QUADRATIC FORMS

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1. Talks

This section contains a rough description of the material you should cover in your talk. It also contains the relevant reference to the literature.

1.1. **Basic concepts 1.** Basics on symmetric bilinear forms and isometries. The matrix associated to a bilinear form and the congruence relation. Regular spaces and orthogonal decompositions. Cover the material in Sections 1 to 3 of Chapter 1 of [1].

1.2. **Basic concepts 2.** Introduce the notion of isotropic vector and subspace, hyperbolic spaces and their basic properties. Prove Witt's theorem on the extension of isometries, and deduce Witt's cancellation and Witt's decomposition theorems. The material is essentially contained in Sections 4 to 5 of Chapter 1 of [1]. Compare with Chapter 1 of [3].

1.3. **Grothendieck-Witt and Witt Rings.** Define the Grothendieck-Witt and Witt rings of a field, and introduce the basic invariants of the classes of quadratic spaces. This is contained in Section 1 and 2 of Chapter 2 of [1].

1.4. **Finite fields.** Describe the structure of the group of units in a finite field. Go on with proving that every quadratic space of dimension at least three over a finite field k has an isotropic vector. Use these results to describe $W(K)$ and $GW(K)$. The material is contained in Section 1 and 2 of Chapter 1 of [2] and in Section 3 of Chapter 2 of [1].

1.5. **Quadratic reciprocity and real quadratic spaces.** State and prove the quadratic Reciprocity Law: the reference is Section 3 of Chapter 1 of [2]. This concludes the study of quadratic forms over finite fields. Go on with the classification of quadratic spaces over ordered fields and explain the consequences for W and GW of such fields. The reference for this is Section 4 of Chapter 2 of [1].

1.6. **p-adic integers and fields.** Cover Chapter 2 of [2]. Introduce the ring of p-adic integers $\mathbb{Z}_p$ and its field of fractions $\mathbb{Q}_p$. Explain how to approximate solutions to p-adic equations and prove Hensel's lemma. Explain the structure of the group of units of a p-adic fields and of the subgroup of squares.

1.7. **The Hilbert symbol.** Cover chapter 3 of [2]

1.8. **p-adic and rational quadratic spaces.** Chapter 4 of [2] and its appendix on sums of squares.

1.9. **Pfister forms 1.** Introduce the fundamental ideal I in the (Grothendieck-)Witt ring. Prove that Pfister forms can be rescaled by any non-zero value that they represent. This is contained in Section 2.1 of Chapter 2 of [3].

1.10. **Pfister forms 2.** State and prove the applications to sums of squares and to the level of a field. Explain the relation with the existence of real embeddings, the torsion of $W(K)$ and the nilpotency of $I \subset W(K)$. Prove the classification theorem of the prime ideals of $W(K)$. Proceed with introducing the notion of linked Pfister form and the Elman-Lam theorem. Conclude with the Cassels-Pfister theorem and its consequences. This is contained in Sections 2.2, 2.3 and 2.4 up to 2.4.6 of Chapter 2 of [3].

1.11. **Function fields of quadrics.** The reference for this talk is Chapter 3 of [3]. Introduce what is the function field of an integral algebraic variety. State and prove Arason's theorem describing the rank of quadratic forms having a zero over the function field of some quadric. State and prove the Arason-Pfister theorem on the rank of anisitropic Pfister forms.

1.12. **More on Pfister forms?** In alternative one can talk about the directions of the theory and relate to some other fields.

2. Structure of the talks

The talks should last approximatively 90 minutes. There is some tolerance here, but you shouldn't plan in advance to use 30 minutes more (or less) then your given time slot.

Remember that examples and counterexamples to your statements are very much appreciated, and they are essential for a solid knowledge of the material, so we all wich to hear some, hopefully many.

Prepare notes for your talk. Handwritten notes are surely accepted, TeXed notes are better. Please do not count on your memory for a successful talk. Similarly do not count on bringing with you the reference texsts: reading the book is something that has to take place when you prepare, while at the board we expect an already digested material and exposition.

3. Calendar

Number	Date	Speaker
0	September 20th	Lorenzo Mantovani
/	September 27th (to be rescheduled? maybe on October 29th?) /	
1	October 4th	Martin Bergamin
2	October 11th	Livio P. Zehnder
3	October 18th	Esther Schmid
4	October 25th	Danijela Nikodijevic
5	November 1st	Felix
6	November 8th	
7	November 15th	
8	November 22nd	Felix
9	November 29th	
10	December 6th	
11	December 13th	
12	December 20th (is the semester still on?)	

4. Contact

For any question of scientific or organizational nature, you can send me an email at `lorenzo.mantovani@math.uzh.ch`. In alternative you can find me in my office: Y27K04.

References

[1] W. Scharlau, *Quadratic and Hermitian forms*. Springer, 1984.
[2] J.-P. Serre, *A course in Arithmetic*. Springer GTM 7, 1973.
[3] B. Kahn, *Formes quadratiques sur un corps*. SMF Cours spécialisés, 2009.