Algebraic Geometry 4 - Homework 4

Note that the first two problems are those from last week.

Problem 1:

(a) Let fgAb be the abelian category of finitely generated abelian groups. Show that $\mathrm{K}_0(\mathrm{fgAb}) \cong \mathbb{Z}$, the isomorphism being induced by the rank $\mathrm{rk}(A) := \dim_{\mathbb{Q}}(A \otimes_{\mathbb{Z}} \mathbb{Q})$ of the abelian group A.

(b) Let finAb be the abelian category of finite abelian groups. Show that $\mathrm{K}_0(\mathrm{finAb}) \cong \mathbb{Q}_+^\times$ (the group of positive rationals under multiplication).

(c) Let countAb be the abelian category of abelian groups generated by at most countably many elements. Show that $\mathrm{K}_0(\mathrm{countAb}) \cong 0$. Hint: $\infty + 1 = \infty$.

Problem 2: Let $A_\bullet$ be a bounded complex

$$0 \longrightarrow A_m \longrightarrow A_{m+1} \longrightarrow \cdots \longrightarrow A_n \longrightarrow 0$$

in a small abelian category $\mathcal{A}$ (if you are not familiar with abelian categories: let $\mathcal{A} = \mathrm{Coh}(X)$, X a noetherian scheme). We define its *Euler characteristic* as

$$\chi(A_\bullet) := \sum_{i=m}^{n} (-1)^i [A_i] \in \mathrm{K}_0(\mathcal{A}).$$

(a) Show that

$$\chi(A_\bullet) = \sum_{i=m}^{n} (-1)^i [\mathrm{H}^i(A_\bullet)].$$

(b) Let $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ be a short exact sequence of bounded complexes in $\mathcal{A}$. Show that $\chi(B_\bullet) = \chi(A_\bullet) + \chi(C_\bullet)$.

Problem 3: Let R be an integral domain and let $\mathbb{P}$ be the category of finitely generated projective R-modules. Recall that for $P \in \mathbb{P}$ we have the rank defined as $\mathrm{rank}(P) = \dim_{\mathrm{Quot}(R)}(P \otimes_R \mathrm{Quot}(R))$ and the determinant $\det(P) := \Lambda_R^{\mathrm{rank}(P)} P$, i.e. the highest non-trivial exterior power of P. Note that the latter is automatically a line bundle on $\mathrm{Spec}(R)$.

(a) Show that $\mathrm{rank}\colon \mathbb{P} \to \mathbb{N}_0$ induces a well-defined surjective morphism $\mathrm{K}_0(\mathrm{Spec}(R)) \to \mathbb{Z}$ of rings.

(b) Show that $\det\colon \mathbb{P} \to \mathrm{Pic}(\mathrm{Spec}(R))$ induces a well-defined surjective morphism $\mathrm{K}_0(\mathrm{Spec}(R)) \to \mathrm{Pic}(\mathrm{Spec}(R))$ of abelian groups.

(c) If A is any abelian group, check that $\mathbb{Z} \oplus A$ becomes a ring by $(m, a) \cdot (n, b) := (mn, na + mb)$ for all $m, n \in \mathbb{Z}$ and $a, b \in A$.

(d) Show that $(\text{rank}, \det) : \mathbb{P} \to \mathbb{N}_0 \oplus \text{Pic}(\text{Spec}(R))$ induces a well-defined surjective morphism $\text{K}_0(\text{Spec}(R)) \to \mathbb{Z} \oplus \text{Pic}(\text{Spec}(R))$ of rings, where the multiplication on the second ring is defined as in the previous part.

Problem 4: We want to show the *Devissage Theorem*, which states: Let $\mathcal{A}$ be an abelian subcategory of the category of coherent $\mathcal{O}_X$-modules (or any other small abelian category), X a scheme. Let $\mathcal{B}$ be a subcategory of $\mathcal{A}$ such that

- $\mathcal{B}$ is closed under subobjects in $\mathcal{A}$, i.e. if $A \hookrightarrow B$ is a monomorphism in $\mathcal{A}$ with $B \in \mathcal{B}$, then $A \in \mathcal{B}$.
- $\mathcal{B}$ is closed under quotients in $\mathcal{A}$, i.e. if $B \twoheadrightarrow A$ is an epimorphism in $\mathcal{A}$ with $B \in \mathcal{B}$, then $A \in \mathcal{B}$.
- $\mathcal{B}$ is an abelian category with kernels, cokernels and sums being those of $\mathcal{A}$.
- Every element of $\mathcal{A}$ has a *suitable* filtration
$$A = A_0 \supseteq A_1 \supseteq \ldots \supseteq A_n = 0$$
where each quotient A_i/A_{i+1} is in $\mathcal{B}$.

Then the inclusion induces an isomorphism $\text{K}_0(\mathcal{B}) \cong \text{K}_0(\mathcal{A})$.

(a) Show that the inclusion defines a well-defined morphism $i\colon \text{K}_0(\mathcal{B}) \to \text{K}_0(\mathcal{A})$ of abelian groups.

(b) Show that i is surjective, a preimage of $[A] \in \text{K}_0(\mathcal{A})$, where $A \in \mathcal{A}$, being given by $f(A) := \sum_{i=0}^{\infty} [A_i/A_{i+1}] \in \text{K}_0(\mathcal{B})$, where $A_\bullet$ is a suitable filtration on A.

(c) Schreier Refinement Theorem: Let $A_\bullet$ and $A'_\bullet$ be two filtrations of $A \in \mathcal{A}$. Define $A_{i,j} := (A_i \cap A'_j) + A_{i+1}$ and $A'_{j,i} := (A_i \cap A'_j) + A'_{j+1}$. Show that the lexicographic ordering of the pairs (x, y) turns $A_{i,j}$ into a refinement of $A_\bullet$ and $A'_{j,i}$ into a refinement of $A'_\bullet$. Here we say that $B_\bullet$ refines $A_\bullet$ as filtrations on $A \in \mathcal{A}$ if there are integers $n_0 = 0, n_1, \ldots$ such that $B_{n_i} = A_i$.

(d) Show that $f(A)$ does, as an element of $\text{K}_0(\mathcal{B})$, not depend on the choice of the filtration used to define it. Hint: With the notation as above recall (without proof) Zassenhaus' Lemma: $A_{i,j}/A_{i,j+1} \cong A'_{j,i}/A'_{j,i+1}$.

(e) Let $0 \to A \to B \to C \to 0$ be an exact sequence in $\mathcal{B}$. Use appropriate filtrations to show that $f(A) - f(B) + f(C) = 0$ in $\text{K}_0(\mathcal{B})$.

(f) Prove the theorem.

Problem 5: Use the Devissage Theorem, or mimic its proof for those special cases, to show the following two results:

(a) Let R be a noetherian ring and let I be a nilpotent ideal of R. Then $\mathrm{G}_0(\mathrm{Spec}(R)) \cong \mathrm{G}_0(\mathrm{Spec}(R/I))$. Similarly, if X is a noetherian scheme, then $\mathrm{G}_0(X) \cong \mathrm{G}_0(X^{\mathrm{red}})$.

(b) Let $Z \hookrightarrow X$ be a closed immersion of noetherian schemes. Let S_Z be the subcategory of $\mathrm{Coh}(X)$ of sheaves with support in Z. Then $\mathrm{G}_0(Z) \cong \mathrm{K}_0(S_Z)$.