Algebraic Geometry 4-Homework 1

1) Let $L \to X$ be a line bundle on some finite type k-scheme X and let $s : X \to L$ be the 0-section. Show that the operator $\tilde{c}_1(L)$ is equal to $s^* s_* : \mathrm{CH}_n(X) \to \mathrm{CH}_{n-1}(X)$.

2) Let $L \to X$ be a line bundle on some smooth integral finite type k-scheme X. Recall that the Chern class $c_1(L) \in \mathrm{CH}^1(X)$ is defined as $c_1(L) := \tilde{c}_1(L)([X])$, where $[X] \in \mathrm{CH}_{\dim_k X}(X)$ is the fundamental class $1 \cdot X$. Show that for line bundles L and M on X,

$$c_1(L) \cdot c_1(M) = \tilde{c}_1(L)(\tilde{c}_1(M)([X])).$$

3) a) Show that the ring $\mathrm{CH}^*(\mathbb{P}^n_k)$ is isomorphic to $\mathbb{Z}[t]/t^{n+1}$, with t mapping to $c_1(\mathcal{O}(1))$.

4) Let $i : X \to Y$ be a regular embedding of codimension c (in $\mathbf{Sch}/k$), let $f : V \to Y$ be a morphism with V an integral finite type k-scheme of dimension d, and form the Cartesian diagram

$$\begin{array}{ccc} W & \xrightarrow{i'} & V \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[i]{} & Y \end{array}$$

a) Show that each integral component of the cone $C_{W/V}$ has dimension d over k. *Hint* First show that $Def(i')$ is integral and has dimension $d+1$ over k.
b) Suppose that W is irreducible and has dimension $d-c$. Show that the closed immersion $C_{W/V} \to f'^*(N_i)$ induces an isomorphism on the underlying reduced subschemes.
c) With the assumptions as in (b), show $C_{W/V}$ is irreducible. Letting $C = C_{W/V\mathrm{red}}$, write $\mathrm{cyc}_C(C_{W/V}) = m \cdot C$. Show that $(i, i')^!([V]) = m \cdot [W]$.
d) With the assumptions and notations as in (b) and (c),

$$0 < m \leq lng_{\mathcal{O}_{Y,W}} \mathcal{O}_{X,W} \otimes_{\mathcal{O}_{Y,W}} \mathcal{O}_{V,W}$$

(you can assume that f is a closed immersion, if you wish).
e) With the assumptions as in (b), show that if $lng_{\mathcal{O}_{Y,W}} \mathcal{O}_{X,W} \otimes_{\mathcal{O}_{Y,W}} \mathcal{O}_{V,W} = 1$, then $\mathcal{O}_{V,W}$ and $\mathcal{O}_{X,W}$ are regular local rings and the multiplicity m is one.

5) Let X be a smooth k-scheme and let $F \subset X$ be a smooth closed subscheme of codimension $c+1$. Let $q : \tilde{X} \to X$ be the blowup of X along F and let $E \subset \tilde{X}$ be the exceptional divisor $q^{-1}(F)$, giving the Cartesian

diagram

$$\begin{array}{ccc} E & \xrightarrow{i'} & \tilde{X} \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle q} \\ F & \xrightarrow[i]{} & X \end{array}$$

a) Recall that

$$E = \mathrm{Proj}_{\mathcal{O}_F}(\oplus_{n\geq 0}\mathcal{I}_F^n/\mathcal{I}_F^{n+1}) \cong \mathrm{Proj}_{\mathcal{O}_F}(\mathrm{Sym}^*_{\mathcal{O}_F}(\mathcal{I}_F/\mathcal{I}_F^2)) \cong \mathbb{P}(N_i);$$

and

$$\tilde{X} = \mathrm{Proj}_{\mathcal{O}_X}(\oplus_{n\geq 0}\mathcal{I}_F^n).$$

This gives the invertible sheaf $\mathcal{O}(1)$ on $\tilde{X}$ with restriction $\mathcal{O}_E(1)$ on E and its dual $\mathcal{O}_E(-1)$. Show that the invertible sheaf $\mathcal{O}_{\tilde{X}}(E)\otimes_{\mathcal{O}_{\tilde{X}}}\mathcal{O}_E$ is isomorphic to $\mathcal{O}_E(-1)$. *Hint*: Reduce to the case $X = \mathrm{Spec}\, A$, $I_F = (a_0, \ldots, a_c)$. Let $T_i \in H^0(\tilde{X}, \mathcal{O}(1))$ be the element corresponding to $a_i \in I_F$ and let $U_i \subset \tilde{X}$ be the open subscheme defined by $T_i \neq 0$. Show that the U_i cover $\tilde{X}$ and that $E \cap U_i$ is defined by a_i. Use this to show define $\mathcal{O}_{\tilde{X}}(E)\otimes_{\mathcal{O}_{\tilde{X}}}\mathcal{O}_E$ and $\mathcal{O}_E(-1)$ by cocycles.
b) Use the projective bundle formula to compute $\mathrm{CH}^*(E)$ in terms of $\mathrm{CH}^*(F)$ and show that q'_* induces an isomorphism

$$\ker(i'_*) \to \ker(i_*).$$

Hint: if $i'_*(x) = 0$, then $c_1(\mathcal{O}_E(1))\cdot x = -i_E^*(i'_*(x)) = 0$. Then show that

$$q'_*(c_1(O_E(1)^c \cdot q'^*(x))) = x$$

for $x \in \mathrm{CH}^*(F)$ (*Hint*: Reduce to the case $F = \mathrm{Spec}\, K$, K a field by a dimension count.
c) Show that $\mathrm{CH}^*(\tilde{X}) \cong \mathrm{CH}^*(X) \oplus \oplus_{i=0}^{c-1}\mathrm{CH}^{*-i-1}(F)$.