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→ Problem

still sparse

Hard to control subcodes of smaller ranks

Generalized Filtration Weight

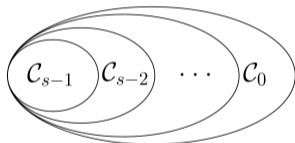
Filtration

For $\mathcal{C} \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n$, define for all $i \in \{0, \dots, s-1\}$: $\mathcal{C}_i = \mathcal{C} \cap \langle p^i \rangle$
maximal Lee weight in \mathcal{C}_i is $M_i = \lfloor \frac{p^{s-i}}{2} \rfloor p^i$

Generalized Filtration Weight

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$$\mathcal{C}_{s-1} \subseteq \mathcal{C}_{s-2} \subseteq \dots \subseteq \mathcal{C}_1 \subseteq \mathcal{C}_0 = \mathcal{C}$$

$$\rightarrow d_L(\mathcal{C}) \leq d_L(\mathcal{C}_1) \leq \dots \leq d_L(\mathcal{C}_{s-1}) \leq d_L(\mathcal{C}_{s-1})$$

Generalized Filtration Weight

Filtration

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Generalized Lee Weights

$\mathcal{C} \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n$. For all $r \in \{1, \dots, s\}$: $d_L^r(\mathcal{C}) = d_L(\mathcal{C}_{r-1})$

Generalized Filtration Weight

Filtration

For $\mathcal{C} \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n$, define for all $i \in \{0, \dots, s-1\}$: $\mathcal{C}_i = \mathcal{C} \cap \langle p^i \rangle$
maximal Lee weight in \mathcal{C}_i is $M_i = \lfloor \frac{p^{s-i}}{2} \rfloor p^i$

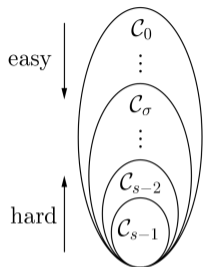
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Properties

- $d_L(\mathcal{C}) = d_L^1(\mathcal{C})$
- $d_L^r(\mathcal{C}) \leq d_L^{r+1}(\mathcal{C})$ for $r < s$
- $d_L^r(\mathcal{C}) \leq p^{r-1} + (n-K)M_{r-1}$

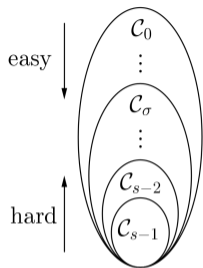
Generalized Filtration Weight



using information from G

how far down should we go?

Generalized Filtration Weight

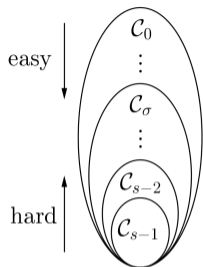


← stop at \mathcal{C}_σ

using information from G

how far down should we go?

Generalized Filtration Weight



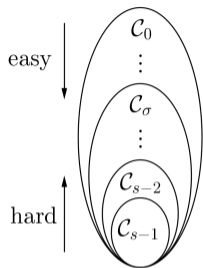
← stop at \mathcal{C}_σ

← stop at \mathcal{C}_{s-1}

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how far down should we go?

Generalized Filtration Weight



← stop at \mathcal{C}_σ

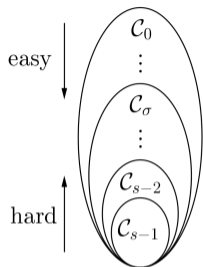
← stop at \mathcal{C}_{s-1}

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how far down should we go?

→ Singleton bound with
several conditions

Generalized Filtration Weight



← stop at \mathcal{C}_σ

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← stop at \mathcal{C}_{s-1}

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New Lee-Metric Singleton Bound

$\mathcal{C} \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n$, subtype (k_0, \dots, k_σ) , ℓ : max prime power $\ell \neq \sigma, s$ in G , appears n' times:

$$d_L(\mathcal{C}) \leq p^{s-\ell+\sigma} + (n - K - n') \lfloor \frac{p^{\ell-\sigma}}{2} \rfloor p^{s-\ell+\sigma}$$

Examples

$$G_1 = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

$$\mathbb{Z}/9\mathbb{Z}, \quad d_L(\langle G_1 \rangle) = 3$$

- Shiromoto: $d_L \leq 5$
- Join: $d_L \leq 3$
- Column weight: $d_L \leq 5$
- Filtration: $d_L \leq 3$

Examples

$$G_1 = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

$$G_2 = \begin{pmatrix} 1 & 10 & 4 & 20 & 9 \\ 0 & 3 & 9 & 18 & 9 \end{pmatrix}$$

$$\mathbb{Z}/9\mathbb{Z}, \quad d_L(\langle G_1 \rangle) = 3$$

$$\mathbb{Z}/27\mathbb{Z}, \quad d_L(\langle G_2 \rangle) = 9$$

- Shiromoto: $d_L \leq 5$
- Join: $d_L \leq 3$
- Column weight: $d_L \leq 5$
- Filtration: $d_L \leq 3$
- Shiromoto: $d_L \leq 40$
- Join: $d_L \leq 36$
- Column weight: $d_L \leq 38$
- Filtration: $d_L \leq 9$

Examples

$$G_1 = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

$$G_2 = \begin{pmatrix} 1 & 10 & 4 & 20 & 9 \\ 0 & 3 & 9 & 18 & 9 \end{pmatrix}$$

$$G_3 = \begin{pmatrix} 1 & 0 & 25 & 50 & 75 & 100 \\ 0 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

$$\mathbb{Z}/9\mathbb{Z}, \quad d_L(\langle G_1 \rangle) = 3$$

$$\mathbb{Z}/27\mathbb{Z}, \quad d_L(\langle G_2 \rangle) = 9$$

$$\mathbb{Z}/125\mathbb{Z}, \quad d_L(\langle G_3 \rangle) = 5$$

- Shiromoto: $d_L \leq 5$
- Join: $d_L \leq 3$
- Column weight: $d_L \leq 5$
- Filtration: $d_L \leq 3$

- Shiromoto: $d_L \leq 40$
- Join: $d_L \leq 36$
- Column weight: $d_L \leq 38$
- Filtration: $d_L \leq 9$

- Shiromoto: $d_L \leq 249$
- Join: $d_L \leq 200$
- Column weight: $d_L \leq 247$
- Filtration: $d_L \leq 5$

Are the optimal codes dense?

Are the optimal codes dense?

$$p \rightarrow \infty$$

- \mathcal{C} is a free code: $\sigma = 0$, $\ell = 0$
- Recover Shiromoto:
$$d_L(\mathcal{C}) \leq 1 + (n - K) \lfloor \frac{p^s}{2} \rfloor$$
- Optimal codes are sparse



E. Byrne, A.-L. Horlemann, K. Khathuria, V. Weger “Density of free modules over finite chain rings.”, *Linear Algebra and its Applications*, 2022.

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- Only way to not get sparse:
 $n' = n - K$:
- \mathbb{P} whole row is $p^\ell : \frac{1}{p^{n-K}}$
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E. Byrne, A.-L. Horlemann, K. Khathuria, V. Weger “Density of free modules over finite chain rings.”, *Linear Algebra and its Applications*, 2022.

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 $n' = n - K$:
- \mathbb{P} whole row is $p^\ell : \frac{1}{p^{n-K}}$
- Optimal codes are sparse

But for $s \rightarrow \infty$: not sparse!

$$\text{density} \left(\frac{p-1}{p} \right)^{K(n-K)}$$



E. Byrne, A.-L. Horlemann, K. Khathuria, V. Weger “Density of free modules over finite chain rings.”, *Linear Algebra and its Applications*, 2022.

Questions?

Summary

- several definitions of generalized Lee weights
- new (tighter) Lee-metric Singleton bounds
- new Singleton bound for which MLD codes are not sparse for $s \rightarrow \infty$

Open Question

- MLD codes construction
- technique for Lee-metric bounds with dense optimal codes

Questions?

Summary

- several definitions of generalized Lee weights
- new (tighter) Lee-metric Singleton bounds
- new Singleton bound for which MLD codes are not sparse for $s \rightarrow \infty$

Open Question

- MLD codes construction
- technique for Lee-metric bounds with dense optimal codes

Thank you!

Generalized Filtration Weight

$$\mathcal{C} = \langle G_{sys} \rangle$$
$$\max \sigma : k_\sigma \neq 0$$

$$G_{sys} = \begin{pmatrix} \text{Id}_{k_1} & & & & \star \\ 0 & p\text{Id}_{k_2} & & & p\star \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & p^\sigma \text{Id}_{k_\sigma} & p^\sigma \star \end{pmatrix}$$

Generalized Filtration Weight

$$\mathcal{C} = \langle G_{sys} \rangle$$
$$\max \sigma : k_\sigma \neq 0$$

↓

$$\mathcal{C}_\sigma$$

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↓

$$G_\sigma = (p^\sigma \text{Id}_K \quad p^\sigma A_\sigma)$$

Generalized Filtration Weight

$$\mathcal{C} = \langle G_{sys} \rangle$$
$$\max \sigma : k_\sigma \neq 0$$

↓

$$\mathcal{C}_\sigma$$

ℓ : max prime power in $p^\sigma A_\sigma$

$$G_{sys} = \begin{pmatrix} \text{Id}_{k_1} & & & & \star \\ 0 & p\text{Id}_{k_2} & & & p\star \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & p^\sigma \text{Id}_{k_\sigma} & p^\sigma \star \end{pmatrix}$$

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$$G_\sigma = (p^\sigma \text{Id}_K \quad p^\sigma A_\sigma)$$

n' : max number of p^ℓ in one row

Generalized Filtration Weight

$$\mathcal{C} = \langle G_{sys} \rangle$$

$$\max \sigma : k_\sigma \neq 0$$

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$$\mathcal{C}_\sigma$$

ℓ : max prime power in $p^\sigma A_\sigma$

1. If $\ell = \sigma$

→

$$G_{sys} = \begin{pmatrix} \text{Id}_{k_1} & & & & \star \\ 0 & p\text{Id}_{k_2} & & & p\star \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & p^\sigma \text{Id}_{k_\sigma} & p^\sigma \star \end{pmatrix}$$

↓

$$G_\sigma = (p^\sigma \text{Id}_K \quad p^\sigma A_\sigma)$$

n' : max number of p^ℓ in one row

$$d_L(\mathcal{C}_\sigma) \leq p^\sigma + (n - K)M_\sigma$$

Generalized Filtration Weight

$$\mathcal{C} = \langle G_{sys} \rangle$$

$$\max \sigma : k_\sigma \neq 0$$

↓

$$\mathcal{C}_\sigma$$

ℓ : max prime power in $p^\sigma A_\sigma$

1. If $\ell = \sigma$ →

2. If $\ell = s$ →

$$G_{sys} = \begin{pmatrix} \text{Id}_{k_1} & & & & \star \\ 0 & p\text{Id}_{k_2} & & & p\star \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & p^\sigma \text{Id}_{k_\sigma} & p^\sigma \star \end{pmatrix}$$

↓

$$G_\sigma = (p^\sigma \text{Id}_K \quad p^\sigma A_\sigma)$$

n' : max number of p^ℓ in one row

$$d_L(\mathcal{C}_\sigma) \leq p^\sigma + (n - K)M_\sigma$$

$$d_L(\mathcal{C}_\sigma) \leq p^\sigma + (n - K - n')M_\sigma$$

Generalized Filtration Weight

$$\mathcal{C} = \langle G_{sys} \rangle$$

$$\max \sigma : k_\sigma \neq 0$$

↓

$$\mathcal{C}_\sigma$$

ℓ : max prime power in $p^\sigma A_\sigma$

1. If $\ell = \sigma$ →

2. If $\ell = s$ →

3. If $\ell \neq \sigma, \ell \neq s$ →

$$G_{sys} = \begin{pmatrix} \text{Id}_{k_1} & & & & \star \\ 0 & p\text{Id}_{k_2} & & & p\star \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & p^\sigma \text{Id}_{k_\sigma} & p^\sigma \star \end{pmatrix}$$

↓

$$G_\sigma = (p^\sigma \text{Id}_K \quad p^\sigma A_\sigma)$$

n' : max number of p^ℓ in one row

$$d_L(\mathcal{C}_\sigma) \leq p^\sigma + (n - K)M_\sigma$$

$$d_L(\mathcal{C}_\sigma) \leq p^\sigma + (n - K - n')M_\sigma$$

go to $\mathcal{C}_{s-\ell+\sigma}$: multiply with $p^{s-\ell}$

Generalized Filtration Weight

$$\mathcal{C}_\sigma \qquad G_\sigma = (p^\sigma \text{Id}_K \quad p^\sigma A_\sigma)$$

↓

↓

$$\mathcal{C}_{s-\ell+\sigma} \qquad G_{s-\ell+\sigma} = (p^{s-\ell+\sigma} \text{Id}_K \quad p^{s-\ell+\sigma} A_{s-\ell+\sigma})$$

Generalized Filtration Weight

 \mathcal{C}_σ

$$G_\sigma = (p^\sigma \text{Id}_K \quad p^\sigma A_\sigma)$$

$$\underbrace{(0p^\sigma 0)}_K \quad \underbrace{p^\ell \cdots p^\ell}_{n'} \quad \underbrace{*\cdots*}_{n-K-n'}$$

 \downarrow
 \downarrow
 \downarrow
 $\mathcal{C}_{s-\ell+\sigma}$

$$G_{s-\ell+\sigma} = (p^{s-\ell+\sigma} \text{Id}_K \quad p^{s-\ell+\sigma} A_{s-\ell+\sigma})$$

$$\underbrace{(0p^{s-\ell+\sigma} 0)}_K \quad \underbrace{0 \cdots 0}_{n'} \quad \underbrace{*\cdots*}_{n-K-n'}$$

Generalized Filtration Weight

$$\begin{array}{ccc}
 \mathcal{C}_\sigma & G_\sigma = (p^\sigma \text{Id}_K & p^\sigma A_\sigma) & (\underbrace{0p^\sigma 0}_K \underbrace{p^\ell \cdots p^\ell}_{n'} \underbrace{\star \cdots \star}_{n-K-n'}) \\
 \downarrow & & \downarrow & \downarrow \\
 \mathcal{C}_{s-\ell+\sigma} & G_{s-\ell+\sigma} = (p^{s-\ell+\sigma} \text{Id}_K & p^{s-\ell+\sigma} A_{s-\ell+\sigma}) & (\underbrace{0p^{s-\ell+\sigma} 0}_K \underbrace{0 \cdots 0}_{n'} \underbrace{\star \cdots \star}_{n-K-n'})
 \end{array}$$

New Lee-Metric Singleton Bound

$\mathcal{C} \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n$, subtype (k_0, \dots, k_σ) , max prime power $\ell \neq \sigma, s$, appears n' times:

$$d_L(\mathcal{C}) \leq p^{s-\ell+\sigma} + (n - K - n') \lfloor \frac{p^{\ell-\sigma}}{2} \rfloor p^{s-\ell+\sigma}$$

Support and Weights of Codes: Lee Metric

Support and Weight of Code

$$\begin{aligned}x \in (\mathbb{Z}/p^s\mathbb{Z})^n : \quad \text{supp}_H(x) &= \{i \in \{1, \dots, n\} \mid x_i \neq 0\} \subseteq \{1, \dots, n\} & \rightarrow \text{wt}_H(x) &= |\text{supp}_H(x)| \\ \mathcal{C} \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n : \quad \text{supp}_H(\mathcal{C}) &= \{i \in \{1, \dots, n\} \mid \exists x \in \mathcal{C} : x_i \neq 0\} & \rightarrow \text{wt}_H(\mathcal{C}) &= |\text{supp}_H(\mathcal{C})|\end{aligned}$$

Support and Weights of Codes: Lee Metric

Support and Weight of Code

$$\begin{array}{lll} x \in (\mathbb{Z}/p^s\mathbb{Z})^n : & \text{supp}_H(x) = (\text{wt}_H(x_1), \dots, \text{wt}_H(x_n)) \subset \mathbb{N}^n & \rightarrow \text{wt}_H(x) = |\text{supp}_H(x)| \\ \mathcal{C} \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n : & \text{supp}_H(\mathcal{C}) = \bigvee_{c \in \mathcal{C}} \text{supp}_H(c) & \rightarrow \text{wt}_H(\mathcal{C}) = |\text{supp}_H(\mathcal{C})| \end{array}$$

$$s, t \in \mathbb{N}^n : \quad \bullet \text{ size } |s| = \sum_{i=1}^n s_i \quad \bullet \text{ join } s \vee t = (\max\{s_1, t_1\}, \dots, \max\{s_n, t_n\})$$

Support and Weights of Codes: Lee Metric

Support and Weight of Code

$$\begin{aligned}x \in (\mathbb{Z}/p^s\mathbb{Z})^n : & \quad \text{supp}_L(x) = (\text{wt}_L(x_1), \dots, \text{wt}_L(x_n)) & \rightarrow \text{wt}_L(x) = |\text{supp}_L(x)| \\ \mathcal{C} \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n : & \quad \text{supp}_L(\mathcal{C}) = \bigvee_{c \in \mathcal{C}} \text{supp}_L(c) & \rightarrow \text{wt}_L(\mathcal{C}) = |\text{supp}_L(\mathcal{C})|\end{aligned}$$

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Generalized Lee Weights

$\mathcal{C} \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n$ of rank K . For all $r \in \{1, \dots, K\}$:

$$d_L^r(\mathcal{C}) = \min\{\text{wt}_L(\mathcal{D}) \mid \mathcal{D} \subseteq \mathcal{C} \text{ of rank } r\}$$

Generalized Lee Weights

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Example

$\mathcal{C} \subseteq (\mathbb{Z}/9\mathbb{Z})^4$ generated by $\begin{pmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 3 & 3 \end{pmatrix}$

$$d_L(\mathcal{C}) = 2$$

$$d_L^1(\mathcal{C}) = 6$$

$$d_L^2(\mathcal{C}) = 9$$

$$d_L^3(\mathcal{C}) = 12$$

$$\text{wt}_L(\mathcal{C}) = 16$$

Generalized Lee Weights

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Properties

- $d_L(\mathcal{C}) \leq d_L^1(\mathcal{C})$
- $d_L^r(\mathcal{C}) \leq d_L^{r+1}(\mathcal{C})$ for $r < K$
- $d_L^K(\mathcal{C}) \leq \text{wt}_L(\mathcal{C})$

Generalized Lee Weights

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$\mathcal{C} \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n$ of rank K . For all $r \in \{1, \dots, K\}$:

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socle: $\mathcal{C}_{s-1} = \mathcal{C} \cap \langle p^{s-1} \rangle$ of maximal Lee weight $M_{s-1} = \lfloor \frac{p}{2} \rfloor p^{s-1}$

Properties

All subcodes attaining the r th generalized Lee weights are in the socle: $d_L^r(\mathcal{C}) = d_H^r(\mathcal{C})M_{s-1}$

Generalized Lee Weights

Generalized Lee Weights

$\mathcal{C} \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n$ of rank K . For all $r \in \{1, \dots, K\}$:

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New Lee-Metric Singleton Bound

$$d_L(\mathcal{C}) \leq M_{s-1}(n - K + 1)$$

Better than previous $d_L(\mathcal{C}) \leq M(n - K + 1)$

Generalized Lee Weights

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New Lee-Metric Singleton Bound

$$d_L(\mathcal{C}) \leq M_{s-1}(n - K + 1)$$

Better than previous $d_L(\mathcal{C}) \leq M(n - K + 1)$

only codes with $p = 3$ can attain it

Need different approach

Lee Column Weight

Example

$\mathcal{C} \subseteq \mathbb{F}_2^4$ generated by

$$G = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$G_1 = (0 \ 0 \ 1 \ 0) \rightarrow d_H^1(\mathcal{C}) = 1$$

$$G_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \rightarrow d_H^2(\mathcal{C}) = 3$$

$$G \rightarrow d_H^3(\mathcal{C}) = 4$$

Lee Column Weight

Example

$\mathcal{C} \subseteq \mathbb{F}_2^4$ generated by

$$G = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$G_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \rightarrow d_H^1(\mathcal{C}) = 1 = \text{colwt}(G_1)$$

$$G_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \rightarrow d_H^2(\mathcal{C}) = 3 = \text{colwt}(G_2)$$

$$G \rightarrow d_H^3(\mathcal{C}) = 4 = \text{colwt}(G)$$

Lee Column Weight

Example

$\mathcal{C} \subseteq \mathbb{F}_2^4$ generated by

$$G = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$G_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \rightarrow d_H^1(\mathcal{C}) = 1 = \text{colwt}(G_1)$$

$$G_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \rightarrow d_H^2(\mathcal{C}) = 3 = \text{colwt}(G_2)$$

$$G \rightarrow d_H^3(\mathcal{C}) = 4 = \text{colwt}(G)$$

Lee Column Weight

$$A = \begin{pmatrix} \vdots & & \vdots \\ a_1^\top & \cdots & a_n^\top \\ \vdots & & \vdots \end{pmatrix} \rightarrow \text{colwt}_L(A) = |(\max \text{supp}_L(a_1), \dots, \max \text{supp}_L(a_n))|$$

Lee Column Weight

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$$A = \begin{pmatrix} \vdots & & \vdots \\ a_1^\top & \cdots & a_n^\top \\ \vdots & & \vdots \end{pmatrix} \rightarrow \text{colwt}_L(A) = |(\max \text{supp}_L(a_1), \dots, \max \text{supp}_L(a_n))|$$

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$$\text{Example: } G = \begin{pmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 3 & 3 \end{pmatrix} \rightarrow \text{colwt}_L(G) = |(1, 1, 3, 3)| = 8$$

Lee Column Weight

Lee Column Weight

$$A = \begin{pmatrix} \vdots & & \vdots \\ a_1^\top & \cdots & a_n^\top \\ \vdots & & \vdots \end{pmatrix} \rightarrow \text{colwt}_L(A) = |(\max \text{supp}_L(a_1), \dots, \max \text{supp}_L(a_n))|$$

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Lee Column Weight

$$\mathcal{C} \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n: \text{colwt}_L(\mathcal{C}) = \min\{\text{colwt}(G) \mid \langle G \rangle = \mathcal{C}\}$$

Lee Column Weight

Lee Column Weight

$$A = \begin{pmatrix} \vdots & & \vdots \\ a_1^\top & \cdots & a_n^\top \\ \vdots & & \vdots \end{pmatrix} \rightarrow \text{colwt}_L(A) = |(\max \text{supp}_L(a_1), \dots, \max \text{supp}_L(a_n))|$$

$$\text{Example: } G = \begin{pmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 3 & 3 \end{pmatrix} \rightarrow \text{colwt}_L(G) = |(1, 1, 3, 3)| = 8$$

Lee Column Weight

$$\mathcal{C} \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n: \text{colwt}_L(\mathcal{C}) = \min\{\text{colwt}(G) \mid \langle G \rangle = \mathcal{C}\}$$

Highly depends on the choice of generator matrix

Lee Column Weight

Generalized Lee Column Weights

$\mathcal{C} \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n$ of rank K . For all $r \in \{1, \dots, K\}$:

$$d_L^r(\mathcal{C}) = \min\{\text{colwt}_L(\mathcal{D}) \mid \mathcal{D} \subseteq \mathcal{C} \text{ of rank } r\}$$

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Properties

- $d_L(\mathcal{C}) = d_L^1(\mathcal{C})$
- $d_L^r(\mathcal{C}) < d_L^{r+1}(\mathcal{C})$ for $r < K$
- $d_L^K(\mathcal{C}) = \text{colwt}_L(\mathcal{C})$

Lee Column Weight

support subtype of a code is (n_0, \dots, n_{s-1}) , where

$$n_i = |\{j \in \{1, \dots, n\} \mid \langle c_j \rangle = \langle p^i \rangle\}|$$

→ Remainder support subtype $(\mu_0, \dots, \mu_{s-1})$ is support subtype in $C_{n-K, \dots, n}$

$$\text{colwt}_L(\mathcal{C})m \leq \sum_{i=0}^{s-1} p^i k_i + \sum_{i=0}^{s-1} \mu_i M_i,$$

where $M_i = \lfloor \frac{p^{s-1}}{2} \rfloor p^i$

Singleton Bound

$\mathcal{C} \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n$ with subtype (k_0, \dots, k_{s-1}) , σ largest with $k_\sigma \neq 0$, support subtype in redundant part $(\mu_{n-K}, \dots, \mu_n)$,

$$d_L(\mathcal{C}) \leq \sum_{i=0}^{s-1} p^i k_i + \sum_{i=n-K}^n \mu_i M_i - \left(\sum_{i=0}^{\sigma-1} \left(\sum_{j=0}^i k_j \right) \lfloor \frac{p}{2} \rfloor p^i + (k_\sigma - 1)p^\sigma \right)$$

Much better than previous bound $d_L(\mathcal{C}) \leq M(n - K + 1)$