Algebraic Geometry 4-Homework 2

1. Representation theory and vector bundles. We fix a base-field k. The group scheme GL_r/k represents the functor from k-algebras to groups: $A \mapsto \mathrm{GL}_r(A)$, where $\mathrm{GL}_r(A)$ is the multiplicative group of units in the ring of $n \times n$ matrices $M_{n\times n}(A)$ with coefficients in A. As a scheme GL_r/k is the open subscheme $\det(X_{ij}) \neq 0$ of $\mathbb{A}_k^{n^2} := \operatorname{Spec} k[\{X_{ij} \mid 1 \leq i, j \leq n\}$; the usual matrix multiplication and inverse define the group structure on $\mathrm{GL}_r/k(A)$. A *rational representation* is a morphism of k-schemes $\rho : \mathrm{GL}_r/k \to \mathrm{GL}_N/k$ such that $\rho(A) : \mathrm{GL}_r/k(A) \to \mathrm{GL}_N/k(A)$ is a group homomorphism for all k-algebras A.

For a k-scheme X, a rank r vector bundle $E \to X$ is defined by an open cover $\{U_i\}$ of X and a cocycle $\{\xi_{ij} \in \mathrm{GL}_r(\mathcal{O}_X(U_i \cap U_j))\}$, $\xi_{ij}\xi_{jk} = \xi_{ik}$ after passing to $\mathrm{GL}_r(\mathcal{O}_X(U_i \cap U_j \cap U_k))$. Given a rational representation $\rho : \mathrm{GL}_r/k \to \mathrm{GL}_N/k$, one has the cocycle $\{\rho(\xi_{ij}) \in \mathrm{GL}_N(\mathcal{O}_X(U_i \cap U_j))\}$, satsifying the cocycle condition, and thus defining a rank N vector bundle $\rho(E) \to X$.

For example, we have the determinant representation $\det : \mathrm{GL}_r/ \to \mathrm{GL}_1/k$, so for every vector bundle $E \to X$, we have the determinant line bundle $\det E \to X$.

a) Let $E \to X$ be a vector bundle, isomorphic to a direct sum of line bundles $E \cong \oplus_{i=1}^r L_i$. Show that $\det E \cong L_1 \otimes_{\mathcal{O}_X} \ldots \otimes_{\mathcal{O}_X} L_r$.

b) Let $0 \to E' \to E \to E'' \to 0$ be an exact sequence of vector bundles on X. Define an isomorphism $\det E \cong \det E' \otimes_{\mathcal{O}_X} \det E''$.

c) Let $E \to X$ be a vector bundle. Show that $\tilde{c}_1(E) = \tilde{c}_1(\det E)$.

Other examples of representations include

i) the nth tensor power $(-)^{\otimes n} : \mathrm{GL}_r \to \mathrm{GL}_{r^n}$: For an A-linear automorphism $g : A^r \to A^r$ we have the A-linear automorphism $g^{\otimes n} : (A^r)^{\otimes_A n} \to (A^r)^{\otimes_A n}$, $g(v_1 \otimes \ldots \otimes v_n) := g(v_1) \otimes \ldots \otimes g(v_n)$. For $e_1, \ldots, e_r$ the standard basis of A^r, we have the standard basis of $(A^r)^{\otimes_A n}$, $e_{i_1} \otimes \ldots \otimes e_{i_n}$, $1 \leq i_1, \ldots, i_n \leq r$. Using this basis for $(A^r)^{\otimes_A n}$, sending g to $g^{\otimes n}$ gives the rational representation

$$(-)^{\otimes n} : \mathrm{GL}_r/k \to \mathrm{GL}_{r^n}/k$$

ii) the nth exterior power: $g(v_1 \wedge \ldots \wedge v_n) := g(v_1) \wedge \ldots \wedge g(v_n)$. Using the basis $e_{i_1} \wedge \ldots \wedge e_{i_n}$, $1 \leq i_1 < \ldots < i_n \leq r$ gives the rational representation

$$\Lambda^n : \mathrm{GL}_r/k \to \mathrm{GL}_{\binom{n}{r}}/k.$$

ii) the nth symmetric power: $g(v_1 \wedge \ldots \wedge v_n) := g(v_1) \cdot \ldots \cdot g(v_n)$. Using the basis $e_{i_1} \cdot \ldots \cdot e_{i_n}$, $1 \leq i_1 le i_2 \leq \ldots \leq i_n \leq r$ gives the rational representation

$$\mathrm{Sym}^n : \mathrm{GL}_r/k \to \mathrm{GL}_{\binom{n+r-1}{r-1}}/k.$$

2. Symmetric functions and Chern classes. Consider the sequence of polyonomial rings $\mathbb{Z}[\xi_1, \ldots, \xi_n]$, with homomorphisms $\pi_n : \mathbb{Z}[\xi_1, \ldots, \xi_{n+1}] \to$

$\mathbb{Z}[\xi_1, \dots, \xi_n]$ sending ξ_{n+1} to 0 and ξ_i to ξ_i for $i \leq n$. Define

$$\mathbb{Z}[\xi_1, \xi_2, \dots] := \lim_{\substack{\leftarrow \\ n}} \mathbb{Z}[\xi_1, \dots, \xi_n]$$

An element of $\mathbb{Z}[\xi_1, \xi_2, \dots]$ is thus a sequence of polynomials $f_n(\xi_1, \dots, \xi_n)$ with $f_{n+1}(\xi_1, \dots, \xi_n, 0) = f_n(\xi_1, \dots, \xi_n)$ for all $n \geq 0$. Call $f = (f_n)_n$ homogeneous of degree d if each f_n is homogeneous of degree d.

We have the formal product

$$\prod_{n=1}^{\infty}(1 + \xi_n T) = 1 + \sigma_1(\xi_1, \xi_2, \dots)T + \dots + \sigma_m(\xi_1, \xi_2, \dots)T^m + \dots$$

with each σ_m a well-defined element of $\mathbb{Z}[\xi_1, \xi_2, \dots]$, called the mth elementary symmetric function in $\xi_1, \xi_2, \dots$. The truncated version $\sigma_m(\xi_1, \xi_2, \dots, \xi_n)$ is the classical mth elementary symmetric function in $\xi_1, \xi_2, \dots, \xi_n$.

The nth symmetric group Σ_n acts on $\mathbb{Z}[\xi_1, \xi_2, \dots]$ by permuting $\xi_1, \dots, \xi_n$ and leaving ξ_m fixed for all $m > n$:

$$f^\sigma(\xi_1, \dots, \xi_n, \xi_{n+1}, \dots) = f(\xi_{\sigma(1)}, \xi_{\sigma(2)}, \dots, \xi_{\sigma(n)}, \xi_{n+1}, \dots).$$

The ring of *symmetric functions in* $\xi_1, \xi_2, \dots$, $\mathbb{Z}[\xi_1, \xi_2, \dots]^{\Sigma_\infty}$, is by definition the subring of $\mathbb{Z}[\xi_1, \xi_2, \dots]$ of elements invariant under Σ_n for all n. Clearly $\sigma_m(\xi_1, \dots)$ is in $\mathbb{Z}[\xi_1, \xi_2, \dots]^{\Sigma_\infty}$ for each m; we let $\mathbb{Z}[\sigma_1(\xi_1, \dots), \sigma_2(\xi_1, \dots), \dots]$ be the subring of $\mathbb{Z}[\xi_1, \xi_2, \dots]$ generated by the elements $\sigma_m(\xi_1, \xi_2, \dots)$ for all m. An element $f(\sigma_1, \sigma_2, \dots) \in \mathbb{Z}[\sigma_1(\xi_1, \dots), \sigma_2(\xi_1, \dots), \dots]$ is homogeneous of degree d in $\xi_1, \xi_2, \dots$ if and only if f is homogeneous of weighted degree d in $\sigma_1, \sigma_2, \dots$, where we give σ_m degree m.

A basic theorem of symmetric functions is

Theorem 1. $\mathbb{Z}[\xi_1, \xi_2, \dots]^{\Sigma_\infty} = \mathbb{Z}[\sigma_1(\xi_1, \dots), \sigma_2(\xi_1, \dots), \dots]$. *Each subring* $\mathbb{Z}[\sigma_1(\xi_1, \dots), \sigma_2(\xi_1, \dots), \dots, \sigma_m(\xi_1, \xi_2, \dots)]$ *is a polynomial ring in the generators* $\sigma_1(\xi_1, \dots), \sigma_2(\xi_1, \dots), \dots, \sigma_m(\xi_1, \xi_2, \dots)$, *and* $\mathbb{Z}[\sigma_1(\xi_1, \dots), \sigma_2(\xi_1, \dots), \dots]$ *is the limit of polynomial rings*

$$\mathbb{Z}[\sigma_1(\xi_1, \dots), \sigma_2(\xi_1, \dots), \dots] = \lim_{\substack{\leftarrow \\ m}} \mathbb{Z}[\sigma_1(\xi_1, \dots), \sigma_2(\xi_1, \dots), \dots, \sigma_m(\xi_1, \xi_2, \dots)].$$

Furthermore, the restriction map

$$\mathbb{Z}[\sigma_1(\xi_1, \dots), \sigma_2(\xi_1, \dots), \dots] \to \mathbb{Z}[\sigma_1(\xi_1, \dots, \xi_n), \dots, \sigma_n(\xi_1, \dots, \xi_n)]$$

is an isomorphism when restricted to the respective subgroups of homogeneous elements of degree d *for all* $n \geq d$.

a) Suppose that $E = \oplus_{i=1}^r L_i$, $L_1, \dots, L_r$ line bundles. Show that

$$E^{\otimes n} \cong \oplus_{1 \leq i_1, \dots, i_n \leq r} L_{i_1} \otimes \dots \otimes L_{i_n}$$
$$\Lambda^n E \cong \oplus_{1 \leq i_1 < \dots < i_n \leq r} L_{i_1} \otimes \dots \otimes L_{i_n}$$
$$\mathrm{Sym}^n E \cong \oplus_{1 \leq i_1 \leq \dots \leq i_n \leq r} L_{i_1} \otimes \dots \otimes L_{i_n}$$

b) Show there are universal polynomials $T_n^i(X_1, \ldots, X_i)$, $L_n^i(X_1, \ldots, X_i)$ and $S_n^i(X_1, \ldots, X_i)$, of weighted degree i (with $\deg X_j = j$) such that for each vector bundle $E \to X$, we have

$$\begin{aligned} T_n^i(\tilde{c}_1(E), \ldots, \tilde{c}_i(E)) &= \tilde{c}_i(E^{\otimes n}), \\ L_n^i(\tilde{c}_1(E), \ldots, \tilde{c}_i(E)) &= \tilde{c}_i(\Lambda^n E) \\ T_n^i(\tilde{c}_1(E), \ldots, \tilde{c}_i(E)) &= \tilde{c}_i(\mathrm{Sym}^n E). \end{aligned}$$

Hint: use (a), theorem 1 and the splitting principle.
c) Let $E \to X$ be a rank 2 vector bundle. Find a formula for $\tilde{c}_i(\mathrm{Sym}^i E)$, $i = 2, 3$.

3. Let $i : Z \to X$ be a codimension c regular embedding, let $\pi : \tilde{X} \to X$ be the blow-up $Bl_Z X$ and $E \subset \tilde{X}$ the exceptional divisor $\pi^{-1}(Z)$, giving the cartesian diagram

$$\begin{array}{ccc} E & \xrightarrow{i_E} & \tilde{X} \\ {\scriptstyle\bar{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} \\ Z & \xrightarrow[i_Z]{} & X \end{array}$$

We have $E = \mathrm{Proj}_{\mathcal{O}_Z}(\mathrm{Sym}^* \mathcal{I}_Z/\mathcal{I}_Z^2)$; let $\mathcal{O}_E(1)$ be the corresponding tautological line bundle with surjection $\bar{\pi}^*(\mathcal{I}_Z/\mathcal{I}_Z^2) \to \bar{\pi}^*\mathcal{O}_E(1)$. Show that

$$(i_Z, i_E)^! = \tilde{c}_1(\mathcal{O}_E(1))^{c-1} \circ i_E^*.$$

Hint: Use HW1, 5(a): $N_{E/\tilde{X}} = \mathcal{O}_E(1)^{-1}$.