Algebraic Geometry 4 - Homework 5

Problem 1: Let $\mathcal{E}$ be a *trivial* vector bundle of rank $r+1$ on a quasi-compact scheme X. (Thus in particular $\mathbb{P}\mathcal{E} \approx \mathbb{P}^r_X$.)

(a) Show that for any vector bundle $\mathcal{F}$ on $\mathbb{P}\mathcal{E}$ we have $\sum_{i=0}^{r+1}(-1)^i\binom{r+1}{i}[\mathcal{F}(i)] = 0 \in K_0(\mathbb{P}\mathcal{E})$.

(b) Deduce that as a ring, $K_0(\mathbb{P}\mathcal{E}) \approx K_0(X)[z]/z^{r+1}$, for a well-chosen $z \in K_0(\mathbb{P}\mathcal{E})$.

(c) Given an example of a scheme X and a (non-trivial) vector bundle $\mathcal{E}$ such that the analog of (b) fails.

Problem 2: Let A be a Dedekind domain. Recalling the classification of finitely generated projective modules if necessary, show that the homomorphism $(\mathrm{rank}, \det) : K_0(A) \to \mathbb{Z} \oplus \mathrm{Pic}(A)$ (from Problem 3 of the previous sheet) is an isomorphism.

Problem 3: Let X be a non-singular connected curve over a field k which has a rational point. We wish to show that $K_0(X) = \mathbb{Z} \oplus \mathrm{Pic}(X)$.

(a) Show that for any scheme Y, there is a ring homomorphism $(\mathrm{rank}, \det) : K_0(Y) \to \mathbb{Z} \oplus \mathrm{Pic}(Y)$.

(b) Let $P \in X(k)$ and $i : \{P\} \to X$ the inclusion. Show that $[i_*\mathcal{O}_P] = [\mathcal{O}] - [\mathcal{O}(P)]$.

(c) Show that $K_0(X)$ is generated by line bundles.

(d) Let L_1, L_2 be line bundles on X. Show that $[L_1] + [L_2] = [L_1 \otimes L_2] + 1$. [*Hint:* Recall that if $\bar{X}$ is a smooth proper curve, then the complement of any finite non-empty set of rational points is affine.]

(e) Conclude that $(\mathrm{rank}, \det)$ is an isomorphism.