Counting in a fixed conjugacy class in Teichmüller space

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Abstract

Let γ be a pseudo-Anosov homeomorphism of a compact orientable surface and let X be an element of the Teichmüller space of the same surface. In this paper, we find asymptotics for the number of pseudo-Anosov homeomorphisms that are conjugate to γ and the axis of their action on Teichmüller space intersects the ball of radius R centered at X, as R tends to infinity.

1 Introduction

1.1 Statement of results

Let M be a compact manifold of variable negative curvature, and x, y belong to the universal cover of M. In [Mar04], Margulis obtained asymptotics for the number of elements in the $\pi_1(M)$ orbit of y that lie in a ball of large radius centered at x. As a consequence, he was able to compute assymptotics for the Riemannian volume of such balls. In [ABEM12], these results were extended to the setting of Teichmüller theory. Let us introduce some notation before stating the results of [ABEM12].

Fix S_g to be a compact surface of genus $g \geq 2$. We denote the Teichmüller space, moduli space, and the mapping class group (or modular group) of S_g by $\mathcal{T}_g$, $\mathcal{M}_g$ and Γ respectively. For a point $X \in \mathcal{T}_g$, let $\mathfrak{B}(X, R)$ denote the ball of radius R centered at X, where the distance is measured with respect to the Teichmüller metric. We denote the orbit of X under the action of Γ by $\Gamma \cdot X$. It is proved in [ABEM12] that, given $X, Y \in \mathcal{T}_g$, as $R \to \infty$,

$$|\Gamma \cdot Y \cap \mathfrak{B}(X, R)| \sim \frac{\Lambda^2}{h \operatorname{Vol}(\mathcal{M}_g)} e^{hR}, \quad \text{and}$$

$$\operatorname{Vol}(\mathfrak{B}(X, R)) \sim \frac{\Lambda^2}{h} e^{hR}.$$

Here, $h = 6g - 6$ is the entropy of Teichmüller geodesic flow with respect to Masur-Veech measure, and Λ is the Hubbard-Masur constant [ABEM12, Dum15]. $\mathcal{M}_g$ is equipped with the push-forward of the normalized Masur-Veech measure on the moduli space of unit area quadratic differentials, and $\operatorname{Vol}(\mathcal{M}_g)$ stands for the total mass of $\mathcal{M}_g$ with respect to this measure. (See Section 2.3 of [ABEM12].) The cardinality of a finite set S is denoted by $|S|$, and $A(R)$ is said to be asymptotic to $B(R)$, written $A(R) \sim B(R)$, if $A(R)/B(R) \to 1$ as $R \to \infty$.

The goal of this paper is to prove asymptotics similar to ones mentioned above for large balls in a quotient of the Teichmüller space. To state our results, we need a few definitions first. Fix $\gamma \in \Gamma$ to be a pseudo-Anosov homeomorphism and let $\mathcal{L}_\gamma$ be the axis of its action on Teichmüller space, namely, the unique geodesic that is kept fixed by γ. The cyclic group generated by γ, denoted by $\langle\gamma\rangle$, acts on $\mathcal{T}_g$ properly discontinuously, hence we can form the quotient to be the cylinder $\mathcal{C}_\gamma = \langle\gamma\rangle\backslash\mathcal{T}_g$. The corresponding covering map is denoted by Π_γ. Since the action of γ on $\mathcal{L}_\gamma$ is by translation, the quotient $\overline{\mathcal{L}}_\gamma = \langle\gamma\rangle\backslash\mathcal{L}_\gamma$ is a closed geodesic in $\mathcal{C}_\gamma$. Define the ball of radius R in $\mathcal{C}_\gamma$ around $\overline{\mathcal{L}}_\gamma$ to be

$$\mathfrak{B}(\overline{\mathcal{L}}_\gamma, R) = \{\overline{Y} \in \mathcal{C}_\gamma : d(\overline{Y}, \overline{\mathcal{L}}_\gamma) \leq R\},$$

where the distance in $\mathcal{C}_\gamma$ is the one induced by the Teichmüller distance on its cover $\mathcal{T}_g$. Let $\mathcal{MF}$ denote the space of measured foliations, and for $\zeta \in \mathcal{MF}$ and $X \in \mathcal{T}_g$, denote the extremal length of ζ in X by $\operatorname{Ext}(\zeta, X)$. Define the unit extremal ball around $\mathcal{L}_\gamma$ by

$$\mathfrak{B}_{\operatorname{Ext}}(\mathcal{L}_\gamma) = \{\zeta \in \mathcal{MF} : \inf_{X \in \mathcal{L}_\gamma} \operatorname{Ext}(\zeta, X) \leq 1\}.$$

This set is fixed by γ, and the action of $\langle\gamma\rangle$ on it is proper and discontinuous once we remove the two rays in $\mathcal{MF}$, denoted by $[\gamma^{\pm}]$, that are fixed by this action. Hence we can form the quotient

$$\mathcal{C}_{\operatorname{Ext},\gamma} = \langle\gamma\rangle\backslash\big(\mathfrak{B}_{\operatorname{Ext}}(\mathcal{L}_\gamma)\smallsetminus[\gamma^{\pm}]\big).$$

The Thurston measure ν on $\mathcal{MF}$ induces a measure on $\mathcal{C}_{\operatorname{Ext},\gamma}$, which we denote by ν as well. The orbit counting asymptotics is given by the following theorem.

Theorem A. *Let γ and $\overline{\mathcal{L}}_\gamma$ be as above and $X \in \mathcal{T}_g$. Then as $R \to \infty$,*

$$|\Pi_\gamma(\Gamma \cdot X) \cap \mathfrak{B}(\overline{\mathcal{L}}_\gamma, R)| \sim \frac{\Lambda}{h \operatorname{Vol}(\mathcal{M}_g)} \nu(\mathcal{C}_{\operatorname{Ext},\gamma}) e^{hR}. \tag{1}$$

The volume asymptotics is given by the following theorem.

Theorem B. *Let γ and $\overline{\mathcal{L}}_\gamma$ be as above. Then as $R \to \infty$,*

$$\operatorname{Vol}\big(\mathfrak{B}(\overline{\mathcal{L}}_\gamma, R)\big) \sim \frac{\Lambda}{h} \nu(\mathcal{C}_{\operatorname{Ext},\gamma}) e^{hR}.$$

Given $X \in \mathcal{T}_g$, the following sets are in one-to-one correspondence.

- $\Pi_\gamma(\Gamma \cdot X) \cap \mathfrak{B}(\overline{\mathcal{L}}_\gamma, R)$
- $\{\mathbf{g}.\mathcal{L}_\gamma : \mathbf{g} \in \Gamma, \text{ and } \mathfrak{B}(X, R) \cap \mathbf{g}.\mathcal{L}_\gamma \neq \varnothing\}$

For an element $\mathbf{g} \in \Gamma$ we denote the conjugacy class of $\mathbf{g}$ by $\operatorname{Conj}(\mathbf{g}) = \{\mathbf{g}' \in \Gamma : \mathbf{g}' \text{ is conjugate to } \mathbf{g}\}$. If $\mathbf{g}$ is pseudo-Anosov, we denote the axis of $\mathbf{g}$ by $\mathcal{L}_{\mathbf{g}}$. We say that $\mathbf{g}$ is primitive if for every $\mathbf{h} \in \Gamma$ such that $\mathbf{g} = \mathbf{h}^n$, we have $n = 1$. If γ is a primitive pseudo-Anosov, then the above two sets are in one-to-one correspondence with

- $\{\mathbf{g} \in \Gamma : \mathbf{g} \in \operatorname{Conj}(\gamma), \text{ and } \mathfrak{B}(X, R) \cap \mathcal{L}_{\mathbf{g}} \neq \varnothing\}$

Therefore we have the following corollary.

Corollary C. *Let $\gamma \in \Gamma$ be a primitive pseudo-Anosov, and $X \in \mathcal{T}_g$. Then as $R \to \infty$,*

$$|\{\mathbf{g} \in \Gamma : \mathbf{g} \in \mathrm{Conj}(\gamma), \quad \text{and} \ \ \mathfrak{B}(X,R) \cap \mathcal{L}_{\mathbf{g}} \neq \varnothing\}| \sim \frac{\Lambda}{h\,\mathrm{Vol}(\mathcal{M}_g)} \nu(\mathcal{C}_{\mathrm{Ext},\gamma}) e^{hR}.$$

Note that when γ is replaced by γ^k for an integer $k \in \mathbb{Z}$, the left hand side of the above equation remains fixed, while the right hand side gets multiplied by $|k|$.

1.2 Remarks and the relation to other works

Before explaining the idea of the proof, let us mention the history of the problem and make a few remarks.

- Theorem A and Theorem B were proved in the setting of hyperbolic surfaces in [EM93]. To explain this, let Σ be a compact surface of constant negative curvature -1, and let Γ be its fundamental group. Fixing $\gamma \in \Gamma$, let ℓ_γ be the axis of its action on $\widetilde{\Sigma} \simeq \mathbb{D}$. Define $C_\gamma = \langle\gamma\rangle\backslash\widetilde{\Sigma}$, let $\pi_\gamma \colon \mathbb{D} \to C_\gamma$ be the corresponding covering map, and set $\overline{\ell}_\gamma = \pi_\gamma(\ell_\gamma)$. For a given $x \in \widetilde{\Sigma}$ Theorem 2.5 of [EM93] gives

$$|\pi_\gamma(\Gamma \cdot x) \cap \mathfrak{B}(\overline{\ell}_\gamma, R)| \sim \frac{\mathrm{Length}(\overline{\ell}_\gamma)}{\mathrm{Area}(\Sigma)} e^R, \qquad \text{as} \ \ R \to \infty,$$

where $\mathfrak{B}(\overline{\ell}_\gamma, R)$ is defined as before. A calculation in hyperbolic metric shows Theorem B in this setting, namely, as $R \to \infty$,

$$\mathrm{Area}\left(\mathfrak{B}(\overline{\ell}_\gamma, R)\right) \sim \mathrm{Length}(\overline{\ell}_\gamma) e^R.$$

- In the setting of manifolds of variable negative curvature, Theorem A can be obtained as a special case of *common perpendicular counting*, proved in [PP17]. To explain this, let M be a compact manifold of (variable) negative curvature, and $\pi \colon \widetilde{M} \to M$ be its covering map. Let Γ, γ, ℓ_γ, π_γ, $\overline{\ell}_\gamma$ be as above, and note that $\pi(\ell_\gamma)$ is a closed geodesic in M. Then $\pi_\gamma(\Gamma\cdot x) \cap \mathfrak{B}(\overline{\ell}_\gamma, R)$ is in one-to-one correspondence with $\mathrm{Perp}(\pi(x), \pi(\ell_\gamma), R)$, the perpendiculars from $\pi(x)$ to $\pi(\ell_\gamma)$ of length less than or equal R, where such a perpendicular is defined as a locally geodesic path that starts from $\pi(x)$ and arrives perpendicularly at $\pi(\ell_\gamma)$. It follwos from Theorem 1 of [PP17] that for some constant $c_\gamma > 0$,

$$|\pi_\gamma(\Gamma \cdot x) \cap \mathfrak{B}(\overline{\ell}_\gamma, R)| \sim c_\gamma e^{\delta R}, \qquad \text{as} \ \ R \to \infty.$$

Here, δ is the topological entropy of the geodesic flow on $T^1 M$, the unit tangent bundle of M. Under some additional conditions (for example, if M is a surface of variable negative curvature), an exponentially small error term is obtained for the above asymptotics in Theorem 3 of the same paper.

1.3 The outline of the proof

Theorem B, proved at the end of Section 5.1, follows by integrating the counting function of Theorem A over the moduli space $\mathcal{M}_g$. This is in fact how volume asymptotics are usually obtained from counting asymptotics. Hence the main task is to prove Theorem A. We adapt the notation introduced in Section 1.1. Define

$$\mathfrak{B}(\mathcal{L}_\gamma, R) = \{X \in \mathcal{T}_g : d(X, \mathcal{L}_\gamma) \le R\}.$$

Fix a point $P \in \mathcal{T}_g$, and note that $\Pi_\gamma(\Gamma \cdot P) \cap \mathfrak{B}(\overline{\mathcal{L}}_\gamma, R)$ is in one-to-one correspondence with

$$\langle\gamma\rangle\backslash\big(\Gamma \cdot P \cap \mathfrak{B}(\mathcal{L}_\gamma, R)\big) = \{\langle\gamma\rangle \cdot (\mathbf{g}.P) : d(\mathbf{g}.P, \mathcal{L}_\gamma) \le R\}. \tag{2}$$

Note that since $\Gamma \cdot P \cap \mathfrak{B}(\mathcal{L}_\gamma, R)$ is $\langle\gamma\rangle$–invariant, taking its quotient by $\langle\gamma\rangle$ is justified.

Fix a point $O \in \mathcal{L}_\gamma$, and for an $\epsilon > 0$ let $O = X_0$, X_1, ..., $X_N = \gamma.O$ be a sequence of consecutive points on the geodesic segment $[O, \gamma.O]$ such that $d(X_i, X_{i+1}) < \epsilon$, for $0 \le i < N$. This is called an ϵ–net in $[O, \gamma.O]$. Translating this net by the powers γ, we obtain a γ–invariant ϵ–net (..., X_{-1}, X_0, X_1, ...) of $\mathcal{L}_\gamma$. Define the map $\mathcal{P}\colon \mathcal{PMF} \to \mathbb{Z}$ by

$$\mathcal{P}[\zeta] = i \quad \text{if} \quad \operatorname{Ext}(\zeta, X_i) = \inf_{j\in\mathbb{Z}} \operatorname{Ext}(\zeta, X_j).$$

For $i \in \mathbb{Z}$, define $[\mathcal{A}_i] = \mathcal{P}^{-1}(i)$, and note that these sets form a γ–invariant partition of $\mathcal{PMF}$. Let $S(X_i, [\mathcal{A}_i], R)$ denote the sector of radius R centered at X_i and observing $[\mathcal{A}_i]$, namely, all the points $Y \in \mathcal{T}_g$ such that $d(X_i, Y) \le R$ and the geodesic connecting X_i to Y hits the boundary at an element of $[\mathcal{A}_i]$. (see 5.1 for a precise definition.)

The main geometric idea of this paper is that the sets $S(X_i, [\mathcal{A}_i], R)$ are almost disjoint and they almost cover all of $\mathfrak{B}(\mathcal{L}_\gamma, R)$. This, and the fact that $\gamma.S(X_i, [\mathcal{A}_i], R) = S(X_{i+N}, \mathcal{A}_{i+N}, R)$ implies that

$$\sum_{i=0}^{N-1} |\Gamma \cdot P \cap S(X_i, [\mathcal{A}_i], R)|$$

gives a good approximation for $|\langle\gamma\rangle\backslash(\Gamma \cdot P \cap \mathfrak{B}(\mathcal{L}_\gamma, R))|$. The asymptotics of $|\Gamma \cdot P \cap S(X_i, [\mathcal{A}_i], R)|$ as $R \to \infty$ is given in [ABEM12]. Summing up these asymptotics as the ϵ–net $(X_i)_{i\in\mathbb{Z}}$ in $\mathcal{L}_\gamma$ gets finer, namely, as $\epsilon \to 0$, we obtain the right hand side of (1).

For technical reasons, we need the boundary of the sets $[\mathcal{A}_i]$ to be of measure 0. This is a consequence of Proposition 3.1, proved in Section 3. Assuming this proposition, the rest of the paper can be read independently of Section 3. Section 4 is devoted to the statement and proof of Proposition 4.13, which is the main tool we use to compare extremal and Teichmüller lengths. In Section 5, we carry out the sector approximation scheme that we mentioned earlier. Both facts that the sets $S(X_i, [\mathcal{A}_i], R)$ are almost disjoint, and that they almost cover $\mathfrak{B}(\mathcal{L}_\gamma, R)$ are applications of Proposition 4.13.

Remark 1.1. Using the same method, Theorem A can be proved for an arbitrary compact set $\mathcal{K}_\gamma \subset \mathcal{C}_\gamma$ replacing $\overline{\mathcal{L}}_\gamma$. In this case $\mathcal{C}_{\mathrm{Ext},\gamma}$ should be replaced by a similarly defined subset of $\langle\gamma\rangle\backslash\mathcal{MF}$. The only change in the proof would be to replace the ϵ–net in $\mathcal{L}_\gamma$ by a γ–invariant ϵ–net in $\Pi_\gamma^{-1}(\mathcal{K}_\gamma)$. This is defined as $\Pi_\gamma^{-1}(S_\epsilon)$, where the finite set $S_\epsilon \subset \mathcal{K}_\gamma$ is such that the balls of radius ϵ around its elements cover $\mathcal{K}_\gamma$.

1.4 A formula for the derivative of extremal length

We end this introduction by stating a formula that we obtained in Section 3 in the course of proving Proposition 3.1. For $X \in \mathcal{T}_g$, let $\mathcal{Q}(X)$ denote the space of marked quadratic differentials based at X, and define the homeomorphism $\mathcal{V}_X \colon \mathcal{Q}(X) \to \mathcal{MF}$ by sending a quadratic differential to its vertical measured foliation. Fixing $\zeta \in \mathcal{MF}$, we can define $\operatorname{Ext}_\zeta \colon \mathcal{T}_g \to \mathbb{R}$ by $\operatorname{Ext}_\zeta(X) = \operatorname{Ext}(\zeta, X)$. This function is differentiable, and its derivative is given by the Gardiner's formula ([Gar84]) to be

$$\mathrm{d}_X \operatorname{Ext}_\zeta(\mu) = 2\Re\langle \mathcal{V}_X^{-1}(\zeta), \mu\rangle.$$

Here, $\mu \in \mathrm{T}_X(\mathcal{T}_g)$, the tangent space to $\mathcal{T}_g$ at X, is a Beltrami differential, and

$$\langle \mathcal{V}_X^{-1}(\zeta), \mu\rangle = \int_X \mu.\mathcal{V}_X^{-1}(\zeta)$$

is the usual inner product between a quadratic differential and a Beltrami differential.

If, instead of fixing a measured foliation, we fix an element X of the Teichmüller space, we can define $\operatorname{Ext}_X \colon \mathcal{MF} \to \mathbb{R}$ by

$$\operatorname{Ext}_X(\zeta) = \operatorname{Ext}(\zeta, X).$$

In order to compute the derivative of Ext_X, we need a differential structure on $\mathcal{MF}$. However, $\mathcal{MF}$, equipped with train-track charts, is only piecewise linear. Despite this, if ζ is generic, meaning that it does not have a leaf connecting any two of its singularities and all of its singularities are simple, then $\mathcal{MF}$ is smooth at ζ (in the sense of Definition 3.3). For such a ζ, we denote the tangent space to $\mathcal{MF}$ at ζ by $\mathrm{T}_\zeta \mathcal{MF}$, and we denote the Thurston symplectic form on $\mathrm{T}_\zeta \mathcal{MF}$ by ω_{Th}. The derivative of Ext_X at the generic points of $\mathcal{MF}$ is given by the following theorem.

Theorem D. *Fix $X \in \mathcal{T}_g$, and let $\zeta \in \mathcal{MF}$ be generic. Then Ext_X is smooth at ζ, and there exists an element $\alpha \in \mathrm{T}_\zeta \mathcal{MF}$ such that*

$$\mathrm{d}_\zeta \operatorname{Ext}_X(\cdot) = \omega_{\mathrm{Th}}(\alpha, \cdot)$$

This is proved as Theorem 3.10. This theorem, moreover, gives an explicit construction of α in certain train-track charts around ζ.

Notation. Throughout this paper, for real numbers a, b, c, we write $a \simeq_c b$ if $|a-b| < c$.

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2 Background on Teichmüller space

Teichmüller space. Let S_g be a compact oriented topological surface of genus $g \geq 2$. The genus g and the surface S_g are fixed throughout the paper. For a Riemann surface X, an orientation preserving homeomorphism $f \colon S_g \to X$ is called a *marking*. Let

$f_i\colon S_g \to X_i$, $i = 1, 2$ be two markings. f_1 and f_2 are called *equivalent*, if there exists a biholomorphism $h\colon X_1 \to X_2$ such that $h \circ f_1$ is isotope to f_2. The Teichmüller space of S_g, denoted by $\mathcal{T}_g$, is defined as the space of all markings up to the equivalence mentioned above. For two elements $[f_i\colon S_g \to X_i]$, $i = 1, 2$ of $\mathcal{T}_g$, the *change of marking* from $[f_1]$ to $[f_2]$ is defined to be the isotopy class of $f_2 \circ f_1^{-1}\colon X_1 \to X_2$. From now on, we denote an element $[f\colon S_g \to X]$ of $\mathcal{T}_g$ by X, and keep the marking in the back of our mind.

The mapping class group (or modular group) of S_g is denoted by Γ, and is defined to be the group of orientation preserving homeomorphisms of S_g up to isotopy. An element $\mathbf{g} = [\varphi\colon S_g \to S_g]$ of the mapping class group acts on $[f\colon S_g \to X] \in \mathcal{T}_g$ by change of marking, namely, $\mathbf{g}.[f] = [f \circ \varphi^{-1}]$. Note that we defined the action in a way that for all $X \in \mathcal{T}_g$ and $\mathbf{g}, \mathbf{h} \in \Gamma$ we have $(\mathbf{g}.\mathbf{h}).X = \mathbf{g}.(\mathbf{h}.X)$. Taking the quotient of $\mathcal{T}_g$ by Γ, we obtain the moduli space $\mathcal{M}_g = \Gamma\backslash\mathcal{T}_g$, hence the elements of $\mathcal{M}_g$ are of the form $\Gamma \cdot X$ for $X \in \mathcal{T}_g$.

Quadratic differentials. For a Riemann surface X, we denote the space of holomorphic quadratic differentials (or quadratic differentials for short) on X by $\mathcal{Q}(X)$. For a quadratic differential $q \in \mathcal{Q}(X)$, define the norm of q to be

$$|q| = \int_X |q(z)||dz|^2.$$

We define $\mathcal{QT}_g$ to be the space of marked quadratic differentials, more precisely, this is the space of equivalence classes $[f\colon S_g \to X,\, q \in \mathcal{Q}(X)]$, where the equivalence relation is the natural one. We denote $[f]$ by (X, q), or sometimes only by q. Sending (X, q) to X gives a projection map $\pi\colon \mathcal{QT}_g \to \mathcal{T}_g$. The principal stratum is defined to be the subset of $\mathcal{QT}_g$ consisting of quadratic differentials with only simple zeros, and is denoted by $\mathcal{QT}_g(\mathbf{1})$.

Let $q \in \mathcal{Q}(X)$ be a quadratic differential. A flat chart for q is a holomorphic chart from the complex plane $\mathbb{C}$ to X, on which the pullback of q is dz^2. The change of coordinates between two flat charts is of the form $z \to \pm z + c$ for some $c \in \mathbb{C}$. Denoting the zeros of q by Σ_q, we can cover $X \smallsetminus \Sigma_q$ by flat charts, giving X the structure of a flat surface (or a half-translation surface). Note that $q \in \mathcal{QT}_g(\mathbf{1})$ if and only if the angle around every element of Σ_q is 3π.

The group $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathcal{QT}_g$ by acting by matrix multiplication on the corresponding flat charts. More precisely, let $q \in \mathcal{QT}_g$ and $A \in \mathrm{SL}_2(\mathbb{R})$. Then q gives an atlas of flat charts, $\varphi_i\colon U_i \to V_i$, $i \in I$, from $\mathbb{C}$ to S_g. $A.q$ is defined to be the quadratic differential given by the charts $\varphi_i \circ A^{-1}\colon A.U_i \to V_i$. Note that we defined this action in a way that $A.(B.q) = (A.B).q$ for $A, B \in \mathrm{SL}_2(\mathbb{R})$. With this definition, the Teichmüller flow $g_t\colon \mathcal{QT}_g \to \mathcal{QT}_g$ is given by

$$g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

For every $X \in \mathcal{T}_g$, the cotangent space to $\mathcal{T}_g$ at X is naturally identified with $\mathcal{Q}(X)$, hence the norm on $\mathcal{Q}(X)$ induces a norm on the tangent space to $\mathcal{T}_g$ at X, denoted by $\mathrm{T}_X\,\mathcal{T}_g$. This is proved to be a Finsler metric on $\mathcal{T}_g$, and the resulting distance is called the Teichmüller distance. We denote the distance between two points $X, Y \in \mathcal{T}_g$ by $d(X, Y)$, and show the geodesic connecting X to Y by $[X, Y]$. Given $X \in \mathcal{T}_g$ and $\zeta \in \mathcal{MF}$, by

[HM79] there exists a unique $q \in \mathcal{Q}(X)$ such that q has ζ as its vertical measured foliation. We define the geodesic ray connecting X to ζ by

$$[X, \zeta) = \{\pi(g_t.q),\, 0 \leq t < \infty\}.$$

Extremal length. Given $X \in \mathcal{T}_g$, let $\mathcal{V}_X\colon\ \mathcal{Q}(X)\smallsetminus\{0\} \to \mathcal{MF}$ be the function that sends a nonzero quadratic differential to its vertical measured foliation. By [HM79], $\mathcal{V}_X$ is a homeomorphism. We can define the extremal length of $\zeta \in \mathcal{MF}$ at $X \in \mathcal{T}_g$ by

$$\mathrm{Ext}(\zeta, X) = |\mathcal{V}_X^{-1}(\zeta)|. \tag{3}$$

To see that this coincides with the usual definition of extremal length, see [Ker80].

Let $X, Y \in \mathcal{T}_g$ and $\zeta \in \mathcal{MF}$. The Busemann functions are defined by

$$\beta(\zeta, X) = \frac{1}{2}\log \mathrm{Ext}(\zeta, X);$$
$$\beta(\zeta, X, Y) = \beta(\zeta, Y) - \beta(\zeta, X).$$

For a measured foliation ζ, we denote the projective class of ζ by $[\zeta] \in \mathcal{PMF}$. Since for $\lambda > 0$ we have $\beta(\lambda.\zeta, X) = \beta(\zeta, X) + \log\lambda$, the value of $\beta(\zeta, X, Y)$ only depends on the projective class of ζ. This common value is denoted by $\beta([\zeta], X, Y)$.

For X, Y and ζ as before, Kerschoff inequality ([Ker80]) states that

$$\frac{\mathrm{Ext}(\zeta, Y)}{\mathrm{Ext}(\zeta, X)} \leq e^{2d(X,Y)}.$$

Taking logarithms, we obtain

$$\beta(\zeta, Y) \leq \beta(\zeta, X) + d(X, Y).$$

The equality holds if and only if Y, X and ζ appear in this order on a geodesic line, or equivalently, $X \in [Y, \zeta)$. If we think of $\beta(\zeta, X)$ as the "length at infinity" of $[X, \zeta)$, the above can be thought of as the triangle inequality in $\Delta(\zeta, X, Y)$. (note that, unlike the usual length, $\beta(\zeta, X)$ can be negative.)

3 Equidistant measured foliations are negligible

Given $X, Y \in \mathcal{T}_g$, define $\mathcal{E}_{X,Y} \subset \mathcal{MF}$ by

$$\mathcal{E}_{X,Y} = \{\zeta \in \mathcal{MF} : \mathrm{Ext}(\zeta, X) = \mathrm{Ext}(\zeta, Y)\}.$$

The goal of this section is to prove the following.

Proposition 3.1. *Let $X, Y \in \mathcal{T}_g$ be distinct. Then $\mathcal{E}_{X,Y}$ is of Thurston measure zero.*

In Section 3.1 we recall some background on train tracks. Train tracks are important for us since they give nice charts on $\mathcal{MF}$. In Section 3.2 we compute the derivative of $\mathrm{Ext}(\boldsymbol{\cdot}, X)\colon\ \mathcal{MF} \to \mathbb{R}$ in certain train track charts. Finally, we prove Proposition 3.1 at the end of Section 3.3.

3.1 Background on train tracks

3.1.1 Definitions and notation

A train track on S_g is an embedded 3–regular graph in S_g such that its vertices are locally modeled on Figure 1. The vertices of this graph are called *switches* and the edges are called *branches* of the train track. In the same figure, the branch a is called an *outgoing* branch and b, c are called *incoming* branches. A branch is said to be *large* if it is the outgoing branch for both of its endpoints. A train track τ is said to be *complete* if all the components of $S_g - \tau$ are cusped triangles.

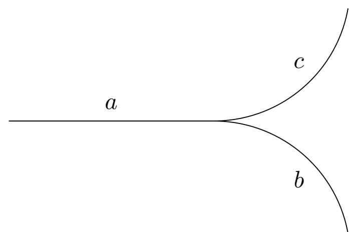

Figure 1: A switch

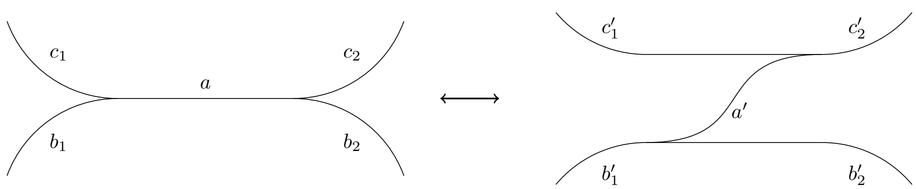

Figure 2: A splitting along the branch a, when $\mu(c_2) > \mu(c_1)$.

We denote the set of vertices of a train track τ by $V(\tau)$, and the set of edges of this train track by $E(\tau)$. A function $w\colon E(\tau) \to \mathbb{R}$ is called a *weight* on τ, if we have $w(a) = w(b) + w(c)$ for every switch as in Figure 1. Let $W(\tau)$ denote the set of all weights on τ. This is a linear subspace of $\mathbb{R}^{E(\tau)}$, cut out by $\#V(\tau)$ equations. A weight $w \in W(\tau)$ is said to be *positive*, or a *measure* on the train track τ, if $w(e) > 0$ for all the branches e of τ. We denote the set of all measures on τ by $W^+(\tau)$. A train track τ with a measure μ on it is denoted by the pair (τ, μ), and is called a *measured train track.*

For a measured train track (τ, μ), a *splitting* along a large branch a is shown in Figure 2 . Note that the branch a splits differently according to whether $\mu(c_1)$ is greater, less than, or equal to $\mu(c_2)$. A *collision* is the reverse of a splitting, i.e., going from the right hand side of Figure 2 to the left hand side.

Given a measured train track (τ, μ), we can foliate a rectangular neighborhood of τ, and define a transverse measure on it according to μ. Shrinking the components of the complement of this neighborhood, we get a measured foliation, denoted by $\mathcal{F}_\tau(\mu)$. A measured foliation ζ is said to be *carried* by (τ, μ) if $\zeta = \mathcal{F}_\tau(\mu)$. ζ is said to be carried by τ if there exists a measure $\mu \in W^+(\tau)$ such that ζ is carried by (τ, μ). We denote the set of all measured foliations carried by τ by $\mathcal{U}(\tau)$. If τ is *bi-recurrent*, then

$$\mathcal{F}_\tau\colon\ W^+(\tau) \to \mathcal{U}(\tau)$$

is a homeomorphism ([PH92] Theorem 1.7.12). If τ is bi-recurrent and complete, then $W^+(\tau)$ is of full dimension $6g-6$, then $\mathcal{U}(\tau)$ is an open subset of $\mathcal{MF}$, and $\mathcal{F}_\tau$ is called a *train track chart.*

For a measured foliation $\zeta \in \mathcal{MF}$ the following are equivalent:

- All the leaves of ζ are dense.
- ζ has only simple singularities and does not have a leaf connecting any two of its singularities. (or a leaf connecting a singularity to itself)
- Every train track carrying ζ is complete.

A measured foliation is called *generic* if it satisfies one (and hence all) of the above conditions. The set of all generic measured foliations is denoted by $\mathrm{Gen}(S_g)$. Observe that in each train track chart, the nongeneric measured foliations are cut out by the union of countably many linear subspaces of codimension one. Since we can cover $\mathcal{MF}$ by finitely many train track charts, we deduce that nongeneric measured foliations are of Thurston measure 0, or in other words, $\mathrm{Gen}(S_g)$ has full measure. Note that despite being measure 0, nongeneric measured foliations form a dense subset of $\mathcal{MF}$. (consider the rational multiples of simple closed multicurves.)

For two train track charts $\mathcal{F}_{\tau_i}\colon\ W^+(\tau_i) \to \mathcal{U}(\tau_i)$, $i = 1, 2$, define

$$\mathcal{F}_{\tau_2\tau_1} = \mathcal{F}_{\tau_2}^{-1} \circ \mathcal{F}_{\tau_1},$$

and denote the domain of $\mathcal{F}_{\tau_2\tau_1}$ by $W^+(\tau_1, \tau_2)$. This is a subset of $W^+(\tau_1)$. The set $W^+(\tau_2, \tau_1) \subset W^+(\tau_2)$ can be defined similarly. By [PH92] Theorem 3.1.4, the change of coordinates map

$$\mathcal{F}_{\tau_2\tau_1}\colon\ W^+(\tau_1, \tau_2) \to W^+(\tau_2, \tau_1)$$

is piecewise linear. This means that there exists an integer $N = N(\tau_1, \tau_2)$, and a pair of partitions $\mathcal{P}_1$ of $W^+(\tau_1, \tau_2)$ and $\mathcal{P}_2$ of $W(\tau_2, \tau_1)$ into polytopes, both of cardinality N, such that the restriction of $\mathcal{F}_{\tau_2\tau_1}$ to each of the polytopes in $\mathcal{P}_1$ is a linear isomorphism to a polytope in $\mathcal{P}_2$. Note that by a polytope in a linear space V, we mean the intersection of finitely many half-spaces in V. It follows from the proof of Theorem 3.1.4 of [PH92] that the faces of the polytopes in $\mathcal{P}_1$ lie in $\mathcal{F}_{\tau_1}^{-1}(\mathcal{MF} \setminus \mathrm{Gen}(S_g))$. Hence we have the following lemma.

Lemma 3.2. *Let τ_1 and τ_2 be bi-recurrent complete train tracks, and assume that $\zeta \in \mathcal{U}(\tau_1) \cap \mathcal{U}(\tau_2)$ is a generic measured foliation. Then $\mathcal{F}_{\tau_2}^{-1} \circ \mathcal{F}_{\tau_1}$ is linear in a neighborhood of $\mathcal{F}_{\tau_1}^{-1}(\zeta)$.*

Inspired by this lemma, we make the following definition.

Definition 3.3. Let M be a topological manifold given by the charts

$$\varphi_i\colon\ U_i \to V_i, \quad U_i \subset \mathbb{R}^n,\ V_i \subset M,$$

for i belonging to some index set I. M is said to be *smooth* at a point $x \in M$ if the transition maps are smooth near x. This means that for all indices $i, j \in I$ such that $x \in V_i \cap V_j$, the change of coordinates map

$$\varphi_{ji} = \varphi_j^{-1} \circ \varphi_i\colon\ U_{ij} \to U_{ji}$$

is smooth on a neighbourhood of $\varphi_i^{-1}(x)$, where U_{ij} is the domain of definition of φ_{ji}.

If a manifold M is smooth at x, the tangent space to M at x, denoted by $\mathrm{T}_x M$, can be defined in the usual way. We can define a manifold to be *linear* or *analytic* at a point in a similar way. With these definitions, we have the following corollary of Lemma 3.2.

Corollary 3.4. *The train track charts on $\mathcal{MF}$ give it the structure of a topological manifold that is linear at all the points in* $\mathrm{Gen}(S_g)$*.*

Fix a train track τ, and for $w_1, w_2 \in W(\tau)$ define

$$\omega_\tau(w_1, w_2) = \frac{1}{2} \sum_{v \in V(\tau)} \det \begin{pmatrix} w_1(b_v) & w_1(c_v) \\ w_2(b_v) & w_2(c_v) \end{pmatrix}, \tag{4}$$

where at each switch v of τ, the outgoing branch is labeled by a_v, and the incoming branches are labeled by b_v, c_v, in a way that $a_v b_v c_v$ is counter-clockwise. (see Figure 1.) With this definition, ω_τ gives a nondegenarate antisymmetric bilinear pairing on $W(\tau)$. Since $W(\tau)$ is a vector space, $\mathrm{T}_\mu W^+(\tau)$ is naturally identified with $W(\tau)$ for $\mu \in W^+(\tau)$, hence (4) gives an antisymmetric form on $W^+(\tau)$, denoted by ω_τ as well.

Let τ_1 and τ_2 be complete bi-recurrent train tracks. Adapting the notation introduced in the discussion before Lemma 3.2, it can be proved that the restriction of $\mathcal{F}_{\tau_2\tau_1}$ to the interior of each polytope in $\mathcal{P}_1$ sends ω_{τ_1} to ω_{τ_2}. (See [PH92] Theorem 3.2.4.) Hence, gluing together the forms ω_τ for different train tracks τ, we obtain a bilinear pairing on $\mathrm{T}_\zeta\mathcal{MF}$ for $\zeta \in \mathrm{Gen}(S_g)$. This is called the *Thurston symplectic form*, and is denoted by ω_{Th}.

3.1.2 Train tracks and flat structures

Let X be a Riemann surface and q a quadratic differential on X. Denoting the set of zeros of q by Σ_q, we define a *saddle triangulation* (or a *triangulation* for short) of q to be a triangulation Δ of X such that the vertices of the triangles belong to Σ_q, and the edges are straight lines in the flat metric induced by q. Such a triangluation always exists by [MS91]. For a triangle $ABC \in \Delta$, a *comparison triangle* is defined as a flat model of ABC, namely, this is a Euclidean triangle $A'B'C'$ in the plane, together with a flat chart $\varphi\colon\ A'B'C' \to ABC$ that sends A' to A, B' to B and C' to C. (note that φ has a continuous extension to the edges of $A'B'C'$.) The comparison triangle for a given triangle in Δ is unique up to translation and reflection from the origin.

We say that a triangulation Δ of q is *non-vertical* if none of the triangles in Δ have a vertical side. Given a non-vertical triangulation Δ of q, we can turn the dual graph of Δ into a measured train track in the following way. Let ABC be a triangle in Δ, and let $A'B'C'$ be its comparison triangle. Since $A'B'C'$ does not have a vertical side, up to relabeling the vertices, we may assume

$$\Re(\overrightarrow{B'C'}),\ \Re(\overrightarrow{B'A'}),\ \Re(\overrightarrow{A'C'}) > 0.$$

Denoting the edges dual to the sides BC, AC, AB of the triangle ABC by a, b, c respectively, we announce a to be the outgoing edge, and b, c to be incoming edges, and define

$$\mu(a) = \Re(\overrightarrow{B'C'}),\ \mu(c) = \Re(\overrightarrow{B'A'}),\ \mu(c) = \Re(\overrightarrow{A'C'}).$$

(τ, μ) constructed in this way is a measured train track that carries $\mathcal{V}(q)$. This is called the train track *dual* to Δ. In the case that Δ contains triangles with vertical sides, the dual measured train track can be defined by first removing those edges of the dual graph that are dual to the vertical sides, and then defining the measure on the remaining edges as before.

Let Δ be an arbitrary triangulation of q, and assume that BC is the common side of two triangles ABC, $DBC \in \Delta$. We define a *flip* along BC in Δ to be a triangulation Δ' obtained from Δ by replacing the triangles $\{ABC, DBC\}$ by $\{ABD, ACD\}$. Let (τ, μ) and (τ', μ') be the train tracks dual to Δ and Δ', respectively. Denoting the branch of τ that is dual to BC by e, we can see that (τ', μ') is obtained from (τ, μ) by either a split, collision or *shift* along e. (see [PH92] Section 2.1 for the definition of shift.) The situation where a splitting (or collision) happens is shown in Figure 3. The converse to this is also true, namely, we have the following lemma.

Lemma 3.5. *Let q be a quadratic differential, Δ a triangulation of q, and (τ, μ) the train track that is dual to Δ. Then every split, collision or shift in (τ, μ) corresponds to a flip in Δ, and vice versa.*

Recall that, starting with a Riemann surface X, a quadratic differential q on X, and a triangulation Δ of q, we arrived at (τ, μ), the dual of Δ, which is a train track on X. If in addition to this data, we have a marking $f\colon S_g \to X$ as well, then pulling back (τ, μ) by this marking, we obtain a train track on the topological surface S_g, which is also called the train track dual to Δ. Note that if f changes in its isotopy class, the pull-back of (τ, μ) also changes by isotopy. Thus, for a (equivalence class of) marked quadratic differential $[f\colon S_g \to X,\ q \in \mathcal{Q}(X)]$, and a triangulation Δ of q, we can think of the dual of Δ as an isotopy class of train tracks on S_g.

3.2 A formula for the derivative of Ext_X

In this section, given $X \in \mathcal{T}_g$, we compute the derivative of $\mathrm{Ext}(X, \zeta)$ considered as a function of $\zeta \in \mathcal{MF}$. (see theorem 3.10.) To simplify the notation, we denote $\mathrm{Ext}(\zeta, X)$ by $\mathrm{Ext}_X(\zeta)$ throughout this section and Section 3.3. Define

$$N_X\colon\ \mathcal{Q}(X) \to \mathbb{R}, \quad \text{by} \quad N_X(q) = |q|.$$

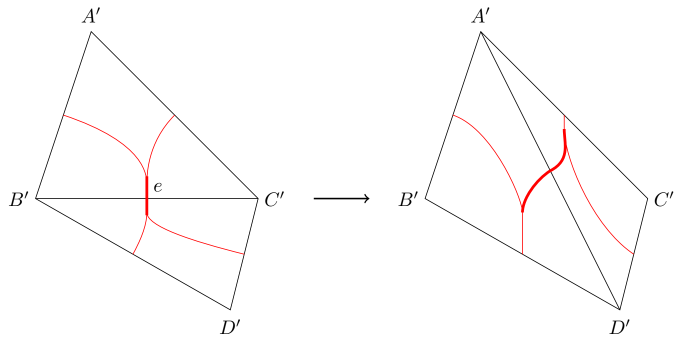

Figure 3: If e is large, then a flip along BC corresponds to a splitting along e.

The idea behind computing the derivative of Ext_X is to write Equation (3) as

$$\operatorname{Ext}_X = N_X \circ \mathcal{V}_X^{-1}.$$

The derivative of N_X is given in [Roy71], and the derivative of $\mathcal{V}_X$ can be computed as a consequence of Douady-Hubbard formula (see Lemma 3.9). The chain rule then gives a formula for the derivative of Ext_X. Using the results of Section 5 of [Dum15], we can write this derivative in a more compact form (see Equation 6).

Note that $\mathcal{MF}$, equipped with train track charts, is not a smooth manifold, however, by Corollary 3.4, it is smooth on its full measure subset $\operatorname{Gen}(S_g)$. Hence, in order to compute the derivative of Ext_X, it makes sense to restrict ourselves to these points. Inspired by this situation, we make the following definition:

Definition 3.6. Let M and φ_i, $i \in I$, be as in Definition 3.3. A function $f\colon M \to \mathbb{R}$ is said to be *smooth* at a smooth point $x \in M$, if $f \circ \varphi_i$ is smooth in a neighborhood of $\varphi_i^{-1}(x)$ for a chart φ_i that covers x.

A *linear* (*analytic*) function at a linear (analytic) point can also be defined in a similar way. If f is smooth at x, we can define the derivative of f at x, $\mathrm{d}_x f\colon \mathrm{T}_x M \to \mathbb{R}$, in the usual way.

Fix $X \in \mathcal{T}_g$, and define

$$\mathcal{Q}_{\mathbf{1}}(X) = \mathcal{Q}(X) \cap \mathcal{QT}_g(\mathbf{1})$$

to be the quadratic differentials on X that have only simple zeros. For $q \in \mathcal{Q}_{\mathbf{1}}(X)$, and $\phi, \psi \in \mathcal{Q}(X)$ define

$$\omega_q(\phi, \psi) = \frac{1}{4}\Im\Big(\int_X \frac{\phi\bar{\psi}}{|q|}\Big).$$

For $q \in \mathcal{Q}_{\mathbf{1}}(X)$, identifying $\mathrm{T}_q\,\mathcal{Q}(X)$ with $\mathcal{Q}(X)$, ω_q gives an antisymmetric pairing on $\mathrm{T}_q\,\mathcal{Q}(X)$. Varying q in $\mathcal{Q}_{\mathbf{1}}(X)$, we obtain a 2–from on $\mathcal{Q}_{\mathbf{1}}(X)$, which we denote by ω_X. By [Dum15] Theorem 5.3, ω_X is symplectic. Moreover, we have the following theorem, which is a consequence of [Dum15] Theorem 5.8.

Theorem 3.7. *$\mathcal{V}_X$ gives a symplectomorphism from $(\mathcal{Q}(X), \omega_X)$ to $(\mathcal{MF}, \omega_{\mathrm{Th}})$, at the points where both of these forms are defined.*

Lemma 3.8. *The function N_X restricted to $\mathcal{Q}_{\mathbf{1}}(X)$ is smooth, and its derivative at a point $q \in \mathcal{Q}_{\mathbf{1}}(X)$ is given by*

$$\mathrm{d}_q N_X(\boldsymbol{\cdot}) = 4\omega_q(iq, \boldsymbol{\cdot}).$$

Proof. The fact that N_X is smooth on $\mathcal{Q}_{\mathbf{1}}(X)$ follows from the proof of Theorem 5.3 of [Dum15]. In that proof, in a neighborhood of $q \in \mathcal{Q}_{\mathbf{1}}(X)$, $N = N_X$ is written as $N = N_0^\epsilon + N_1^\epsilon$, and it is proved that both N_0^ϵ and N_1^ϵ are smooth on this neighborhood.

The derivative of N_X is computed in [Roy71] Lemma 1. □

Lemma 3.9 computes the derivative of $\mathcal{V}_X$ in certain train track charts on $\mathcal{MF}$. For this computation we need the following construction:

Construction. Fix $X \in \mathcal{T}_g$, a quadratic differntial $q \in \mathcal{Q}(X)$, and a non-vertical triangulation Δ of q. Let (τ, μ) be the train track dual to Δ. For every $\phi \in \mathcal{Q}(X)$, define

$$w_\Delta(\phi)\colon\ E(\tau) \to \mathbb{R}$$

as follows. If the branch e of τ is dual to the side AB of a triangle in Δ, set

$$w_\Delta(\phi)(e) = \frac{1}{2}\Re\Big(\int_A^B \frac{\phi}{\sqrt{q}}\Big), \tag{5}$$

where the integral is taken over the saddle connection AB, and we use the branch of $\sqrt{q}$ that makes $\Re(\int_A^B \sqrt{q})$ positive. It can be checked that $w_\Delta(\phi)$ is a weight on τ, hence w_Δ is a linear function from $\mathcal{Q}(X)$ to $W(\tau)$. For the next lemma, recall the definition of train track charts $\mathcal{F}_\tau\colon\ W^+(\tau) \to \mathcal{MF}$, given in Section 3.1.

Lemma 3.9. *Fix $X \in \mathcal{T}_g$ and a quadratic differntial $q \in \mathcal{Q}(X)$ such that $\mathcal{V}_X(q)$ is generic. Let Δ be a triangulation of q, and let (τ, μ) be the train track dual to Δ. Then $\mathcal{F}_\tau^{-1} \circ \mathcal{V}_X$ is real analytic in a neighborhood of q, and its derivative at q,*

$$\mathrm{D}_q(\mathcal{F}_\tau^{-1} \circ \mathcal{V}_X)\colon\ \mathrm{T}_q\,\mathcal{Q}(X) \to \mathrm{T}_\mu\,W^+(\tau),$$

is given by w_Δ. (Note that we identified $\mathrm{T}_q\,\mathcal{Q}(X)$ and $\mathrm{T}_\mu\,W^+(\tau)$ with $\mathcal{Q}(X)$ and $W(\tau)$ respectively).

Proof. Note that since $\mathcal{V}_X(q)$ is generic, $q \in \mathcal{Q}_{\mathbf{1}}(X)$ and Δ is non-vertical (i.e., none of the triangles in Δ have a vertical side). First, we give a description of the function $\mathcal{F}_\tau^{-1} \circ \mathcal{V}_X$ in a neighborhood of q. If $q' \in \mathcal{Q}(X)$ is near q, the zero set of q' is near the zero set of q, hence we can choose a triangulation of q', denoted by $\Delta(q')$, that is close to $\Delta = \Delta(q)$. Let $(\tau(q'), \mu(q'))$ be the measured train track dual to $\Delta(q')$, hence $\mathcal{V}_X(q')$ is carried by $(\tau(q'), \mu(q'))$. Since $\Delta = \Delta(q)$ is non-vertical, $\tau(q')$ can be moved slightly to coincide with $\tau = \tau(q)$. As a result, $\mu(q')$ can be conisdered as a measure on τ, i.e., an element of $\mathcal{W}^+(\tau)$. Since $(\tau, \mu(q'))$ carries $\mathcal{V}_X(q')$, we have $\mathcal{F}_\tau(\mu(q')) = \mathcal{V}_X(q')$, or

$$\mathcal{F}_\tau^{-1} \circ \mathcal{V}_X(q') = \mu(q').$$

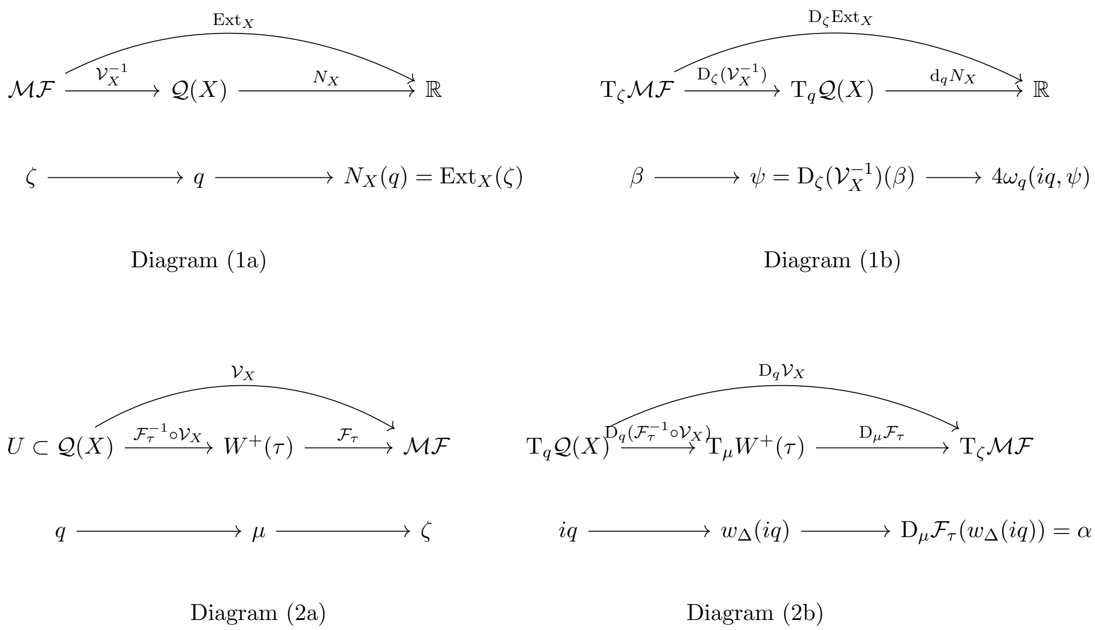

Figure 4: Diagrams for Theorem 3.10.

Now, let A and B be two of the zeros of q such that AB is the side of a triangle in Δ, and assume e is the branch of τ that is dual to AB. For q' near q, denote the zeros of q' that are close to A and B by $A(q')$ and $B(q')$ respectively. Then, by the definition of dual train track, we have

$$\mu(q')(e) = \Re\Big(\int_{A(q')}^{B(q')} \sqrt{q'} \Big),$$

where, as before, the integral is taken over the saddle connection in q' that connects $A(q')$ to $B(q')$, and the sign for $\sqrt{q'}$ is chosen so that $\Re\left(\int_{A(q')}^{B(q')} \sqrt{q'}\right) > 0$.

Recall that for a small enough neighborhood U of q, $\mathrm{P}_{AB}\colon U \to \mathbb{C}$ given by

$$\mathrm{P}_{AB}(q') = \int_{A(q')}^{B(q')} \sqrt{q'}$$

is called a *period function*, and its derivative at $q \in U$ is given by Douady-Hubbard formula ([DH75], Proposition 1) to be

$$\mathrm{d}_q \, \mathrm{P}_{AB}(\phi) = \frac{1}{2}\int_A^B \frac{\phi}{\sqrt{q}}, \quad \text{for } \phi \in \mathrm{T}_q \, \mathcal{Q}(X).$$

This completes the proof of the lemma. □

Theorem 3.10. *Fix $X \in \mathcal{T}_g$, and let $\zeta \in \mathcal{MF}$ be a generic measured foliation. Then* Ext_X *is real analytic at ζ, and there exists $\alpha = \alpha(X, \zeta) \in \mathrm{T}_\zeta \, \mathcal{MF}$ such that*

$$\mathrm{d}_\zeta \, \mathrm{Ext}_X(\bullet) = 4\omega_{\mathrm{Th}}(\alpha, \bullet). \tag{6}$$

To find a formula for α, let $q = \mathcal{V}_X^{-1}(\zeta)$, fix an arbitrary triangulation Δ of q, and let (τ, μ) be the train track dual to Δ. then α can be computed in the train track chart $\mathcal{F}_\tau$ to be $w_\Delta(iq)$. More precisely, we have

$$\alpha = \mathrm{D}_\mu \, \mathcal{F}_\tau\big(w_\Delta(iq)\big). \tag{7}$$

Proof. In Diagram (1a), $\mathrm{Ext}_X = N_X \circ \mathcal{V}_X^{-1}$ because of Equation (3). Taking the derivative of Diagram (1a), we obtain Diagram (1b). The upper row in Diagram (1b) commutes by chain rule, and the lower right arrow is by Lemma 3.8. In the same diagram, β is an arbitrary element of $\mathrm{T}_\zeta \, \mathcal{MF}$, and ψ is defined to be $\mathrm{D}_\zeta(\mathcal{V}_X^{-1})(\beta)$. by Theorem 3.7, $\mathcal{V}_X$ is a symplectomorphism from $(\mathcal{Q}(X), \omega_X)$ to $(\mathcal{MF}, \omega_{\mathrm{Th}})$, hence we have

$$\omega_q(iq, \psi) = \omega_{\mathrm{Th}}\big(\, \mathrm{D}_q \, \mathcal{V}_X(iq), \mathrm{D}_q \, \mathcal{V}_X(\psi)\big).$$

Set $\alpha = \mathrm{D}_q \, \mathcal{V}_X(iq)$, and note that, by the definition of ψ we have $\beta = \mathrm{D}_q \, \mathcal{V}_X(\psi)$, hence we have

$$\omega_q(iq, \psi) = \omega_{\mathrm{Th}}(\alpha, \beta).$$

Since the upper row of Diagram (1b) commutes, we have

$$\mathrm{D}_\zeta \, \mathrm{Ext}_X(\beta) = 4\omega_q(iq, \psi).$$

This proves the first statement of the theorem.

Now we use Diagrams (2a) and (2b) to find the stated formula for α. Let Δ and its dual train track (τ, μ) be as in the statement of the theorem. The lower row of Diagram (2a) holds since (τ, μ) carries $\zeta = \mathcal{V}_X(q)$. Taking the derivative of this diagram, we obtain Diagram (2b). The lower left arrow in this diagram is by Lemma 3.9. The upper row of this diagram commutes by chain rule, hence we have

$$\alpha = \mathrm{D}_q \, \mathcal{V}_X(iq) = \mathrm{D}_\mu \, \mathcal{F}_\tau\big(w_\Delta(iq)\big),$$

which is what we wanted. □

3.3 Proof of Proposition 3.1

To prove Proposition 3.1, we need three preliminary lemmas. The proof of the proposition is given at the end of this section. The first lemma is an immediate consequence of the defining equation for w_Δ (Equation 5).

Lemma 3.11. *Let $X \in \mathcal{T}_g$, $q \in \mathcal{Q}(X)$, and Δ be a non-vertical triangluation of q. Denote the train track dual to Δ by (τ, μ). For an arbitrary triangle ABC in Δ, let $A'B'C'$ be its flat comparison triangle, and e be the branch of τ that is dual to AB. By reflecting $A'B'C'$ through the origin if necessary, we can assume $\Re(\overrightarrow{A'B'}) > 0$. Under this assumption, we have*

$$w_\Delta(iq)(e) = -\frac{1}{2} \Im(\overrightarrow{A'B'}).$$

By the discussion at the end of Section 3.1.2, the dual train track to a triangulation of a marked quadratic differential can be treated as an isotopy class of measured train tracks on the topological surface S_g. This is how we treat the dual train track in the following lemma.

Lemma 3.12. *Let X, q, Δ, τ, μ be as in Lemma 3.11, then (X, q) is uniquely determined by the data $(\tau, \mu, w_\Delta(iq))$. More precisely, the following holds. For $j = 1, 2$, let $X_j \in \mathcal{T}_g$ and $q_j \in \mathcal{Q}(X_j)$. Let Δ_j be a non-vertical triangulation of q_j, with dual train track (τ_j, μ_j), and set $w_j = w_\Delta(iq_j)$. Furthermore, assume that there exists an isotopy of S_g that superimposes τ_2 on τ_1 in such a way that the corresponding measures μ_1, μ_2 and the corresponding weights w_1, w_2 coincide. Then the change of marking from X_1 to X_2 is isotopic to a biholomorphism that sends q_1 to q_2.*

Proof. We will only use this lemma under the additional assumption that $q \in \mathcal{Q}_{\mathbf{1}}(X)$ (see the proof of Proposition 3.1, given at the end of this section), hence we give the proof only in this case. The general case can be proved in a similar way.

Let $\mathbf{A}$ denote the set of triples (X, q, Δ) where $X \in \mathcal{T}_g$, $q \in \mathcal{Q}_{\mathbf{1}}(X)$, and Δ is a non-vertical triangulation of q, up to the following equivalence relation. We say $(X, q, \Delta) \sim (X', q', \Delta')$ if the change of marking from X to X' is isotopic to a biholomorphism that sends q to q' and Δ to Δ'. Let $\mathbf{B}$ denote the set of triples (τ, μ, w), where τ is a complete train track in S_g, μ is a measure on τ, and w is a weight on this train track, up to the following equivalence relation. We say $(\tau, \mu, w) \sim (\tau', \mu', w')$ if they are the same up to isotopy. We claim that the function

$$\mathbf{F}\colon \mathbf{A} \to \mathbf{B}, \quad \text{given by} \quad \mathbf{F}[(X, q, \Delta)] = [(\tau, \mu, w_\Delta(iq))],$$

has an inverse, where (τ, μ) is the train track dual to Δ, and $[\cdot]$ denotes the equivalence class of $\cdot$. The lemma follows from this claim.

To prove the claim, let (τ, μ) be a complete train track, and $w \in W(\tau)$ a weight on τ. Then at each switch as in Figure 1, we can construct a Euclidean triangle $A'B'C'$ such that

$$\overrightarrow{B'C'} = \big(\mu(a), -w(a)\big),\ \overrightarrow{B'A'} = \big(\mu(c), -w(c)\big),\ \overrightarrow{A'C'} = \big(\mu(b), -w(b)\big),$$

where $\vec{v} = (x, y)$ means that the vector $\vec{v} \in \mathbb{R}^2$ has coordinates (x, y). Gluing these triangles along the edges according to τ, we obtain flat charts on S_g. These charts turn S_g into a marked Riemann surface X, where the marking is given by the identity map from S_g to itself. Since the charts constructed above are flat, they also give a quadratic differential $q \in \mathcal{Q}(X)$. This quadratic differential belongs to $\mathcal{Q}_{\mathbf{1}}(X)$ because τ is complete.

Denoting the dual to the underlying graph of τ by Δ, we see that Δ is realized as a saddle triangulation in the flat structure considered above, and Δ is non-vertical since $\mu > 0$. Thus we have arrived at an element of $\mathbf{A}$. Checking that this element does not depend on the equivalence class of (τ, μ, w), and that the map obtain in this way is in fact the inverse of $\mathbf{F}$ are left to the reader. $\square$

Lemma 3.13. *Let $X, X' \in \mathcal{T}_g$, and let $\zeta \in \mathrm{Gen}(S_g)$ be a generic measrued foliation. By Theorem 3.10 there are $\alpha, \alpha' \in \mathrm{T}_\zeta \mathcal{MF}$ such that*

$$\mathrm{d}_\zeta \operatorname{Ext}_X = 4\omega_{\mathrm{Th}}(\alpha, \boldsymbol{\cdot}), \quad \textit{and} \;\; \mathrm{d}_\zeta \operatorname{Ext}_{X'} = 4\omega_{\mathrm{Th}}(\alpha', \boldsymbol{\cdot}).$$

Then $\alpha = \alpha'$ implies $X = X'$.

Proof. Let $q \in \mathcal{Q}(X)$ be such that $\mathcal{V}_X(q) = \zeta$. Choose an arbitrary triangulation Δ of q, and let (τ, μ) be the train track that is dual to Δ. Note that since ζ is generic, $q \in \mathcal{Q}_{\mathbf{1}}(X)$ and Δ is non-vertical. Define $q' \in \mathcal{Q}(X')$, Δ', (τ', μ') in a similar way. Since both (τ, μ) and (τ', μ') carry ζ, Theorem 2.8.5 of [PH92] implies that they are *equivalent*, i.e., there exists a sequence of shifts, splits and collapses that take one to the other. Hence we can find a positive integer n, and a sequence of train tracks $(\tau_i, \mu_i), 1 \le i \le n$ such that

$$(\tau, \mu) = (\tau_1, \mu_1) \rightharpoonup (\tau_2, \mu_2) \rightharpoonup \ldots \rightharpoonup (\tau_n, \mu_n) = (\tau', \mu'),$$

where $(\tau_i, \mu_i) \rightharpoonup (\tau_{i+1}, \mu_{i+1})$ means that (τ_{i+1}, μ_{i+1}) is obtained from (τ_i, μ_i) by a shift, a split, or a collapse.

By Lemma 3.5, a sequence

$$\Delta = \Delta_1 \rightharpoonup \Delta_2 \rightharpoonup \ldots \rightharpoonup \Delta_n$$

of triangulations of q can be constructed in a way that (τ_j, μ_j) is the dual train track to Δ_j. Indeed, each Δ_{j+1} is obtained from Δ_j by a flip.

To summarize, we obtained a triangulation Δ_n of q such that its dual measured train track (τ_n, μ_n) is isotope to (τ', μ'). Theorem 3.10 applied to the Riemann surface X, quadratic differential q, and the triangulation Δ_n of q gives

$$\mathrm{D}_\zeta \mathcal{F}_{\tau'}^{-1}(\alpha) = \mathrm{D}_\zeta \mathcal{F}_{\tau_n}^{-1}(\alpha) = w_{\Delta_n}(iq).$$

The same theorem applied to X', q', Δ' gives

$$\mathrm{D}_\zeta \mathcal{F}_{\tau'}^{-1}(\alpha') = w_{\Delta'}(iq').$$

If $\alpha = \alpha'$ then $w_{\Delta_n}(iq) = w_{\Delta'}(iq')$, hence $(\tau_n, \mu_n, w_{\Delta_n}(iq))$ is isotopic to $(\tau', \mu', w_{\Delta'}(iq'))$, thus Lemma 3.12 implies $X = X'$. □

Proof of Proposition 3.1. Define the function $E_{X,Y} \colon \mathcal{MF} \to \mathbb{R}$ by

$$E_{X,Y}(\zeta) = \operatorname{Ext}_X(\zeta) - \operatorname{Ext}_Y(\zeta),$$

and note that we have $\mathcal{E}_{X,Y} = E_{X,Y}^{-1}(0)$. We claim that for every $\zeta \in \mathcal{E}_{X,Y} \cap \mathrm{Gen}(S_g)$ there exists an open neighborhood $\mathcal{U}_\zeta$ of ζ such that $\nu(\mathcal{E}_{X,Y} \cap \mathcal{U}_\zeta) = 0$. To prove this claim, fix such a ζ, and note that by Theorem 3.10 there are $\alpha_X, \alpha_Y \in \mathrm{T}_\zeta \mathcal{MF}$ such that

$$\mathrm{d}_\zeta \operatorname{Ext}_X = 4\omega_{\mathrm{Th}}(\alpha_X, \boldsymbol{\cdot}), \quad \text{and} \quad \mathrm{d}_\zeta \operatorname{Ext}_Y = 4\omega_{\mathrm{Th}}(\alpha_Y, \boldsymbol{\cdot}), \text{ hence}$$
$$\mathrm{d}_\zeta E_{X,Y} = 4\omega_{\mathrm{Th}}(\alpha_X - \alpha_Y, \boldsymbol{\cdot})$$

Since $X \neq Y$, Lemma 3.13 implies $\alpha_X - \alpha_Y \neq 0$. Non-degeneracy of the Thurston form ([PH92] Theorem 3.2.4) then implies that $\mathrm{d}_\zeta\, E_{X,Y} \neq 0$. Since $\zeta \in \mathrm{Gen}(S_g)$, Theorem 3.10 implies that $E_{X,Y}$ is smooth at ζ, hence the set $\mathcal{E}_{X,Y} = E_{X,Y}^{-1}(0)$ is locally a submanifold of codimension 1 around ζ. This proves the claim.

Since $\mathrm{Gen}(S_g)$ is of full measure, to prove $\nu(\mathcal{E}_{X,Y}) = 0$, it is enough to show that $\nu(\mathcal{E}_{X,Y} \cap \mathrm{Gen}(S_g)) = 0$. Since ν is (a multiple of) the Lebesgue measure on train track charts, it is a Radon measure, that is, for every measurable subset $\mathcal{A} \subset \mathcal{MF}$, we have

$$\nu(\mathcal{A}) = \sup\{\nu(\mathcal{K}) : \mathcal{K} \subset \mathcal{A} \text{ is compact}\}$$

As a result, to prove that $\nu(\mathcal{E}_{X,Y} \cap \mathrm{Gen}(S_g)) = 0$, it is enough to prove $\nu(\mathcal{K}) = 0$ for every compact set $\mathcal{K} \subset \mathcal{E}_{X,Y} \cap \mathrm{Gen}(S_g)$. Fix $\mathcal{K}$ to be such a set. As a consequence of the above claim, we can cover $\mathcal{K}$ by finitely many sets $\mathcal{U}_\zeta$ with $\nu(\mathcal{K} \cap \mathcal{U}_\zeta) = 0$. This proves that $\nu(\mathcal{K}) = 0$, and hence completes the proof of the proposition.

□

4 Comparing Teichmüller and extremal lengths

4.1 Projection to a thick geodesic

In this section, we state a few general facts about those geodesics in the Teichmüller space that lie completely in the thick part, where by a geodesic we always mean a bi-infinite geodesic, unless otherwise stated. We denote the covering map from Teichmüller space to the moduli space by $\Pi\colon \mathcal{T}_g \to \mathcal{M}_g$.

Definition 4.1. Let $\mathcal{K} \subset \mathcal{M}_g$ be compact. A Teichmüller geodesic $\mathcal{G}$ is said to be $\mathcal{K}$*–thick* if $\mathcal{G} \subset \Pi^{-1}(\mathcal{K})$.

Recall that we denote the projective class of $\zeta \in \mathcal{MF}$ by $[\zeta] \in \mathcal{PMF}$. Assume $\mathcal{G}$ is a $\mathcal{K}$–thick geodesic for some compact set $\mathcal{K} \subset \mathcal{M}_g$, and let $X \in \mathcal{T}_g$ and $\zeta \in \mathcal{MF}$ be arbitrary. Define

$$\begin{aligned} \mathrm{proj}(X, \mathcal{G}) &= \{Y \in \mathcal{G} : d(X, Y) = d(X, \mathcal{G})\}; \\ \mathrm{proj}([\zeta], \mathcal{G}) = \mathrm{proj}(\zeta, \mathcal{G}) &= \{Y \in \mathcal{G} : \mathrm{Ext}(\zeta, Y) = \mathrm{Ext}(\zeta, \mathcal{G})\}, \end{aligned}$$

where

$$d(X, \mathcal{G}) = \inf\{d(X, Y) : Y \in \mathcal{G}\}, \quad \text{and} \quad \mathrm{Ext}(\zeta, \mathcal{G}) = \inf\{\mathrm{Ext}(\zeta, Y) : Y \in \mathcal{G}\}.$$

Both $\mathrm{diam}\big(\mathrm{proj}(X, \mathcal{G})\big)$ and $\mathrm{diam}\big(\mathrm{proj}(\zeta, \mathcal{G})\big)$ are bounded by constants depending only on $\mathcal{K}$, where diam stands for the diameter of a set. The boundedness of $\mathrm{diam}\big(\mathrm{proj}(X, \mathcal{G})\big)$ is a consequence of the contraction theorem of [Min96], and the boundedness of $\mathrm{diam}\big(\mathrm{proj}(\zeta, \mathcal{G})\big)$ is also standard and follows, say, from Proposition 4.5.

The following theorem roughly says that the Teichmüller metric behaves like a δ–hyperbolic (Gromov hyperbolic) metric in the thick part.

Theorem 4.2. ***([Raf14] Theorem 8.1)*** *Let $\mathcal{K} \subset \mathcal{M}_g$ be compact and $X, Y, Z \in \mathcal{T}_g$. Then there are constants C and D, only depending on $\mathcal{K}$, such that the following holds. If $U, V \in [X, Y]$ are such that $[U, V] \subset \Pi^{-1}(\mathcal{K})$ and $d(U, V) > C$, then for every $W \in [U, V]$ we have*

$$\min\left\{d(W, [Z, X]),\ d(W, [Z, Y])\right\} < D.$$

Recall the definition of the geodesic connecting a point in the Teichmüller space to a (projective) measrued foliation, given in Section 2, and note that the above theorem remains true if one (or several) of X, Y, Z is replaced by a measured foliation. The following is a consequence of Theorem 4.2.

Proposition 4.3. *Let $\mathcal{K} \subset \mathcal{M}_g$ be compact and $\mathcal{G}$ be a $\mathcal{K}$–thick geodesic. Then there exists a constant $C = C(\mathcal{K})$ such that for every $X \in \mathcal{T}_g$, $H \in \mathrm{proj}(X, \mathcal{G})$, and $Y \in \mathcal{G}$ the geodesic connecting X to Y passes through $\mathfrak{B}(H, C)$, the ball of radius C centered at H.*

Proof. Let X, H, Y be as in the proposition. Applying Theorem 4.2 to the triangle $\Delta(X, H, Y)$, we obtain $C' = C'(\mathcal{K})$ such that for every $Z \in [H, Y]$ there is $W \in [X, H] \cup [X, Y]$ such that $d(Z, W) < C'$. We claim that $C = 3C' + 1$ statisfies the statement of the proposition.

If $d(H, Y) \leq 2C' + 1$ then we are done, otherwise let $Z \in [H, Y]$ be such that $d(H, Z) = 2C' + 1$. By the choice of C', we can find $W \in [X, H] \cup [X, Y]$ such that $d(Z, W) < C'$. If $W \in [X, H]$, then by triangle inequality in $\Delta(W, H, Z)$ we obtain $d(W, H) > C' + 1$, hence

$$d(X, Z) \leq d(X, W) + d(W, Z) = d(X, H) - d(H, W) + d(W, Z) < d(X, H) - 1,$$

which contradicts the choice of H. This contradiction implies $W \in [X, Y]$. Triangle inequality in $\Delta(H, Z, W)$ then implies $d(H, W) < 3C' + 1$, proving the claim. □

Corollary 4.4. *Let $\mathcal{K} \subset \mathcal{M}_g$ be compact and $\mathcal{G}$ be a $\mathcal{K}$–thick geodesic. Then there exists $C = C(\mathcal{K})$ such that for every $X \in \mathcal{T}_g$, $H \in \mathrm{proj}(X, \mathcal{G})$ and $Y \in \mathcal{G}$, we have*

$$d(X, Y) \simeq_C d(X, H) + d(H, Y).$$

Proof. Let $\mathcal{G}$, X and H be as in the corollary, and let $C' = C'(\mathcal{K})$ be the constant given by Proposition 4.3. If $[X, Y]$ intersects $\mathfrak{B}(H, C')$ at Z, then

$$d(X, Z) \simeq_{C'} d(X, H), \qquad \text{and} \qquad d(Z, Y) \simeq_{C'} d(H, Y).$$

The corollary follows from summing up these two estimates. □

Proposition 4.3 remains true if X in its statement is replaced by a measured foliation $\zeta \in \mathcal{MF}$, and the proof parallels the one given above. In fact, replacing the symbol X by ζ throughout this proof, replacing $d(X, \cdot)$ by $\beta(\zeta, \cdot)$, and using Kerschoff inequality instead of triangle inequality for triangles with a vertex at infinity, we obtain a proof for the version of this proposition in which ζ replaces X. (see the discussion at the end of Section 2.) Corollary 4.4 can be proved in this setting as well, hence we have the following.

Proposition 4.5. *Let $\mathcal{K} \subset \mathcal{M}_g$ be compact and $\mathcal{G}$ be a $\mathcal{K}$–thick geodesic. Then there exists a constant $C = C(\mathcal{K})$ such that for every $\zeta \in \mathcal{MF}$, $H \in \mathrm{proj}(\zeta, \mathcal{G})$ and $Y \in \mathcal{G}$, we have*

$$\beta(\zeta, Y) \simeq_C \beta(\zeta, H) + d(H, Y).$$

Moreover, the geodesic connecting ζ to Y passes through $\mathfrak{B}(H, C)$.

4.2 Busemann approximation

The goal of this section is to prove Proposition 4.13, which plays a major role in Section 5. We start with the following definition.

Definition 4.6. Let $\mathcal{K} \subset \mathcal{M}_g$ be compact. The geodesic segment $[X, Y]$ connecting two points $X, Y \in \mathcal{T}_g$ is called *$\mathcal{K}$–typical* if it spends at least half of its time in $\Pi^{-1}(\mathcal{K})$.

A typical geodesic segment remains typical if its endpoints are moved by a bounded distance. This is the content of Corollary 4.8, which itself is a direct consequence of Theorem 4.7. Before stating this theorem, recall that we say two geodesics $\mathcal{G}_1\colon [0, a] \to \mathcal{T}_g$ and $\mathcal{G}_2\colon [0, b] \to \mathcal{T}_g$, parametrized with respect to arc length, *D–fellow travel* for some $D > 0$, if $|a - b| < D$ and for all $0 \le t \le \min\{a, b\}$ we have $d\big(\mathcal{G}_1(t), \mathcal{G}_2(t)\big) < D$.

Theorem 4.7. ***([Raf14] Theorem 7.1)*** *For every compact set $\mathcal{K} \subset \mathcal{M}_g$ and constant $C > 0$, there exists $D = D(\mathcal{K}, C)$ such that the following holds. For every $X, Y \in \Pi^{-1}(\mathcal{K})$ and $\bar{X}, \bar{Y} \in \mathcal{T}_g$ such that $d(X, \bar{X})$ and $d(Y, \bar{Y})$ are both less than C, the geodesics $[X, Y]$ and $[\bar{X}, \bar{Y}]$ D*–fellow travel.

Corollary 4.8. *For every compact set $\mathcal{K} \subset \mathcal{M}_g$ and constant $C > 0$, there exists an enlargement $\mathcal{K}' \supset \mathcal{K}$ such that the following holds. Let $X, Y \in \Pi^{-1}(\mathcal{K})$ be such that $[X, Y]$ is $\mathcal{K}$–typical, and assume $\bar{X}, \bar{Y} \in \mathcal{T}_g$ are such that $d(X, \bar{X})$ and $d(Y, \bar{Y})$ are both less than C, then $[\bar{X}, \bar{Y}]$ is $\mathcal{K}'$–typical.*

Proof. Let D be the constant given by Theorem 4.7, and set $\mathcal{K}' = \mathfrak{B}(\mathcal{K}, D + 2C)$. □

Remark 4.9. Note that the conclusion of Theorem 4.7 remains valid if X and $\bar{X}$ are both replaced by the same measured foliation $\zeta \in \mathcal{MF}$ ([Raf14] Remark 7.2). In this case, we should allow $a = b = +\infty$ in the definition of fellow traveling. We will use this version of the theorem later.

In general, two geodesic rays going to the same point in the boundary of Teichmüller space can stay a bounded distance apart (see [Mas75]). However, they do get exponentially close to each other during the times that they are both in the thick part of Teichmüller space. This is the content of the following theorem.

Theorem 4.10. ***([EMR21])*** *For every compact set $\mathcal{K} \subset \mathcal{M}_g$ and constant $C > 0$, there are positive numbers $\alpha = \alpha(\mathcal{K})$ and $D = D(\mathcal{K}, C)$ such that the following holds. If $X, Y \in \Pi^{-1}(\mathcal{K})$ and $\zeta \in \mathcal{MF}$ are such that*

$$d(X, Y) < C, \quad \text{and} \quad \mathrm{Ext}(\zeta, X) = \mathrm{Ext}(\zeta, Y);$$

and

$$Z_1 \in [X, \zeta) \cap \Pi^{-1}(\mathcal{K}) \quad \text{and} \quad Z_2 \in [Y, \zeta) \cap \Pi^{-1}(\mathcal{K})$$

are such that $d(X, Z_1) = d(Y, Z_2)$; and the geodesic segment $[X, Z_1]$ is $\mathcal{K}$–typical, then

$$d(Z_1, Z_2) < D e^{-\alpha d(X, Z_1)}.$$

Remark 4.11. We can define several norms on the tangent space to the principal stratum of quadratic differentials. (See [ABEM12] for Euclidean and Hodge norms, and [AGY06] for the AGY norm.) The proof of the above theorem uses the relation between these norms, and also Theorem 1.1 of [Fra18]. Theorem 4.10 enables us to avoid arguments involving the above mentioned norms altogether, hence simplifying the proof of our main theorem (Theorem A).

The following lemma is used in the proof of Proposition 4.13.

Lemma 4.12. ***([Min96] Corollary 4.1)*** *Let $\mathcal{K} \subset \mathcal{M}_g$ be compact and $\mathcal{G}$ be a $\mathcal{K}$–thick geodesic. Then there exists a constant $C = C(\mathcal{K})$ such that for every $X, Y \in \mathcal{T}_g$ we have*

$$\operatorname{diam}\big(\operatorname{proj}(X,\mathcal{G}) \cup \operatorname{proj}(Y,\mathcal{G})\big) < d(X,Y) + C.$$

We are now ready to prove the main proposition of this section:

Proposition 4.13. *Let $\mathcal{K} \subset \mathcal{M}_g$ be compact and $\mathcal{G}$ be a $\mathcal{K}$–thick geodesic. Then for every $\epsilon > 0$ there exists a constant $C = C(\mathcal{K}, \epsilon)$ such that the following holds. If $\zeta \in \mathcal{MF}$; $X, Y \in \mathcal{G}$; $Z \in [X, \zeta) \cap \Pi^{-1}(\mathcal{K})$, and $H \in \operatorname{proj}(Z, \mathcal{G})$ are such that the goedesic segment $[Z, H]$ is $\mathcal{K}$–typical and of length greater than C, then*

$$d(Z,Y) - d(Z,X) \simeq_\epsilon \beta(\zeta, Y) - \beta(\zeta, X) = \beta([\zeta], X, Y).$$

Proof. Recall the definition of $\operatorname{proj}(\zeta, \mathcal{G})$ given at the beginning of this section. Let $H_\zeta \in \operatorname{proj}(\zeta, \mathcal{G})$ and assume $C' = C'(\mathcal{K})$ is the constant given by Proposition 4.5, so there is $X' \in (\zeta, X]$ such that $d(X', H_\zeta) < C'$. By Theorem 4.7 there is a constant D, depending only on C' and $\mathcal{K}$, such that $(\zeta, X']$ and $(\zeta, H_\zeta]$ D–fellow travel (see Remark 4.9). As a result, we can find $Z_\zeta \in (\zeta, H_\zeta]$ with $d(Z, Z_\zeta) < D$. Since $H_\zeta \in \operatorname{proj}(Z_\zeta, \mathcal{G})$, by Lemma 4.12 there exists D' depending on D and $\mathcal{K}$ such that $d(H, H_\zeta) < D'$. Triangle inequality then implies $d(X', H) < D' + C'$, hence $[Z, H]$ and $[Z, X']$ fellow travel, so $[Z, X']$ is $\mathcal{K}'$–typical for some enlargement $\mathcal{K}'$ of $\mathcal{K}$ (Corollary 4.8), and since the constants C', D, D' depend on $\mathcal{K}$, the enlargement $\mathcal{K}'$ only depends on $\mathcal{K}$ as well.

Applying Proposition 4.5 once again gives $Y' \in (\zeta, Y]$ with $d(Y', H_\zeta) < C'$, hence

$$d(X', Y') \le d(X', H_\zeta) + d(H_\zeta, Y') < 2C'.$$

Using Kerckhoff inequality we get

$$|\beta(\zeta, X', Y')| \le d(X', Y') < 2C',$$

thus by moving Y' along $(\zeta, Y]$ by at most $2C'$ we may obtain $Y'' \in [Y, \zeta)$ such that $\beta(\zeta, X') = \beta(\zeta, Y'')$. Let Z' be the point obtained by flowing Y'' along $[Y'', \zeta)$ by time $t = d(X', Z)$. Since $[X', Z]$ is $\mathcal{K}'$–typical and $d(X', Y'') < 4C'$, Theorem 4.10 implies that there exists $C'' = C''(\mathcal{K}', C', \epsilon)$ such that if $d(X', Z) > C''$, we have $d(Z, Z') < \epsilon$. Set $C = C'' + D' + C'$. We claim that C satisfies the conclusion of the proposition. If $d(Z, H) > C$, we have

$$d(X', Z) \ge d(Z, H) - d(H, X') \ge d(Z, H) - (D' + C') > C''.$$

By the choice of C'' we have $d(Z,Z') < \epsilon$, hence

$$d(Z,Y) - d(Z,X) \simeq_\epsilon d(Z',Y) - d(Z,X).$$

Since $d(X',Z) = d(Y'',Z')$, we get

$$\begin{aligned} d(Z',Y) - d(Z,X) &= d(Y'',Y) - d(X',X) \\ &= \big(\beta(\zeta,Y) - \beta(\zeta,Y'')\big) - \big(\beta(\zeta,X) - \beta(\zeta,X')\big) \\ &= \beta(\zeta,Y) - \beta(\zeta,X). \end{aligned}$$

This concludes the proof of the proposition. □

5 Proof of the main theorem

The goal of this section is to give the proof of Theorem A. In Section 5.1 we set up the notation and introduce the two main ingredients of the proof. We also show how to derive Theorem B from Theorem A at the end of this section. Section 5.2 contains the proof of Theorem A, and starts with a summary of the steps involved in this proof.

5.1 Preliminary discussion

Given $X \in \mathcal{T}_g$, and a compact set $\mathcal{K} \subset \mathcal{M}_g$, define

$$\mathrm{Typ}(X,\mathcal{K}) = \{Y \in \mathcal{T}_g : [X,Y] \text{ is } \mathcal{K}\text{–typical}\}.$$

The following is a direct consequence of Theorem 1.7 of [EMR19]:

Theorem 5.1. *There exist a constant $\kappa_1 > 0$ and a compact set $\mathcal{K}_1 \subset \mathcal{M}_g$, both of which only depend on the genus g of the surface, such that the following holds. For every compact set $\mathcal{K} \subset \mathcal{M}_g$, there exist a constant $C = C(\mathcal{K})$ such that for all $X, Y \in \Pi^{-1}(\mathcal{K})$ we have*

$$|\Gamma \cdot Y \cap \mathfrak{B}(X,R) \smallsetminus \mathrm{Typ}(X,\mathcal{K}_1)| \le C e^{(h-\kappa_1)R}.$$

Proof. Fix σ to be the principal stratum of quadratic differentials, and recall the definitions of $\mathcal{Q}_{j,\epsilon}$ and $G\colon \mathcal{T}_g \to \mathbb{R}^+$, given in the introduction of [EMR19]. Applying Theorem 1.7 of the same paper to $j = 1$, $\theta = \frac{1}{2}$ and $\delta = \frac{1}{4}$, we obtain an $\epsilon > 0$ such that for all R large enough and $X, Y \in \mathcal{T}_g$ we have

$$N_{\frac{1}{2}}(\mathcal{Q}_{1,\epsilon}, X, Y, R) \le G(X)G(Y)e^{(h-\frac{1}{4})R},$$

where $N_{\frac{1}{2}}(\mathcal{Q}_{1,\epsilon}, X, Y, R)$ denotes the number of elements $\mathbf{g} \in \Gamma$ such that $\mathbf{g}.Y \in \mathfrak{B}(X,R)$ and the geodesic segment connecting X to $\mathbf{g}.Y$ spends at least half of its time in $\mathcal{Q}_{1,\epsilon}$. (here, a geodesic is considered as a subset of $\mathcal{QT}_g$.) By the definition of $\mathcal{Q}_{1,\epsilon}$, if X belongs to the ϵ–thin part of the Teichmüller space (i.e., X has a curve with extremal length less than ϵ), then every $q \in \mathcal{Q}_{\mathbf{1}}(X)$ lies in $\mathcal{Q}_{1,\epsilon}$. Thus, setting $\mathcal{K}_1$ to be the ϵ–thick part of the moduli space, each element of $\Gamma \cdot Y \cap \mathfrak{B}(X,R) \smallsetminus \mathrm{Typ}(X,\mathcal{K}_1)$ contributes to $N_{\frac{1}{2}}(Q_{1,\epsilon}, X, Y, R)$, hence

$$|\Gamma \cdot Y \cap \mathfrak{B}(X,R) \smallsetminus \mathrm{Typ}(X,\mathcal{K}_1)| \le N_{\frac{1}{2}}(Q_{1,\epsilon}, X, Y, R).$$

The function G descends to a continuous function from $\mathcal{M}_g$ to $\mathbb{R}^+$, hence it is bounded by some constant C' on $\Pi^{-1}(\mathcal{K})$. The current theorem then follows for $C = C'^2$ and $\kappa_1 = \frac{1}{4}$. □

For a measured foliation $\zeta \in \mathcal{MF}$, denote the projective class of ζ by $[\zeta] \in \mathcal{PMF}$, and for $\mathcal{U} \subset \mathcal{MF}$, define $[\mathcal{U}] = \{[\zeta] : \zeta \in \mathcal{U}\}$. Recall that, for $X \in \mathcal{T}_g$ and $q \in \mathcal{Q}(X)$, the vertical measured foliation of q is denoted by $\mathcal{V}_X(q)$. Given $[\mathcal{U}] \subset \mathcal{PMF}$, define

$$S(X, [\mathcal{U}], R) = \big\{\pi(g_t.q) : q \in \mathcal{Q}(X),\ [\mathcal{V}_X(q)] \in [\mathcal{U}], \ \text{ and } \ 0 \le t \le R\big\};$$
$$S_{\mathrm{Ext}}(X, [\mathcal{U}]) = \{\zeta \in \mathcal{MF} : [\zeta] \in [\mathcal{U}], \ \text{ and } \ \mathrm{Ext}(\zeta, X) \le 1\}.$$

For $\mathcal{U} \subset \mathcal{MF}$, we define $S(X,\mathcal{U}, R) = S(X, [\mathcal{U}], R)$. The following is an equivalent form of [ABEM12] Theorem 2.9.

Theorem 5.2. *Let $X, Y \in \mathcal{T}_g$ and let $[\mathcal{U}]$ be an open subset of $\mathcal{PMF}$ with negligible boundary. Then, as $R \to \infty$,*

$$|\Gamma \cdot Y \cap S(X, [\mathcal{U}], R)| \sim \frac{\Lambda}{h \,\mathrm{Vol}(\mathcal{M}_g)} \nu\big(S_{\mathrm{Ext}}(X, [\mathcal{U}])\big) e^{hR}. \tag{8}$$

Proof. Given $X \in \mathcal{T}_g$, let $\mathcal{Q}^1(X) \subset \mathcal{Q}(X)$ be the set of quadratic differentials of unit norm on X. Define

$$S^1_{\mathrm{Ext}}(X) = \{\zeta \in \mathcal{MF} : \mathrm{Ext}(\zeta, X) = 1\},$$

and let $\mathcal{V}^1_X \colon \mathcal{Q}^1(X) \to S^1_{\mathrm{Ext}}(X)$ be the restriction of $\mathcal{V}_X$ to $\mathcal{Q}^1(X)$. Let $\bar{\nu}$ be the measure on $S^1_{\mathrm{Ext}}(X)$ obtained by coning off the Thurston measure, that is, for $E \subset S^1_{\mathrm{Ext}}(X)$ set

$$\bar{\nu}(E) = \nu\big(\{\lambda.\zeta : \zeta \in E, \ \text{ and } \ 0 < \lambda \le 1\}\big).$$

Let μ denote the Masur-Veech measure on $\mathcal{Q}^1\mathcal{T}_g$, the space of unit norm quadratic differentials, and let s_X denote the conditional of μ on $\mathcal{Q}^1(X)$. For $q \in \mathcal{Q}^1(X)$, define $\lambda^-(q)$ by the following Radon-Nikodym derivative:

$$\lambda^-(q) = \frac{d(\mathcal{V}^1_X)^*\bar{\nu}}{ds_X}(q). \tag{9}$$

Theorem 2.9 of [ABEM12] states that, as $R \to \infty$,

$$|\Gamma \cdot Y \cap S(X, [\mathcal{U}], R)| \sim \frac{\Lambda}{h \,\mathrm{Vol}(\mathcal{M}_g)} \Big(\int_{\overline{\mathcal{U}}} \lambda^-(q) ds_X(q) \Big) e^{hR},$$

where $\overline{\mathcal{U}} = \{q \in \mathcal{Q}^1(X) : [\mathcal{V}_X(q)] \in [\mathcal{U}]\}$. By (9) we have

$$\int_{\overline{\mathcal{U}}} \lambda^-(q) ds_X(q) = \bar{\nu}(\mathcal{V}^1_X(\overline{\mathcal{U}})) = \nu(S_{\mathrm{Ext}}(X, [\mathcal{U}])).$$

This concludes the proof of the theorem. □

Now we show how to deduce Theorem B from Theorem A. The argument is parallel to the one given at the beginning of Section 5 of [ABEM12]. The key in both arguments is Theorem 5.1 of [ABEM12].

Proof of Theorem B assuming Theorem A.. Fix an element $O \in \mathcal{L}_\gamma$, and let L be the translation length of γ, that is, $L = d(O, \gamma.O)$. Given $X \in \mathcal{T}_g$ and $R > 0$, we can define a function

$$h\colon \langle\gamma\rangle\backslash\big(\Gamma \cdot X \cap \mathfrak{B}(\mathcal{L}_\gamma, R)\big) \to \Gamma \cdot X \cap \mathfrak{B}(O, R+L) \tag{10}$$

as follows. Given an element a in the domain of h, fix a representative $\mathbf{g}.X$ of a, i.e., fix an element $\mathbf{g} \in \Gamma$ such that $a = \langle\gamma\rangle \cdot (\mathbf{g}.X)$. Choose an element $H \in \operatorname{proj}(\mathbf{g}.X, \mathcal{L}_\gamma)$, and let $k \in \mathbb{Z}$ be such that $d(O, \gamma^k.H) < L$. We then define

$$h(a) = \gamma^k \mathbf{g}.X.$$

Since h is an injection, we have

$$|\langle\gamma\rangle\backslash\big(\Gamma \cdot X \cap \mathfrak{B}(\mathcal{L}_\gamma, R)\big)| \leq |\Gamma \cdot X \cap \mathfrak{B}(O, R+L)|. \tag{11}$$

Since X and R were arbitrary, the above inequality is true for every $X \in \mathcal{T}_g$ and $R > 0$.

By [ABEM12] Theorem 5.1, there exists a constant C, only depending on the base point O, such that for all $X \in \mathcal{T}_g$ and $R > 0$ we have

$$|\Gamma \cdot X \cap \mathfrak{B}(O, R)| < Ce^{hR}.$$

This, combined with (11), implies that for $C' = Ce^{hL}$ and every $X \in \mathcal{T}_g$ we have

$$|\langle\gamma\rangle\backslash\big(\Gamma \cdot X \cap \mathfrak{B}(\mathcal{L}_\gamma, R)\big)| < C'e^{hR}. \tag{12}$$

Recall the definitions of the covering maps $\Pi\colon \mathcal{T}_g \to \mathcal{M}_g$ and $\Pi_\gamma\colon \mathcal{T}_g \to \mathcal{C}_\gamma$ from the introduction, and define the covering map

$$\Pi_{\gamma,\Gamma}\colon \mathcal{C}_\gamma \to \mathcal{M}_g \quad \text{by} \quad \langle\gamma\rangle \cdot X \mapsto \Gamma \cdot X.$$

Since $\Pi_{\gamma,\Gamma}$ is a local diffeomorphism, we have

$$\operatorname{Vol}\big(\mathfrak{B}(\overline{\mathcal{L}}_\gamma, R)\big) = \int_{\mathcal{M}_g} |\Pi_{\gamma,\Gamma}^{-1}(X) \cap \mathfrak{B}(\overline{\mathcal{L}}_\gamma, R)| \, d\operatorname{Vol}(X)$$

We multiply both sides of this equation by e^{-hR} and take the limit as $R \to \infty$. Since C' in (12) does not depend on X, we can apply Lebesgue's dominated convergence theorem to take the limit inside the integral, obtaining

$$\lim_{R\to\infty} e^{-hR} \operatorname{Vol}\big(\mathfrak{B}(\overline{\mathcal{L}}_\gamma, R)\big) = \int_{\mathcal{M}_g} \lim_{R\to\infty} e^{-hR} |\Pi_{\gamma,\Gamma}^{-1}(X) \cap \mathfrak{B}(\overline{\mathcal{L}}_\gamma, R)| \, d\operatorname{Vol}(X)$$

Theorem A then concludes the proof. ∎

5.2 Proof of Theorem A

Fix a pseudo-Anosov homeomorphism γ, and fix two points $P \in \mathcal{T}_g$ and $O \in \mathcal{L}_\gamma$ for the rest of this section. Let κ_1 and $\mathcal{K}_1$ be as in Theorem 5.1, and let C_1 be the constant C given by the same theorem for $\mathcal{K} = \Pi(\mathcal{L}_\gamma) \cup \Pi(P)$. To lighten the notation, we will not show the dependance of constants on γ, O, P, $\mathcal{K}_1$. For example, we will write $C = C(\mathcal{K})$ instead of $C = C(\gamma, O, \mathcal{K})$. Recall the definitions of Π_γ and $\mathcal{C}_{\mathrm{Ext},\gamma}$ from the introduction. The goal of this section is to prove

$$\left|\Pi_\gamma\big(\Gamma \cdot P \cap \mathfrak{B}(\mathcal{L}_\gamma, R)\big)\right| \sim \frac{\Lambda}{h\,\mathrm{Vol}(\mathcal{M}_g)} \nu(\mathcal{C}_{\mathrm{Ext},\gamma}) e^{hR}, \quad \text{as } R \to \infty. \tag{13}$$

The proof is given in two parts: Lemma 5.6 and Lemma 5.9.

Definition 5.3. Given $\epsilon > 0$, a sequence of points $\mathcal{X} = (X_i)_{i \in \mathbb{Z}} \subset \mathcal{L}_\gamma$ is called an ϵ*–net* in $\mathcal{L}_\gamma$ (or an ϵ*–net* for short) if the following holds.

- $X_0 = O$.
- $d(X_i, X_{i+1}) < \epsilon$, for $i \in \mathbb{Z}$.
- The geodesic segments $[X_{i-1}, X_i]$ and $[X_i, X_{i+1}]$ intersect only at X_i, for every $i \in \mathbb{Z}$.
- There exists $N = N(\mathcal{X}) \in \mathbb{N}$ such that $X_N = \gamma.O$.
- $X_{i+N} = \gamma.X_i$ for every $i \in \mathbb{Z}$.

For an ϵ–net $\mathcal{X}$ in $\mathcal{L}_\gamma$, and for all $i \in \mathbb{Z}$, define

$$\mathcal{A}_i(\mathcal{X}) = \{\zeta \in \mathcal{MF} : \mathrm{Ext}(\zeta, X_i) \le 1 \text{ and } \beta(\zeta, X_i) = \inf_{j \in \mathbb{Z}} \beta(\zeta, X_j)\}.$$

Since $\mathcal{X}$ is γ–invariant, the sets $\mathcal{A}_i(\mathcal{X})$ are γ*–equivariant*, meaning that for all $i \in \mathbb{Z}$, $\gamma.\mathcal{A}_i(\mathcal{X}) = \mathcal{A}_{i+N(\mathcal{X})}(\mathcal{X})$.

We need to introduce some notation before stating the next lemma. Let $[\gamma^\pm]$ be the set containing the two elements of $\mathcal{PMF}$ that are fixed by γ. $[\gamma^\pm]$ can be considered as a subset of $\mathcal{MF}$, namely, the union of the two rays that are fixed by γ. Define the unit extremal ball around $\mathcal{L}_\gamma$ by

$$\mathfrak{B}_{\mathrm{Ext}}(\mathcal{L}_\gamma) = \{\zeta \in \mathcal{MF} : \mathrm{Ext}(\zeta, \mathcal{L}_\gamma) \le 1\} \smallsetminus [\gamma^\pm],$$

where $\mathrm{Ext}(\zeta, \mathcal{L}_\gamma)$ is as defined in Section 4.1. By Theorem 6.9 of [MP89], $\langle\gamma\rangle$ acts totally discontinuously on $\mathcal{PMF} \smallsetminus [\gamma^\pm]$, hence it also acts totally discontinuously on $\mathcal{MF} \smallsetminus [\gamma^\pm]$. Let $\Pi_{\mathrm{Ext},\gamma}$ denote the covering map of the latter action. Since $\mathfrak{B}_{\mathrm{Ext}}(\mathcal{L}_\gamma)$ is γ–invariant, we can form the quotient

$$\mathcal{C}_{\mathrm{Ext},\gamma} = \langle\gamma\rangle \backslash \mathfrak{B}_{\mathrm{Ext}}(\mathcal{L}_\gamma).$$

Note that the Thurston measure ν on $\mathcal{MF}$ descends to a measure on $\mathcal{C}_{\mathrm{Ext},\gamma}$, which we denote by ν as well.

Lemma 5.4. *Let $\mathcal{X}$ be an ϵ–net in $\mathcal{L}_\gamma$, then we have*

$$\frac{1}{e^{\epsilon(6g-6)}}\nu(\mathcal{C}_{\mathrm{Ext},\gamma}) \leq \sum_{i=0}^{N(\mathcal{X})-1} \nu(\mathcal{A}_i(\mathcal{X})) \leq \nu(\mathcal{C}_{\mathrm{Ext},\gamma}).$$

Proof. Set $\mathcal{A}_i = \mathcal{A}_i(\mathcal{X})$ and $N = N(\mathcal{X})$.

Right hand inequality. Given $i \in \mathbb{Z}$, define

$$\mathcal{A}_i^{<} = \{\zeta \in \mathcal{A}_i : \beta(\zeta, X_i) < \inf_{j\neq i} \beta(\zeta, X_j)\};$$
$$\mathcal{A}_i^{=} = \mathcal{A}_i \smallsetminus \mathcal{A}_i^{<}.$$

Thus $\mathcal{A}_i^{=}$ consists of those $\zeta \in \mathcal{A}_i$ such that $\beta(\zeta, X_i) = \beta(\zeta, X_j)$ for some $j \neq i$. Recall that given $X, Y \in \mathcal{T}_g$, $\mathcal{E}_{X,Y}$ is defined by

$$\mathcal{E}_{X,Y} = \{\zeta \in \mathcal{MF} : \mathrm{Ext}(\zeta, X) = \mathrm{Ext}(\zeta, Y)\}.$$

By Proposition 3.1, $\mathcal{E}_{X,Y}$ has Thurston measure zero, and by the definition of $\mathcal{A}_i^{=}$, we have

$$\mathcal{A}_i^{=} \subset \bigcup_{j\neq i} \mathcal{E}(X_i, X_j),$$

hence

$$\nu(\mathcal{A}_i^{=}) = 0. \tag{14}$$

Note that the image of $\mathcal{A}_i$ under the covering map $\Pi_{\mathrm{Ext},\gamma}$ lies in $\mathcal{C}_{\mathrm{Ext},\gamma}$. We make the following claims:

1. For all $i \in \mathbb{Z}$, $\Pi_{\mathrm{Ext},\gamma}\colon\ \mathcal{A}_i^{<} \to \mathcal{C}_{\mathrm{Ext},\gamma}$ is an injection.
2. The sets $\Pi_{\mathrm{Ext},\gamma}(\mathcal{A}_i^{<})$ are pairwise disjoint for $0 \leq i < N$.

To prove the first claim, fix $i \in \mathbb{Z}$ and assume that $\zeta_1, \zeta_2 \in \mathcal{A}_i^{<}$ have the same image under $\Pi_{\mathrm{Ext},\gamma}$. This means that ζ_2 lies in the $\langle\gamma\rangle$–orbit of ζ_1, i.e., there is $k \in \mathbb{Z}$ such that $\gamma^k.\zeta_1 = \zeta_2$. Hence ζ_2 belongs to the intersection of $\mathcal{A}_i^{<}$ and $\gamma^k.\mathcal{A}_i^{<} = \mathcal{A}_{i+kN}^{<}$. However, by definition we have

$$\mathcal{A}_i^{<} \cap \mathcal{A}_j^{<} = \varnothing, \quad \text{for} \quad i \neq j.$$

This means that we should have $k = 0$, hence $\zeta_1 = \zeta_2$. This proves the first claim. The second claim follows by a similar argument.

The above claims imply that

$$\sum_{i=0}^{N-1} \nu(\mathcal{A}_i^{<}) \leq \nu(\mathcal{C}_{\mathrm{Ext},\gamma}).$$

Since $\nu(\mathcal{A}_i^{=}) = 0$, we have $\nu(\mathcal{A}_i^{<}) = \nu(\mathcal{A}_i)$. This proves the right hand inequality of the lemma.

Left hand inequality. Given $\zeta \in \mathfrak{B}_{\text{Ext}}(\mathcal{L}_\gamma)$, let $H_\zeta \in \text{proj}(\zeta, \mathcal{L}_\gamma)$. (see Section 4.1 for the definition of proj.) By the definition of $\mathfrak{B}_{\text{Ext}}(\mathcal{L}_\gamma)$ we have $\text{Ext}(\zeta, H_\zeta) \leq 1$, and since $\mathcal{X}$ is an ϵ–net, if X_{i_0} is the element of the net that is closest to H_ζ, we have $d(H_\zeta, X_{i_0}) < \epsilon$. Kerckhoff inequality in $\Delta(\zeta, H_\zeta, X_{i_0})$ then implies

$$\text{Ext}(\zeta, X_{i_0}) \leq e^{2d(H_\zeta, X_{i_0})}\,\text{Ext}(\zeta, H_\zeta) < e^{2\epsilon}.$$

Hence

$$\inf_{i\in\mathbb{Z}} \text{Ext}(\zeta, X_i) < e^{2\epsilon}. \tag{15}$$

By Proposition 4.5 the infimum above is attained at some $i = i_1$. Then we have $\zeta/e^\epsilon \in \mathcal{A}_{i_1}$, hence $\zeta \in e^\epsilon \mathcal{A}_{i_1}$, where $e^\epsilon \mathcal{A}_{i_1} = \{e^\epsilon \zeta : \zeta \in \mathcal{A}_{i_1}\}$. Since $\zeta \in \mathfrak{B}_{\text{Ext}}(\mathcal{L}_\gamma)$ was arbitrary, this means that

$$\mathfrak{B}_{\text{Ext}}(\mathcal{L}_\gamma) \subset \bigcup_{i\in\mathbb{Z}} e^\epsilon \mathcal{A}_i \implies \mathcal{C}_{\text{Ext},\gamma} \subset \bigcup_{i=0}^{N-1} \Pi_{\text{Ext},\gamma}(e^\epsilon \mathcal{A}_i).$$

Since Thurston measure is the $6g-6$ dimensional Lebesgue measure in train track charts, we have $\nu(e^\epsilon \mathcal{A}_i) = e^{\epsilon(6g-6)}\nu(\mathcal{A}_i)$, hence

$$\nu(\mathcal{C}_{\text{Ext},\gamma}) \leq \sum_{i=0}^{N-1} e^{\epsilon(6g-6)}\nu(\mathcal{A}_i).$$

which is equivalent to the left hand inequality of the lemma.

□

Given an ϵ–net $\mathcal{X}$, for $\delta > 0$ and $i \in \mathbb{Z}$ define

$$\mathcal{A}_i^{-\delta}(\mathcal{X}) = \{\zeta \in \mathcal{MF} : \text{Ext}(\zeta, X_i) \leq 1, \quad \text{and} \quad \beta(\zeta, X_i) < \inf_{j\neq i} \beta(\zeta, X_j) - \delta\};$$
$$\mathcal{A}_i^{+\delta}(\mathcal{X}) = \{\zeta \in \mathcal{MF} : \text{Ext}(\zeta, X_i) \leq 1, \quad \text{and} \quad \beta(\zeta, X_i) < \inf_{j\neq i} \beta(\zeta, X_j) + \delta\}.$$

These sets enjoy the following properties:

- They are open and γ–eqivariant,i.e., $\gamma.\mathcal{A}_i^{\pm\delta} = \mathcal{A}_{i+N(\mathcal{X})}^{\pm\delta}$.
- $\mathcal{A}_i^{-\delta}(\mathcal{X}) \subset \mathcal{A}_i(\mathcal{X}) \subset \mathcal{A}_i^{+\delta}(\mathcal{X})$.
- $\mathcal{A}_i^{-\delta}(\mathcal{X}) \uparrow \mathcal{A}_i^{<}(\mathcal{X})$ as $\delta \downarrow 0$, i.e., the following holds. For $0 < \delta_1 < \delta_2$, $\mathcal{A}_i^{-\delta_2} \subset \mathcal{A}_i^{-\delta_1}$; and $\cup_{\delta>0}\mathcal{A}_i^{-\delta} = \mathcal{A}_i^{<}(\mathcal{X})$.
- $\mathcal{A}_i^{+\delta}(\mathcal{X}) \downarrow \mathcal{A}_i(\mathcal{X})$ as $\delta \downarrow 0$.

Since the boundary of $\mathcal{A}_i^{\pm\delta}(\mathcal{X})$ has nonzero measure for at most countably many pairs (i, δ), we can (and will) work under the additional assumption that $\mathcal{A}_i^{\pm\delta}(\mathcal{X})$ has negligible boundary. The following lemma contains the main observation about the sets $\mathcal{A}_i^{-\delta}(\mathcal{X})$. (comapre with Lemma 5.8.)

Lemma 5.5. *Let $\mathcal{X} = (X_i)_{i\in\mathbb{Z}}$ be an ϵ–net, $\mathcal{K} \subset \mathcal{M}_g$ be compact and $\delta > 0$. Then there exists a constant $C = C(\epsilon, \delta, \mathcal{K})$ such that the sets*

$$\Gamma \cdot P \cap S(X_i, \mathcal{A}_i^{-\delta}(\mathcal{X}), +\infty) \cap \mathrm{Typ}(X_i, \mathcal{K}) \smallsetminus \mathfrak{B}(X_i, C), \quad i \in \mathbb{Z}$$

are mutually disjoint.

Proof. Let

$$Y \in \Gamma \cdot P \cap S(X_i, \mathcal{A}_i^{-\delta}(\mathcal{X}), +\infty) \cap \mathrm{Typ}(X_i, \mathcal{K}) \smallsetminus \mathfrak{B}(X_i, C), \tag{16}$$

for a constant C to be determined below, and let $H \in \mathrm{proj}(Y, \mathcal{L}_\gamma)$. We make the following claim.

Claim. There exists a compact set $\mathcal{K}'$, only depending on $\mathcal{K}$, such that $[Y, H]$ is $\mathcal{K}'$–typical.

To prove the claim, let $\zeta \in \mathcal{A}_i^{-\delta}(\mathcal{X})$ be such that Y lies on $[X_i, \zeta)$. Since $\mathcal{X}$ is an ϵ–net, we have

$$\beta(\zeta, X_i) = \inf_{k\in\mathbb{Z}} \beta(\zeta, X_k) \simeq_\epsilon \inf_{Y\in\mathcal{L}_\gamma} \beta(\zeta, Y).$$

Because of the shape of $\beta(\zeta, X)$ as a function of $X \in \mathcal{L}_\gamma$, described in Proposition 4.5, the latter infimum should be attained near X_i, namely, there exists $C' = C'(\mathcal{K})$ such that if $H_\zeta \in \mathrm{proj}(\zeta, \mathcal{L}_\gamma)$, then $d(X_i, H_\zeta) < C'$. Thus, by Theorem 4.7, $(\zeta, X_i]$ and $(\zeta, H_\zeta]$ D–fellow travel for some $D = D(\mathcal{K}, C')$. As a result, if $Z_\zeta \in (\zeta, H_\zeta]$ is such that $d(Y, X_i) = d(Z_\zeta, H_\zeta)$ then $d(Y, Z_\zeta) < D$. Since $H_\zeta = \mathrm{proj}(Z_\zeta, \mathcal{L}_\gamma)$, Lemma 4.12 implies $d(H_\zeta, H) < D'$ for some $D' = D'(\mathcal{K}, D)$. Setting $C'' = C' + D'$, triangle inequality then implies $d(X_i, H) < C''$, and since $[Y, X_i]$ is $\mathcal{K}$–typical, Corollary 4.8 implies that $[Y, H]$ is $\mathcal{K}'$–typical for an enlargement $\mathcal{K}' \supset \mathcal{K}$ that only depends on $\mathcal{K}$ and C''. This proves the claim.

By Proposition 4.13, we can find $C^{(3)} = C^{(3)}(\mathcal{K}', \delta)$ such that if $d(Y, \mathcal{L}_\gamma) = d(Y, H) > C^{(3)}$, then for all $j \in \mathbb{Z}$ we have

$$d(Y, X_i) - d(Y, X_j) \simeq_{\delta/2} \beta(\zeta, X_i) - \beta(\zeta, X_j). \tag{17}$$

Note that by triangle inequality in $\Delta(Y, H, X_i)$,

$$d(Y, H) \geq d(Y, X_i) - d(X_i, H) > d(Y, X_i) - C'',$$

Hence, letting $C = C'' + C^{(3)}$ in (16), we get $d(Y, H) > C^{(3)}$, hence (17) holds. Let $j \neq i$. Since $\zeta \in \mathcal{A}_i^{-\delta}(\mathcal{X})$, we have $\beta(\zeta, X_i) - \beta(\zeta, X_j) < -\delta$, hence (17) implies $d(Y, X_i) < d(Y, X_j)$. If

$$Y \in \Gamma \cdot P \cap S(X_j, \mathcal{A}_j^{-\delta}(\mathcal{X}), +\infty) \cap \mathrm{Typ}(X_j, \mathcal{K}) \smallsetminus \mathfrak{B}(X_j, C)$$

as well, then by changing the role of X_i and X_j in the argument above we obtain $d(Y, X_j) < d(Y, X_i)$. This contradiction shows that the two sets mentioned in the lemma are disjoint. □

Lemma 5.6. *With the same notation as before, we have*

$$\frac{\Lambda}{h \operatorname{Vol}(\mathcal{M}_g)} \nu(\mathcal{C}_{\mathrm{Ext},\gamma}) \leq \liminf_{R\to\infty} e^{-hR} |\langle\gamma\rangle \backslash \big(\Gamma \cdot P \cap \mathfrak{B}(\mathcal{L}_\gamma, R)\big)|.$$

Proof. Recall that $\mathcal{K}_1$ is the compact set given by Theorem 5.1. Fix $\epsilon, \delta > 0$ for now, and let $\mathcal{X}$ be an arbitrary ϵ–net in $\mathcal{L}_\gamma$. Let $C = C(\epsilon, \delta, \mathcal{K}_1)$ be the constant given by Lemma 5.5. This means that the sets $\mathsf{A}_i^{-\delta}(\mathcal{X}, R; C)$, defined by

$$\mathsf{A}_i^{-\delta}(\mathcal{X}, R; C) = \Gamma \cdot P \cap S(X_i, \mathcal{A}_i^{-\delta}(\mathcal{X}), R) \cap \mathrm{Typ}(X_i, \mathcal{K}_1) \setminus \mathfrak{B}(X_i, C),$$

are mutually disjoint. Recall that Π_γ is the covering map from $\mathcal{T}_g$ to $\langle\gamma\rangle\backslash\mathcal{T}_g$. We claim that for every $i \in \mathbb{Z}$,

$$\Pi_\gamma \colon \mathsf{A}_i^{-\delta}(\mathcal{X}, R; C) \to \langle\gamma\rangle\backslash\mathcal{T}_g$$

is an injection. (compare with the first claim in the proof of Lemma 5.4.)

To prove this claim, assume that $\Pi_\gamma(X) = \Pi_\gamma(Y)$ for $X, Y \in \mathsf{A}_i^{-\delta}(\mathcal{X}, R; C)$. This implies that there exists $k \in \mathbb{Z}$ such that $\gamma^k.X = Y$, so Y belongs to the intersection of $\mathsf{A}_i^{-\delta}(\mathcal{X}, R; C)$ with $\gamma^k.\mathsf{A}_i^{-\delta}(\mathcal{X}, R; C) = \mathsf{A}_{i+kN(\mathcal{X})}^{-\delta}(\mathcal{X}, R; C)$. By the choice of C we have $k = 0$, hence $X = Y$. This proves the claim.

A similar argument implies that the sets $\Pi_\gamma\big(\mathsf{A}_i^{-\delta}(\mathcal{X}, R; C)\big)$ are disjoint for $0 \le i < N(\mathcal{X})$, so

$$\sum_{i=0}^{N(\mathcal{X})-1} |\mathsf{A}_i^{-\delta}(\mathcal{X}, R; C)| \le |\langle\gamma\rangle\backslash\big(\Gamma \cdot P \cap \mathfrak{B}(\mathcal{L}_\gamma, R)\big)|.$$

Multiplying both sides by e^{-hR} and taking $\liminf$ as $R \to \infty$, we obtain

$$\sum_{i=0}^{N(\mathcal{X})-1} \liminf_{R\to\infty} e^{-hR} |\mathsf{A}_i^{-\delta}(\mathcal{X}, R; C)| \le \liminf_{R\to\infty} e^{-hR} |\langle\gamma\rangle\backslash\big(\Gamma \cdot P \cap \mathfrak{B}(\mathcal{L}_\gamma, R)\big)|. \tag{18}$$

Since $|\Gamma \cdot P \cap \mathfrak{B}(X_i, C)|$ is bounded by a constant that only depends on C, we have

$$\liminf_{R\to\infty} e^{-hR} |\mathsf{A}_i^{-\delta}(\mathcal{X}, R; C)| = \liminf_{R\to\infty} e^{-hR} |\Gamma \cdot P \cap S(X_i, \mathcal{A}_i^{-\delta}, R) \cap \mathrm{Typ}(X_i, \mathcal{K}_1)|$$

By Theorem 5.1, the number of $Y \in \Gamma \cdot P \cap \mathfrak{B}(X_i, R)$ such that $[X_i, Y]$ is not $\mathcal{K}_1$–typical is of order $e^{(h-\kappa_1)R}$, hence

$$\begin{aligned} &\liminf_{R\to\infty} e^{-hR} |\Gamma \cdot P \cap S(X_i, \mathcal{A}_i^{-\delta}, R) \cap \mathrm{Typ}(X_i, \mathcal{K}_1)| \\ &= \liminf_{R\to\infty} e^{-hR} |\Gamma \cdot P \cap S(X_i, \mathcal{A}_i^{-\delta}, R)| = \frac{\Lambda}{h\,\mathrm{Vol}(\mathcal{M}_g)} \nu(\mathcal{A}_i^{-\delta}(\mathcal{X})), \end{aligned}$$

where the last equality is by Theorem 5.2. The above equalities and (18) shows that

$$\frac{\Lambda}{h\,\mathrm{Vol}(\mathcal{M}_g)} \sum_{i=0}^{N(\mathcal{X})-1} \nu(\mathcal{A}_i^{-\delta}(\mathcal{X})) \le \liminf_{R\to\infty} e^{-hR} |\langle\gamma\rangle\backslash\big(\Gamma \cdot P \cap \mathfrak{B}(\mathcal{L}_\gamma, R)\big)|.$$

Note that the above is valid for every $\epsilon > 0$, ϵ–net $\mathcal{X}$, and $\delta > 0$. Keeping ϵ and the ϵ–net $\mathcal{X}$ fixed in the above expression, we let $\delta \downarrow 0$. Recall that as $\delta \downarrow 0$, $\mathcal{A}_i^{-\delta}(\mathcal{X}) \uparrow \mathcal{A}_i^{<}(\mathcal{X})$

(see the discussion before Lemma 5.5), hence $\nu(\mathcal{A}_i^{-\delta}(\mathcal{X})) \uparrow \nu(\mathcal{A}_i^{<}(\mathcal{X}))$. Equation (14) implies that $\nu(\mathcal{A}_i^{<}(\mathcal{X})) = \nu(\mathcal{A}_i(\mathcal{X}))$, thus

$$\frac{\Lambda}{h\,\mathrm{Vol}(\mathcal{M}_g)} \sum_{i=0}^{N(\mathcal{X})-1} \nu(\mathcal{A}_i(\mathcal{X})) \leq \liminf_{R\to\infty} e^{-hR} |\langle\gamma\rangle\backslash\big(\Gamma \cdot P \cap \mathfrak{B}(\mathcal{L}_\gamma, R)\big)|.$$

Now, using Lemma 5.4 and making the ϵ–net $\mathcal{X}$ finer, i.e., letting $\epsilon \to 0$, proves the lemma. □

The above lemma proves "half" of Theorem A, the other half is proved as Lemma 5.9. The proof of Lemma 5.9 parallels the proof of the above lemma, this time we use Lemma 5.8 instead of Lemma 5.5. For a compact set $\mathcal{K} \subset \mathcal{M}_g$, define

$$\mathrm{Typ}(\mathcal{L}_\gamma, \mathcal{K}) = \{Y \in \mathcal{T}_g : [Y, H] \text{ is } \mathcal{K}\text{–typical for some } H \in \mathrm{proj}(Y, \mathcal{L}_\gamma)\}.$$

We need the following consequence of Theorem 5.1.

Lemma 5.7. *There exists a constant C_2 and a compact set $\mathcal{K}_2 \subset \mathcal{M}_g$ such that for all $R > 0$,*

$$|\langle\gamma\rangle\backslash\big(\Gamma.P \cap \mathfrak{B}(\mathcal{L}_\gamma, R) \smallsetminus \mathrm{Typ}(\mathcal{L}_\gamma, \mathcal{K}_2)\big)| < C_2 e^{(h-\kappa_1)R}.$$

Proof. Note that $\mathrm{Typ}(\mathcal{L}_\gamma, \mathcal{K}_2)$ is γ–invariant, hence taking the quotient by $\langle\gamma\rangle$ above is justified. Let L be the translation length of γ, and let

$$h\colon \langle\gamma\rangle\backslash\big(\Gamma \cdot P \cap \mathfrak{B}(\mathcal{L}_\gamma, R)\big) \to \Gamma \cdot P \cap \mathfrak{B}(O, R+L)$$

be the map defined in the proof of Theorem B, given at the end of Section 5.1.

Let a be an arbitrary element in the domain of h. It follows from the definition of this function that if $h(a) = \mathbf{g}.P$, then $d(O, H) < L$ for some $H \in \mathrm{proj}(\mathbf{g}.P, \mathcal{L}_\gamma)$. By Corollary 4.8, there exists an enlargement $\mathcal{K}_2 \supset \mathcal{K}_1$, only depending on $\mathcal{K}_1$ and L, such that if $[\mathbf{g}.P, O]$ is $\mathcal{K}_1$–typical, then $[\mathbf{g}.P, H]$ has to be $\mathcal{K}_2$–typical. Hence h sends

$$\langle\gamma\rangle\backslash\big(\Gamma \cdot P \cap \mathfrak{B}(\mathcal{L}_\gamma, R) \smallsetminus \mathrm{Typ}(\mathcal{L}_\gamma, \mathcal{K}_2)\big) \quad \text{to} \quad \Gamma \cdot P \cap \mathfrak{B}(O, R+L) \smallsetminus \mathrm{Typ}(O, \mathcal{K}_1).$$

By the choice of C_1 (see the beginning of this section),

$$|\Gamma \cdot P \cap \mathfrak{B}(O, R+L) \smallsetminus \mathrm{Typ}(O, \mathcal{K}_1)| \leq C_2 e^{(h-\kappa_1)R}.$$

for $C_2 = C_1 e^{hL}$. The injectivity of h then concludes the proof. □

Lemma 5.8. *For every $\epsilon > 0$; ϵ–net $\mathcal{X} = (X_i)_{i\in\mathbb{Z}}$; $\delta > 0$, and compact set $\mathcal{K} \subset \mathcal{M}_g$ there exists $C = C(\epsilon, \delta, \mathcal{K})$ such that the following holds. If*

$$Y \in \Gamma \cdot P \cap \mathfrak{B}(\mathcal{L}_\gamma, R) \cap \mathrm{Typ}(\mathcal{L}_\gamma, \mathcal{K}) \smallsetminus \mathfrak{B}(\mathcal{L}_\gamma, C),$$

and $i_0 \in \mathbb{Z}$ is such that $d(Y, X_{i_0}) = \inf_{i\in\mathbb{Z}} d(Y, X_i)$, then

$$Y \in S(X_{i_0}, \mathcal{A}_{i_0}^{+\delta}(\mathcal{X}), R + \epsilon).$$

Proof. Let C be a constant to be determined later, and fix

$$Y \in \Gamma \cdot P \cap \mathfrak{B}(\mathcal{L}_\gamma, R) \cap \operatorname{Typ}(\mathcal{L}_\gamma, \mathcal{K}) \setminus \mathfrak{B}(\mathcal{L}_\gamma, C). \tag{19}$$

Note that by Corollary 4.4, $\inf_{i \in \mathbb{Z}} d(Y, X_i)$ is attained for some $i_0 \in \mathbb{Z}$. Let the geodesic ray connecting X_{i_0} to Y hit the boundary at $[\zeta]$, i.e., ζ is such that $Y \in [X_{i_0}, \zeta)$. By Proposition 4.13 there is a constant $C = C(\mathcal{K}, \delta)$ such that if $d(Y, \mathcal{L}_\gamma) > C$, then for all $i \in \mathbb{Z}$,

$$\beta(\zeta, X_i) - \beta(\zeta, X_{i_0}) \simeq_\delta d(Y, X_i) - d(Y, X_{i_0}).$$

We choose C in (19) to be the same constant, hence the above holds for Y. Let $i \in \mathbb{Z}$ be arbitrary. By the choice of i_0 we have $d(Y, X_i) - d(Y, X_{i_0}) \geq 0$, hence

$$\beta(\zeta, X_i) - \beta(\zeta, X_{i_0}) > -\delta \implies \beta(\zeta, X_{i_0}) < \beta(\zeta, X_i) + \delta.$$

Since i was arbitrary, we conclude that $\zeta \in \mathcal{A}_{i_0}^{+\delta}(\mathcal{X})$. The only thing left to show is that $d(X_{i_0}, Y) \leq R + \epsilon$.

To show this, (compare with the proof of left hand inequality in Lemma 5.4) let $H \in \operatorname{proj}(Y, \mathcal{L}_\gamma)$ and note that since $Y \in \mathfrak{B}(\mathcal{L}_\gamma, R)$, we have $d(Y, H) \leq R$. Choose X_{i_1} to be the point of $\mathcal{X}$ that is closest to H. Since the mesh of $\mathcal{X}$ is less than ϵ we have $d(H, X_{i_1}) < \epsilon$, hence by triangle inequality we get $d(Y, X_{i_1}) < R + \epsilon$. So

$$d(Y, X_{i_0}) = \inf_{j \in \mathbb{Z}} d(Y, X_i) < R + \epsilon.$$

This proves the lemma. $\square$

Lemma 5.9. *With the same notation as before, we have*

$$\limsup_{R \to \infty} e^{-hR} |\langle \gamma \rangle \backslash \Gamma \cdot P \cap \mathfrak{B}(\mathcal{L}_\gamma, R)| \leq \frac{\Lambda}{h \operatorname{Vol}(\mathcal{M}_g)} \nu(\mathcal{C}_{\mathrm{Ext},\gamma}).$$

Proof. Let the compact set $\mathcal{K}_2$ be as in Lemma 5.7. Fix $\epsilon, \delta > 0$ for now, and let $\mathcal{X} = (X_i)_{i \in \mathbb{Z}}$ be an arbitrary ϵ–net in $\mathcal{L}_\gamma$. Let $C = C(\epsilon, \delta, \mathcal{K}_2)$ be the constant given by Lemma 5.8 for the ϵ–net $\mathcal{X}$, δ, and the compact set $\mathcal{K}_2$. For $R > 0$ write

$$\Gamma \cdot P \cap \mathfrak{B}(\mathcal{L}_\gamma, R) = \mathsf{B}_1(R; C) \cup \mathsf{B}_2(R) \cup \mathsf{B}_3(C),$$

where

$$\begin{aligned} \mathsf{B}_1(R; C) &= \Gamma \cdot P \cap \mathfrak{B}(\mathcal{L}_\gamma, R) \cap \operatorname{Typ}(\mathcal{L}_\gamma, \mathcal{K}_2) \setminus \mathfrak{B}(\mathcal{L}_\gamma, C); \\ \mathsf{B}_2(R) &= \Gamma \cdot P \cap \mathfrak{B}(\mathcal{L}_\gamma, R) \setminus \operatorname{Typ}(\mathcal{L}_\gamma, \mathcal{K}_2); \\ \mathsf{B}_3(C) &= \Gamma \cdot P \cap \mathfrak{B}(\mathcal{L}_\gamma, C). \end{aligned}$$

Note that all the three sets above are γ–invariant, hence we can form the quotients

$$\overline{\mathsf{B}}_1(R; C) = \langle \gamma \rangle \backslash \mathsf{B}_1(R, C), \quad \overline{\mathsf{B}}_2(R) = \langle \gamma \rangle \backslash \mathsf{B}_2(R), \quad \overline{\mathsf{B}}_3(C) = \langle \gamma \rangle \backslash \mathsf{B}_3(C),$$

and obtain

$$\begin{aligned}|\langle\gamma\rangle\backslash\big(\Gamma\cdot P\cap\mathfrak{B}(\mathcal{L}_\gamma,R)\big)| \le |\overline{\mathsf{B}}_1(R;C)|+|\overline{\mathsf{B}}_2(R)|+|\overline{\mathsf{B}}_3(C)| \implies \\ \limsup_{R\to\infty} e^{-hR}|\langle\gamma\rangle\backslash\big(\Gamma\cdot P\cap\mathfrak{B}(\mathcal{L}_\gamma,R)\big)| \le \\ \limsup_{R\to\infty} e^{-hR}|\overline{\mathsf{B}}_1(R;C)| + \limsup_{R\to\infty} e^{-hR}|\overline{\mathsf{B}}_2(R)| + \limsup_{R\to\infty} e^{-hR}|\overline{\mathsf{B}}_3(C)|\end{aligned} \tag{20}$$

Since $|\overline{\mathsf{B}}_3(C)|$ is a constant only depending on C, the third term in (20) is zero. By Lemma 5.7, the middle term of (20) is zero as well. Thus

$$\limsup_{R\to\infty} e^{-hR}|\langle\gamma\rangle\backslash\Gamma\cdot P\cap\mathfrak{B}(\mathcal{L}_\gamma,R)| \le \limsup_{R\to\infty} e^{-hR}|\overline{\mathsf{B}}_1(R;C)|$$

To find an upper bound for $|\overline{\mathsf{B}}_1(R;C)|$, note that by Lemma 5.8 and the choice of C,

$$\begin{aligned}\mathsf{B}_1(R;C) &\subset \bigcup_{i\in\mathbb{Z}}(\Gamma\cdot P\cap S(X_i,\mathcal{A}_i^{+\delta}(\mathcal{X}),R+\epsilon)) \implies \\ |\overline{\mathsf{B}}_1(R;C)| &\le \sum_{i=0}^{N(\mathcal{X})-1} |\Gamma\cdot P\cap S(X_i,\mathcal{A}_i^{+\delta}(\mathcal{X}),R+\epsilon)| \implies \\ \limsup_{R\to\infty} e^{-hR}|\overline{\mathsf{B}}_1(R;C)| &\le \sum_{i=0}^{N(\mathcal{X})-1} \limsup_{R\to\infty} e^{-hR}|\Gamma\cdot P\cap S(X_i,\mathcal{A}_i^{+\delta}(\mathcal{X}),R+\epsilon)| \\ &= \sum_{i=0}^{N(\mathcal{X})-1} \frac{\Lambda}{h\,\mathrm{Vol}(\mathcal{M}_g)}e^{h\epsilon}\nu(\mathcal{A}_i^{+\delta}(\mathcal{X})). \qquad \text{(by Theorem 5.2)}\end{aligned}$$

Using the above upper bound for the first term of (20), and recalling that the second and third terms of (20) are zero, we obtain

$$\limsup_{R\to\infty} e^{-hR}|\langle\gamma\rangle\backslash\Gamma\cdot P\cap\mathfrak{B}(\mathcal{L}_\gamma,R)| \le \frac{\Lambda}{h\,\mathrm{Vol}(\mathcal{M}_g)}e^{h\epsilon}\sum_{i=0}^{N(\mathcal{X})-1}\nu(\mathcal{A}_i^{+\delta}(\mathcal{X})).$$

As in the proof of the lower bound, we first let $\delta\downarrow 0$ and use $\bigcap\mathcal{A}_i^{+\delta}(\mathcal{X})=\mathcal{A}_i(\mathcal{X})$, then let $\epsilon\downarrow 0$ and use Lemma 5.4 to conclude the proof.

□

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