

Measure classification and (non)-escape of mass for horospherical actions on regular trees

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joint work with Corina Ciobotaru and Vladimir Finkelshtein

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-Nowadays, they play important role in the structure theory of tdlc groups.

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- Bruhat decomposition: $G = G_\xi w G_\xi \sqcup G_\xi$, where w is an element with $w^2 \xi = \xi$ (Weyl group element).

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This gives an embedding $(P)SL_2(\hat{K}) \hookrightarrow \text{Aut}(T_{p^n+1})$, (the latter is much larger)

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So our geometric subgroup G_ξ^0 corresponds to a compact extension of the unipotent group when we specialize to $SL_2(\hat{K})$.

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Why not looking at the genuine unipotent subgroups as in the classical results of Dani, Margulis, Ratner, ..?

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Theorem (CFS 19' (a Dani theorem))

Let $G = \text{Aut}(T)$, $\xi \in \partial T$, $\Gamma < G$ a discrete subgroup. Let μ be a G_ξ^0 -invariant and ergodic probability measure on G/Γ . Then, either

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2. Theorem (a Furstenberg theorem): If Γ is a cocompact lattice, then G_ξ^0 -action on G/Γ is uniquely ergodic.

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- 3) One concludes by adapting an ending argument due to Ghys' that uses Howe–Moore property which in this setting were proven by Lubotzky–Mozes.

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An example is *the modular ray* which is the edge-indexed graph associated to the Nagao lattice $\text{PSL}_2(F_q[T])$ in $\text{PSL}_2(F_q((T^{-1})))$.

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Remark

Note that this result applies to simple linear algebraic groups over non-archimedean local fields.

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Let $G < \text{Aut}(T)$ be a non-compact, closed and topologically simple subgroup acting transitively on ∂T . Let $\Gamma < G$ be a geometrically finite lattice, $\xi \in \partial T$ and G_ξ^0 the horospherical subgroup in G . Then,

1. every orbit of G_ξ^0 is either compact or dense, and
2. The family of (discrete) one-parameter compact orbits is in bijection with the cusps of $\Gamma \backslash T$.

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Note that this result applies to simple linear algebraic groups over non-archimedean local fields. In this case, for characteristic zero, it is a specialization of Ratner's results. For non-zero characteristic, it can be deduced by combining works of Mohammadi and Ghosh.

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* * *

In the second part of the talk, we will be interested in *statistical properties of dense orbits* in the previous theorem. Namely, their equidistribution property.

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We want a result saying that most of the elements in F_i keeps a point $x \in G/\Gamma$ in a compact set (analogue of classical Dani-Margulis results).

Results: non-escape of mass and equidistribution

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Theorem (CFS, 19')

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Theorem (CFS, 19')

For every $x \in G/\Gamma$ not belonging to a compact G_η^0 -orbit, the sequence of orbits $F_n x$ endowed with their orbital probability measures, equidistributes to the Haar measure. In other words, writing $\Psi_x : G_\eta^0 \rightarrow G/\Gamma$, $\Psi(g) := gx$, we have $\Psi_{x*} m_{F_n} \rightarrow m_X$ as $n \rightarrow \infty$.

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Escape of mass:

Theorem (CFS, 19')

There exist a lattice Γ in $\text{Aut}(T_6)$ and geodesically non-divergent $x \in \text{Aut}(T_6)$ whose G_η^0 orbit exhibits escape of mass.

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So, we want to find a compact set in $\Gamma \backslash G$ with the previous property. We might as well look for it in $\Gamma \backslash G/K$ where K is a compact subgroup of G . In particular, we can take K to be the stabilizer of a vertex $v \in T$.

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To define such a projection map, let us choose a lift of $\Gamma \setminus T$ in T , i.e. a ray $0, 1, \dots$ (by abuse of notation identified also with $\Gamma \setminus T$). Let $\pi : T \rightarrow \Gamma \setminus T$ be the projection.

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Consider

$$\begin{aligned} \text{Label} : G/\Gamma &\longrightarrow \Gamma \setminus T \simeq \Gamma \setminus G/K \\ g\Gamma &\mapsto \pi(g^{-1}0) \end{aligned} \tag{3.1}$$

This is a continuous map with compact fibres.

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For every $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that every $g\Gamma$ not contained in a compact G_η^0 -orbit satisfies

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We want to see this distribution it as the n^{th} -step distribution of a judiciously constructed Markov chain.

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Corollary

Let $\Gamma < G$ be a lattice, $v \in VT$. Then, the label distribution on $S(v, n)$ (sphere) converges to a probability measure.

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Let $C(n) := \text{Label}(y_n) \in \mathbb{N}$ denote the label of y_n . So we are interested in non-escape of mass in the sequence of distributions of Markov chain obtained by starting the chain at $C(n)$ and running it up to time n .

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It follows from our considerations in the proof of Hedlund's theorem that the first case occurs if and only if G_η^0 orbit of $g\Gamma$ is compact. (this is the same situation with the geodesic flow on the modular surface).

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For example, in the diophantine approximation problem in $\mathbb{F}_q((T^{-1}))$ (studied already by Artin and Mahler); the size of the excursions of labels in the quotient $\mathrm{PSL}_2(\mathbb{F}_q[T]) \backslash T_{q+1}$ of the projection of a geodesic ray in T_{q+1} converging to some $\xi \in \partial T_{q+1} \simeq \mathbb{F}_q((T^{-1}))\mathbb{P}^1$ is precisely the degrees of the best-approximating polynomials obtained by the continued fraction expansion in $\mathbb{F}_q((T^{-1}))$.

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This completes the proof.

However, it naturally suggests the problem of whether this can be extended beyond geometrically finite case which is answered in the negative by the last escape of mass theorem we had stated.

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Use iteratively subgaussian concentration inequalities for Markov chains for these tube-parts of $\Gamma \backslash T$ each time ignoring exponentially small proportion of trajectories.

Deduce that most of the trajectories of the Markov chain does not have time to leave the tube which yields the escape of mass.

Thank you

Thank you

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Thank you for keeping the trees alive.