Code Equivalence

Violetta Weger

Finite Geometry and Friends Summer School 2025

September 2025

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Finite Friends and Geometry, 2023

Classic Heroes vs. Quantum Avengers

Classical attackers  → Classical heroes 

X ~~RSA~~
X ~~DLP~~
X ~~EC-DLP~~

Quantum attackers  → Quantum heroes  post-quantum

 Lattice-based

 Hash-based

 Multivariate

 Isogeny-based

 Code-based

Finite Friends and Geometry, 2023

Motivation

Put differently

Technology Innovation Institute

On the mathematics of post-quantum cryptography, 2025


LESS
CF-LESS

THANK YOU VERY MUCH FOR THE ATTENTION!

Quantum attackers  → Quantum heroes  post-quantum

 Lattice-based

 Hash-based

 Multivariate

 Isogeny-based

Code-based Cryptography

 Decoding-based

 Code-Equivalence

Code Equivalence - The Problem

Given two codes $\mathcal{C}, \mathcal{C}'$, find a linear isometry φ such that $\varphi(\mathcal{C}) = \mathcal{C}$.

"Is code equivalence easy to decide?" Petrank, Roth. 2002.

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Plan

- Basics of Coding Theory
- Hardness of Code Equivalence
- Connections to other Problems
- LESS Signature Scheme
- Solvers
- Some new Results
- Introduction to Complexity Theory
- Finite Friends
- Summary

Code Equivalence - Organization

Material:

Lecture Notes

Exercises

$\mathbb{F}_q$ finite field of order q a prime power

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Definition ∘ $[n,k]_q$ linear code $\mathcal{C}$: $\mathbb{F}_q$-linear subspace of $\mathbb{F}_q^n$ of dimension k

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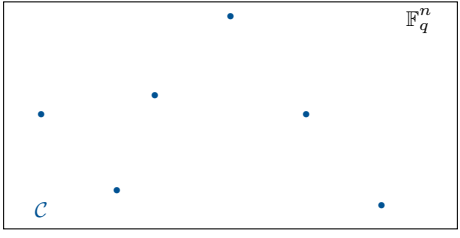

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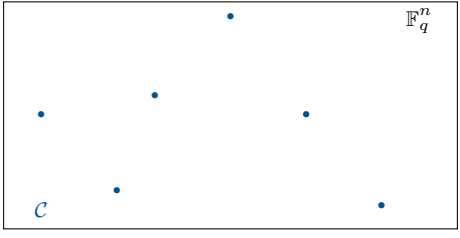

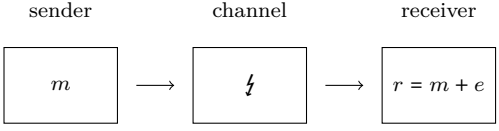

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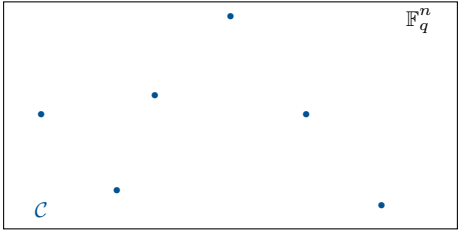

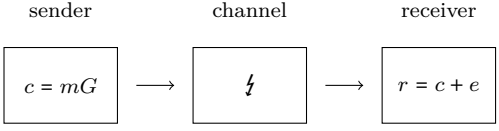

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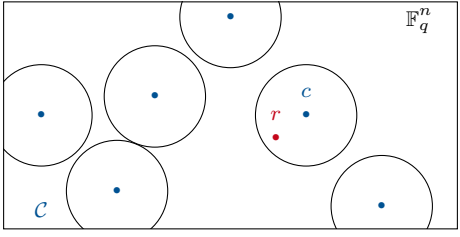

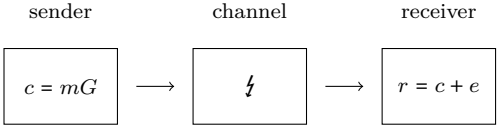

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- The Hamming weight of $x \in \mathbb{F}_q^n$ is $\mathrm{wt}(x) = |\{i \mid x_i \neq 0\}|$
- The Hamming distance between $x, y \in \mathbb{F}_q^n$ is
$$d(x, y) = \mathrm{wt}(x - y) = |\{i \mid x_i \neq y_i\}|$$
- The minimum Hamming distance of $\mathcal{C} \subseteq \mathbb{F}_q^n$ is
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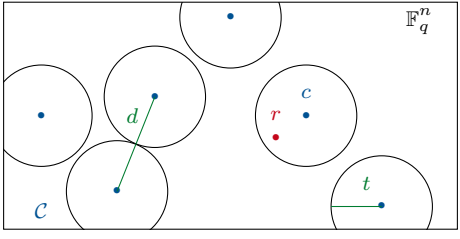

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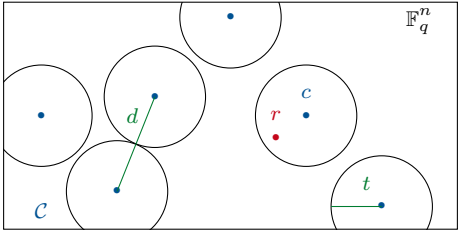

A $[n, k, d]_q$ code $\mathcal{C}$ can correct $t = \lfloor \frac{d-1}{2} \rfloor$ errors

$\mathcal{C} = \langle G \rangle = \ker(H^\top) \subseteq \mathbb{F}_q^n$ of dimension k

Definition

- G is in systematic form if $G = \begin{pmatrix} \mathrm{Id}_k & A \end{pmatrix}$
- H is in systematic form if $H = \begin{pmatrix} B & \mathrm{Id}_{n-k} \end{pmatrix}$

Properties

- For $S \in \mathrm{GL}_q(k)$ also $\langle SG \rangle = \mathcal{C}$
- For some permutation matrix P, SGP is in systematic form
- For $S \in \mathrm{GL}_q(n-k)$ also $\ker((SH)^\top) = \mathcal{C}$
- For some permutation matrix P, SHP is in systematic form
- If $G = \begin{pmatrix} \mathrm{Id}_k & A \end{pmatrix}$, then $H = \begin{pmatrix} -A^\top & \mathrm{Id}_{n-k} \end{pmatrix}$

$\mathcal{C} = \langle G \rangle = \ker(H^\top) \subseteq \mathbb{F}_q^n$ of dimension k

Definition

- The dual code of $\mathcal{C}$ is
$$\mathcal{C}^\perp = \{x \in \mathbb{F}_q^n \mid \langle x, y \rangle = 0 \ \forall \ c \in \mathcal{C}\}$$
- $\mathcal{C}^\perp = \langle H \rangle = \ker(G^\top) \subseteq \mathbb{F}_q^n$ of dimension $n - k$
- If $\mathcal{C} = \mathcal{C}^\perp$ then $\mathcal{C}$ is called self-dual
- If $\mathcal{C} \subset \mathcal{C}^\perp$ then $\mathcal{C}$ is called self-orthogonal
- The hull of $\mathcal{C}$ is $\mathcal{H}(\mathcal{C}) = \mathcal{C} \cap \mathcal{C}^\perp$

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Exercises

- Show that $\langle H \rangle = \mathcal{C}^\perp$
- Show that $(\mathcal{C}^\perp)^\perp = \mathcal{C}$.
- Show that if $GG^\top = 0$, then $\mathcal{C}$ is self-orthogonal
- Show that $\mathcal{C}$ is self-dual if and only if $\mathcal{C}$ is self-orthogonal and $n = 2k$
- Show that $\mathcal{H}(\mathcal{C}) = \ker\left(\begin{pmatrix} G \\ H \end{pmatrix}^\top\right)$

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Theorem If $\mathcal{C}$ is random, then

$$\mathbb{P}(\dim(\mathcal{H}(\mathcal{C})) = h) = \prod_{i=1}^{\infty} q^i \frac{q^i - 1}{q^{2i} - 1} \prod_{i=1}^{n} (q^i - 1)^{-1} \sim (1 - 1/q) q^{-h(h+1)/2}$$

"On the dimension of the hull" N. Sendrier, 1997

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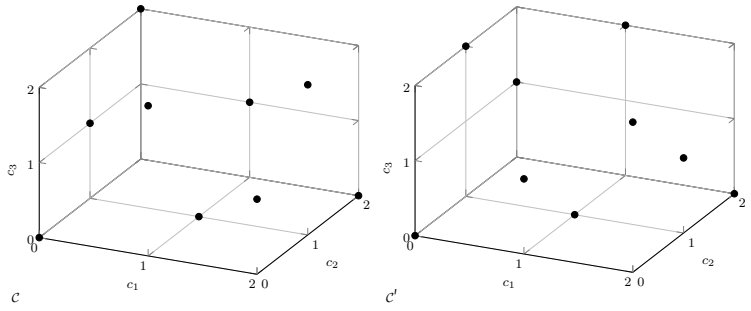
Exercise

- If $G = \begin{pmatrix} \mathrm{Id}_k & A \end{pmatrix}$ and $AA^\top + \mathrm{Id}_{n-k}$ has full rank, then $\mathcal{H}(\mathcal{C}) = \{0\}$
- If $GG^\top$ has full rank, then $\mathcal{H}(\mathcal{C}) = \{0\}$

$$G = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}, \quad G' = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$



$$\mathcal{C} = \{(0,0,0),(1,0,2),(2,0,1),(1,1,0),(2,1,2),(0,1,1),(0,2,2),(1,2,1),(2,2,0)\}$$
$$\mathcal{C}' = \{(0,0,0),(0,1,2),(0,2,1),(1,1,0),(1,2,2),(1,0,1),(2,0,2),(2,1,1),(2,2,0)\}$$

Definition ∘ A linear isometry for a distance function d is a linear map

$$\varphi : \mathbb{F}_q^n \to \mathbb{F}_q^n \text{ s.t. } \forall\ x, y \in \mathbb{F}_q^n\colon\ d(x, y) = d(\varphi(x), \varphi(y))$$

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- $\varphi = (d, \sigma) \in (\mathbb{F}_q^\star)^n \rtimes S_n$ called monomial transformation
- $D = \mathrm{diag}(d)$, permutation matrix P, DP called monomial matrix
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Theorem

If $\varphi : \mathcal{C} \to \mathcal{C}'$ linear isometry, then exists $\mu \in (\mathbb{F}_q^\star)^n \rtimes S_n$ s.t. $\mu_{|\mathcal{C}} = \varphi$

"Combinatorial problems of elementary abelian groups" F.J. MacWilliams, 1962

Let $\mathcal{C}, \mathcal{C}'$ be $[n,k]_q$ linear codes

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Proposition

If $\mathcal{C} = \langle G \rangle$ is linearly equivalent to $\mathcal{C}' = \langle G' \rangle$, then there exist $S \in \mathrm{GL}_q(k), D = \mathrm{diag}(d)$, permutation matrix P, s.t. $SGDP = G'$

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Property

- If $\mathcal{C}$ is random, then $\mathrm{Aut}(\mathcal{C}) = \{\mathrm{id}\}$ with high probability for large n

"Rigid linear binary codes" H. Lefmann, K. Phelps, V. Rödl, 1993

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Proposition If $\varphi \in S_n$ is s.t. $\varphi(\mathcal{C}) = \mathcal{C}'$ then $\varphi(\mathcal{C}^\perp) = \mathcal{C}'^\perp$

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Exercises

- If $\varphi \in S_n$ is s.t. $\varphi \in \mathrm{Aut}(\mathcal{C})$ then $\varphi \in \mathrm{Aut}(\mathcal{H}(\mathcal{C}))$
- If $\varphi \in (\mathbb{F}_q^\star)^n \rtimes S_n$ is s.t. $\varphi(\mathcal{C}) = \mathcal{C}'$ then $\exists \varphi' \in (\mathbb{F}_q^\star)^n \rtimes S_n : \varphi'(\mathcal{C}^\perp) = \mathcal{C}'^\perp$

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- The support of $\mathcal{C}$ is $\text{supp}(\mathcal{C}) = \{i \mid \exists c \in \mathcal{C} : c_i \neq 0\}$
- The weight of $\mathcal{C}$ is $\text{wt}(\mathcal{C}) = |\text{supp}(\mathcal{C})|$

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- Let $r \in \{1, \ldots, k\}$, the rth generalized weight of $\mathcal{C}$ is

$$d_r(\mathcal{C}) = \min\{\text{wt}(\mathcal{D}) \mid \mathcal{D} \subseteq \mathcal{C}, \dim(\mathcal{D}) = r\}$$

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Exercises

- Show that $d_r(\mathcal{C}) = d_r(\mathcal{C}')$
- For $r \in \{1, \dots, k-1\}$ show that $d_r(\mathcal{C}) < d_{r+1}(\mathcal{C})$

Let $\mathcal{C}$ be an $[n, k, d]_q$ linear code. $\langle G \rangle = \mathcal{C} = \ker(H^\top)$

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- n is called

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- H is called

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Signature scheme

Code Equivalence - Cryptography

Signer

Signature Scheme

Verifier

secret key

public key

message m

Code Equivalence - Cryptography

Signer | Signature Scheme | Verifier

secret key → public key

message m

, $m \rightarrow$ signature s

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Code Equivalence - Cryptography

Signer | Signature Scheme | Verifier

??? ⟶ ♀ public key

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probability of getting accepted:

cheating probability α

Code Equivalence - Cryptography

Prover

Zero-Knowledge Protocol

Verifier

secret key

public key

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c_0, c_1 commitments

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challenge $b \in \{0, 1\}$

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response r_b

Code Equivalence - Cryptography

Prover — Zero-Knowledge Protocol — Verifier

secret key → public key

c_0, c_1 commitments →

← challenge $b \in \{0, 1\}$

response r_b →

check $\varphi, r_b \to c_b$

Code Equivalence - Cryptography

Zero-Knowledge Protocol

Prover		Verifier
		
???	$\longrightarrow$	public key
c_0, c_1 commitments	$\longrightarrow$	
	$\longleftarrow$	challenge $b \in \{0,1\}$
response r_b	$\longrightarrow$	check $, r_b \to c_b$

- zero-knowledge
- complete
- soundness error α

Code Equivalence - Cryptography

Prover — Zero-Knowledge Protocol — Verifier

secret key → public key

c_0, c_1 commitments →

← challenge $b \in \{0, 1\}$

response r_b → check $\mathrm{pk}, r_b \to c_b$

t rounds

- zero-knowledge
- complete
- soundness error $\alpha \to \alpha^t$

Signer

Fiat-Shamir Transform

Verifier

ZK Protocol → Signature Scheme

secret key

public key

c_0, c_1 commitments

$b = \mathsf{Hash}(m, c_0, c_1)$

response r_b

$m, s = (c_0, c_1, r_b)$

$b = \mathsf{Hash}(m, c_0, c_1)$

check $\mathrm{pk}, r_b \to c_b$

Main motivation: LESS

Code Equivalence - Cryptography

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Problem:

- Standardizations take time
- All based on novel problems: secure?

Code Equivalence - Cryptography

Prover | LESS ZK-Protocol | Verifier

$\varphi = D, P$ ⟶ G, G' s.t. $SGDP = G'$

Code Equivalence - Cryptography

Prover | LESS ZK-Protocol | Verifier

$\varphi = D, P$

commitment $\tilde{G} = \tilde{\varphi}(G)$

$\longrightarrow$

$\longrightarrow$

G, G' s.t. $SGDP = G'$

Code Equivalence - Cryptography

Prover

LESS ZK-Protocol

Verifier

♀ $\varphi = D, P$

commitment $\tilde{G} = \tilde{\varphi}(G)$

♀ G, G^{l} s.t. $SGDP = G^{l}$

challenge $b \in \{0, 1\}$

Code Equivalence - Cryptography

Prover | LESS ZK-Protocol | Verifier

$\varphi = D, P$ →

G, G^I s.t. $SGDP = G^I$

commitment $\tilde{G} = \tilde{\varphi}(G)$ →

←

challenge $b \in \{0, 1\}$

response $r_0 = \tilde{\varphi}, r_1 = \tilde{\varphi} \circ \varphi^{-1}$ →

Code Equivalence - Cryptography

Prover | LESS ZK-Protocol | Verifier

$\varphi = D, P$ →

commitment $\tilde{G} = \tilde{\varphi}(G)$ →

← challenge $b \in \{0, 1\}$

response $r_0 = \tilde{\varphi}, r_1 = \tilde{\varphi} \circ \varphi^{-1}$ →

G, G' s.t. $SGDP = G'$

check $\tilde{\varphi}(G) = \tilde{G}$ or $\tilde{\varphi} \circ \varphi^{-1}(G') = \tilde{G}$

Code Equivalence - Cryptography

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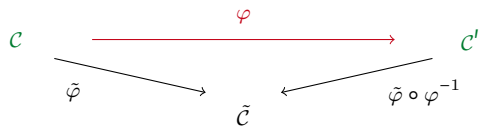

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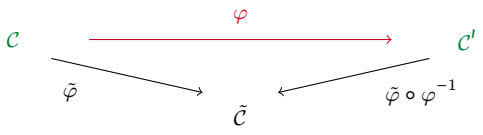

soundness error $\frac{1}{2}$

Set up $\mathcal{P}$ a decisional problem, I an instance, s a solution

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Example Syndrome Decoding Problem:

Given H, s, t, does there exist a e s.t. $eH^\top = s, \mathrm{wt}(e) \leq t$

Instance $=(H, s, t)$ Solution $=$ yes/no

Set up $\mathcal{P}$ a computational problem, I an instance, s a solution

Example Syndrome Decoding Problem:

Given H, s, t, find error vector e s.t. $eH^\top = s, \mathrm{wt}(e) \leq t$

Instance $=(H, s, t)$ Solution $= e$

Set up $\mathcal{P}$ a computational problem, I an instance, s a solution

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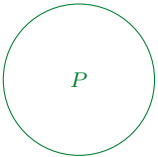
Given H, s, t, find error vector e s.t. $eH^\top = s, \mathrm{wt}(e) \leq t$

Instance $=(H, s, t)$ Solution $= e$

Aim complexity theory: How hard are such problems?

Is SDP harder than sorting / determining minimum distance/ code equivalence?

Complexity Classes



- $\mathcal{P} \in P$ if can solve $\mathcal{P}$ in poly. time by a deterministic Turing machine

- polynomial time: $\mathcal{O}(n^c)$ for some constant c

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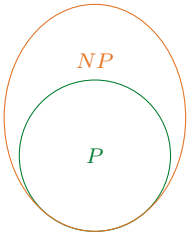

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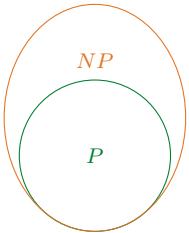

- $\mathcal{P} \in P$ if can solve $\mathcal{P}$ in poly. time by a deterministic Turing machine
- $\mathcal{P} \in NP$ if can solve $\mathcal{P}$ in poly. time by a non-deterministic Turing machine

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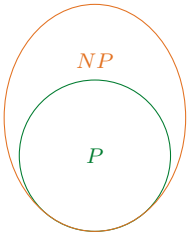
Complexity Classes



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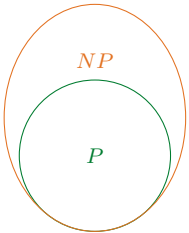

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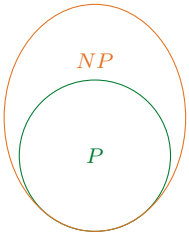

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- quasi- polynomial time: $\mathcal{O}(2^{\log(n)^c})$ for some constant c
- exponential time: $\mathcal{O}(2^{nc})$ for some constant c

How to compare hardness of problems?

Polynomial-time reduction form $\mathcal{R}$ to $\mathcal{P}$

1.	take any instance I of $\mathcal{R}$	$\rightarrow$	2.	transform to a instance I' of $\mathcal{P}$
			$\downarrow$	
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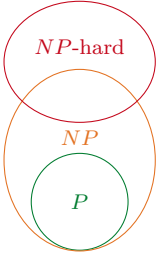
→ hardness($\mathcal{P}$) ≥ hardness($\mathcal{R}$)



- $\mathcal{P} \in NP$-hard if $\exists$ poly. time reduction from every $\mathcal{R} \in NP$ to $\mathcal{P}$
- NP-complete = $NP \cap NP$-hard
- if $\mathcal{R} \in NP$-hard and $\mathcal{R} \to \mathcal{P}$ then $\mathcal{P} \in NP$-hard

Code Equivalence - Coffee break

Code Equivalence - Exercises

$\mathcal{C} = \langle G \rangle = \ker(H^\top)$ a $[n,k]_q$ linear code

1. Show that $\langle H \rangle = \mathcal{C}^\perp$.
2. Show that $(\mathcal{C}^\perp)^\perp = \mathcal{C}$.
3. Show that if $GG^\top = 0$, then $\mathcal{C}$ is self-orthogonal.
4. Show that $\mathcal{C}$ is self-dual iff $\mathcal{C}$ is self-orthogonal and $n = 2k$.
5. Show that $\mathcal{H}(\mathcal{C}) = \ker\left(\begin{pmatrix} G \\ H \end{pmatrix}^\top\right)$.
6. Let G be in systematic form, i.e., $G = \begin{pmatrix} \mathrm{Id}_k & A \end{pmatrix}$ for $A \in \mathbb{F}_q^{k\times(n-k)}$.
   Show that if $AA^\top + \mathrm{Id}_{n-k}$ is full rank, then $\dim(\mathcal{H}(\mathcal{C})) = 0$.
7. Show that if $GG^\top$ has full rank, then $\dim(\mathcal{H}(\mathcal{C})) = 0$.

$\mathcal{C} = \langle G \rangle$ and $\mathcal{C}' = \langle G' \rangle$

1. Show that the linear isometries form a group with respect to the composition.
2. Give the automorphism group of $\mathcal{C} = \langle (1,0,0), (0,1,1) \rangle \subseteq \mathbb{F}_2^3$.
3. Let $\varphi \in \mathrm{Aut}(\mathcal{C})$ be a permutation. Show that $\varphi \in \mathrm{Aut}(\mathcal{C} \cap \mathcal{C}^\perp)$.
4. Show that $\mathcal{C}^\perp$ is linearly equivalent to $\mathcal{C}'^\perp$.
5. Show that for all $w \in \{1, \ldots, n\}$ we have that $A_w(\mathcal{C}) = A_w(\mathcal{C}')$.
6. Show that for $r \in \{1, \ldots, k-1\}$ we have $d_r(\mathcal{C}) < d_{r+1}(\mathcal{C})$.
7. Show that for all $r \in \{1, \ldots, k\}$ we have that $d_r(\mathcal{C}) = d_r(\mathcal{C}')$.
8. Consider the code $\mathcal{C}_1 \subseteq \mathbb{F}_3^3$ generated by

$$G_1 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}$$ and the code $\mathcal{C}_2 \subseteq \mathbb{F}_3^3$ generated by $G_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

Are the two codes linear equivalent, permutation equivalent or not equivalent?

How hard is code equivalence?

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Linear Equivalence Problem (LEP)

Given $\mathcal{C}, \mathcal{C}'$ two $[n,k]_q$ linear codes, find $\varphi \in (\mathbb{F}_q^\star)^n \rtimes S_n$ s.t. $\varphi(\mathcal{C}) = \mathcal{C}'$

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Are they NP-hard?

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Merlin

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- if PH $\neq$ AM: $\mathcal{P} \in$ co-AM is not NP-hard

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$I = (\mathcal{C}_1, \mathcal{C}_2)$

find $\mathcal{C}_b$ equivalent to $\mathcal{C}$

$\mathcal{C}$ ←

→ b

choose $b \in \{1, 2\}$ and φ

compute $\mathcal{C} = \varphi(\mathcal{C}_b)$

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soundness error: $1/2$ t rounds $\rightarrow 1/2^t$ $\rightarrow$ LEP $\in$ co-AM

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→ other solvers using canonical forms → cost $\in \mathcal{O}\left(\sqrt{\binom{n}{k}}\right)$

"On linear equivalence, canonical forms, and digital signatures", T. Chou, E. Persichetti, P. Santini, 2025

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Code Equivalence - Different View Point

Finite Geometry

Finite Geometry

Definition Finite projective geometry of dimension k and order q

$$\mathrm{PG}(k,q) = (\mathbb{F}_q^{k+1} \setminus \{0\})/\sim$$

where $u \sim v$ iff $u = \lambda v$ for some $\lambda \in \mathbb{F}_q^\star$

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Definition $\mathcal{M}$ is a projective $[n,k,d]_q$ system if $\mathcal{M}$ is a finite set of n points of $\mathrm{PG}(k-1,q)$ not all on a hyperplane and

$$d = n - \max\{|H \cap \mathcal{M}| \mid H \subseteq \mathrm{PG}(k-1,q), \dim(H) = k-2\}$$

Code Equivalence - Different View Point

Connection

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$$G = \begin{pmatrix} g_{1,1} & \cdots & g_{1,n} \\ \vdots & & \vdots \\ g_{k,1} & \cdots & g_{k,n} \end{pmatrix}$$

$\mathcal{C}$ a $[n, k, d]_q$ linear non-degenerate code

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Code Equivalence - Different View Point

Matroids

Matroids

Definition A matroid M is a pair (E, I) where E is a finite set and I is a collection of subsets of E, called independent sets, s.t.

1. $\varnothing \in I$
2. if $A \in I, B \subseteq A$ then $B \in I$
3. if $A, B \in I$, $|A| < |B|$, then $\exists b \in B \setminus A$ s.t. $A \cup \{b\} \in I$

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Connection

$G \in \mathbb{F}_q^{k \times n}$ generator matrix $\rightarrow$ representable matroid $M(G) = (E, I)$ where $E = \{1, \ldots, n\}$ and $I = \{S \subset E \mid G_S \text{ has full rank }\}$

Code Equivalence - Different View Point

Matroids

Matroids

Definition

A matroid M is a pair (E, r) where E is a finite set and $r : \mathcal{P}(E) \to \mathbb{N}_0$ is a rank function, s.t.

1. $0 \leq r(X) \leq |X|$ for all $X \subseteq E$
2. if $X \subseteq Y \subseteq E$ then $r(X) \leq r(Y)$
3. for all $X, Y \subseteq E$: $r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$

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$G \in \mathbb{F}_q^{k \times n}$ generator matrix $\to$ representable matroid $M(G) = (E, I)$ where $E = \{1, \ldots, n\}$ and for all $S \in \mathcal{P}(E)$: $r(S) = \dim(\langle G_S \rangle)$

Code Equivalence - Different View Point

Designs

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Definition A $t-(v,k,\lambda)$ design is a pair (X,B), where X = set of v points
B = collection of k-elements subsets of X (blocks), s.t.
every t-element subset of X is contained in exactly λ blocks

Code Equivalence - Different View Point

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Connection

$\mathcal{C}$ a $[n,k,d]_q$ linear code $\rightarrow$ $X = \{1,\ldots,n\}$ and

$$B = \{\mathrm{supp}(c_1),\ldots,\mathrm{supp}(c_N) \mid c_i \in \mathcal{C}, \mathrm{wt}(c_i) = d\}$$

Designs

Assmus-Mattson Theorem

$\mathcal{C}$ a $[n, k, d]_q$ linear code with weight enumerators A_i

$\mathcal{C}^{\perp}$ a $[n, n-k, d']_q$ linear code with weight enumerators A'_i

For $t < d$, s the number of $i < n - t$ s.t. $A'_i \neq 0$

If $s \leq d - t$, then the supports of all codewords in $\mathcal{C}$ of weight u with $d \leq u \leq n$ form a t-design

"New 5-designs" E.F. Assmus, H.F. Mattson, 1969

Reductions

- PEP → LEP ✓

Reduction $\mathcal{R} \to \mathcal{P}$
if can solve $\mathcal{P} \to$ can solve $\mathcal{R}$
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favorite finite friend: graphs

Reduction from LEP to PEP

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Proposition $\mathcal{C}, \mathcal{C}'$ $[n,k]_q$ linear codes, $\varphi \in (\mathbb{F}_q^\star)^n \rtimes S_n$ s.t. $\varphi(\mathcal{C}) = \mathcal{C}'$
Then exists $\sigma \in S_{n(q-1)}$ s.t. $\sigma(\lambda \otimes \mathcal{C}) = \lambda \otimes \mathcal{C}'$

Reduction from PEP to GI

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Definition A graph $\mathcal{G}$ is a pair (V, E) with vertices V and edges $E \subseteq V \times V$

Reduction from PEP to GI

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Definition Two graphs $\mathcal{G} = (V, E)$ and $\mathcal{G}' = (V', E')$ are isomorphic if
$\exists f : V \to V' \ \{u, v\} \in E \leftrightarrow \{f(u), f(v)\} \in E'$

Reduction from PEP to GI

Definition A undirected, weighted graph $\mathcal{G} = (V, E)$ is s.t.
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Given $\mathcal{G} = (V, E), \mathcal{G}' = (V, E')$ with $V = \{1, \ldots, n\}$

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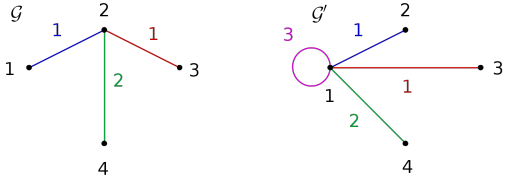

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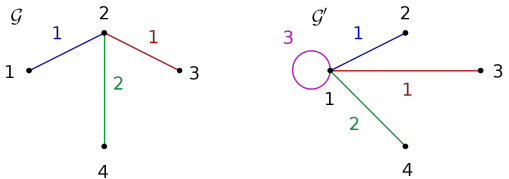

Babai's algorithm: GI is quasi-polynomial time! cost in $\mathcal{O}(2^{\log(n)^c})$

"Graph isomorphism in quasipolynomial time" L. Babai, 2016

Reduction from PEP to GI

Reduction from PEP to GI

Definition The adjacency matrix A of a weighted graph $\mathcal{G}$ is

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Theorem If $\mathcal{H}(\mathcal{C}) = \{0\}$ then PEP can be reduced to GI $\rightarrow$ PEP is easier than GI

"Permutation code equivalence is not harder than GI" M. Bardet, A. Otmani, M. Saeed-Taha, 2019

Reduction from GI to PEP

Reduction from GI to PEP

Definition The incidence matrix B of a graph $\mathcal{G}$ with $|V| = v, |E| = e$ is

$$B_{i,j} = \begin{cases} 1 & \text{if } i = \{\ell, j\} \in E \\ 0 & \text{else} \end{cases} \qquad B \in \mathbb{F}_2^{e \times v}$$

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Theorem We can reduce GI to PEP

"Is code equivalence easy to decide?" E. Petrank, M. Roth, 2002

If LEP → PEP and PEP → GI then LEP → GI

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Under the rug

We can only reduce PEP to GI if $\mathcal{H}(\mathcal{C}) = \{0\}$ is $\mathcal{H}(\lambda \otimes \mathcal{C}) = \{0\}$?

Code Equivalence - Connections

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Exercise Show that $\sum_{\alpha \in \mathbb{F}_q^\star} \alpha^\ell = \begin{cases} 0 & \text{if } (q-1) \nmid \ell \\ -1 & \text{if } (q-1) \mid \ell \end{cases}$

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Proposition If $q \geq 4$, then $\lambda \otimes \mathcal{C}$ is self-orthogonal

A bit of hope

$q = p^{2m}$

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Definition

- Let $x, y \in \mathbb{F}_q^n$. The Hermitian inner product is

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- A Hermitian parity-check matrix $H^\star$ is s.t. $\langle H^\star \rangle = \mathcal{C}^\star$

"How easy is code equivalence over $\mathbb{F}_q$?" N. Sendrier, D. Simos, 2013

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Two New results

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Let $A \in \mathbb{F}_q^{k \times n}$ of rank r and $b \in \mathbb{F}_q^k$

The system $Ax^\top = b^\top$ has a solution iff $\mathrm{rk}([A \mid b]) = r$

→ only 2! (with some heuristics)

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How many pairs $(\mathcal{C}, \varphi(\mathcal{C}))$ needed to recover φ?

→ only 2!

"Don't use it twice!" A. Budroni, J. Chi-Domínguez, D. D'Alconzo, A. Di Scala, M. Kulkarni, 2024

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Exercise

$\langle G \rangle = \mathcal{C}$. Show that $\langle G^{(2)} \rangle = \mathcal{C}^{(2)}$

where $G^{(2)} = \begin{pmatrix} g_1 * g_1 \\ \vdots \\ g_1 * g_k \\ \vdots \\ g_k * g_k \end{pmatrix} \in \mathbb{F}_q^{\binom{k+1}{2} \times n}$

Theorem Let $\mathcal{C}$ be a $[n,k]_q$ linear code. Then $\dim(\mathcal{C}^{(2)}) = \min\left\{n, \binom{k+1}{2}\right\}$

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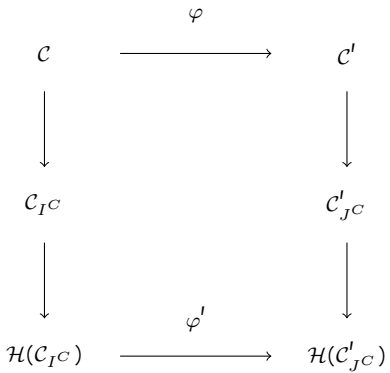
Exercises

- Let $\mathcal{C}, \mathcal{C}'$ be $[n,k]_q$ linear codes and $\varphi = (D,P) \in (\mathbb{F}_q^\star)^n \rtimes S_n$ s.t. $\varphi(\mathcal{C}) = \mathcal{C}'$. Then $\varphi' = (D^2, P) \in (\mathbb{F}_q^\star)^n \rtimes S_n$ is s.t. $\varphi'(\mathcal{C}^{(2)}) = \mathcal{C}'^{(2)}$
- Show that $\mathcal{H}(\mathcal{C}^{(2)}) \neq \mathcal{H}(\mathcal{C})^{(2)}$

Recall SSA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$\mathcal{C}$
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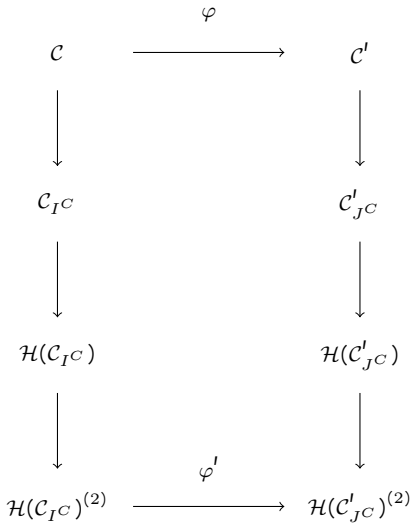
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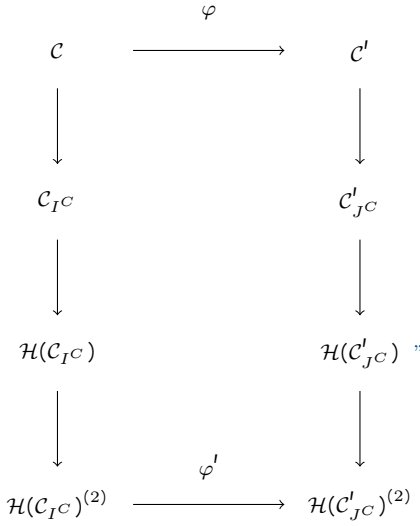


"Using the Schur Product to Solve the Code Equivalence Problem"
M. Battagliola, R. Mora, Rocco, P. Santini, 2025

recent attack on

"Hollow LWE" M. Albrecht, B. Benčina, R. Lai, 2025

Definition Let $\mathcal{C}$ be an $[n,k]_q$ linear code. The ℓ power code is

$$\mathcal{C}^{(\ell)} = \underbrace{\mathcal{C} \star \cdots \star \mathcal{C}}_{\ell}$$

Theorem Let $\mathcal{C}$ be an $[n,k]_q$ linear code. If $\ell < q$, then

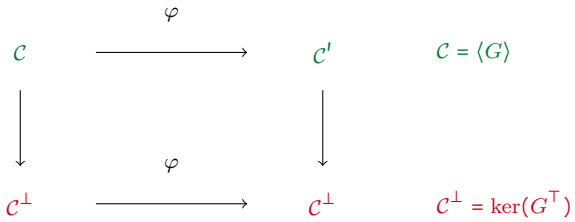
$$\dim(\mathcal{C}^{(\ell)}) = \min\left\{\binom{k+\ell-1}{\ell}, n\right\}$$

Exercises

- Show that $(\lambda \otimes \mathcal{C})^{(2)} \neq \lambda \otimes \mathcal{C}^{(2)}$
- Show that $(\lambda \otimes G)^{(\ell)} = \lambda^{\ell} \otimes G^{(\ell)}$

Why $A(G, G^\top) = G^\top (GG^\top)^{-1} G$?

If $\varphi \in S_n$



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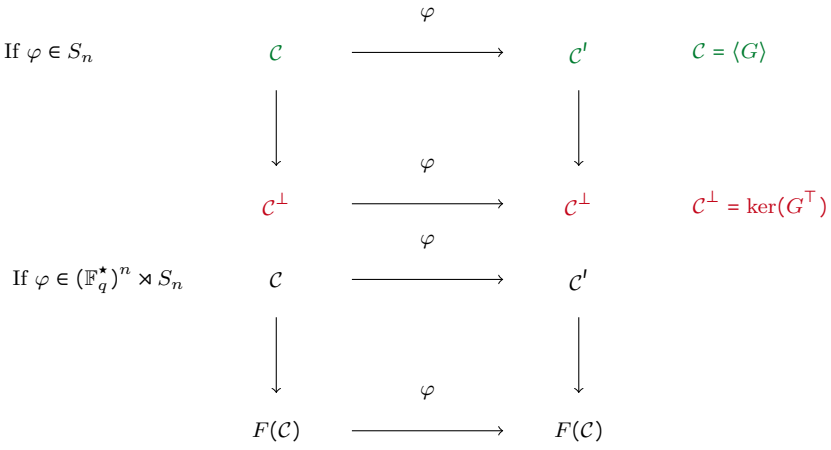

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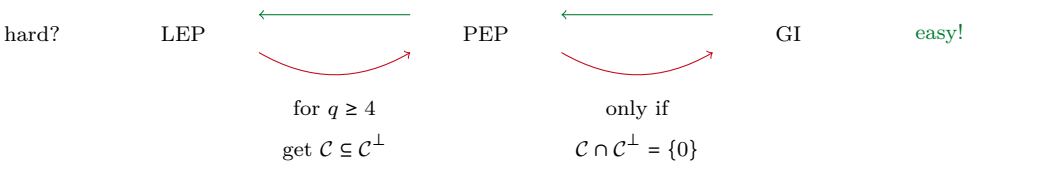

Other metrics?

Code Equivalence - Bonus Round

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→ easy!

"On the hardness of code equivalence problems in rank metric" A. Couvreur, T. Debris-Alazard, P. Gaborit, 2020

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→ easier than Hamming

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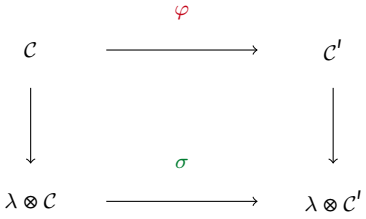

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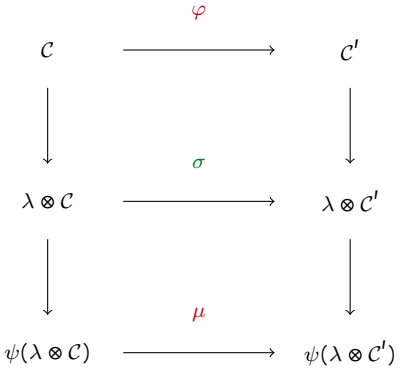

Computational → decisional

"A search-to-decision reduction for the permutation code equivalence problem" J.-F. Biasse, G. Micheli, 2023

Workshop on the Mathematics of Post-Quantum Cryptography

Munich, September 7–11, 2026

https://mathpqc26.cry.cit.tum.de/

Code Equivalence - Exercises

$\mathcal{C} = \langle G \rangle = \ker(H^\top)$ a $[n,k]_q$ linear code

1. Let $H^\star \in \mathbb{F}_q^{(n-k)\times n}$ be a Hermitian parity-check matrix of $\mathcal{C}$.
   Show that $H^\star (G^{p^m})^\top = 0$. That is $\mathcal{C}^\star = \ker((G^{p^m})^\top)$.
2. Show that $H^\star = H^{p^m}$ is a Hermitian parity-check matrix.
3. Show that $(\mathcal{C}^\star)^\star = \mathcal{C}$.
4. Show that $\mathcal{H}^\star(\mathcal{C}) = \ker\left(\begin{pmatrix} G^{p^m} \\ H \end{pmatrix}^\top\right)$.
5. Let $\mathcal{C}$ be linearly equivalent to $\mathcal{C}'$. Show that $\mathcal{C}^\star$ is linearly equivalent to $(\mathcal{C}')^\star$.
6. Show that if $\varphi \in S_n$ is such that $\varphi(\mathcal{C}) = \mathcal{C}'$,
   then $\mathcal{H}^\star(\mathcal{C})$ is permutation equivalent to $\mathcal{H}^\star(\mathcal{C}')$.
7. Show that $A^\star$ is independent on the choice of G.
   Show that if $G(G^{p^m})^\top$ has full rank, then $\dim(\mathcal{H}^\star(\mathcal{C})) = 0$.

$\mathcal{C} = \langle G \rangle = \ker(H^\top)$ a $[n,k]_q$ linear code

1. Show that $\sum_{\alpha \in \mathbb{F}_q^\star} \alpha^\ell = \begin{cases} 0 & \text{if } (q-1) \nmid \ell, \\ -1 & \text{if } (q-1) \mid \ell. \end{cases}$
2. Show that $\mathcal{C}^{(2)}$ is generated by $G^{(2)}$.
3. Show that if $\varphi = (D, P) \in (\mathbb{F}_q^\star)^n \rtimes S_n$ is such that $\varphi(\mathcal{C}) = \mathcal{C}'$, then $\varphi' = (D^2, P) \in (\mathbb{F}_q^\star)^n \rtimes S_n$ is such that $\varphi'(\mathcal{C}^{(2)}) = \mathcal{C}'^{(2)}$.
4. Show that $\mathcal{H}(\mathcal{C})^{(2)} \neq \mathcal{H}(\mathcal{C}^{(2)})$.
5. Reduce the following LEP instance to GI using the square code:

$$G = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 0 \end{pmatrix} \in \mathbb{F}_5^{2\times 4} \text{ and } G' = \begin{pmatrix} 4 & 1 & 0 & 2 \\ 0 & 4 & 2 & 0 \end{pmatrix}.$$

6. Show that $(\lambda \otimes \mathcal{C})^{(2)} \neq \lambda \otimes \mathcal{C}^{(2)}$.
7. Show that $(\lambda \otimes G)^{(\ell)} = \lambda^\ell \otimes G^{(\ell)}$.

Code Equivalence - Solutions

Solutions

Slides