University of Mumbai - Department of Atomic Energy
Centre for Excellence in Basic Sciences

MPr 1001
Project Report

Hilbert-Samuel Polynomial and Cowsik's Conjecture

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April 14, 2016

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Introduction

Let R be a 3-dimensional regular local ring and p be a prime ideal of height two in R. We denote by $\mathcal{R}_s(p) = \bigoplus_{n=0}^{\infty} p^{(n)} t^n$, the symbolic Rees algebra of p. Here $p^{(n)} = p^n R_p \cap R$ is the n^{th} symbolic power of p. The problem when $\mathcal{R}_s(p)$ is Noetherian was raised by Cowsik [1]. He showed a relationship between set-theoretic complete intersection and Noetherian propery of $\mathcal{R}_s(p)$. He showed that p is a set-theoretic complete intersection in a Noetherian local ring R if $\dim(R/p) = 1$ and if $\mathcal{R}_s(p)$ is Noetherian. Thus it became important to know when $\mathcal{R}_s(p)$ is Noetherian.

Many researchers proved the conjecture for special cases. In [6, 7] Roberts gave counterexamples to Cowsik's conjecture. Later Goto, Nishida and Shimoda [2] constructed infinitely many examples of the defining primes of monomial space curves, whose Symbolic Rees algebra is not Noetherian.

The main aim of this project is to study Huneke's criterion of $\mathcal{R}_s(p)$ to be Noetherian for a height two prime ideal p in 3-dimensional regular local ring. This report is divided into three chapters.

In Chapter 1, we provide a quick introduction to the theory of reductions of ideals and superficial elements. We also study stability of $\operatorname{Ass}(R/I^n)$ and Burch's inequality.

Chapter 2 discusses Hilbert-Samuel polynomial of an $\mathfrak{m}$-primary ideal in a Cohen-Macaulay local ring. Using superficial sequences and reductions, we prove a result by Huneke [4], which states that if $(R, \mathfrak{m})$ is a d-dimensional Cohen-Macaulay ring and I is an $\mathfrak{m}$-primary ideal, then $e_0(I) - e_1(I) = \lambda(R/I)$ if and only if $r(I) \leq 1$ (This was also proved independently by A. Ooishi). Here $r(I)$ is the reduction number of I.

In Chapter 3, we study Huneke's criterion for $\mathcal{R}_s(p)$ to be Noetherian for height two primes in a 3-dimensional regular local ring. As a consequence, we show that if p is a height two prime in a 3-dimensional regular local ring and $e(R/p) = 3$, then $\mathcal{R}_s(p)$ is Noetherian. We also discuss an example of a height two prime ideal p in a 3-dimensional regular local ring with $e(R/p) = 6$, for which $\mathcal{R}_s(p)$ is Noetherian.

1. Preliminaries

1.1 Reductions of ideals

Let R be a Noetherian ring. Let I be an ideal of R. An ideal $J \subseteq I$ is called a **reduction** of I if $JI^n = I^{n+1}$ for some n. If J does not properly contain a reduction of I, then it is called a **minimal reduction** of I.

The **reduction number** of I with respect to J, denoted by $\mathrm{r}_J(I)$, is the least non-negative integer n such that $I^{n+1} = JI^n$. The reduction number of I is the minimum of the reduction numbers $\mathrm{r}_J(I)$ where J varies over all minimal reductions of I.

Let R be a ring and $F = \{I\}_{n=0}^{\infty}$ be a filtration of ideals, i.e. $I_0 = R$ and $I_i I_j \subseteq I_{i+j}$. Then the Rees ring of R with respect to F is

$$R[Ft] = I_0 \oplus I_1 t \oplus I_2 t^2 \oplus \cdots$$

Proposition 1.1. *Let $J \subseteq I$ be ideals of a Noetherian ring R. Then J is a reduction of I if and only if $R[It] = \bigoplus_{n=0}^{\infty} I^n t^n$ is a finite $R[Jt]$-module.*

Proof. Let J be a reduction of I. Then for some positive integer r, $JI^r = I^{r+1}$. Thus for all $k \geq 1$ and $n \geq r$,

$$\begin{aligned}(J^k t^k)(I^n t^n) &= J^{k-1} I^{n+1} t^{n+k} \\ &= I^{n+k} t^{n+k}.\end{aligned}$$

Hence

$$\begin{aligned} R[It] &= R \oplus It \oplus I^2t^2 \oplus \cdots \oplus I^r t^r \oplus (Jt)I^r t^r \oplus (J^2t^2)I^r t^r \oplus \cdots \\ &= R \oplus It \oplus \cdots \oplus R[Jt]I^r t^r \\ &= R[Jt] \oplus R[Jt]It \oplus \cdots \oplus R[Jt]I^r t^r \end{aligned}$$

Hence $R[It]$ is a finite $R[Jt]$-module.

Conversely, Let $R[It]$ be a finite $R[Jt]$-module. Since $R[It]$ is a graded $R[Jt]$-module, there is a finite set of homogeneous generators of $R[It]$ as an $R[Jt]$-module. We write $R[It] = R[Jt] \oplus R[Jt]It \oplus \cdots \oplus R[Jt]I^r t^r$ for some r. Now equate homogeneous components of degree $r+1$ on both the sides,

$$I^{r+1}t^{r+1} = J^{r+1}t^{r+1} + J^r I t^{r+1} + \cdots + JI^r t^{r+1}.$$

Thus $I^{r+1} = JI^r$. Hence J is a reduction of I. □

Corollary 1.2. *Let $K \subseteq J \subseteq I$ be ideals of a Noetherian ring R. Then K is a reduction of I if and only if K is a reduction of J and J is a reduction of I.*

Proof. Let K be a reduction of I. Then $R[It]$ is a finite $R[Kt]$-module. Hence $R[Jt]$ is a finite $R[Kt]$-module and $R[It]$ is a finite $R[Jt]$-module. Therefore K is a reduction of J and J is a reduction of I.

Conversely, let K be a reduction of J and J be a reduction of I. Then $R[Jt]$ is a finite $R[Kt]$-module and $R[It]$ is a finite $R[Jt]$-module. Hence $R[It]$ is a finite $R[Kt]$-module. Therefore K is a reduction of I. □

Let $(R, \mathfrak{m})$ be a local ring and I be an ideal of R. The **fiber cone** of I, denoted by $F(I)$, is the graded algebra $R[It]/\mathfrak{m}R[It] = \bigoplus_{n=0}^{\infty} I^n/\mathfrak{m}I^n$. The **analytic spread** of I, denoted by $l(I)$, is the Krull dimension of $F(I)$.

Proposition 1.3. *Let I be an ideal of a local ring $(R, \mathfrak{m})$. For $x \in I$, put $x^o = x + \mathfrak{m}I \in I/\mathfrak{m}I$. Let $J = (x_1, \ldots, x_s) \subseteq I$. Then J is a reduction of I if and only if $(x_1^o, \ldots, x_s^o)$ is primary for the maximal homogeneous ideal $F(I)_+$. In particular, $l(I) \leq \mu(J)$.*

Proof. Let J be a reduction of I. Then for some positive integer r, $JI^r = I^{r+1}$. Note that $(x_1^o, \ldots, x_s^o) = (J + \mathfrak{m}I)/\mathfrak{m}I$ is a graded ideal of $F(I)$. So $(x_1^o, \ldots, x_s^o)_n = (JI^{n-1} + \mathfrak{m}I^n)/\mathfrak{m}I^n$ for all $n \geq 1$. Hence for $n \geq r + 1$,

$$(x_1^o, \ldots, x_s^o)_n = I^n/\mathfrak{m}I^n = F(I)_n.$$

Thus $(x_1^o, \ldots, x_s^o)$ is $F(I)_+$-primary.

Conversely, Let $(x_1^o, \ldots, x_s^o)$ be $F(I)_+$-primary. Then there is a positive integer r such that $(x_1^o, \ldots, x_s^o)_n = F(I)_n$ for $n \geq r$. Hence $JI^{r-1} + \mathfrak{m}I^r = I^r$. This implies $I^r/JI^{r-1} = \mathfrak{m}\left(I^r/JI^{r-1}\right)$. So, by Nakayama's Lemma, $I^r = JI^{r-1}$. Hence J is a reduction of I. Since $(x_1^o, \ldots, x_s^o)$ is $F(I)_+$-primary, $l(I) \leq \mu(J)$. □

Proposition 1.4. *Let $J \subseteq I$ be a reduction of an ideal I in a local ring $(R, \mathfrak{m})$. Then J contains a minimal reduction of I. Let $x_1, \ldots, x_s \in J$ be such that $x_1^o, x_2^o, \ldots, x_s^o \in I/\mathfrak{m}I$ are linearly independent and s is minimal with respect to the property that $K = (x_1, \ldots, x_s)$ is a reduction of I contained in J. Then K is a minimal reduction of I contained in J.*

Proof. Let $K' \subseteq K$ be a reduction of I. Let $f : K/\mathfrak{m}K \longrightarrow I/\mathfrak{m}I$ be the natural map of $k := R/\mathfrak{m}$-vector spaces. Since $x_1^o, x_2^o, \ldots, x_s^o \in I/\mathfrak{m}I$ are linearly independent, $x_1 + \mathfrak{m}K, x_2 + \mathfrak{m}K, \ldots, x_s + \mathfrak{m}K$ are linearly independent on $K/\mathfrak{m}K$. Hence $\ker(f) = (K \cap \mathfrak{m}I)/\mathfrak{m}K = 0$. Thus $K \cap \mathfrak{m}I = \mathfrak{m}K$.

We claim that $K + \mathfrak{m}I = K' + \mathfrak{m}I$. Suppose $K' + \mathfrak{m}I \subsetneq K + \mathfrak{m}I$. Then $(K' + \mathfrak{m}I)/\mathfrak{m}I$ is a proper subspace of $(K + \mathfrak{m}I)/\mathfrak{m}I$. Let $t = \dim((K' + \mathfrak{m}I)/\mathfrak{m}I)$ and $b_1, \ldots, b_t \in K$ be such that $b_1^o, \ldots, b_t^o \in I/\mathfrak{m}I$ are linearly independent. Note that $t < \dim((K + \mathfrak{m}I)/\mathfrak{m}I) = s$. Since K' is a reduction of I, $\dim(F(I)/(b_1^o, \ldots, b_t^o)) = 0$. This contradicts the minimality of s.

Thus $K \subseteq (K' + \mathfrak{m}I) \cap K = K' + (\mathfrak{m}I \cap K) = K' + \mathfrak{m}K$. By Nakayama's Lemma, $K = K'$. Therefore K is a minimal reduction of I. □

Proposition 1.5. *Let $(R, \mathfrak{m})$ be a local ring with infinite residue field. Let I be an ideal of R and $x_1, \ldots, x_s \in I$. Then $J = (x_1, \ldots, x_s)$ is a minimal reduction of I if and only if $x_1^o, \ldots, x_s^o$ is a homogeneous system of parameters of $F(I)$.*

Proof. Let J be a minimal reduction of I. Let $l = l(I)$. We claim that $s = l$. Since s is smallest with respect to the property that $\dim(F(I)/(x_1^o, \ldots, x_s^o)) = 0$, $s \geq l$. Suppose $s > l$. Since $k := R/\mathfrak{m}$ is infinite, by Noether Normalization Lemma, there exists $y_1, \ldots, y_l \in I$ such that $F(I)$ is integral over the polynomial ring $k[y_1^o, \ldots, y_l^o]$. Hence $F(I)/(y_1^o, \ldots, y_l^o)$ is zero-dimensional. This is a contradiction to the minimality of s. Hence $s = l$.

Conversely, let $x_1, \ldots, x_s \in I$ such that $x_1^o, \ldots, x_s^o$ is a homogeneous system of parameters of $F(I)$. Then $J = (x_1, \ldots, x_s)$ is a reduction of I. By the above proposition it is a minimal reduction of I. □

Let I be an ideal of a local ring $(R, \mathfrak{m})$. Then the **altitude** of I is

$$\operatorname{alt}(I) = \sup\{\operatorname{ht}(p) : p \text{ is a minimal prime of } I\}.$$

Corollary 1.6. *Let I be an ideal of a local ring $(R, \mathfrak{m})$. Then*

$$\operatorname{alt}(I) \leq l(I) \leq \dim(R).$$

Proof. We may assume that $R/\mathfrak{m}$ is infinite. Let J be a minimal reduction of I. Then there is a positive integer n such that $I^{n+1} = JI^n$. Hence $V(I) = V(J)$. For any minimal prime p of J, $\operatorname{ht}(p) \leq \mu(J) = l(I)$. Hence $\operatorname{alt}(I) = \operatorname{alt}(J) \leq l(I)$. Since $\lambda\left(I^n/\mathfrak{m}I^{n+1}\right) \leq \lambda\left(I^n/I^{n+1}\right)$ for all n, $\dim(F(I)) \leq \dim(G(I))$. Hence $l(I) = \dim(F(I)) \leq \dim(G(I)) = \dim(R)$. □

1.2 Stability of $\operatorname{Ass}(R/I^n)$ and Burch's inequality

Lemma 1.7. *Let $R = \bigoplus_{n=0}^{\infty} R_n$ be a Noetherian graded ring. Let I be a homogeneous ideal and x be a homogeneous element. Let $(I : x) \cap S = \emptyset$ for a multiplicative set S of R_0. Then there is a homogeneous element y such that $(I : xy)$ is a prime and $(I : xy) \cap S = \emptyset$.*

Proof. Let $\mathcal{P} = \{(I : xy') : (I : xy') \text{ is prime for some } y' \in R \text{ and } (I : xy') \cap S = \emptyset\}$. Then $\mathcal{P}$ has a maximal element, say $(I : xy)$. We show that $(I : xy)$ is a prime and $(I : xy) \cap S = \emptyset$. Let ab be homogeneous elements of R such that $a, b \notin (I : xy)$ and suppose $ab \in (I : xy)$. Then $a \in (I : xyb) \setminus (I : xy)$. Thus $(I : xyb) \cap S \neq \emptyset$, say $s \in (I : xyb) \cap S$. Similarly $t \in (I : xya) \cap S$. Hence $st \in (I : xy) \cap S$, which is a contradiction. Hence $ab \notin (I : xy)$ and thus $(I : xy)$ is prime. □

Theorem 1.8. *Let I be an ideal of a Noetherian ring R. Then the sequence* $\operatorname{Ass}(R/I^n)$ *stabilizes.*

Proof. Let $\operatorname{Ass}^*(I) = \{p \in \operatorname{Spec}(R) : p \in \operatorname{Ass}(I^n/I^{n+1}) \text{ for some } n\}$. We show that $\operatorname{Ass}^*(I)$ is a finite set. Let $p \in \operatorname{Ass}^*(I)$. Then $p = (0 : c^*)_R$ for some $c \in I^k \setminus I^{k+1}$. Let $G = \bigoplus_{n=0}^{\infty} I^n/I^{n+1}$ and let $G_n = I^n/I^{n+1}$. Then $p = (0 : c)_G \cap R$. By Lemma 1.7, there is homogeneous $d^* \in G$ such that $p' = (0 : cd)_G$ is a prime in G and $p = p' \cap R$. Note that $p' \in \operatorname{Ass}(G)$. Since $\operatorname{Ass}(G)$ is finite, $\operatorname{Ass}^*(I)$ is a finite set.

Next, we show that $\operatorname{Ass}(I^{n-1}/I^n) \subseteq \operatorname{Ass}(I^n/I^{n+1})$ for large n. Let $(0 : G_1)_G = (a_1^*, a_2^*, \ldots, a_s^*)$ and $l = 1 + \max(\deg(a_i^*))$. Then we get $G_n \cap (0 : G_1)_G = 0$ for all $n \geq l$. Let $p = (0 : c^*)_R$ for some $c^* \in G_n$ and $n \geq l$. Then $p = (0 : c^* G_1)_R$. But $c^* G_1 \subseteq G_{n+1}$, hence $p \in \operatorname{Ass}(G_{n+1})$. Thus $\operatorname{Ass}(I^{n-1}/I^n) \subseteq \operatorname{Ass}(I^n/I^{n+1})$ for large n. Since $\operatorname{Ass}^*(I)$ is a finite set, $\operatorname{Ass}(I^{n-1}/I^n) = \operatorname{Ass}(I^n/I^{n+1})$ for large n.

From the exact sequence

$$0 \longrightarrow I^n/I^{n+1} \longrightarrow R/I^{n+1} \longrightarrow R/I^n \longrightarrow 0,$$

we get

$$\text{Ass}\,(R/I^{n+1}) \subseteq \text{Ass}\,(R/I^n) \cup \text{Ass}\,(I^n/I^{n+1}).$$

But $\text{Ass}\,(I^{n-1}/I^n) = \text{Ass}\,(I^n/I^{n+1}) \subseteq \text{Ass}\,(R/I^n)$ for large n. Hence $\text{Ass}\,(R/I^n) = \text{Ass}\,(R/I^{n+1})$ for large n.

□

Let I be an ideal of a local ring $(R, \mathfrak{m})$. We denote $A^*(I)$ by the stable value of $\text{Ass}\,(R/I^n)$. Then the depth (R/I^n) also stabilizes to a value $\beta(I)$.

Theorem 1.9 (Burch's Inequality). *Let I be an ideal of a local ring $(R, \mathfrak{m})$. Then*

$$l(I) \leq \dim(R) - \beta(I).$$

Proof. Apply induction on $\beta(I)$. If $\beta(I) = 0$, then the inequality follows from Corollary 1.6. Let $\beta(I) \geq 1$. Then there is an r such that $\mathfrak{m} \notin Ass(R/I^r)$ for all $n \geq r$. Let $x \in \mathfrak{m} \setminus \left(\bigcup_{p \in A^*(I)} p \right)$. Then $\beta((I,x)/(x)) = \beta(I) - 1$. By induction hypothesis,

$$l((I,x)/(x)) \leq \dim(R/(x)) - \beta((I,x)/(x)).$$

Then by the choice of x, $(I^n : x) = I^n$ for large all $n \geq r$. We claim that $l((I,x)/(x)) = l(I)$. Consider the n^{th} graded component of $F((I,x)/(x))$,

$$\frac{(I^n, x)}{(\mathfrak{m}I^n, x)} \simeq \frac{I^n}{I^n \cap (\mathfrak{m}I^n + (x))} = \frac{I^n}{\mathfrak{m}I^n + I^n \cap (x)}.$$

Since $(I^n : x) = I^n$ for large n, we get $I^n \cap (x) \subseteq xI^n \subseteq \mathfrak{m}I^n$. Thus $F((I,x)/(x))$ and $F(I)$ has same Hilbert polynomial, hence equal dimension. Hence

$$\begin{aligned} l(I) = l((I,x)/(x)) &\leq \dim(R/(x)) - \beta((I,x)/(x)) \\ &= \dim(R) - \beta(I). \end{aligned}$$

□

1.3 Superficial elements

Let I be an ideal of a local ring $(R, \mathfrak{m})$. We say $x \in I$ is **superficial** for I if there is a non-negative integer c such that for all $n > c$,

$$(I^n : x) \cap I^c = I^{n-1}.$$

Proposition 1.10. *Let I be an ideal of a local ring $(R, \mathfrak{m})$.*

(i) If I is nilpotent, then every $x \in I$ is superficial for I.

(ii) If I is not nilpotent, then a superficial element x of I satisfies $x \in I \setminus I^2$.

Proof. (i) Let $I^r = 0$ for some r. Then for $c = r$ and $n > c$, $(I^n : x) \cap I^c = 0 = I^{n-1}$ for any $x \in I$. Thus every $x \in I$ is superficial element.

(ii) Suppose I is not nilpotent and let x be a superficial element for I. Then there is a non-negative integer c such that $(I^n : x) \cap I^c = I^{n-1}$ for all $n > c$. Suppose $x \in I^2$. Then $xI^c \subseteq (I^{c+2})$ and for $n = c+2$,

$$(I^{c+2} : x) \cap I^c = I^{c+1}.$$

Hence $I^c = I^{c+1}$. By Nakayama's Lemma, $I^c = 0$, which is a contradiction as I is not nilpotent. Hence $x \notin I^2$.

□

Proposition 1.11. *Let I be an ideal of a local ring $(R, \mathfrak{m})$. Let $x \in I \backslash I^2$ and $x^* = x + I^2$. Then x is superficial for I if and only if the multiplication map $x^* : I^n/I^{n+1} \longrightarrow I^{n+1}/I^{n+2}$ is injective for large n.*

Proof. Let x be a superficial element for I. Then there is a non-negative integer c such that $(I^n : x) \cap I^c = I^{n-1}$ for all $n > c$. Suppose $n > c$ and $b \in I^n$ and $b^* x^* = 0$. Then $b \in (I^{n+2} : x) \cap I^c = I^{n+1}$. Thus $b^* = 0$. Hence the map x^* is injective for large n.

Conversely, let the multiplication map $x^* : I^n/I^{n+1} \longrightarrow I^{n+1}/I^{n+2}$ be injective for $n > c$. We show that $(I^n : x) \cap I^c = I^{n-1}$ for all $n > c$. Let $b \in (I^n : x) \cap I^c$ and let $b \in I^m \setminus I^{m+1}$ for some $m > c$. Since $b^* \neq 0$, $b^* x^* \neq 0$. Thus $xb \notin I^{m+2}$. But $xb \in I^n$, so $n < m+2$. So $b \in I^m \subseteq I^{n-1}$. Hence $(I^n : x) \cap I^c = I^{n-1}$. □

Existence of superficial element

Proposition 1.12. *Let $(R, \mathfrak{m})$ be a local ring with infinite residue field . Let M be an R-module. If $N_1, \ldots, N_t$ are proper submodules of M, then $N_1 \cup N_2 \cup \cdots \cup N_t \subsetneq M$.*

Proof. Apply induction on t. For $t = 1$, it is trivial. Let $t \geq 2$. Suppose $M = N_1 \cup N_2 \cup \cdots \cup N_t$. We may assume that $N_1 \nsubseteq (N_2 \cup \cdots \cup N_t)$ and $(N_2 \cup \cdots \cup N_t) \nsubseteq N_1$. Let $a \in N_1 \setminus (N_2 \cup \cdots \cup N_t)$ and $b \in (N_2 \cup \cdots \cup N_t) \setminus N_1$. Since there are infinitely many units in R, by Pigeon-Hole Principle, there exist distinct units $u, w \in R$ such that for some j

$$a + ub, a + wb \in N_j$$

Since $(w-u)b \in N_j$ and $(w-u)$ is a unit, $b \in N_j$ and thus $j \neq 1$. Similarly $(w-u)a \in N_j$. Hence $a \in N_j$, which is a contradiction as $j \neq 1$. Thus $N_1 \cup N_2 \cup \cdots \cup N_t \subsetneq M$. □

Theorem 1.13. *Let $(R, \mathfrak{m})$ be a local ring with infinite residue field. Let $I, J_1, \ldots, J_t$ be ideals of R such that $I \nsubseteq J_1 \cup \ldots \cup J_t$. Then there exist $x \in I \setminus (J_1 \cup \ldots \cup J_t)$ such that x is superficial for I.*

Proof. Consider the R-modules $M = I/I^2$ and $N_i = \dfrac{(J_i \cap I) + I^2}{I^2}$ for $i = 1, \ldots, t$. Note that $(J_i \cap I) + I^2 \subsetneq I$, because if $(J_i \cap I) + I^2 = I$, then by Nakayama's Lemma $J_i \cap I = I$, which is a contradiction as $I \nsubseteq J_i$. Thus each N_i is a proper submodule of M.
Let $G(I)$ denote the associated gradation of I and let $G_N = I^N/I^{N+1}$. Let

$$(0) = Q_1 \cap \cdots \cap Q_s \cap Q_{s+1} \cap \cdots \cap Q_g$$

be the reduced primary decomposition of (0) in $G(I)$, where each Q_i is a P_i-primary ideal of $G(I)$. Suppose $G_1 \nsubseteq P_i$ for $i = 1, \ldots, s$ and $G_1 \subseteq P_j$ for $j = s+1, \ldots, g$. Then

$G_1 \cap P_1, \ldots, G_1 \cap P_s$ are proper G_0-submodules of G_1. By previous proposition, there exists $x \in I \setminus I^2$ such that

$$x^* \in G_1 \setminus \left(\{\cup_{i=1}^{s} P_i\} \cup \{\cup_{i=1}^{t} N_i\} \right).$$

We show that x is superficial for I. By Proposition 1.11, it is enough to show $(0 : x^*) \cap G_n = 0$ for large n. Suppose $b^* x^* = 0$. Since $x^* \notin (P_1 \cap \cdots \cap P_s)$, $b^* \in (Q_1 \cap \cdots \cap Q_s)$. Since each Q_j is P_j-primary, for some large N, $P_j^N \subseteq Q_j$ for all j. Hence $G_1^N = G_N \subseteq Q_j$ for each $j = s+1, \ldots, g$. Thus

$$G_N \cap (0 : x^*) \subseteq Q_1 \cap \cdots \cap Q_s \cap Q_{s+1} \cap \cdots \cap Q_g = 0.$$

Hence x is superficial for I.

□

2. Hilbert-Samuel polynomial

Let $(R, \mathfrak{m})$ be a d-dimensional Noetherian local ring and let q be an $\mathfrak{m}$-primary ideal of R. Let M be a finitely generated R-module. The **Hilbert-Samuel function** of M with respect to q is defined as the numerical function $H_{q,M}(n) = \lambda\left(M/q^n M\right)$. For large n, $H_{q,M}(n)$ is given by a polynomial $P_{q,M} \in \mathbb{Q}[x]$, called **Hilbert-Samuel polynomial** of M with respect to q, with $\deg\left(P_{q,M}(n)\right) = d$. It is written in terms of binomial coefficients as

$$P_{q,M}(x) = e_0(q, M)\binom{x+d-1}{d} - e_1(q, M)\binom{x+d-2}{d-1} + \cdots + (-1)^d e_d(q, M).$$

where $e_i(q, M)$ for $i = 0, \ldots, d$ are integers. The leading coefficient $e_0(q, M)$, denoted by $e(q, M)$, is called the **multiplicity** of M with respect to q.

Let I be an $\mathfrak{m}$-primary ideal of R. The **Hilbert function** $H_I(n)$ of I is defined as $H_I(n) = \lambda\left(R/I^n\right)$. For large n, $H_I(n)$ is a polynomial function of n of degree d. In other words, there is a polynomial $P_I(x) \in \mathbb{Q}[x]$, called the **Hilbert polynomial** of I, such that $H_I(n) = P_I(n)$ for all large n. It is written in terms of the binomial coefficients as:

$$P_I(x) = e_0(I)\binom{x+d-1}{d} - e_1(I)\binom{x+d-2}{d-1} + \cdots + (-1)^d e_d(I).$$

where $e_i(I)$ for $i = 0, 1, \ldots, d$ are integers, called the Hilbert coefficients of I. The leading coefficient $e_0(I)$, called the multiplicity of I.

2.1 Hilbert polynomial of one dimensional Cohen-Macaulay local rings

Let $(R, \mathfrak{m})$ be a one dimensional Cohen-Macaulay local ring. Let I be an $\mathfrak{m}$-primary ideal. The Hilbert polynomial of I, $P_I(n)$ has degree 1. Write

$$P_I(n) = e_0 n - e_1.$$

The **postulation number** of I is defined to be

$$n(I) = \max\{n | H_I(n) \neq P_I(n)\}.$$

Theorem 2.1. *Let I be an $\mathfrak{m}$-primary ideal of a 1-dimensional Cohen-Macaulay local ring $(R, \mathfrak{m})$. Then*

(i) $P_I(n+1) - H_I(n+1) \geq P_I(n) - H_I(n)$ *for all* $n \geq 0$

(ii) $e_0 - e_1 \leq \lambda\left(R/I\right)$

(iii) $e_1 \geq 0$ *and* $e_1 = 0$ *if and only if* I *is principal.*

Proof. (i) Since $\operatorname{ht}(I) = \dim(R) = 1$, $l(I) = 1$. Since the residue field is infinite, there exists $x \in I$ such that (x) is a reduction of I. Thus $xI^{n-1} = I^n$ for all large n. Hence for large n, $\lambda(R/I^n) = \lambda\left(R/xI^{n-1}\right) = \lambda(R/(x)) + \lambda\left(R/I^{n-1}\right)$. Thus $e_0 = \lambda(R/(x))$. Now note for all $n \geq 0$,

$$\begin{aligned} P_I(n+1) - H_I(n+1) &= e_0(n+1) - e_1 - \lambda\left(R/I^{n+1}\right) \\ &= ne_0 - e_1 + \lambda(R/(x)) - \lambda\left(R/I^{n+1}\right) \\ &= ne_0 - e_1 + \lambda((x)/xI^n) + \lambda\left(I^{n+1}/xI^n\right) \\ &= P_I(n) - H_I(n) + \lambda\left(I^{n+1}/xI^n\right). \end{aligned}$$

Hence $P_I(n+1) - H_I(n+1) \geq P_I(n) - H_I(n)$.

(ii) For large n, $H_I(n) = P_I(n)$. Thus for $n \geq 0$, $H_I(n) \geq P_I(n)$. In particular, for $n = 1$, $e_0 - e_1 \leq \lambda(R/I)$.

(iii) From (ii) it follows that $e_1 \geq e_0 - \lambda(R/I) = \lambda(R/(x)) - \lambda(R/I) = \lambda(I/(x)) \geq 0$. Clearly if $e_1 = 0$, $I = (x)$. Conversely, if $I = (x)$, then for all $n \geq 1$,

$$\begin{aligned} \lambda(R/(x)^n) &= \lambda(R/(x)) + \lambda\left((x)/(x)^2\right) + \cdots + \lambda\left((x)^{n-1}/(x)^n\right). \\ &= ne_0. \end{aligned}$$

Hence $e_1 = 0$.

□

Proposition 2.2. *Let* I *be an* $\mathfrak{m}$*-primary ideal of a* 1*-dimensional Cohen-Macaulay local ring* $(R, \mathfrak{m})$*. Let* (x) *be a minimal reduction of* I*. Then*

$$\mathrm{r}_{(x)}(I) = \mathrm{n}(I) + 1.$$

Proof. By previous theorem, for all $n \geq 0$

$$H_I(n+1) - P_I(n+1) = H_I(n) - P_I(n) - \lambda\left(I^{n+1}/xI^n\right).$$

Let $r = \mathrm{r}_{(x)}(I)$ and $k = \mathrm{n}(I)$. Then $\lambda\left(I^{n+1}/xI^n\right) = 0$ for all $n \geq r$. So by previous theorem, $H_I(n) - P_I(n) = H_I(r) - P_I(r)$ for all $n \geq r$. But $H_I(n) = P_I(n)$ for large n. This implies $H_I(r) = P_I(r)$ and hence $r \geq k+1$. Now for $n = k+1$, by previous theorem,

$$H_I(k+2) - P_I(k+2) = H_I(k+1) - P_I(k+1) - \lambda\left(I^{k+2}/xI^{k+1}\right).$$

Hence $\lambda\left(I^{k+2}/xI^{k+1}\right) = 0$. Thus $I^{k+2} = xI^{k+1}$ and hence $r \leq k+1$. □

Proposition 2.3. *Let* I *be an* $\mathfrak{m}$*-primary ideal of a* 1*-dimensional Cohen-Macaulay local ring* $(R, \mathfrak{m})$*. Let* (x) *be a minimal reduction of* I*. Then* $e_0 - e_1 = \lambda(R/I)$ *if and only if* $\mathrm{r}_{(x)}(I) \leq 1$*.*

Proof. Let $e_0 - e_1 = \lambda(R/I)$. Then by Theorem 2.1, $P_I(1) - H_I(1) = e_0 - e_1 - \lambda(R/I) = 0$. Hence $\mathrm{n}(I) \leq 0$. By previous proposition. $\mathrm{r}_{(x)}(I) \leq 1$.

Conversely, let $\mathrm{r}_{(x)}(I) \leq 1$. Then $\mathrm{n}(I) \leq 0$. Hence $H_I(1) = P_I(1)$. Thus $e_0 - e_1 = \lambda(R/I)$. □

2.2 Superficial sequence and Hilbert polynomial

Let $(R, \mathfrak{m})$ be a local ring, and let I be an ideal of R. A sequence $x_1, \ldots, x_s \in I$ is called a **superficial sequence** for I if x_i is superficial for $I/(x_1, \ldots, x_{i-1})$ for $i = 1, 2, \ldots, s$.

Lemma 2.4. *Let $x_1, \ldots, x_s$ be a superficial sequence for I. Then for $n >> 0$,*

$$I^n \cap (x_1, \ldots, x_s) = (x_1, \ldots, x_s)I^{n-1}.$$

Proof. Clearly, $(x_1, \ldots, x_s)I^{n-1} \subseteq I^n \cap (x_1, \ldots, x_s)$. We prove the forward implication by applying induction on s. Let $s = 1$ and put $x_1 = x$. Then there is a positive integer c such that for $n > c$, $(I^n : x) \cap I^c = I^{n-1}$. By Artin-Rees Lemma, there is a positive integer p such that for all $n \geq p$,

$$I^n \cap (x) = I^{n-p}(I^p \cap (x)) \subseteq (x)I^{n-p}.$$

We show that $I^n \cap (x) \subseteq xI^{n-1}$ for all $n > p + c$. Let $y \in I^n \cap (x)$, say $y = bx$ for some $b \in R$. Then $y \in xI^{n-p}$, so that $y = dx$ for some $d \in I^{n-p} \subseteq I^c$. Thus $(b - d)x = 0$. Hence $(b-d) \in (0 : x) \subseteq (I^n : x)$. So $d \in (I^n : x)$. Since $d \in I^c$, $d \in I^c \cap (I^n : x) = I^{n-1}$. Thus $d \in I^{n-1}$ and $y = dx \in xI^{n-1}$.

Now let $s \geq 2$ and assume that for large n, $I^n \cap (x_1, \ldots, x_{s-1}) = (x_1, \ldots, x_{s-1})I^{n-1}$. Since $\overline{x_s}$ is superficial for $I/(x_1, \ldots, x_{s-1})$, by induction, for large n,

$$(\overline{x_s}) \cap [I/(x_1, \ldots, x_{s-1})]^n = (\overline{x_s})\ [I/(x_1, \ldots, x_{s-1})]^{n-1}.$$

Hence for large n,

$$\begin{aligned} x_sI^{n-1} + (x_1, \ldots, x_{s-1}) = &\ (x_1, \ldots, x_s) \cap [I^n + (x_1, \ldots, x_{s-1})] \\ = &\ I^n \cap (x_1, \ldots, x_s) + (x_1, \ldots, x_{s-1}). \end{aligned}$$

Therefore,

$$\begin{aligned} I^n(x_1, \ldots, x_s) &\subseteq x_sI^{n-1} + (x_1, \ldots, x_{s-1}) \cap I^n \\ &= x_sI^{n-1} + (x_1, \ldots, x_{s-1})I^{n-1} \\ &= (x_1, \ldots, x_s)I^{n-1}. \end{aligned}$$

□

Lemma 2.5. *Let $J \subseteq I$ be ideal of local ring $(R, \mathfrak{m})$. Let $x \in J$ be superficial for I. If $J/(x)$ is reduction of $I/(x)$, then J is a reduction of I.*

Proof. Since x is superficial for I, for all large n

$$I^n \cap (x) = xI^{n-1}.$$

As $J/(x)$ is reduction of $I/(x)$, for all large n

$$(J/(x))(I/(x))^{n-1} = (I/(x))^n.$$

Equivalently, $JI^{n-1} + (x) = I^n + (x)$. Hence

$$I^n \subseteq JI^{n-1} + (x) \cap I^{n-1} = JI^{n-1} + (x)I^{n-1} = JI^{n-1}.$$

Thus $I^n = JI^n$. Hence J is a reduction of I. □

Theorem 2.6. *Let $(R,\mathfrak{m})$ be a d-dimensional local ring and let I be an $\mathfrak{m}$-primary ideal of R. Let $x_1,\ldots,x_d$ be a superficial sequence for I. Then $J=(x_1,\ldots,x_d)$ is a minimal reduction for I.*

Proof. We prove by induction on d. Let $d=1$. Then $\dim(R/(x_1))=0$. Hence (x_1) is $\mathfrak{m}$-primary. Therefore $I^n\subseteq(x_1)$ for large n. By Lemma 2.4, $I^n\cap(x_1)=x_1I^{n-1}$ for large n. Hence $I^n=x_1I^{n-1}$ for large n. Thus (x_1) is a reduction of I.

Let $d\geq 2$. Since $\overline{x_2},\ldots,\overline{x_d}$ is a superficial sequence for $I/(x_1)$ in the $d-1$-dimensional local ring, $\overline{a_2},\ldots,\overline{a_d}$ is a reduction of $I/(x_1)$. By previous lemma, J is a reduction of I. □

Theorem 2.7. *Let I be an $\mathfrak{m}$-primary ideal of a d-dimensional local ring $(R,\mathfrak{m})$. Let x be a superficial element for I. Let $\overline{R}=R/(x)$ and $\overline{I}=I/(x)$. Then*

(i) $P_{\overline{I}}(n)=\Delta P_I(n)+\lambda(0:x)$. *Hence* $\dim(R/(x))=d-1$.

(ii) For $i=0,1,\ldots,d-2$, $e_i(\overline{I})=e_i(I)$ *and* $e_{d-1}(\overline{I})=e_{d-1}(I)+\lambda(0:x)$.

Proof. (i) Consider the following exact sequence

$$0\longrightarrow\frac{(I^n:x)}{I^{n-1}}\longrightarrow\frac{R}{I^{n-1}}\longrightarrow\frac{R}{I^n}\longrightarrow\frac{R}{(I^n,x)}\longrightarrow 0.$$

We get $\lambda(R/(x,I^n))=\lambda(R/I^n)-\lambda(R/I^{n-1})+\lambda((I^n:x)/I^{n-1})$. Thus for large n,

$$P_{\overline{I}}(n)=\Delta P_I(n)+\lambda\left((I^n:x)/I^{n-1}\right).$$

So we need to show that $\lambda\left((I^n:x)/I^{n-1}\right)=\lambda(0:x)$. Since x is a superficial element for I, there exists a positive integer c such that for all $n>c$,

$$(I^n:x)\cap I^c=I^{n-1}.$$

Equivalently, the map $\dfrac{(I^n:x)}{I^{n-1}}\longrightarrow\dfrac{R}{I^c}$, defined by $\bar{b}\mapsto b+I^c$, is injective for all $n>c$. Hence for $n>c$, $\lambda\left((I^n:x)/I^{n-1}\right)\leq\lambda(R/I^c)$. Now consider the exact sequence

$$0\longrightarrow\frac{R}{I^c\cap(I^n:x)}\longrightarrow\frac{R}{I^c}\oplus\frac{R}{(I^n:x)}\longrightarrow\frac{R}{I^c+(I^n:x)}\longrightarrow 0.$$

Then for all $n>c$ we get $\lambda(R/I^c)+\lambda(R/(I^n:x))=\lambda\left(R/I^{n-1}\right)+\lambda(R/(I^c+(I^n:x)))$. Hence

$$\lambda\left(R/I^{n-1}\right)-\lambda(R/(I^n:x))=\lambda(R/I^c)-\lambda(R/\left(I^c+(I^n:x)\right)).$$

This implies

$$\lambda\left(\frac{(I^n:x)}{I^{n-1}}\right)=\lambda\left(\frac{I^c+(I^n:x)}{I^c}\right). \tag{2.1}$$

Next we claim that $I^c+(I^n:x)=(0:x)+I^c$ for all $n>c$. Clearly $(0:x)\subseteq(I^n:x)$. So let $b\in(I^n:x)$. By Artin Rees Lemma, there is a positive integer p such that for all $n\geq p$

$$I^n\cap(x)=I^{n-p}(I^p\cap(x)).$$

So $bx\in I^n\cap(x)=I^{n-p}(I^p\cap(x))\subseteq xI^{n-p}$. Hence $bx=xy$ for some $y\in I^{n-p}$. So $b-y\in(0:x)$ and hence $b\in(0:x)+I^{n-p}\subseteq(0:x)+I^c$ for all large n, which proves

the claim.
By (2.1) and claim,

$$\lambda\left(\frac{(I^n:x)}{I^{n-1}}\right)=\lambda\left(\frac{(0:x)+I^c}{I^c}\right)=\lambda\left(\frac{(0:x)}{I^c\cap(0:x)}\right).$$

Note that $I^c\cap(0:x)\subseteq I^c\cap(I^n:x)=I^{n-1}$ for all large n. Hence, by Krull intersection theorem, $I^c\cap(0:x)=0$. Thus we have proved that $\lambda\left((I^n:x)/I^{n-1}\right)=\lambda\,(0:x)$ and hence

$$P_{\overline{I}}(n)=\Delta P_I(n)+\lambda\,(0:x). \tag{2.2}$$

(ii) The above equation gives $\deg\left(P_{\overline{I}}(n)\right)=\deg\left(P_I(n)\right)-1$. Hence $\dim\left(R/(x)\right)=d-1$ and x is a parameter for R. For $d=1$, $e_0(\overline{I})=e_0(I)+\lambda\,(0:x)$. Furthermore for $d\geq 2$, by comparing the coefficients of the polynomials in (2.2), $e_i(\overline{I})=e_i(I)$ for $i=0,1,\ldots,d-2$ and $e_{d-1}(\overline{I})=e_{d-1}(I)+\lambda\,(0:x)$. □

Proposition 2.8. *Let I be an ideal of a local ring $(R,\mathfrak{m})$. Let x be a superficial element for I and let $\overline{x}=x+I^2$. Then for large n,*

$$\left(\frac{G(I)}{(\overline{x})}\right)_n\simeq G(I/(x))_n.$$

Proof. For sufficiently large n, we have

$$\left(\frac{G(I)}{(\overline{x})}\right)_n=\frac{I^n/I^{n+1}}{(xI^{n-1}+I^{n+1})/I^{n+1}}\simeq\frac{I^n}{xI^{n-1}+I^{n+1}}.$$

On the other hand for large n

$$G(I/(x))_n=\frac{I^n+(x)}{I^{n+1}+(x)}\simeq\frac{I^n}{I^{n+1}+(x)\cap I^n}=\frac{I^n}{I^{n+1}+xI^{n-1}}.$$

□

Theorem 2.9 (Sally-Machine)**.** *Let I be an ideal of a local ring $(R,\mathfrak{m})$. Let x be a superficial element for I. Suppose* $\operatorname{depth} G(I/(x))>0$*. Then x^* is $G(I)$-regular.*

Proof. (B. Singh) Let $G(I)=\bigoplus_{n=0}^{\infty}G_n$, where $G_n=I^n/I^{n+1}$. We will show that $(x^*)^s$ is $G(I)$-regular for all s. For each s we need to show that $G_n\cap(0:(x^*)^s)=0$ for all $n\geq 0$. Let $f:G(I)\longrightarrow G(I/(x))$ be the natural map.

We claim that $f((0:(x^*)^s))=0$. Since x is a superficial element for I, the multiplication map $(x^*)^s:G_n\longrightarrow G_{n+1}$ is injective for large n and $s\geq 1$. Hence $(0:(x^*)^s)G_n\subseteq(0:(x^*)^s)\cap G_n=0$ for large n. This implies, $f(G_n)f(0:(x^*)^s)=f(G_n(0:(x^*)^s))=0$ for large n. Since $\operatorname{depth}(G(I/(x))>0$, $f(G_n)=G(I/(x))_n$ has an $G(I/(x))$-regular element for large n. Thus, $f(0:(x^*)^s)=0$.

Apply induction on n. For $n=0$. Let $\overline{a}\in G_0\cap(0:(x^*)^s)$. Note that $f(G_0)=G(I/(x))_0=R/I=G_0$. Hence $f(\overline{a})=\overline{a}\in f(0:(x^*)^s)$. Thus $\overline{a}=0$. Now, let $n\geq 1$. Let $b\in I^n\setminus I^{n+1}$ and $b^*\in G_n\cap(0:(x^*)^s)$. Then $bx^s\in I^{n+s+1}$. Since $f(b^*)=0$, $b\in I^{n+1}+(x)$. Let $b=c+dx$ for some $c\in I^{n+1}$ and $d\in R$. If $d\in I^t\setminus I^{t+1}$ for some $t<n$, then $cx^s=bx^s-dx^{s+1}\in I^{n+s+1}$. This implies $dx^{s+1}\in I^{n+s+1}\subseteq I^{t+s+1}$. Hence $d^*(x^*)^{s+1}=0$. By induction, $d^*=0$, which is a contradiction. Hence $d\in I^n$ and thus $b^*=0$. □

Proposition 2.10. *Let I be an $\mathfrak{m}$-primary ideal of a d-dimensional local ring $(R, \mathfrak{m})$. Let $x_1, \ldots, x_r$ be a superficial sequence for I. Suppose that* depth $(R) \geq r$. *Then $x_1, \ldots, x_r$ is an R-regular sequence.*

Proof. We prove by induction on r. Let $r = 1$ and put $x_1 = x$. Then there exists a positive integer c such that for $n > c$,

$$(I^n : x) \cap I^c = I^{n-1}.$$

Hence for all large n, $(0 : x) \cap I^c \subseteq I^{n-1}$. By Krull Intersection Theorem, $(0 : x) \cap I^c = 0$. Since depth $(R) = \text{depth}_I (R) > 0$, I^c has a regular element, say a. Then $(0 : x)a \subseteq (0 : x) \cap I^c = 0$. Hence $(0 : x) = 0$, and thus x is an R-regular element.

Let $r \geq 2$ and let $x_1, \ldots, x_r$ be a superficial sequence for I. Then $\overline{x_2}, \ldots, \overline{x_r}$ is a superficial sequence for $I/(x_1)$. By induction, $\overline{x_2}, \ldots, \overline{x_r}$ is an $R/(x_1)$-regular sequence. Hence $x_1, \ldots, x_r$ is an R-regular sequence. □

Theorem 2.11. *Let I be an ideal of a local ring $(R, \mathfrak{m})$. Let $x_1, x_2, \ldots, x_s \in I/I^2$. Then $x_1^*, x_2^*, \ldots, x_s^*$ is a $G(I)$-regular sequence if and only if $x_1, x_2, \ldots, x_s$ is an R-regular sequence and for all $n \geq 1$*

$$(x_1 \ldots, x_s) \cap I^n = (x_1, \ldots, x_s)I^{n-1}.$$

Proof. Apply induction on s. Let $s = 1$ and put $x_1 = x$. Let x^* be $G(I)$-regular. Let $a \in R$ such that $ax = 0$. If $a \neq 0$, then by Krull intersection theorem there is m such that $a \in I^m \setminus I^{m+1}$. Then $a^*x^* = 0$ and thus $a^* = 0$, which is a contradiction as x^* is nonzerdivisor. Therefore $a = 0$, and hence x is R-regular. Now let $b \in I^m \setminus I^{m+1}$ and $bx \in I^n$. Then $b^*x^* \in I^{m+1}/I^{m+2}$ and $b^*x^* \neq 0$. As $bx \notin I^{m+2}$, $n \leq m + 1$. Hence $b \in I^{n-1}$.
Conversely, let x be R-regular and $(x) \cap I^n = xI^{n-1}$ for all $n \geq 1$. Let $b^* \in I^m/I^{m+1}$ and $b^*x^* = 0$. Then $bx \in I^{m+2} \cap (x) = xI^{m+1}$. As x id R-regular, $b \in I^{m+1}$. Hence $b^* = 0$.

Inductive step: Let $s \geq 2$ and assume the result for $s - 1$. Let $x_1^*, \ldots, x_s^*$ be a $G(I)$-regular sequence. Let $S = R/(x_1)$ and $J = I/(x_1)$. Since x_1^* is $G(I)$-regular, $G(I/(x_1)) \simeq G(I)/(x_1^*)$. This implies $x_2^*, \ldots, x_s^*$ is a $G(I/(x_1))$-regular sequence. By induction hypothesis, $\overline{x_2}, \ldots, \overline{x_s}$ is $R/(x_1)$-regular sequence and $J^n \cap (\overline{x_2}, \ldots, \overline{x_s}) = (\overline{x_2}, \ldots, \overline{x_s})J^{n-1}$ for all $n \geq 1$. Since x_1^* is $G(I)$-regular, x_1 is R-regular. Hence $x_1, \ldots, x_s$ is an R-regular sequence. It remain to show that for $n \geq 1$

$$I^n \cap (x_1, \ldots, x_s) = (x_1, \ldots, x_s)I^{n-1}.$$

Let $r_1x_1 + \cdots + r_sx_s \in I^n$ for some $r_1, \ldots, r_s \in R$. Then $\overline{r_2x_2} + \cdots + \overline{r_sx_s} \in J^n \cap (\overline{x_2}, \ldots, \overline{x_s}) = J^{n-1}(\overline{x_2}, \ldots, \overline{x_s})$. Hence $\overline{r_2x_2} + \cdots + \overline{r_sx_s} = \overline{t_2}\overline{x_2} + \cdots + \overline{t_s}\overline{x_s}$ for some $t_2, \ldots, t_s \in I^{n-1}$. So for some $t_1 \in R$,

$$(r_2 - t_2)x_2 + \cdots + (r_s - t_s)x_s = t_1x_1.$$

This implies,

$$(r_1 + t_1)x_1 = (r_1x_1 + \cdots + r_sx_s) - (t_2x_2 + \cdots + t_sx_s) \in I^n.$$

Therefore, $(r_1 + t_1) \in I^{n-1}$. Thus,

$$r_1x_1 + \cdots + r_sx_s = (r_1 + t_1)x_1 + t_2x_2 + \cdots + t_sx_s \in (x_1, \ldots, x_s)I^{n-1}.$$

Conversely, Let $x_1, \ldots, x_s$ be an R-regular sequence and for $n \geq 1$

$$I^n \cap (x_1, \ldots, x_s) = (x_1, \ldots, x_s)I^{n-1}.$$

We claim that $I^n \cap (x_1, \ldots, x_{s-1}) = (x_1, \ldots, x_{s-1})I^{n-1}$ for $n \geq 1$. We prove the claim by induction on n. For $n = 1$, $I \cap (x_1, \ldots, x_{s-1}) = (x_1, \ldots, x_{s-1})$. Let $n \geq 2$ and let $r_1x_1 + \cdots + r_{s-1}x_{s-1} \in I^n$ for some $r_1, \ldots, r_{s-1} \in R$. Then

$$r_1s_1 + \cdots + r_{s-1}x_{s-1} \in I^n \cap (x_1, \ldots, x_s) = (x_1, \ldots, x_s)I^{n-1}.$$

So $r_1x_1 + \cdots + r_{s-1}x_{s-1} = t_1x_1 + \cdots + t_sx_s$ for some $t_1, \ldots, t_s \in I^{n-1}$. Hence $t_sx_s \in (x_1, \ldots, x_{s-1})$. Since $x_1, \ldots, x_s$ is an R-regular sequence, $t_s \in (x_1, \ldots, x_{s-1}) \cap I^{n-1} = (x_1, \ldots, x_{s-1})I^{n-2}$. Hence $t_sx_s \in I^{n-1}(x_1, \ldots, x_{s-1})$. Hence $r_1x_1 + \cdots + r_{s-1}x_{s-1} \in I^{n-1}(x_1, \ldots, x_{s-1})$, which proves the claim.
By inductive hypothesis, $x_1^*, \cdots, x_{s-1}^*$ is a $G(I)$-regular sequence. Thus

$$G(I)/(x_1^*, \ldots, x_{s-1}^*) \simeq G(I/(x_1, \ldots, x_{s-1})).$$

By $s = 1$ case of induction, x_s^* is $G(I)/(x_1^*, \ldots, x_{s-1}^*)$ regular. Hence $x_1^*, \ldots, x_s^*$ is $G(I)$-regular sequence.

□

Theorem 2.12 (D.G. Northcott). *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d > 0$ with infinite residue field and let I be an $\mathfrak{m}$-primary ideal. Then $e_0(I) - e_1(I) \leq \lambda(R/I)$.*

Proof. Apply induction on d. The case $d = 1$ is proved in Theorem 2.1.

Let $d \geq 2$. Let x be a superficial for I. Since depth $(R) > 1$, x is R-regular. Set $S = R/(x)$ and $J = I/(x)$. Then S is Cohen-Macaulay and $\dim(S) = d - 1$. By Theorem 2.7, $e_0(I) = e_0(J)$ and $e_1(I) = e_1(J)$ (because $\lambda(0 : x) = 0$). By induction hypothesis, $e_0(J) - e_1(J) \leq \lambda(S/J)$. Therefore

$$e_0(I) - e_1(I) = e_0(J) - e_1(J) \leq \lambda(S/J) \leq \lambda(R/I).$$

□

Theorem 2.13 (Huneke-Ooishi). *Let $(R, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring with infinite residue field. Let I be an $\mathfrak{m}$-primary ideal. Then $e_0(I) - e_1(I) = \lambda(R/I)$ if and only if $r(I) \leq 1$. In this case $G(I)$ is Cohen-Macaulay and for all $n \geq 1$,*

$$\lambda(R/I^n) = e_0(I)\binom{n+d-1}{d} - e_1(I)\binom{n+d-2}{d-1}.$$

Proof. We prove this by induction on d. The $d = 1$ case is proved in Theorem 2.3.

Now let $d \geq 2$. Let $J = (x_1, \ldots, x_d)$ be a minimal reduction of I, where $x_1, \ldots, x_d$ is a superficial sequence for I. Assume $e_0(I) - e_1(I) = \lambda(R/I)$. We want to show $I^2 = JI$. Let "–" denote images in $\overline{R} = R/(x_1)$. By Theorem 2.7, since x_1 is a superficial element for I, $e_0(I) = e_0(\overline{I})$ and $e_1(I) = e_1(\overline{I})$. This implies

$$\lambda(\overline{R}/\overline{I}) = \lambda(R/I) = e_0(I) - e_1(I) = e_0(\overline{I}) - e_1(\overline{I}).$$

By induction hypothesis, $\overline{I}^2 = \overline{J}\,\overline{I}$ and $G(\overline{I})$ is Cohen-Macaulay. This implies

$$I^2 + (x_1) = JI + (x_1). \tag{2.3}$$

Since x_1 is a superficial element for I and $G(\overline{I})$ is Cohen-Macaulay, by Sally-Machine Theorem, x_1^* is $G(I)$-regular. By Theorem 2.11, $(x_1) \cap I^n = (x_1)I^{n-1}$ for all $n \geq 1$. In particular, $(x_1)I = (x_1) \cap I^2$. Therefore, from (2.3)

$$I^2 \subseteq JI + (x_1) \cap I^2 = JI + (x_1)I = JI.$$

Hence $I^2 = JI$.

Conversely, let $I^2 = JI$. Then $\overline{I}^2 = \overline{J}\,\overline{I}$. By induction hypothesis, $e_0(\overline{I}) - e_1(\overline{I}) = \lambda(\overline{R}/\overline{I}) = \lambda(R/I)$. Since $e_0(I) = e_0(\overline{I})$ and $e_1(I) = e_1(\overline{I})$, $e_0(I) - e_1(I) = \lambda(R/I)$.

We next show that $G(I)$ is Cohen-Macaulay. Since x_1^* is $G(I)$-regular, $G(\overline{I}) \simeq G(I)/(x_1^*)$. Since $G(\overline{I})$ is Cohen-Macaulay, $\overline{x_2}^*, \ldots, \overline{x_d}^*$ is a $G(I)/(x_1^*)$-regular sequence. Therefore $G(I)$ is Cohen-Macaulay. Furthermore, $x_1^*, \ldots, x_d^*$ is a $G(I)$-regular sequence. Hence

$$\begin{aligned} G(I)/(x_1^*, \ldots, x_d^*) \simeq G(I/J) &= \bigoplus_{n=0}^{\infty} \frac{I^n + J}{I^{n+1} + J} \\ &= \frac{R}{I} \oplus \frac{I}{I^2 + J} \oplus \frac{I^2 + J}{I^3 + J} \oplus \cdots \\ &= \frac{R}{I} \oplus \frac{I}{J}. \qquad \text{(since } I^2 \subseteq J\text{)} \end{aligned}$$

Therefore

$$\begin{aligned} H_{G(I)}(t) = \bigoplus_{n=0}^{\infty} \lambda(I^n/I^{n+1})t^n &= \frac{h_0 + h_1 t + \cdots + h_s t^s}{(1-t)^d} \\ &= \frac{H_{G(I/J)}(t)}{(1-t)^d} \\ &= \frac{\lambda(R/I) + \lambda(I/J)t}{(1-t)^d}. \end{aligned}$$

Therefore $h(t) = \lambda(R/I) + (e_0 - \lambda(R/I))\,t$. Thus $e_1(I) = e_0(I) - \lambda(R/I)$ and $e_2(I) = e_3(I) = \cdots = e_d(I) = 0$.

Now we find $\lambda(R/I^{n+1})$.

$$\begin{aligned} \left(\sum_{n=0}^{\infty} \lambda(R/I^{n+1})t^n\right)(1-t) &= \sum_{n=0}^{\infty} \lambda(R/I^{n+1})t^n - \sum_{n=0}^{\infty} \lambda(R/I^{n+1})t^{n+1} \\ &= \sum_{n=0}^{\infty} \lambda(I^n/I^{n+1})t^n. \end{aligned}$$

Hence

$$\begin{aligned} \sum_{n=0}^{\infty} \lambda(R/I^{n+1})t^n &= \frac{\lambda(R/I) + (e_0(I) - \lambda(R/I))\,t}{(1-t)^{d+1}} \\ &= [\lambda(R/I) + (e_0(I) - \lambda(R/I))\,t] \sum_{n=0}^{\infty} \binom{n+d}{d} t^n. \end{aligned}$$

Equating the coefficients of t^n for $n \geq 0$, we get

$$\begin{aligned}\lambda\left(R/I^{n+1}\right) &= \lambda\left(R/I\right)\binom{n+d}{d} + \left(e_0(I) - \lambda\left(R/I\right)\right)\binom{n-1+d}{d} \\ &= e_0(I)\binom{n+d}{d} - e_1(I)\binom{n+d-1}{d}.\end{aligned}$$

□

3. Solution of Cowsik's conjecture for certain primes

3.1 Symbolic Rees algebras

Veronese algebra

Let S be a Noetherian ring. Let $\{I_j\}_{j=0}^{\infty}$ be a family of graded ideals of S with $I_0 = S$ and $I_j I_k \subseteq I_{j+k}$ for all j and k. Consider the Rees ring of S with respect to family $\{I_j\}_{j=0}^{\infty}$,

$$A := \bigoplus_{j=0}^{\infty} I_j t^j.$$

For any positive integer d we denote by $A^{(k)} = \bigoplus_{n=0}^{\infty} A_{kn}$ the k^{th} **Veronese subsalgebra** of A.

Theorem 3.1. *Let the notations be as above. Then A is a finitely generated S-algebra if and only if there exists a positive integer k such that $A_{kn} = A_k^n$ for all $n \geq 1$.*

Proof. Let A be finitely generated S-algebra, i.e., $A = S[f_1 t^{r_1}, f_2 t^{r_2}, \ldots, f_s t^{r_s}]$ with homogeneous $f_i \in I_{r_i}$ for $i = 1, \ldots, s$. Let $r = \operatorname{lcm}(r_1, r_2, \ldots, r_s)$ and set $k = rs$. We show that $A_{kn} = A_k^n$ for all $n \geq 1$.
Let $x \in A_m$. Then x is an S-linear combination of monomials of the form $f_1^{u_1} \cdots f_s^{u_s}$ where $u_1 r_1 + \cdots + u_s r_s \geq m$. If $m \geq rs$, then $u_i r_i \geq r$ for some $i = 1, \ldots, s$. Let $v = r/r_i$. Then $f_1^{u_1} \cdots f_s^{u_s} = (f_1^{u_1} \cdots f_i^{u_i - v} \cdots f_s^{u_s}) f_i^v$. Since $f_1^{u_1} \cdots f_i^{u_i - v} \cdots f_s^{u_s} \in A_{m-r}$ and $a_i^v \in A_r$, $x \in A_{m-r} A_r$. Hence we showed that $A_m \subseteq A_{m-r} A_r$ for $m \geq k$. Therefore, for all $l \geq 1$

$$A_{k+rl} \subseteq A_k A_{rl}.$$

Hence for all $n \geq 1$, $A_{nk} \subseteq A_k^n$. Since the reverse inclusion always holds, the assertion follows.

Conversely, let k be a positive integer such that $A_{kn} = A_k^n$ for all $n \geq 1$. Then $A^{(k)} = \bigoplus_{n=0}^{\infty} A_k^n t^n = S[A_k t]$ is finitely generated S-algebra. Note that $A^{(k;j)} := \bigoplus_{n=0}^{\infty} A_{nk+j}$ is an $A^{(k)}$-module of rank 1. Then $A^{(k;j)}$ is a finitely generated $A^{(k)}$-module. Since $A = \bigoplus_{j=0}^{k-1} A^{(k;j)}$, it follows that A is a finitely generated $A^{(k)}$-module. Hence A is finitely generated S-algebra. □

Symbolic Rees algebra

Let R be a Noetherian ring and p be a prime ideal of R. For a positive integer n, the n^{th} **symbolic power** of p, denoted by $p^{(n)}$, is defined as

$$p^{(n)} := p^n R_p \cap R = \{x \in R : xs \in p^n \text{ for some } s \notin p\}.$$

The graded ring $\bigoplus_{n=0}^{\infty} p^{(n)} t^n$, denoted by $\mathcal{R}_s(p)$, is called the **symbolic Rees algebra** of p.

Proposition 3.2. *Let $(R, \mathfrak{m})$ be a Noethrian local ring and p be a prime ideal of R. Then $\mathcal{R}_s(p)$ is Noetherian if and only if there exists $k \geq 1$ such that $p^{(k)n} = p^{(kn)}$ for all $n \geq 1$.*

Proof. Immediate from Theorem 3.1 for $A = \mathcal{R}_s(p)$. □

Theorem 3.3 (Cowsik, Vasconcelos)**.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring with infinite residue field and dimension d. Let p be a nonmaximal prime ideal of R. If $\mathcal{R}_s(p)$ is Noetherian, then there exists $d-1$ elements $f_1, f_2, \ldots, f_{d-1} \in p$ such that $p = \sqrt{(f_1 \ldots, f_{d-1})}$.*

Proof. Since $\mathcal{R}_s(p)$ is Noetherian, there exists a positive integer k such that for all $n \geq 1$, $p^{(nk)} = p^{(k)n}$. Let $I = p^{(k)}$. Then depth $(R/I^n) \geq 1$ for all $n \geq 1$. By applying Burch's inequality, we get $l(I) \leq \dim(R) - \beta(I) = d-1$. Let $J = (f_1, f_2, \ldots, f_{d-1})$ be a reduction of I. Then $\sqrt{J} = \sqrt{I} = p$, i.e. $p = \sqrt{(f_1 \ldots, f_{d-1})}$. □

3.2 Associativity formula for multiplicities

Lemma 3.4. *Let $(R, \mathfrak{m})$ be a local ring, and let q be an $\mathfrak{m}$-primary ideal of R. Let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be an exact sequence of R-modules. Then $e(q, M) = e(q, M') + e(q, M'')$.*

Proof. We assume that M' is a submodule of M. Then we obtain the exact sequence $0 \longrightarrow M'/(q^n M \cap M') \longrightarrow M/q^n M \longrightarrow M''/q^n M'' \longrightarrow 0$ and

$$\lambda\left(M/q^n M\right) = \lambda\left(M'/(q^n M \cap M')\right) + \lambda\left(M''/q^n M''\right). \tag{3.1}$$

By Artin-Rees Lemma, there is a positive integer r such that for all $n \geq r$, $q^n M \cap M' = q^{n-r}(q^r M \cap M')$. This implies $q^n M' \subseteq M' \cap q^n M \subseteq q^{n-r} M'$ and hence

$$\lambda\left(M'/q^{n-r} M'\right) \leq \lambda\left(M'/(M' \cap q^n M)\right) \leq \lambda\left(M'/q^n M'\right).$$

So that for sufficiently large n, $P_q(M', n-r) \leq \lambda\left(M'/(M' \cap q^n M)\right) \leq P_q(M', n)$. Hence

$$\lim_{n \longrightarrow \infty} \left(\frac{d!}{n^d}\right) \lambda\left(M'/(M' \cap q^n M)\right) = e(q, M').$$

Now multiply (3.1) by $d!/n^d$ and take limit of n to infinity to get the result. □

Theorem 3.5 (Associativity formula)**.** *Let $(R, \mathfrak{m})$ be a local ring, q be an $\mathfrak{m}$-primary ideal. Let $p_1, \ldots, p_r$ be all minimal primes of R such that $\dim(R) = d = \dim(R/p_i)$ for all i. Then for any finitely generated R-module M*

$$e(q, M) = \sum_{i=1}^{r} e(q, R/p_i)\, \lambda\left(M_{p_i}\right).$$

Proof. Choose a filtration $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_k = 0$ such that $M_i/M_{i+1} \simeq R/Q_i$ for each $i = 0, \ldots, k-1$ and $Q_i \in \operatorname{Spec}(R)$. Then by the previous lemma,

$$e(q, M) = \sum_{\substack{i=0 \\ \dim(R/Q_i)=d}}^{k-1} e(q, R/Q_i).$$

Let $p \in \operatorname{Spec}(R)$. Then $M_p \supseteq (M_1)_p \supseteq (M_2)_p \supseteq \cdots \supseteq (M_k)_p = 0$ and

$$(M_i/M_{i+1})_p \simeq (R/Q_i)_p = \begin{cases} 0, & \text{if } p \neq Q_i \\ R_p/Q_{ip}, & \text{if } p = Q_i \end{cases}$$

Therefore,

$$e(q, M) = \sum_{i=1}^{r} e(q, R/p_i)\, \lambda\,(M_{p_i}).$$

□

3.3 Huneke's criterion for Noetherian symbolic Rees algebra for certain primes

Theorem 3.6 (Huneke)**.** *Let $(R, \mathfrak{m})$ be a 3-dimensional regular local ring with $R/\mathfrak{m}$ infinite. Let p be a height two prime ideal of R. Then the following are equivalent:*

(i) $\mathcal{R}_s(p)$ is Noetherian

(ii) there exist elements $f \in p^{(k)}, g \in p^{(l)}$ and $x \in \mathfrak{m} \setminus p$ such that

$$\lambda\,(R/(f, g, x) = kl \cdot \lambda\,(R/(p, x)$$

Proof. Let $\mathcal{R}_s(p)$ be a Noetherian ring. Then, by Proposition 3.2, there exists a positive integer k such that for every positive integer n, $p^{(k)n} = p^{(kn)}$. Let $I = p^{(k)}$. As $p^{(n)}$ is p-primary ideal, $\operatorname{depth}(R/p^{(n)}) = 1$ for all $n \geq 1$. In particular, $\operatorname{depth}(R/I^n) = 1$ for all $n \geq 1$. Thus the stable value of $\operatorname{depth}(R/I^n) = \beta(I) = 1$. Hence by Burch's inequality, $l(I) \leq \dim(R) - \beta(I) = 2$. Also $l(I) \geq \operatorname{ht}(I) = 2$. Thus $l(I) = 2$. Let $J = (f, g)$ be a reduction of I. Let $x \in \mathfrak{m} \setminus p$. Now, since $R/(f, g)$ is Cohen-Macaulay of dimension 1,

$$e(x, R/(f, g)) = \lambda\,(R/(f, g, x)) = e(J, R/(x)).$$

Further, as J is a reduction of I, $e(J, R/(x)) = e(I, R/(x))$. So, to find $e(I, R/(x))$ we look at $\lambda\,(R/(I^n, x)) = e(x, R/I^n)$ since R/I^n is Cohen-Macaulay for all n. Let S denote the ring R/xR. Then by associativity formula,

$$\lambda\,(S/I^nS) = e(x, R/I^n) = e(x, R/p)\, \lambda(R_p/p^{nk}R_p) = e(x, R/p)\binom{nk+1}{2}.$$

The coefficient of $n^2/2$ gives us,

$$e(I, S) = k^2 e(x, R/p),$$

which implies $\lambda\,(R/(f, g, x) = k^2 e(x, R/p)$.

Conversely, suppose $f \in p^{(k)}$, $g \in p^{(l)}$, $x \notin p$, and $\lambda(R/(f,g,x)) = kl\,\lambda(R/(p,x))$. Then $f^l, g^k \in p^{(kl)}$ and $\lambda(R/(f^l,g^k,x)) = kl\,\lambda(R/(f,g,x)) = (kl)^2\,\lambda(R/(p,x))$. Thus we may assume that $k = l$.
Let $I = \left(p^{(k)} + (x)\right)/(x)$. As $\dim(R/(x)) = 2$, we write

$$P_I(n) = e_0\binom{n+1}{2} - e_1 n + e_2.$$

So that $P_I(n) = \lambda(S/I^n)$ for large n. We claim that $e_0 - e_1 = \lambda(S/I)$. Set $J_n = \left(p^{(kn)} + (x)\right)/(x)$. Then, by using associativity formula,

$$\begin{aligned}
\lambda(S/J_n) = \lambda\left(R/(p^{(kn)},x)\right) &= e(x, R/p^{(kn)}) \\
&= e(x,R/p)\cdot\lambda\left(R_p/p^{kn}R_p\right) \\
&= e\cdot\binom{kn+1}{2} \\
\lambda(S/J_n) &= ek^2\binom{n+1}{2} - e\cdot\binom{k}{2}n \qquad (3.2)
\end{aligned}$$

where $e = \lambda(R/(p,x))$. By our assumption, $\lambda\left(S/(\overline{f},\overline{g})\right) = \lambda(R/(f,g,x)) = ek^2$. Since $(\overline{f},\overline{g}) \subseteq I$, $e(I) \le e(\overline{f},\overline{g}) = \lambda\left(S/(\overline{f},\overline{g})\right) = ek^2$. Hence

$$e_0 \le ek^2. \qquad (3.3)$$

Since $p^{(k)n} \subseteq p^{(kn)}$ for all $n \ge 1$, $\lambda(S/I^n) \ge \lambda(S/J_n)$ for all $n \ge 1$. For large n,

$$e_0\binom{n+1}{2} - e_1 n + e_2 \ge ek^2\binom{n+1}{2} - e\cdot\binom{k}{2}n.$$

Hence

$$e_0 \ge ek^2 \text{ and } e\binom{k}{2} \ge e_1. \qquad (3.4)$$

From (3.3) and (3.4), $e_0 = ek^2$ and

$$e_0 - e_1 = ek^2 - e_1 \ge ek^2 - e\binom{k}{2} = e\cdot\binom{k+1}{2} = e(x,R/p)\cdot\lambda\left(R_p/p^kR_p\right) = \lambda(S/I).$$

So $e_0 - e_1 \ge \lambda(S/I)$. By Theorem 2.12, $e_0 - e_1 \le \lambda(S/I)$. Hence $e_0 - e_1 = \lambda(S/I)$. Now by Theorem 2.13, $e_2 = 0$ and $P_I(n) = \lambda(S/I^n)$ for all $n \ge 1$. So for $n \ge 1$,

$$\begin{aligned}
\lambda(S/I^n) &= e_0\binom{n+1}{2} - e_1 n \\
&= ek^2\binom{n+1}{2} - (e_0 - \lambda(S/I))\,n \\
&= ek^2\binom{n+1}{2} - \left(ek^2 - e\cdot\binom{k+1}{2}\right)n \\
&= ek^2\binom{n+1}{2} - e\cdot\binom{k}{2}n \\
&= \lambda(S/J_n).
\end{aligned}$$

Since $I^n \subseteq J_n$, we get $I^n = J_n$ for all $n \geq 1$. Hence $\left(p^{(k)n}, x\right) = \left(p^{(kn)}, x\right)$ for all $n \geq 1$. Thus

$$\begin{aligned} p^{(kn)} &\subseteq p^{(k)n} + (x) \cap p^{(kn)} \\ &= p^{(k)n} + x(p^{(kn)} : x) \\ &= p^{(k)n} + xp^{(kn)}. \end{aligned}$$

By Nakayama's Lemma, $p^{(k)n} = p^{(kn)}$ for all $n \geq 1$. Therefore $\mathcal{R}_s(p)$ is Noetherian. □

Corollary 3.7. *Let $(R, \mathfrak{m})$ be a 3-dimensional regular local ring with $R/\mathfrak{m}$ infinite. Let p be a height two prime ideal with $e(R/p) = 3$. Then $\mathcal{R}_s(p)$ is Noetherian.*

Proof. We may choose $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ such that $e(x, R/p) = 3$. Since $\operatorname{ht}(p) = 2$, R/p is Cohen-Macaulay and hence $e(x, R/p) = \lambda\,(R/(p, x)) = 3$.

If $p \not\subseteq \mathfrak{m}^2$, then A/At is a regular local ring of dimension 2, where $t \in p$ and $t \notin \mathfrak{m}^2$. Also $\operatorname{ht}(p/At) = 1$. Let $\overline{a} \in p/At$. Since every regular local ring is a U.F.D., we write $\overline{a} = \overline{b}_1 \cdots \overline{b}_r$ with each $\overline{b}_i$ prime in A/At. Thus for some i, $0 \subsetneq \overline{b}_i(A/At) \subseteq p/At$. But $\operatorname{ht}(p/At) = 1$, hence $p/At = \overline{b}_i(A/At)$. Thus $p = (t, b_i)$ is a complete intersection and so p^n has only one primary component. Thus $p^n = p^{(n)}$ for all $n \geq 1$. Hence $\mathcal{R}_s(p)$ is Noetherian.

Let $p \subseteq \mathfrak{m}^2$. Let S be the ring $R/(x)$ and $\mathfrak{n}$ be the maximal ideal $\mathfrak{m}/(x)$ of S. Then $\lambda\,(R/(\mathfrak{m}^2, x)) = \lambda\,(S/\mathfrak{n}^2) = 3$. As $\lambda\,(R/(p, x)) = 3$ and $p \subseteq \mathfrak{m}^2$, $(p, x) = (\mathfrak{m}^2, x)$.

Since $R/p^{(2)}$ is Cohen-Macaulay, $\lambda\left(R/(x, p^{(2)})\right) = e\left(x, R/p^{(2)}\right)$. Using Associativity formula, we have

$$\lambda\left(R/(x, p^{(2)})\right) = e\left(x, R/p^{(2)}\right) = e(x, R/p)\,\lambda\left(R_p/p^2R_p\right) = 3 \cdot 3 = 9.$$

On the other hand, $\lambda(R/(x, \mathfrak{m}^4)) = \lambda(S/\mathfrak{n}^4) = 10$. Comparing the lengths of $R/(x, p^{(2)})$ and $R/(x, \mathfrak{m}^4)$, there exists $f \in \mathfrak{m}^3 \cap p^{(2)}$ such that $(p^{(2)}, x) = (x, f, \mathfrak{m}^4)$.

Let "–" denote images in S and $*$ denote the leading coefficients in $\operatorname{gr}_{\mathfrak{n}}(S)$. Then $\deg(\overline{f}^*) = 3$. Choose $g^* \in \operatorname{gr}_{\mathfrak{n}}(S)$ with $\deg(\overline{g}^*) = 4$ such that $g \in p^{(2)}$. Thus for $f, g \in p^{(2)}$,

$$\lambda\,(R/(f, g, x)) = \lambda\,(S/(\overline{f}, \overline{g})) = \deg(\overline{f^*})\deg(\overline{g^*}) = 12 = 2^2 \cdot e(x, R/p)$$

Hence, by Theorem 3.6, $\mathcal{R}_s(p)$ is Noetherian.

□

Example 3.8. Let $R = \mathbb{C}[[X, Y, Z]]$ and $\phi : \mathbb{C}[[X, Y, Z]] \longrightarrow \mathbb{C}[[t]]$ be the homomorphism which sends X to t^6, Y to $t^7 + t^{10}$ and Z to t^8. Let $p = \ker(\phi)$. Then p is a height 2 prime ideal in R and $e(R/p) = 6$. Consider the following elements in R,

$$\begin{aligned} a &= 2xz^3 - 3x^2yz - 2x^4 + y^3 - xyz \\ b &= x^3z - 2yz^2 + xy^2 - x^2z \\ c &= x^2z^2 - 2x^3y + y^2z - xz^2 \\ d &= x^4 - z^3. \end{aligned}$$

Then

$$\begin{aligned} \phi(a) &= 2t^6(t^8)^3 - 3(t^6)^2(t^7 + t^{10})t^8 - 2(t^6)^4 + (t^7 + t^{10})^3 - t^6(t^7 + t^{10})t^8 \\ &= 2t^{30} - 3t^{27} - 3t^{30} - 2t^{24} + t^{21} + 3t^{24} + 3t^{27} + t^{30} - t^{21} - t^{24} \\ &= 0. \end{aligned}$$

$$\begin{aligned}
\phi(b) &= (t^6)^3t^8 - 2(t^7+t^{10})(t^8)^2 + t^6(t^7+t^{10})^2 - (t^6)^2t^8 \\
&= t^{26} - 2t^{23} - 2t^{26} + t^{20} + 2t^{23} + t^{26} - t^{20} \\
&= 0. \\
\phi(c) &= (t^6)^2(t^8)^2 - 2(t^6)^3(t^7+t^{10}) + (t^7+t^{10})^2t^8 - t^6(t^8)^2 \\
&= t^{28} - 2t^{25} - 2t^{28} + t^{22} + 2t^{25} + t^{28} - t^{22} \\
&= 0. \\
\phi(d) &= (t^6)^4 - (t^8)^3 = 0
\end{aligned}$$

Thus $(a, b, c, d) \subseteq p$. We claim that $p = (a, b, c, d)$. Let $I = (a, b, c, d)$. Consider the following matrix

$$M = \begin{pmatrix} x & -y & -2z & 2x \\ z & -2x^2 & -y & 2xz \\ 0 & -z & x & 2y \end{pmatrix}$$

We look at all the 3×3 minors of M,

$$\begin{aligned}
2y^3 + 2x^2yz - 8x^2yz + 4xz^3 - 4x^4 - 2xyz &= 2a \\
-2x^3z + 2x^2z - 2xy^2 + 4yz^2 &= -2b \\
2x^2z^2 - 2xz^2 - 4x^3y + 2y^2z &= 2c \\
-2x^4 - xyz + xyz + 2z^3 &= -2d.
\end{aligned}$$

Thus we observe that, upto unit, I is generated by 3×3 minors of the matrix M.

Now, consider $(I, x) = (a, b, c, d, x) = (y^3, yz^2, y^2z, z^3, x)$. So $\lambda\left(R/(I, x)\right) = 6$. To find $e(x, R/I)$, we find $\lambda\left(R/(I, x^2)\right)$. Now $(I, x^2) = (y^3 - xyz, xy^2 - 2yz^2, y^2z - xz^2, z^3, x^2)$. We get $\lambda\left(R/(I, x^2)\right) = 12$. Since $\dim\left(R/I\right) = 1$, we find $e(x, R/I) = 6$, which implies R/I is Cohen-Macaulay. By associativity formula,

$$e(x, R/I) = \sum_{\substack{q \in \operatorname{Spec}(R/I) \\ \operatorname{ht}(q)=2}} e(x, R/q)\, \lambda\left(R_q/I_q\right).$$

Since we have $p \in \operatorname{Spec}(R/I), \operatorname{ht}(p) = 2$ and $e(x, R/p) = 6$, it follows that $\lambda\left(R_p/I_p\right) = 1$ and I is p-primary. Therefore $I_p = pR_p$ and hence $I = p$.

Now we calculate $p^{(2)}$. It is easy to check that there exists e, f, g such that

$$\begin{aligned}
xe &= 2ad - bc + 4d^2x, \\
xf &= 2c^2 + ab + 2bdx, \quad \text{and} \\
xg &= b^2 + 4cd.
\end{aligned}$$

These equations show that $e, f, g \in p^{(2)}$ and

$$\begin{aligned}
e &\equiv -y^4z \mod (x), \\
f &\equiv y^5 - 2y^2z^3 \mod (x), \quad \text{and} \\
g &\equiv 4z^5 - 4y^3z^2 \mod (x),
\end{aligned}$$

Let $J = (\mathfrak{m}^6, x, y^4z, y^5 - 2y^2z^3, 4z^5 - 4y^3z^2)$. From above equations we obtain $J \subseteq (p^{(2)}, x)$. We claim that $J = (p^{(2)}, x)$. Considering lexicographical ordering we have $\lambda\left(R/J\right) = \lambda\left(R/(\mathfrak{m}^6, x, y^4z, y^5, y^3z^2)\right) = \lambda\left(\mathbb{C}[y, z]/\left((y, z)^6, y^4z, y^5, y^3z^2\right)\right)$. Using the

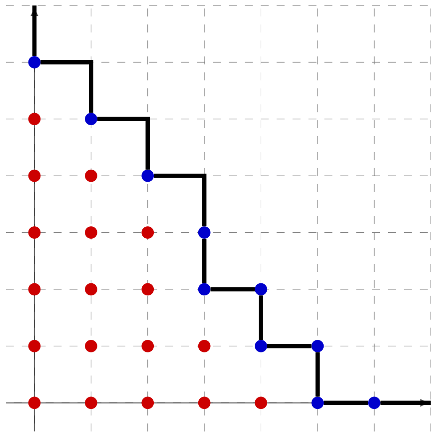

Figure 3.1: The staircase diagram for the ideal $\left((y,z)^6, y^4z, y^5, y^3z^2\right)$

staircase diagram (Figure 3.1), we see that $\lambda\left(\mathbb{C}[y,z]/\left((y,z)^6, y^4z, y^5, y^3z^2\right)\right) = 18$ (number of red nodes).

On the other hand, by associativity formula,

$$\lambda\left(R/(p^{(2)},x)\right) = e(x, R/p^{(2)}) = e(x,R/p)\,\lambda\left(R_p/p^2R_p\right) = 6\cdot 3 = 18.$$

Thus $(p^{(2)},x) = J$.

Now one checks that there exists and element h such that

$$xh = bf^2 + 2ae^2 + geb.$$

So that $h \in p^{(5)}$ and $h \equiv y^{12} - 36y^8z^5 + 72y^5z^8 \mod (x)$. Let $\overline{g}$ and $\overline{h}$ be the images of g and h respectively, in $R/(x)$. The leading coefficient $\overline{g}^*$ of $\overline{g}$ in $\mathrm{gr}_{\mathfrak{m}/(x)}(R/(x)) = \mathbb{C}[y,z]$ is $4z^5 - 4y^3z^2$ and the leading coefficient $\overline{h}^*$ of $\overline{h}$ is $y^{12} - 36y^8z^5 + 72y^5z^8$. Hence $\overline{g}^*$ and $\overline{h}^*$ are relatively prime. So that $(\overline{g}^5)^* = (\overline{g}^*)^5$ and $(\overline{h}^2)^* = (\overline{h}^*)^2$ are relatively prime. Thus for $g^5, h^2 \in p^{(10)}$,

$$\begin{aligned}\lambda\left(R/(g^5,h^2,x)\right) = \lambda\left(\frac{R/(x)}{(\overline{g}^5,\overline{h}^2)}\right) &= \deg\left((\overline{g}^5)^*\right)\cdot\deg\left(\left(\overline{h}^2\right)^*\right)\\ &= 25\cdot 24 = 10^2\cdot 6 = 10^2\cdot e(x,R/p).\end{aligned}$$

By Theorem 3.6, $\mathcal{R}_s(p)$ is Noetherian and the proof shows that $p = \sqrt{(g,h)}$.

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