Algebraic Geometry 4-Homework 3

We use the methods developed so far to count the number of conics in $\mathbb{P}^2$ tangent to 5 lines in general position.

0 (warm-up exercise). a) Let $E \to X$ be a vector bundle, $E^\vee \to X$ the dual bundle. Show that $\tilde{c}_i(E^\vee) = (-1)^i \tilde{c}(E)$.
b) Let $E \to X$ be a rank n vector bundle, $L \to X$ a line bundle. Show that $\tilde{c}_n(E \otimes L) = \sum_{j=0}^{i} \tilde{c}_j(E) \circ \tilde{c}_1(L)^{i-j}$. For extra credit, find a formula for $\tilde{c}_i(E \otimes L)$ in terms of the $\tilde{c}_j(E)$ and $\tilde{c}_1(L)$.
Hint: for both, use the splitting principle.

1. Tangent bundles of $\mathbb{P}^n$. Let $E \to X$ be a rank $n+1$ vector bundle, $\mathbb{P}(E) := \mathrm{Proj}_{\mathcal{O}_X}(\mathrm{Sym}^* E)$ with structure morphism $q : \mathbb{P}(E) \to X$ and tautological surjection $q^*E \to \mathcal{O}_E(1)$. Twisting this by $\mathcal{O}_E(-1)$ gives the exact sequence describing $\Omega_{\mathbb{P}(E)/X}$:

$$0 \to \Omega_{\mathbb{P}(E)/X} \to q^*E(-1) \to \mathcal{O}_{\mathbb{P}(E)} \to 0$$

Let $\tilde{h} = \tilde{c}_1(\mathcal{O}_E(1))$, and let $T_{\mathbb{P}(E)/X}$ be the relative tangent bundle (the dual of $\Omega_{\mathbb{P}(E)/X}$. Conclude that $\tilde{c}(\Omega_{\mathbb{P}(E)/X}) = \tilde{c}(q^*E(-1))$, and $\tilde{c}(T_{\mathbb{P}(E)/X}) = \tilde{c}(q^*E^\vee(1))$. Taking $E = \mathcal{O}_X^{n+1}$, conclude that

$$\tilde{c}(T_{\mathbb{P}^n_X/X}) = (1+\tilde{h})^{n+1}$$

2. Tangent bundles of hypersurfaces. For simplicity, we take as base-scheme $\mathrm{Spec}\, k$, k a field. For X a smooth finite type k-scheme, write T_X for $T_{X/\mathrm{Spec}\, k}$. Let $X \subset \mathbb{P}^n$ be a smooth degree d hypersurface, defined by a section $s \in H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n}(d))$, let $i : X \to \mathbb{P}^n_k$ be the inclusion. Since X is smooth, we use Chern classes instead of Chern class operators. Show that

$$c(T_X) = i^*\left(\frac{(1+h)^{n+1}}{(1+dh)}\right),$$

where $h = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$. *Hint*: Show that $N_i = i^*\mathcal{O}_{\mathbb{P}^n}(d)$ and use the exact sequence

$$0 \to T_X \to i^*T_{\mathbb{P}^n} \to N_i \to 0$$

3. Conics and tangent lines.
a) A *conic* in $\mathbb{P}^2_k$ is a closed subscheme of $\mathbb{P}^2_k$ defined by a section $s \in H^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2}(2))$; there are three types: smooth curves of degree 2, the union of 2 lines, and a double line (i.e., $s = L^2$ for some $L \in H^0(\mathbb{P}^2, \mathcal{O}(1))$). Show that the linear system of conics is isomorphic to a projective space $\mathbb{P}^5_k$ by sending a section $s = \sum_{0 \le i \le j \le 2} a_{ij} X_i X_j$ to the point

$$(a_{00} : a_{01} : a_{02} : a_{11} : a_{12} : a_{22}).$$

b) Let $\ell \subset \mathbb{P}^2_k$ be a line and let C be a conic (not necessarily smooth). Note that the intersection product $[\ell]\cdot[C]$ in $\mathrm{CH}_0(\mathbb{P}^2_k)$ is equivalent to $2\cdot[(0:0:1)]$, so has degree 2. We say that C is tangent to ℓ if either $\ell \subset C$ or $\ell \cap C$

is a single point. Now fix a line ℓ and let $Q_\ell \subset \mathbb{P}^5_k$ be the subscheme defined by the condition $C \in Q_\ell$ iff ℓ is tangent to C. Show that Q_ℓ is a degree 2 hypersurface in $\mathbb{P}^5$. *Hint*: by change of coordinates you may assume that ℓ is the line $X_0 = 0$, and write down the condition for a section $s = \sum_{0 \le i < j \le 2} a_{ij} X_i X_j$ of $H^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2}(2))$ to define a conic $C = C_s$ which is tangent to ℓ. Show that, if $s = L^2$, then the conic $C = C_s$ is tangent to ℓ for any line ℓ.
c) We now assume k has characteristic $\neq 2$. Show that the set of conics C of the form C_s, $s = L^2$, $L \in H^0(\mathbb{P}^2, \mathcal{O}(1))$, is equal to the image of $\mathbb{P}^2$ under the Veronese embedding

$$v : \mathbb{P}^2 \to \mathbb{P}^5$$

$v(x_0 : x_1 : x_2) = (x_0^2 : 2x_0x_1 : 2x_0x_2 : x_1^2 : 2x_1x_2 : x_2^2)$; we let $Ver \subset \mathbb{P}^5$ denote the image of v. Note that $v^*\mathcal{O}_{\mathbb{P}^5}(1) \cong \mathcal{O}_{\mathbb{P}^2}(2)$.
c) Take $\ell_1, \ldots, \ell_5$ five lines in $\mathbb{P}^2$ in general position. Here is a result which is true, but you do not need to prove: Let $Q_i \subset \mathbb{P}^5$ be the quadric corresponding to ℓ_i, $i = 1, \ldots, 5$. Then the scheme-theoretic intersection $Q_1 \cap \ldots \cap Q_5$ is the disjoint union of Ver with finitely many points:

$$Q_1 \cap \ldots \cap Q_5 = Ver \amalg R.$$

The set R correponds to the conics C which are not double lines, and are tangent to all five lines ℓ_i. Show that, if $\ell_{i_1} \cap \ell_{i_2} \cap \ell_{i_3} = \emptyset$ for all $i_1 < i_2 < i_3$, then each $C \in R$ is smooth. Thus, the question of counting the number of smooth conics tangent to five general lines is to find how many distinct points are in R.
d) Let $\pi : \mathbb{P}^n_k \to \operatorname{Spec} k$ be the projection. Show that $\pi_* : \mathrm{CH}_0(\mathbb{P}^n_k) \to \mathrm{CH}_0(\operatorname{Spec} k) = \mathbb{Z} \cdot [\operatorname{Spec} k]$ is an isomorphism; for $x \in \mathrm{CH}_0(\mathbb{P}^n_k)$, we define the integer $\deg_k(x)$ by

$$\pi_*(x) = \deg_k(x) \cdot [\operatorname{Spec} k].$$

The intersection product $[Q_1] \cdots [Q_5]$ is an element of $\mathrm{CH}_0(\mathbb{P}^5_k)$; show that

$$\deg_k([Q_1] \cdots [Q_5]) = 32.$$

e) Let $i_j : Q_j \to \mathbb{P}^5$ be the inclusion, $\Delta : \mathbb{P}^5 \to (\mathbb{P}^5)^5$ the diagonal embedding, $i_{Ver} : Ver \to \mathbb{P}^5$, $i_R : R \to \mathbb{P}^5$ the inclusions. Using the cartesian diagram

$$\begin{array}{ccc} Ver \amalg R & \xrightarrow{\Delta'} & \prod_{j=1}^5 Q_j \\ {\scriptstyle g' = i_{Ver} \amalg i_R} \downarrow & & \downarrow {\scriptstyle \prod_j i_j = g} \\ \mathbb{P}^5 & \xrightarrow[\Delta]{} & \prod_{j=1}^5 \mathbb{P}^5 \end{array}$$

note that $(\Delta, \Delta')^!([\prod_{j=1}^5 Q_j])$ is an element of $\mathrm{CH}_0(Ver \amalg R) = \mathrm{CH}_0(Ver) \times \mathrm{CH}_0(R)$, which we write as

$$(\Delta, \Delta')^!([\prod_{j=1}^5 Q_j]) = (x_{Ver}, x_R)$$

Show that the 0-cycle $x := g'_*((\Delta, \Delta')^!([\prod_{j=1}^5 Q_j])) = i_{Ver*}(x_{Ver}) + i_{R*}(x_R)$ has $\deg_k(x) = 32$, so

$$\deg_k(i_{R*}(x_R)) = 32 - \deg_k(x_{Ver}).$$

f) Let $\Delta'_{Ver} : Ver \to \prod_{j=1}^5 Q_j$ be the restriction of Δ'. We have the vector bundle $\mathcal{E} := i^*_{Ver} N_\Delta / N_{\Delta'_{Ver}}$ on Ver. Show that

$$x_{Ver} = c_2(\mathcal{E}).$$

Hint: Consider the refined Gysin map after removing R from $\mathbb{P}^5$.
g) We identify Ver with $\mathbb{P}^2$ via the map v; let $\alpha_j : \mathbb{P}^2 \to Q_i$ be the resulting inclusion and consider $\mathcal{E}$ and $N_{\Delta'_{Ver}}$ as bundles on $\mathbb{P}^2$. Let $\bar{h} = c_1(\mathcal{O}_{\mathbb{P}^2}(1))$. Show that

$$v^* c(T_{\mathbb{P}^5}) = (1 + 2\bar{h})^6$$
$$\alpha_j^* c(T_{Q_j}) = \frac{(1 + 2\bar{h})^6}{(1 + 4\bar{h})}$$
$$c(N_{\Delta'_{Ver}}) = \frac{(1 + 2\bar{h})^{30}}{(1 + 4\bar{h})^5 (1 + \bar{h})^3}$$
$$c(v^* N_\Delta) = (1 + 2\bar{h})^{24}$$
$$c(\mathcal{E}) = \frac{c(v^* N_\Delta)}{c(N_{\Delta'_{Ver}})} = \frac{(1 + 4\bar{h})^5 (1 + \bar{h})^3}{(1 + 2\bar{h})^6}$$

Noting that $\bar{h}^3 = 0$, compute $c_2(\mathcal{E})$ and find out how many conics are tangent to 5 lines. *Hint*: $\deg c_2(\mathcal{E})$ is a prime number.