Equilibrium measures of affine fractals

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joint works with Ian D. Morris

What is in this talk?

1. Definitions and some overview
2. Affine fractals
3. A dimension gap result and a converse to Hutchinson's theorem

Attractor of an IFS

Let $(X, \mathtt{d})$ be a complete metric space and $\Phi = (T_1, \ldots, T_N)$ be a tuple of contracting transformations of X.

Call such a tuple Φ an *iterated function system* (*IFS*, for short).

One checks that there exists a unique compact set $\mathcal{K}_\Phi \subset X$, called *the attractor* or *the limit set* of the IFS Φ, satisfying $\mathcal{K}_\Phi = \cup_{i=1}^N T_i \mathcal{K}_\Phi$.

The coding map

A fruitful way to look at it, is via the coding map:

let Σ_N be the shift space $\{1, \ldots, N\}^{\mathbb{N}}$, fix some $x \in X$ and consider

$$\begin{aligned} \pi : \Sigma_N &\to X \\ (i_1, i_2, \ldots) &\mapsto \lim_{n \to \infty} T_{i_1} \circ \cdots T_{i_n} x \end{aligned}$$

The limit does not depend on $x \in X$ and this yields a continuous map with image $\mathcal{K}_\Phi$, the limit set.

Similarity and affinity fractals

For the rest of the talk, $X = \mathbb{R}^d$ and the metric $\mathtt{d}$ is the standard Euclidean metric.

The limit set $\mathcal{K}_\Phi$ is called a *similarity fractal* or a *self-similar set* if T_i's are contracting similarities, in other words, they are maps of type $x \mapsto cOx + v$, where $|c| \in (0,1)$, $O \in O_d(\mathbb{R})$ and $v \in \mathbb{R}^d$.

More generally, $\mathcal{K}_\Phi$ is called an *affine fractal* or a *self-affine set*, if T_i's are contracting affine maps of $\mathbb{R}^d$, i.e. $T_i \in \mathrm{GL}_d(\mathbb{R}) \ltimes \mathbb{R}^d$.

Examples

Some familiar ones:

- Middle third Cantor set: $T_1 x = \frac{1}{3}x$ and $T_2 x = \frac{1}{3}x + \frac{2}{3}$;
- Sierpiński triangle, carpet
- von Koch curve, Peano curve, Minkowski curve;
- Menger Sponge,
- Bedford–Mcmullen carpets,
- Your own one (?).

Limit set as the repeller of an expansive map

Recall that we are given an IFS $\Phi = (T_1, \ldots, T_N)$ of contracting affine maps of $\mathbb{R}^d$. Suppose that the images $T_i \mathcal{K}_\Phi$ of the associated self-affine set are all disjoint: $\mathcal{K}_\Phi = \sqcup_{i=1}^N T_i \mathcal{K}_\Phi$. Strong separation condition (SSC).

In this case, the coding map $\pi : \Sigma_N \to \mathcal{K}_\Phi$ is a homeomorphism. Therefore, one can transfer the (non-invertible) action of the shift-map to $\mathcal{K}_\Phi$. Namely,

There exists a unique (N-to-1) map $f : \mathcal{K}_\Phi \to \mathcal{K}_\Phi$ such that $f \circ \pi = \pi \circ \sigma$, where $\sigma : \Sigma_N \to \Sigma_N$ is the shift map. The map f is (can be extend to) a piecewise affine expanding map satisfying $f_{| T_i \mathcal{K}_\Phi} = T_i^{-1}$.

What do we want?

+ From fractal geometry point of view, one is interested in calculating various dimensions of the limit set $\mathcal{K}_\Phi$ as well as dimensions of measures on them (e.g. Bernoulli convolutions).

+ From a dynamical point of view, in a similar spirit to the theory of SRB measures, it is of interest to understand measures of maximal Hausdorff dimension on the repeller $\mathcal{K}_\Phi$. Very often, the study of these measures is also a tool for calculating various dimensions of $\mathcal{K}_\Phi$.

For conformal expanding maps (a particular case of which is similarity fractals with SSC), these questions are the best understood thanks to the works of Bowen, Ruelle and many others: the Hausdorff and Minkowski dimensions of the repeller coincide; this dimension is expressed by Bowen's formula; there exists unique ergodic measure of maximal dimension.

Self-similar/affine measures

The first systematic study of self-similar sets was carried out by Moran (1945) and more generally by Hutchinson (1981).

Before stating it, we remind that given an IFS $\Phi = (T_1, \ldots, T_N)$, for any probability vector $(p_1, \ldots, p_N)$ on N-symbols, there exists a unique probability measure m on $\mathcal{K}_\Phi$ satisfying $m = \sum_{i=1}^N p_i T_{i*} m$. The measure m is called a *self-similar/affine measure*.

Such a measure is the image of a *Bernoulli measure* on Σ_N by the coding map $\pi : \Sigma_N \to \mathcal{K}_\pi$.
+ One can also see it as the stationary measure of a Markov chain on $\mathbb{R}^d$ for the action of random compositions of T_i's.
+ More generally, any shift invariant ergodic measure on Σ_N give rise to an invariant measure on $\mathcal{K}_\pi$ (for the expansive map, under SSC).

Moran–Hutchinson Theorem

Theorem (Moran–Hutchinson)

Let $\Phi = (T_1, \ldots, T_N)$ be a similarity IFS with $T_i x = c_i O_i x + v_i$. Suppose that Φ satisfies the open set condition. Then,

-(H-dimension) the Hausdorff dimension of the associated attractor $\mathcal{K}_\Phi$ is given by $\min\{s_0, d\}$, where $s_0 \geqslant 0$ is the unique real satisfying $\sum_{i=1}^N c_i^{s_0} = 1$.

-(Measure) There exists a unique measure on Σ_N that pushes-forward to a measure of dimension s_0 on $\mathcal{K}_\pi$. One can explicitly describe this measure; it is a self-similar measure on $\mathcal{K}_\pi$.

The number s_0 satisfying the equality above is called the *similarity dimension*.

Two consequent lines of extensions

The satisfactory picture given by the previous result naturally motivates (at least) two sorts of questions.

1) What happens if we drop the separation (open set) assumption?

2) How is the picture for affine fractals?

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For the rest, we will focus on the second question. But let us mention that recent breakthrough results of Solomyak, Hochman, Shmerkin, Varjú and many others' works contributed considerably to the understanding of first question. Nevertheless, fundamental questions still persist: Exact overlaps conjecture, dimensions and absolute continuity of Bernoulli convolutions (and the analogous questions for more general self-similar sets).

First, there were carpets

First study of affine (but not similarity) fractals was carried out by Bedford and Mcmullen in the early '80s, who calculated Hausdorff and Minkowski dimensions of some very particular affine fractals: linear parts consist of diagonal matrices of dimension two and translation parts are some tuned vectors (carpets).

Later in '90s, Bedford and McMullen's results were generalized by Gatzouras, Lalley, Kenyon, Peres who studied sponges, obtained analogues of Ledrappier–Young type dimension formulas, and established exact dimensionality results.

Falconer's singular value potential

Falconer's subsequent important work ('88) found out that the unique zero of the pressure of an appropriate family of potentials (Falconer's singular value potential) gives the Hausdorff dimension of typical affine fractals (without any algebraic assumption on the linear parts).

Let an affine IFS $\Phi = (T_1, \dots, T_N)$ be given and denote by A_i the linear part of the affine map T_i. Denote by Σ_N^* the set of finite words in N symbols $\{1, \dots, N\}$. For $(i_1, \dots, i_n) = \mathtt{i} \in \Sigma_N^*$, let $A_{\mathtt{i}}$ denote the corresponding product $A_{i_1} \dots A_{i_n}$.

For a matrix $A \in \mathrm{GL}_d(\mathbb{R})$, denote by $\alpha_1(A) \geqslant \dots \geqslant \alpha_d(A)$ its singular values. Given $s \in (0, d]$, consider the map

$$\begin{aligned} \phi_s : \Sigma_N^* &\to (0, \infty) \\ \mathtt{i} &\mapsto \alpha_1(A_{\mathtt{i}}) \dots \alpha_{\lfloor s \rfloor}(A_{\mathtt{i}}) \alpha_{\lceil s \rceil}(A_{\mathtt{i}})^{s - \lfloor s \rfloor} \end{aligned}$$

Falconer's result and recent progress (1)

This is map is called *Falconer's singular value potential*. It's easy to see that this map is submultiplicative $\phi_s(\mathtt{ij}) \leqslant \phi_s(\mathtt{i})\phi_s(\mathtt{j})$.

Consider the pressure functional given by $P(\phi_s) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{|\mathtt{i}|=n} \phi_s(\mathtt{i})$. The map $s \mapsto P(\phi_s)$ is strictly decreasing, is positive at 0 and becomes negative as $s \to \infty$ (provided we extend the definition of ϕ_s). Its unique zero s_0 is called *the affinity dimension* of the IFS $\Phi = (T_1, \ldots, T_N)$. Clearly, it only depends on the linear parts and it generalizes the similarity dimension that we had seen earlier.

It is also not hard to see that the affinity dimension s_0 is an upper bound for the Hausdorff dimension of the associated affine fractal $\mathcal{K}_\Phi$.

Falconer's result and recent progresses (2)

Falconer's result ('88) says that typically, it is also a lower bound: for every $(A_1, \ldots, A_N)$ with $||A_i|| < \frac{1}{2}$(Solomyak), for a.e. translation parts v_i, the attactor $\mathcal{K}_\Phi$ of the corresponding IFS $\Phi = (T_1 = (A_1, v_1), T_2, \ldots, T_N)$ has Hausdorff dimension $\min\{d, s_0\}$.

As often, almost sure results are nice but frustrating. It is expected that under SSC and reasonable algebraic assumptions on the linear parts of the affinities of the IFS, Falconer's formula holds true. (This result "contrasts" earlier results of Bedford–McMullen).

This was only recently ('19) verified in dimension two in significant work of Bárány–Hochman–Rapaport: they showed that under SSC assumption, if the action of the linear parts of the IFS is strongly irreducible, then $\dim_H \mathcal{K}_\Phi = s_0$ (Falconer's affinity dimension).

Subadditive thermodynamical formalism: equilibrium measures of affine fractals

The proof of the previous and many other results in dimension theory of fractals is based on analysis of dimensions of equilibrium measures, which is interesting in its own right (e.g. Bernoulli convolutions) and also from the dynamical perspective (e.g. Gatzouras-Peres' conjecture).

The study of equilibrium measures with respect to various potentials on Σ_N^* falls into the realm of non-additive or subadditive thermodynamical formalism considered by Pesin, Pitskel, Falconer*, Barreira and others.

In our setting of affine fractals, Käenmäki ('04) first considered and proved the existence of shift-invariant and ergodic measures μ on Σ_N that maximizes a certain Lyapunov dimension. These are the only candidates for which one can expect $\pi_*\mu$ to have its dimension equal to affinity dimension in $\mathcal{K}_\Phi$, by analogy, they are the candidate SRB measures.

Dimension maximizing equilibrium (DME) measures of affine fractals or Käenmäki measures (2)

Another way to construct these DME measures is via the subadditive variational principle (e.g. Cao–Feng–Huang):

Let $\Phi = (T_1, \dots, T_N)$ be an affine IFS and $\phi_s : \Sigma_N^* \to (0, \infty)$ the associated singular value potential. We have

$$P(\phi_s) = \sup_{\mu \in \mathcal{P}_{\text{erg}}(\Sigma_N)} h(\mu) + \Lambda(\phi_s, \mu),$$

where $h(\mu)$ is the entropy of μ and $\Lambda(\phi_s, \mu)$ is the aggregate Lyapunov exponent $\lim_{n\to\infty} \frac{1}{n} \log \phi_s(\mathtt{i}_n)$ for a.s. $\mathtt{i} \in \Sigma_N$ The supremum is attained= an equilibrium measure of the potential ϕ_s.

DME measures (the candidate “SRB measures”) can be described as the equilibrium measure of ϕ_{s_0}, where s_0 is the affinity dimension.

On uniqueness of DME measures on affine fractals (1)

Note that by definition of s_0, we have $P(\phi_{s_0}) = 0$ and hence, a DME measure satisfies $h(\mu) = -\Lambda(\phi_{s_0}, \mu)$, this is an analogue of Pesin entropy formula.

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Note that the unique Bernoulli measure appearing in Hutchinson's theorem is the unique DME measure of the corresponding similarity fractal.

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More on DME measures: uniqueness

A conjecture of Gatzouras–Peres ('95) anticipated the uniqueness of the ergodic measure with full dimension for an expanding map.

Käenmäki–Vilppolainen ('10) and Barral–Feng ('11) gave counter-examples to this (part of the) conjecture by exhibiting self-affine sponges that supports several measures of full dimension.

On uniqueness of DME measures on affine fractals (2)

Note that in these examples, the linear parts are reducible and it was known that strong irreducibility (an assumption in BHR) of the linear parts imply uniqueness of the DME measure in $\mathbb{R}^d$ with $d \leqslant 3$.

Together with I. Morris ('19), we found examples of affine fractals in dimension 4 with strongly irreducible linear parts, supporting several measures of full dimension.

Comment about this result: if the (projective) linear parts of the IFS generate a Zariski dense group in $PGL_4(\mathbb{R})$ (this is the typical regime), then there exists a unique DME measure. In our construction, we arranged it so that the linear parts live in a (strongly) irreducible representation of $PGL_2(\mathbb{R}) \times PGL_2(\mathbb{R})$ in $PGL_4(\mathbb{R})$ (so, it's strongly irreducible but not Zariski dense).

Dimension gap for self-affine measures

Let $\Phi = (T_1, \dots, T_N)$ be an affine IFS and Σ_N the shift space on N symbols. When T_i's are similarities, recall that Hutchinson's result says that there exists a unique DME measure which is a Bernoulli measure on Σ_N (equivalently, self-similar measure on $\mathcal{K}_\Phi$).

We proved the following alternative

Theorem (Morris–S, '19)

Let $(T_1, \dots, T_N)$ be an affine IFS in $\mathbb{R}^d$. Suppose that the linear parts A_i of T_i's act irreducibly on $\mathbb{R}^d$. Then,

- either

$$\sup_{\mu \text{ self-affine measure}} \dim(\mu) < s_0 = \textit{affinity dimension, or}$$

- All T_i's are similarities.

Some remarks on the dimension gap result

+ In particular, the existence of a Bernoulli DME measure entails that the IFS $(T_1, \ldots, T_N)$ consists of similarities.

+ The result is deduced from a more general result about equilibrium states of a class of submultiplicative potentials (defined with singular values).

+ Using some continuity results of Feng–Shmerkin, it follows that the dimension gap is locally uniform (i.e. one can perturb the affine IFS and the gap persists).

Some words on the proof (1)

The proof makes use of subadditive thermodynamical formalism, recent work of Bochi–Morris giving a qualitative description of DME measures on affine IFS, representations of reductive linear algebraic groups and the work of Benoist on spectral properties of Zariski-dense semigroups in RLAG.

The result is considerably easier in dimension 2. Here are the main lines:

Some words on the proof (2)

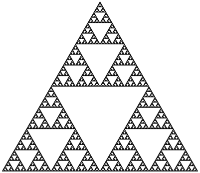
+ The irreducibility assumption implies in dimension two that there exists unique DME measure μ satisfying a Gibbs' inequality (in higher dimensions, this is not the case, but Bochi–Morris description helps).

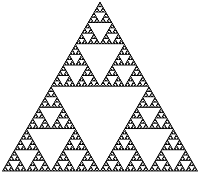
+ The Bernoulli assumption together with Gibbs' inequality force an additivity property of spectral radius of the semigroup generated by linear parts of the IFS (in higher dimensions, one doesn't quite get such an additivity, but a form of it)

+ By non-arithmeticity of length-spectrum in $\mathrm{PGL}_2(\mathbb{R})$ (by Rudolph, Guivarc'h–Raugi), this implies that the Zariski closure of the linear parts is either [irreducible and not strongly irreducible (torus and its conjugate)], [compact] or [reducible]. (in higher dimension this part is handled by representations of reductive groups and Benoist' results)

+ the non-strongly irreducible case (permuted carpet) is handled by another iteration of the first two parts above and shown to be also excluded.

Coffee time





Thanks for your attention!