

Our relations between

R and H

Minimal assumptions (A)

- Y regular integral 1-dim excellent
- X 2-dim
- $f: X \rightarrow Y$ flat and proper
- $f_* \mathcal{O}_X = \mathcal{O}_Y$ (o.f. geom connected and geom. reduced fibres)

Compact assumptions (B)

- Everything over a finite field $k = \mathbb{F}_q$
- Y is smooth proj geom. integral
- \exists smooth generic fibre with o point.

PROPOSITION 1. Under (A31-3) $R^q R_* \mathcal{G}_m = 0$ if $q \geq 2$

As a consequence the Leroy SS $E_2^{p,q} = H^p(Y, R^q f_* \mathcal{G}_m) = H^{p+q}(X, \mathcal{G}_m)$
 collapses at page 3 and it yields

$$0 \rightarrow H^1(Y, \mathcal{G}_m) \rightarrow H^1(X, \mathcal{G}_m) \rightarrow H^0(Y, P) \rightarrow H^2(Y, \mathcal{G}_m) \rightarrow H^2(X, \mathcal{G}_m) \rightarrow \dots$$

$$\rightarrow H^1(Y, P) \rightarrow H^3(Y, \mathcal{G}_m) \rightarrow H^3(X, \mathcal{G}_m) \rightarrow \dots$$

Step 2: By Groth. 's theorem $H^p(X, \mathcal{G}_m) = H_{\mathbb{Z}}^p(X, \mathcal{G}_m)$ so $P := R^1 f_* \mathcal{G}_m$ is the algebra Picard sheaf $\text{Pic}_{X/Y}(\cdot)$.

ii) By the same argument we deduce an exact seq for the pp fibre $X_y \rightarrow y$ when $\gamma: \text{Spec}(k(y)) \xrightarrow{\tilde{\gamma}} Y$ in the an map and we have

$$\begin{array}{ccc} \text{Pic}(X) & \rightarrow & H^0(Y, \text{Pic}_{X/Y}) \\ \downarrow & \subset & \downarrow \\ \text{Pic}(X_y) & \rightarrow & H^0(y, \text{Pic}_{X_y/y}(\cdot)) \end{array} \quad (\square)$$

and we set $\left(\begin{array}{c} S \\ \downarrow \\ S' \\ \downarrow \\ S'' \end{array} \right)$ to be the set of degrees of elements in these four groups. (loc. const. of Euler-Poinc charact)

iii) We deduce an exact sequence

$$0 \rightarrow S \rightarrow B_1(y) \rightarrow B_2(X) \rightarrow H^1(Y, P) \rightarrow T \rightarrow 0 \quad (**)$$

$$\text{where } S = \text{coker}(P_2(y) \rightarrow P_2(X))$$

$$T = \text{ker}(H^3(Y, \mathcal{G}_m) \rightarrow H^3(X, \mathcal{G}_m))$$

PROPOSITION 3 If Y is a curve over a finite field then $B_2(y) = 0$

Step 1: Use $0 \rightarrow \mathcal{G}_m \rightarrow R_y \rightarrow \text{Div}_y \rightarrow 0$ in \mathbb{A}^1_y

in combination with Leroy's SS for i_* and i_{y*} , $y \in Y^{(1)}$ to get a long exact seq. relating $H(Y, \mathcal{G}_m)$ to $H(k(y), \mathcal{G}_m)$ and $H(k(y), \mathbb{Z})$. This is used to deduce that if $p: Y \rightarrow \text{Spec } k \rightarrow R^q p_* \mathcal{G}_m = 0$ if $q \geq 2$ by Leroy's theorem on coh. dim ...

Step 2: By Step 1 the Leray S for $P \times G_m$ collapses giving

$$\dots \rightarrow H^p(k, G_m) \rightarrow H^p(Y, G_m) \rightarrow H^{p-1}(k, \text{Pic}_{Y/k}) \rightarrow \dots$$

Using SES $0 \rightarrow \text{Pic}_{Y/k}^0 \rightarrow \text{Pic}_{Y/k} \rightarrow \mathbb{Z} \rightarrow 0$ on Et/k

together with Wedderburn's theorem on coh dim of k and $\text{Coh}(k) = \hat{\mathbb{Z}}$ we conclude that

$$\begin{aligned} \text{Pic}(Y) &= \Gamma(k, \text{Pic}_{Y/k}) & H^3(Y, G_m) &= \mathbb{Q}/\mathbb{Z} \\ B_2(Y) &= 0 & H^q(Y, G_m) &= 0 \text{ of } q \geq 4. \end{aligned}$$

PROK 4 a) By plugging in this result in $S(\frac{x}{s})$ we see that $B_2(Y) = 0$ implies $S = S'$. In any case, for our final case, the existence of a section $Y \rightarrow X$ implies $S'' = S' = S = 1$.

b) In general we can interpret S and S'' in a geometric way

bii) $S = \text{gcd of degrees of divisors on } X$
of finite flat maps $Y \xrightarrow{R} X \xrightarrow{f} Y$

Such maps induce $h^* h_*$ on cohomology and since $T \subset H^3(Y, G_m) = \mathbb{Q}/\mathbb{Z}$ we deduce the following cor.

COROLLARY 5: i) The groups S and T are killed by 5. In particular they vanish if f has a section

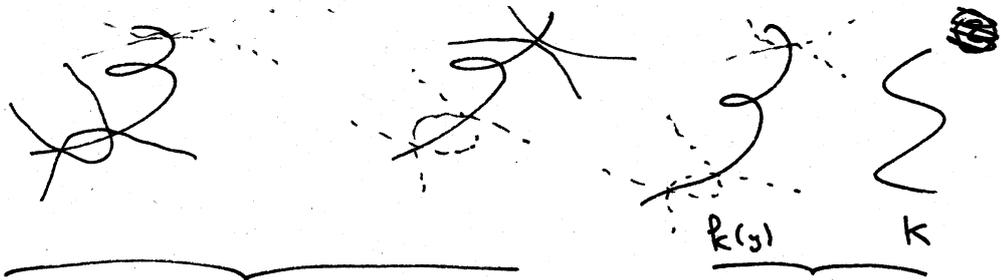
ii) If Y is a curve over a finite field and it's proper, then $S = 0$ and $T \subset H^3(Y, G_m) = \mathbb{Q}/\mathbb{Z}$ is cyclic of order dividing 5;

moreover we have a SES $0 \rightarrow B_2(X) \rightarrow H^1(Y, P) \rightarrow T \rightarrow 0$ (*)

bii) - Also S'' has a geometric interpretation, in general

write: $0 \rightarrow R^1 P^0 \rightarrow P \xrightarrow{\text{deg}} \mathbb{Z} \rightarrow 0$: this is exact.

The deg map has image $\nu \mathbb{Z}$ where ν is the inseparable multiplicity (assume the per. fib is reduced) of the generic fibre X_η when pulled back to Et_η . We associate to ν a torsion $T(\nu) \in H^1(Y, P^0)$ whose order coincides with S'' .



- The fibre X_y decomposes as $\sum_{i=1}^{c_y} m_y^i \mathbb{P}^1$ } $X_y^i = \text{irred comp of } X_y$
 $c_y = \#$
- For every i there is a Galois extension $k^i(y)/k(y)$ of degree v_y^i over which X_y^i splits into geometrically integral pieces. We set $K(y) = \prod_{i=1}^{c_y} k^i(y)$
 $m_y^i = \text{leg } \mathcal{O}_{X_y, X_y^i}$
- If the base field was not perfect one might also have radical multiplicities $\nu_y^i \neq \text{leg } (\mathcal{O}_{X_y, X_y^i})$

We set $P_y := \text{Spec}(K(y)) \rightarrow \text{Spec}(k(y))$ to be the can. map

PROPOSITION 4. Let Y be proper curve over a finite field (and X_y smooth?)

- $E \cong \bigoplus_y E_y$ and $E_y = 0$ iff X_y is geometrically integral
- $\forall y \in Y^{(1)}$ we have a SES $0 \rightarrow \mathbb{Z} \xrightarrow{d} P_y \rightarrow \mathbb{Z} \rightarrow 0$ where d is diagonal to $\bigoplus_{i=1}^{c_y} \mathbb{Z} \xrightarrow{v_y^i} \mathbb{Z} \xrightarrow{(e_i)} \mathbb{Z}$ with $(e_i) \mapsto (m_y^i e_i)$
- $H^1(y, E_y) = \mathbb{Z}/d_y \mathbb{Z}$ where $d_y = \text{gcd}(v_y^i m_y^i)$
- $d_y = \delta_y := \text{gcd of degrees of elts in } \text{Pic}(X_y^{\text{sh}})$

COROLLARY 5. If Y is proper over a finite field, $f: X \rightarrow Y$ has smooth gen fibre and it has no multiple fibres $\Rightarrow H^1(y, E_y) = 0 \forall y \in Y^{(1)}$ and $E_y \neq 0$ iff X_y is not geom. integral, in particular this happens when f has a section, in this case $H^1(y, P) = L(y, B)$

PROOF: i) We are essentially looking at

$$E(y) \rightarrow \text{Pic}(X_y) / \text{Pic}(T_y) \xrightarrow{\phi(y)} \text{Pic}(X_y) / \text{Pic}(T_y)$$

so elts in $k(y)$ must be ~~supp~~ vertical divisors; divisors supp. on different vertical fibres can't have relations. Full vertical fibres are equivalent to zero because they come from $\text{Pic}(T)$.

ii) One has to check the exactness of

$$0 \rightarrow \mathbb{Z} \rightarrow \prod_{i=1}^{c_y} \mathbb{Z} \xrightarrow{(m_y^i)} \text{Pic}(X_y^{\text{sh}}) / \text{Pic}(T_y) \rightarrow \text{Pic}(X_y^{\text{sh}})$$

which amounts to say that each component X_y^i contributes with a generator

of $\text{Pic}(X_y^{\text{sh}})_{\text{vert}}$ with the only relation being given by "being a whole fibre"
 i.e. $\ker \sum_{i=1}^{c_y} m_i v_i^i x_i^i = 0$.

iii) From the LES in cohomology we deduce that

$$H^1(E_y) = \cap \ker (H^1(y, \mathbb{Q}/\mathbb{Z}) \rightarrow \prod_{i=1}^{c_y} H^1(k^i(y), \mathbb{Q}/\mathbb{Z}))$$

and that it ~~must be killed by~~ ^{has order exactly} $\gcd(v_i m_i) = d_y$

iv) Follows from the surjectivity of $\text{Pic}(X_y^{\text{sh}}) \rightarrow \text{Pic}(X_y)$

note. In general one obtains that $S_y = \gcd(\nu_i v_i m_i)$ ν_i are the inseparable multiplicities

the proof tells us more; namely that $\text{Tors}(E_y) = \frac{1}{d_y} [X_y] \cong \mathbb{Z}/d_y \mathbb{Z}$.

§ Relation with the usual LL

$Y = \text{proper}/\mathbb{F}_q$ and smooth periodic fibre X_y .

We have short exact sequences

$$\begin{array}{ccccccc} 0 & \rightarrow & A & \rightarrow & B & \xrightarrow{(\text{div})} & \mathbb{Z}/\mathbb{Z} \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \rightarrow & P^0 & \rightarrow & P & \xrightarrow{(\text{div})} & \mathbb{Z}/\mathbb{Z} \rightarrow 0 \end{array}$$

is surj $\Leftrightarrow S_y = 1 \quad \forall y \in Y^{(1)}$
 is $\hat{\pi}$ surjective because X_y is perim. reduced

We get

$$\begin{array}{ccccccc} 0 \rightarrow \mathbb{Z}/S''\mathbb{Z} \rightarrow H^1(y, A) \rightarrow H^1(y, B) \rightarrow 0 \\ \uparrow \rightarrow \mathbb{L}(y, A) \rightarrow \mathbb{L}(y, B) \\ 0 \rightarrow \mathbb{Z}/S'\mathbb{Z} \rightarrow H^1(y, P^0) \rightarrow H^1(y, P) \rightarrow 0 \end{array}$$

and since we have 0 sections $S = S' = S'' = 1$ and hence \cong

For cohomology we use the Leray spectral seq. for $i_*: \Gamma_{\text{ack}} k \rightarrow Y$.

Indeed $A = i_*(\ker(P_y \rightarrow \mathbb{Z}_y)) = i_*(J(X_y))$

$$0 \rightarrow H^1(y, A) \rightarrow H^1(k, J) \rightarrow H^0(y, R^1 i_* J)$$

$$\begin{array}{ccc} \downarrow & \downarrow & \cong \\ H^1(G_y^{\text{sh}}, A) \rightarrow \bigoplus H^1(\text{Frac } C_{y,y}^{\text{sh}}, J) \rightarrow \bigoplus_y H^1(\text{Frac } G_y^{\text{sh}}, J) \end{array}$$

so we deduce an exact sequence

$$0 \rightarrow \mathbb{L}(y, A) \subseteq H^1(k, J) \rightarrow \bigoplus_{y \in Y^{(1)}} H^1(\text{Frac}(G_y^{\text{sh}}), J)$$

and finally $H^1(\text{Frac}(G_y^{\text{sh}}), J) = H^1(\text{Frac } G_y^{\text{sh}}, J)$

\cong is because of Greenberg's Lemma: $T \in \text{LHS}$ is a torsor and maps to zero on the RHS iff has $\text{pt} \in G_y^{\text{sh}}$

\Rightarrow By Grantham's Lemma i can lift the point to $\mathbb{O}_y^{\mathbb{F}_q}$.

\Rightarrow should hold but I don't know how to prove it: maybe use a density argument. In any case we don't need it

§0. INTRODUCTION

Up to now we have seen that

(BSD)

$$rk(E(K)) = \sum_{s=1}^{\infty} L(E, s)$$

REFINED BSD

$L(E)$ is finite and

$$L^*(E, s) \sim \frac{|L(E)| |\det(a_i, e_j)|}{|E(K)_{tors}|} \cdot (s-1)^2$$

the cycle class map

$$NS(E) \otimes \mathbb{Z} \rightarrow H^2(\bar{E}, \mathbb{Z}_{\ell}(1))^{Gal}$$

is surjective

(2)

$$p(E) = - \sum_{s=1}^{\infty} S(E, s)$$

$p(E)$ # of E

$B_2(E) \rightarrow$ finite and

$$P_2(E, q^{-s}) \sim \frac{|B_2(E)| |\det(D_i \cdot D_j)|}{|NS(E)_{tors}|^2} \cdot (1-q^{-s})^2$$

(TATE)

(ARTIN - TATE)

THEOREM: $k = \mathbb{F}_q$, C/k a smooth proj plane. connected curve
 $K = K(C)$ and $E \rightarrow C$ an elliptic curve, $E \rightarrow C$ it's minimal
 proper cycle model. Then $\# B_2(E) = L(E)$.