

THE GROWTH OF A FIXED CONJUGACY CLASS IN NEGATIVE CURVATURE

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ABSTRACT. Let M be a compact closed manifold of variable negative curvature. Fix an element $\text{id} \neq \gamma$ in the fundamental group Γ of M , and denote the set of elements in Γ that are conjugate to γ by Conj_γ . For two points x, y in the universal cover of M , we obtain asymptotics for the number of Conj_γ -orbits of y that lie in a ball of radius T centered at x , as T tends to infinity. If M is two-dimensional, or of dimension $n \geq 3$ and curvature bounded above by -1 and below by $-\left(\frac{n-1}{n-2}\right)^2$, we find an exponentially small error term for this count.

1. INTRODUCTION

Statement of the main result. Let M be a closed compact manifold of (variable) negative curvature, and let \widetilde{M} be its universal cover. The fundamental group of M , denoted by Γ , acts on \widetilde{M} by deck transformations. We denote the image of a point $x \in \widetilde{M}$ under the action of $g \in \Gamma$ by $g.x$, or simply by gx , and denote the Γ orbit of x by $\Gamma \cdot x$. We denote the ball of radius T centered at a point x by $B_T(x)$. In his Ph.D. thesis (see [Mar04] for an english translation), Margulis proved that for $x, y \in \widetilde{M}$,

$$\#(B_T(x) \cap \Gamma \cdot y) \sim c_{x,y} e^{\delta T}, \quad \text{as } T \rightarrow \infty, \quad (1)$$

where $c_{x,y}$ is a constant only depending on x and y , and δ denotes the topological entropy of geodesic flow on the unit tangent bundle of M .

In [PS98b], using the dynamical zeta function approach, Sharp and Pollicott obtained exponentially small error term for the above count, when M is 2-dimensional and $x = y$. More precisely, for a closed surface M of variable negative curvature, they proved there exists a constant $\kappa > 0$, only depending on the geometry of M , such that for $x \in \widetilde{M}$ we have

$$\#(B_T(x) \cap \Gamma \cdot x) = c_{x,x} e^{\delta T} + \mathbf{O}(e^{(\delta-\kappa)T}).$$

Fixing an element $\text{id} \neq \gamma \in \Gamma$, denote the conjugacy class of γ by Conj_γ , that is,

$$\text{Conj}_\gamma := \{g^{-1}\gamma g : g \in \Gamma\}.$$

Our main theorem, proved in Section 6, is as follows.

Theorem A. *Assume M is two-dimensional, or of dimension $n \geq 3$ and curvature bounded above by -1 and below by $-\left(\frac{n-1}{n-2}\right)^2$. Then there exists a constant κ , only depending on the geometry of M , such that the following holds. For every $\text{id} \neq \gamma \in \Gamma$ and $x, y \in \widetilde{M}$, there exists a constant $\sigma = \sigma(x, y, \gamma)$ such that for $T > 0$,*

$$\#(B_T(x) \cap \text{Conj}_\gamma \cdot y) = \sigma e^{\frac{\delta}{2}T} + \mathbf{O}_{\gamma,x,y}(e^{(\frac{\delta}{2}-\kappa)T}).$$

Previous works. The above theorem was first proved in [Hub56] when M is a compact surface of constant curvature -1 , using Selberg's trace formula. (See also [Hub98]) In [PP15], the same is proved when M is a compact manifold of constant curvature -1 and arbitrary dimension. In Corollary 10 of the same paper, for M a compact manifold of variable negative curvature, it is proved that

$$\#(B_T(x) \cap \text{Conj}_\gamma \cdot y) \asymp_{x,y,\gamma} e^{\frac{\delta}{2}T},$$

that is, there exists a constant $c = c(x, y, \gamma)$, such that for $T > 0$,

$$\frac{1}{c}e^{\frac{\delta}{2}T} \leq \#(B_T(x) \cap \text{Conj}_\gamma \cdot y) \leq ce^{\frac{\delta}{2}T}.$$

In the preprint [Pol18], building on [PS98a], Pollicott obtained asymptotics (without an error term) for Conj_γ orbit counts, when M is a surface of variable negative curvature *with boundary*. At the end of that paper it is mentioned that the methods of the paper are not likely to work for a closed surface (i.e., a surface without boundary).

Adjusted counts. The proof of Theorem A has two main ingredients, the first being Proposition 4.1, and the second being Theorem 6.1. Assuming these two, the proof of Theorem A is given in Section 6. Theorem 6.1 is stated for a point $x \in \widetilde{M}$ and the axis L_γ of an element $\text{id} \neq \gamma \in \Gamma$, however, a similar statement holds (with the exact same proof) in the case where L_γ is replaced by a point $y \in \widetilde{M}$. (See Remark (i)) In this introduction, to better illustrate the idea beneath this theorem, we state it for this latter case as Theorem B.

Fix $x, y \in \widetilde{M}$ and let $h: \Gamma \rightarrow \mathbb{R}$ be a function such that for $g \in \Gamma$,

$$h(g) = d(x, gy) + \mathbf{O}(1).$$

Setting

$$\mathcal{B}^h(T) := \{g \in \Gamma : h(g) \leq T\},$$

it is clear that $\#\mathcal{B}^h(T) \asymp e^{\delta T}$; however, to find more precise estimates on $\#\mathcal{B}^h(T)$ we need more control over the difference $h(g) - d(x, gy)$. Theorem B states that if this difference only depends (upto a small error term) on the manner the geodesic segment $[x, gy]$ departs from x and the manner it arrives at gy , then asymptotics for $\#\mathcal{B}^h(T)$ can be obtained. Moreover, under some additional assumptions, an exponentially small error term may be obtained for this count. To state Theorem B, let $S(x)$ denote the unit tangent sphere based at $x \in \widetilde{M}$, and for $g \in \Gamma$, let $v_x(g) \in S(x)$ (resp. $v_y(g) \in S(gy)$) be the tangent to $[x, gy]$ at x (resp. gy) pointing towards gy (resp. x).

Theorem B. *Assume M is two-dimensional, or of dimension $n \geq 3$ and curvature bounded above by -1 and below by $-\left(\frac{n-1}{n-2}\right)^2$. Then for every $\alpha, \kappa > 0$, there exists a constant κ' , only depending on α, κ , and the geometry of M , such that the following holds. Let $x, y \in \widetilde{M}$, and*

$$F_x: S(x) \rightarrow \mathbb{R}, \quad \text{and} \quad F_y: S(y) \rightarrow \mathbb{R}$$

be α -Hölder functions. Let the function $h: \Gamma \rightarrow \mathbb{R}$ satisfy

$$h(g) = d(x, gy) - F_x(v_x(g)) - F_y(g^{-1} \cdot v_y(g)) + \mathbf{O}(e^{-\kappa d(x, gy)}) \quad (2)$$

for $g \in \Gamma$. Then there exists a constant $\sigma = \sigma(F_x, F_y)$ such that

$$\#\mathcal{B}^h(T) = \sigma e^{\delta T} + \mathbf{O}_{F_1, F_2}(e^{(\delta - \kappa')T}).$$

For $g \in \Gamma$, $h(g)$ can be thought of as the distance between x and gy , adjusted by the functions F_x and F_y on both endpoints of $[x, gy]$. F_x and F_y are called the *adjustment functions*, and $h(g)$ is called the *adjusted distance* between x and gy .

Remarks. We provide the following remarks for the experts.

- (i) Our proof of Theorem B closely follows the proof of Theorem 3 of [PP17] (see the beginning of Section 5 for a discussion), hence we expect that Theorem B holds in the more general setting of [PP17], that is, when x and y are replaced by nonempty proper closed convex subsets C_1 and C_2 of \widetilde{M} such that $\text{Stab}(C_i) \setminus C_i$ has finite skinning measure for $i = 1, 2$; and the role of $[x, gy]$ is played by the common perpendicular between C_1 and $g.C_2$. Indeed, we refrained from writing the proof in this generality to first, keep the notation simple, and second, since we did not have a geometric incentive to do so.

- (ii) As mentioned at the beginning of this section, the error term for (1) is obtained, when $x = y$ and M is a compact surface, in [PS98b]. The proof given in that paper relies on a certain coding of geodesic flow, which is not established (to the best of the author's knowledge) for manifolds of dimension greater than 2. We now describe an alternative approach to obtain error term. By Theorem 7.7, the so-called RHC condition (see Definition 7.4) holds for manifolds satisfying the assumptions of Theorem A, hence, using Theorem 3 of [PP17] for two points $x, y \in \widetilde{M}$, we obtain exponentially small error term in (1) for such manifolds. This is, to the best of the author's knowledge, the best result in this direction.

Notation and conventions. Let $A, B \in \mathbb{R}$ and \bullet be a set of parameters. We write $A = \mathbf{O}_\bullet(B)$ if $|A| \leq cB$ for a constant $c = c(\bullet)$ that only depends on \bullet . We write $A \prec_\bullet B$ (resp. $A \succ_\bullet B$, $A \asymp_\bullet B$) if there exists a constant $c = c(\bullet)$ such that $A \leq cB$ (resp. $A \geq \frac{B}{c}$, $\frac{A}{c} \leq B \leq cA$). For any of these symbols, if a subset of the parameters \bullet are fixed (at the beginning of a section or throughout a proof), we may remove them from the subscript of that symbol.

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2. BACKGROUND

We fix a closed compact manifold of (variable) negative curvature M throughout the text, and denote its universal cover by \widetilde{M} . We further assume that the curvature of M is bounded from above (resp. below) by -1 (resp. $-\mathfrak{b}^2$). We choose, once and for all, a point $o \in \widetilde{M}$, called the *origin*, which remains the same for the rest of this paper. We denote the unit tangent sphere at $x \in \widetilde{M}$ by $S(x)$, and we use the same notation when x is an element of M . The unit tangent bundle of \widetilde{M} (resp. M) is denoted by $T^1\widetilde{M}$ (resp. T^1M), and the *basepoint projection* map, sending a vector in $T^1\widetilde{M}$ (resp. T^1M) to its basepoint in \widetilde{M} (resp. M), is denoted by π in both cases. The natural map from \widetilde{M} to M , denoted by Π , induces a map from $T^1\widetilde{M}$ to T^1M , which we also denote by Π . For a point x in \widetilde{M} (resp. M) and $u \in S(x)$, we denote $-u$ by $\iota(u)$. Flowing $u \in T^1\widetilde{M}$ (resp. $u \in T^1M$) by time t , we obtain an element of $T^1\widetilde{M}$ (resp. T^1M), which we denote by $\mathcal{G}_t(u)$ or u_t , depending on the occasion. The fundamental group $\Gamma := \pi_1(M)$ acts by deck transformations on $T^1\widetilde{M}$, and we denote the image of $u \in T^1\widetilde{M}$ under the action of $g \in \Gamma$ by $g.u$, or simply by gu when no confusion can arise. It is well-known that if $\text{id} \neq \gamma \in \Gamma$, then there exists a bi-infinite geodesic L_γ , called the *axis* of γ , such that γ acts on L_γ by translation.

The boundary at infinity of \widetilde{M} is denoted by $\partial_\infty\widetilde{M}$. For $x, y \in \widetilde{M}$, the geodesic segment connecting x to y is denoted by $[x, y]$, and for $\zeta \in \partial_\infty\widetilde{M}$, the geodesic ray from x to ζ is denoted by $[x, \zeta)$. For $u \in T^1\widetilde{M}$, the geodesic ray $\{\pi(u_t) : t \geq 0\}$ (resp. $\{\pi(u_t) : t \leq 0\}$) hits $\partial_\infty\widetilde{M}$ at a point which we denote by u^+ (resp. u^-). For two distinct points $x, y \in \widetilde{M}$, the unit vector $v \in S(x)$ (resp. $v \in S(y)$) such that $[x, y] = \{\pi(v_t) : 0 \leq t \leq d(x, y)\}$ is called the (unit) tangent vector to $[x, y]$ at x (resp. y), and is denoted by $P_x^1(y)$ (resp. $P_y^1(x)$). Fixing x , the map $P_x^1: \widetilde{M} \rightarrow S(x)$ extends continuously to a map from $\widetilde{M} \cup \partial_\infty\widetilde{M}$ to $S(x)$, which we also denote by P_x^1 .

A bi-infinite geodesic line L , simply called a geodesic from now on, hits the boundary of \widetilde{M} at two points, the set of which we denote by $\partial_\infty L$. Fixing a geodesic L , we denote the foot of perpendicular from a point $x \in \widetilde{M}$ to L by $P_L(x)$, and denote the vector tangent to $[x, P_L(x)]$ at $P_L(x)$ by $P_L^1(x) \in \partial^1 L$, where $\partial^1 L$ denotes the unit normal bundle of L . The map $P_L: \widetilde{M} \rightarrow L$ (resp. $P_L^1: \widetilde{M} \rightarrow \partial^1 L$) extends continuously to a map from $\widetilde{M} \cup \partial_\infty\widetilde{M} \setminus L_\infty$ to L (resp. $\partial^1 L$), which we also denote by P_L (resp. P_L^1).

For $x \in \widetilde{M}$ and $\zeta, \eta \in \partial_\infty \widetilde{M}$, define the *visual distance* between ζ, η as seen by x , by

$$d_x(\zeta, \eta) := e^{-T} \quad \text{for} \quad T := \frac{1}{2} \lim_{t \rightarrow \infty} (d(\zeta_t, \eta_t) - d(x, \zeta_t) - d(x, \eta_t)),$$

where $t \mapsto \zeta_t$ (resp. $t \mapsto \eta_t$) is any geodesic ray ending at ζ (resp. η). One can easily see that for x, ζ, η, T as above, and $u, v \in S(x)$ such that $u^+ = \zeta$ and $v^+ = \eta$, we have

$$d(\pi(u_T), \pi(v_T)) \asymp_b 1.$$

The visual distance remains the same, upto a multiplicative constant, if we change the basepoint. More precisely, for $x, y \in \widetilde{M}$ and $\zeta, \eta \in \partial_\infty \widetilde{M}$ we have

$$e^{-d(x,y)} \leq \frac{d_x(\zeta, \eta)}{d_y(\zeta, \eta)} \leq e^{d(x,y)}.$$

The *Busemann cocycle* $\beta: \partial_\infty \widetilde{M} \times \widetilde{M} \times \widetilde{M} \rightarrow \mathbb{R}$ is defined by

$$\beta(\zeta, x, y) := \lim_{t \rightarrow \infty} (d(\zeta_t, x) - d(\zeta_t, y)),$$

where $t \mapsto \zeta_t$ is any geodesic ray ending at ζ . We summarize the elementary properties of Busemann cocycle as follows. For $x, y, z \in \widetilde{M}$ and $\zeta \in \partial_\infty \widetilde{M}$, we have

$$\begin{aligned} \beta(\zeta, x, y) &= -\beta(\zeta, y, x); \\ \beta(\zeta, x, y) &= \beta(\zeta, x, z) + \beta(\zeta, z, y); \\ |\beta(\zeta, x, y)| &\leq d(x, y). \end{aligned}$$

We consider the following ‘default’ metrics.

- The Riemannian metric $d_{\widetilde{M}}$ (resp. d_M) on \widetilde{M} (resp. M).
- The Sasaki metric d_S on $T^1 \widetilde{M}$ (resp. $T^1 M$), which is the Riemannian metric induced by Sasaki’s inner product on $TT\widetilde{M}$. (resp. TTM .) See Section 2.3 of [PPS15] for more details.
- The visual distance d_o on $\partial_\infty \widetilde{M}$, seen from the origin o .

We may remove the subscript from d_\bullet when it can be inferred from the context.

For metric spaces (X, d_X) and (Y, d_Y) , $X \times Y$ is always equipped with the sup metric

$$d_{X \times Y}((x_1, y_1), (x_2, y_2)) := \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}.$$

A function $F: X \rightarrow Y$ is said to be (α, c) -Hölder for some constants $\alpha, c > 0$, if for all $x_1, x_2 \in X$ we have

$$d_Y(F(x_1), F(x_2)) \leq cd_X(x_1, x_2)^\alpha.$$

We say that the function F is α -Hölder for some $\alpha > 0$, if for every compact set $K \subset X$ there is a constant c_K such that the restriction of F to K is (α, c_K) -Hölder. Finally, we say that F is Hölder-continuous if it is α -Hölder for some $\alpha > 0$. Note that most combinations of Hölder-continuous functions are Hölder-continuous. For example, if X, Y_1, Y_2, Z are metric spaces and $F_1: X \rightarrow Y_1, F_2: X \rightarrow Y_2, H: Y_1 \times Y_2 \rightarrow Z$ are Hölder-continuous, then $x \mapsto H(F_1(x), F_2(x))$ is also Hölder-continuous.

3. HÖLDER-CONTINUOUS FUNCTIONS

The goal of this section is to prove various functions, defined in terms of geometry of \widetilde{M} , are Hölder-continuous. These facts are used throughout the text, but especially in Section 4, to establish the Hölder-continuity of functions introduced in Proposition 4.1. The first few lemmas are well-known; we provided proofs however, since we couldn’t find any in the literature.

Lemma 3.1. *The map from $\mathbb{R} \times T^1 \widetilde{M}$ to $T^1 \widetilde{M}$ sending (t, u) to $\mathcal{G}_t(u)$ is Lipschitz.*

Proof. Fixing $u \in T^1\widetilde{M}$, one can directly check that $\mathcal{G}_t(u)$ is 1-Lipschitz in t (in fact, it is distance preserving). Fixing t , $\mathcal{G}_t(\cdot)$ (the time t flow on T^1M) is a diffeomorphism on the compact manifold T^1M , thus it is Lipschitz with respect to the Riemannian metric d_S on this manifold. It follows that $\mathcal{G}_t(\cdot)$ (the time t flow on $T^1\widetilde{M}$) is also Lipschitz. \square

Lemma 3.2. (i) Fix $A > 0$; let $x, y, z \in \widetilde{M}$ be such that $d(z, x) = d(z, y)$, and let $x' \in [z, x]$ and $y' \in [z, y]$ be such that $d(x', x) = d(y', y) = T$. Then

$$d(x', y') \prec_A e^{-T} d(x, y).$$

(ii) The conclusion of part (i) still holds if $z \in \partial_\infty\widetilde{M}$ and $\beta(z, x, y) = 0$.

(iii) Fix $A > 0$, and let $x, y, z \in \widetilde{M}$ be such that $d(x, y) \leq A$, and let $x' \in [z, x]$. Then, there exists $y' \in [z, y]$ such that

$$d(x', y') \prec_A e^{-d(x', x)} d(x, y).$$

(iv) Part (iii) remains true if $z \in \partial_\infty\widetilde{M}$.

Proof. Part (i) of this lemma directly follows from part (i) of Lemma 2.5. of [PPS15]. Part (ii) can be proved in a similar way. To prove (iii), slide y along the geodesic ray starting from z and passing through y to obtain y'' with $d(z, x) = d(z, y'')$. By triangle inequality we have

$$d(y, y'') = |d(z, y) - d(z, x)| \leq d(x, y),$$

hence

$$d(x, y'') \leq d(x, y) + d(y, y'') \leq 2d(x, y) \leq 2A.$$

Let $T := d(x', x)$, and let $y' \in [y'', z]$ be such that $d(y'', y') = T$. By part (i) we have $d(x', y') \prec_A e^{-T} d(x, y)$, which is what we wanted. (iv) follows from (ii) by a similar argument. \square

In part (4) of Proposition 2.4. of [BH99], it is proved that the projection to a convex subset of a CAT(0) space is distance non-increasing. This result may be improved for a CAT(-1) space, as seen in Proposition 3.3. (At least when the distance between the points being projected and the convex set is large enough.) In what follows, we only use this proposition when the convex set is a geodesic in \widetilde{M} .

Proposition 3.3. Fix $A > 0$, let \mathcal{C} be a convex set in \widetilde{M} , and let $x, y \in \widetilde{M}$ be such that $d(x, y) \leq A$. Then

$$d(P_{\mathcal{C}}(x), P_{\mathcal{C}}(y)) \prec_A e^{-d(x, \mathcal{C})} d(x, y)$$

Proof. Set $h_x := P_{\mathcal{C}}(x)$ and $h_y := P_{\mathcal{C}}(y)$. If $h_x = h_y$ there is nothing to prove, otherwise let $\Delta(\bar{x}, \bar{h}_x, \bar{h}_y)$ and $\Delta(\bar{y}, \bar{x}, \bar{h}_y)$ be comparison triangles in \mathbb{H}^2 for $\Delta(x, h_x, h_y)$ and $\Delta(y, x, h_y)$, such that \bar{h}_x and \bar{y} are on different sides of $[\bar{x}, \bar{h}_y]$. By comparison, we see that $\angle(\bar{x}, \bar{h}_x, \bar{h}_y)$ and $\angle(\bar{y}, \bar{h}_y, \bar{h}_x)$ are both at least $\pi/2$. Let \bar{G} be the geodesic in \mathbb{H}^2 that passes through \bar{h}_x and \bar{h}_y , and let $h_{\bar{x}}$ and $h_{\bar{y}}$ denote the foots of perpendiculars from \bar{x} and \bar{y} to \bar{G} respectively. Let x_1 and y_1 denote the reflections of \bar{x} and \bar{y} across \bar{G} , and let z denote the point that $[x_1, \bar{y}]$ intersects $[h_{\bar{x}}, h_{\bar{y}}]$. Then we have

$$|d(\bar{y}, z) - d(\bar{y}, h_{\bar{y}})| \leq d(z, h_{\bar{y}}) \leq d(h_{\bar{x}}, h_{\bar{y}}) \leq d(\bar{x}, \bar{y}) \leq A,$$

where the second to last inequality holds since \mathbb{H}^2 is a CAT(0) space. Since $|d(\bar{y}, h_{\bar{y}}) - d(\bar{x}, h_{\bar{x}})| \leq d(\bar{x}, \bar{y})$, the above implies $d(\bar{y}, z) = T + \mathbf{O}_A(1)$ for $T := d(\bar{x}, h_{\bar{x}}) = d(x, \mathcal{C})$. By Part (iii) of Lemma 3.2, there exists a point $z' \in [\bar{x}, x_1]$ with $d(z, z') \prec_A e^{-T} d(\bar{x}, \bar{y})$. Since $h_{\bar{x}}$ is the foot of projection from z to $[\bar{x}, x_1]$, we have

$$d(z, h_{\bar{x}}) \leq d(z, z') \prec_A e^{-T} d(x, y)$$

In a similar way, we obtain $d(z, h_{\bar{y}}) \prec_A e^{-T} d(x, y)$, thus $d(h_{\bar{x}}, h_{\bar{y}}) \prec_A e^{-T} d(x, y)$. \square

Recall that $-b^2$ is the lower bound on the curvature of \widetilde{M} . The following lemma is only used in the proof of Lemma 3.5, in which we introduce a useful semi-distance on $T^1\widetilde{M}$ that is Hölder-equivalent to d_S .

Lemma 3.4. Fix $A > 0$, and let $x, y, z \in \widetilde{M}$ be such that $d(z, x) = d(z, y) \geq 1$ and $d(x, y) \leq A$. Let $x_1 \in [z, x]$ and $y_1 \in [z, y]$ be such that $d(z, x_1) = d(z, y_1) = 1$. Then we have

$$d(x_1, y_1) \succ_{\mathfrak{b}, A} e^{-\mathfrak{b}d(z, x)} d(x, y).$$

Proof. By comparison, we may assume that \widetilde{M} is equal to $\mathbb{H}_{\mathfrak{b}}^2$, the hyperbolic plane with constant curvature $-\mathfrak{b}^2$. Let h be the foot of perpendicular from z to $[x, y]$, and let h_1 be the intersection of this perpendicular with $[x_1, y_1]$. Denoting the angle between $[z, h]$ and $[z, y]$ at z by α , the hyperbolic sine formula in $\mathbb{H}_{\mathfrak{b}}^2$ (see the second item of Theorem 7.11.2 of [Bea83] for $\mathfrak{b} = 1$), applied to triangles $\Delta(z, h, y)$ and $\Delta(z, h_1, y_1)$ gives

$$\sin \alpha = \frac{\sinh \mathfrak{b} \frac{d(x, y)}{2}}{\sinh \mathfrak{b} T}, \text{ and } \sin \alpha = \frac{\sinh \mathfrak{b} \frac{d(x_1, y_1)}{2}}{\sinh \mathfrak{b}},$$

where $T := d(z, y)$. Note that, fixing $B > 0$, we have $\sinh x \asymp_B x$ for $x \in [0, B]$ and $\sinh x \asymp_B e^x$ for $x \in [B, +\infty)$. Applying these to the above equalities, we obtain

$$d(x_1, y_1) \asymp_{\mathfrak{b}, A} e^{-\mathfrak{b}T} d(x, y).$$

□

Lemma 3.5. The semi-distance $d_{0, +\infty}$ on $T^1 \widetilde{M}$, defined by

$$d_{0, +\infty}(u, v) := \max\{d(\pi(u), \pi(v)), d_o(u^+, v^+)\}$$

is Hölder-equivalent to Sasaki distance.

Proof. We show that $d_{0, +\infty}$ is Hölder-equivalent to $d_{0, 1}$, the Lemma follows since Sasaki metric is Lipschitz-equivalent to $d_{0, 1}$. Fix $R > 0$, let $u, v \in \pi^{-1}(B_R(o))$, and set

$$x_0 := \pi(u), x_1 := \pi(\mathcal{G}_1(u)), y_0 := \pi(v), y_1 := \pi(\mathcal{G}_1(v)), \zeta := u^+, \eta := v^+.$$

Let $w \in S(y_0)$ be such that $w^+ = \zeta$ and let $y'_1 := \pi(\mathcal{G}_1(w))$. Then a straightforward argument gives

$$d(x_1, y'_1) \asymp_{\mathfrak{b}} d(x_0, y_0).$$

Let $T > 0$ be such that $d_{y_0}(\zeta, \eta) = e^{-T}$, thus $d(\pi(w_T), \pi(v_T)) \asymp_{\mathfrak{b}} 1$. By Lemma 3.4 and Part (i) of Lemma 3.2 we have

$$e^{-\mathfrak{b}T} \prec d(y_1, y'_1) \prec e^{-T}.$$

By triangle inequality we have

$$d(x_1, y_1) \leq d(x_1, y'_1) + d(y'_1, y_1) \prec d(x_0, y_0) + e^{-T} = d(x_0, y_0) + d_{y_0}(\zeta, \eta).$$

Since $d_o(\zeta, \eta) \asymp_R d_{y_0}(\zeta, \eta)$, the above implies $d(x_1, y_1) \prec_R d_{0, +\infty}(u, v)$, hence $d_{0, 1}(u, v) \prec_R d_{0, +\infty}(u, v)$.

On the other hand,

$$e^{-\mathfrak{b}T} \prec d(y_1, y'_1) \leq d(y_1, x_1) + d(x_1, y'_1) \prec_{\mathfrak{b}} d(y_1, x_1) + d(x_0, y_0),$$

thus

$$d_o(\zeta, \eta) \prec_R d_{y_0}(\zeta, \eta) = e^{-T} \prec_{\mathfrak{b}} (d(y_1, x_1) + d(x_0, y_0))^{1/\mathfrak{b}}.$$

This gives $d_o(\zeta, \eta) \prec_R (d_{0, 1}(u, v))^{1/\mathfrak{b}}$, hence $d_{0, +\infty}(u, v) \prec_R (d_{0, 1}(u, v))^{1/\mathfrak{b}}$. □

The following corollary is immediate.

Corollary 3.6. The map from $T^1 \widetilde{M}$ to $\partial_{\infty} \widetilde{M}$ given by $u \mapsto u^+$ is Hölder-continuous.

Lemma 3.7. Fix $A, B > 0$ and let $x, x', y, y' \in \widetilde{M}$ be such that $d(x, x')$ and $d(y, y')$ are both at most A . Let $z \in [x, y]$ be such that the distance between z and the midpoint of $[x, y]$ is at most B . Then there exists a point $z' \in [x', y']$ such that

$$d(z, z') = \mathbf{O}_{A, B}(e^{-d(x, y)/2}).$$

Proof. Let $T := d(x, y)$, and, without loss of generality, assume T is large enough. Since $[x, y]$ and $[x, y']$ get exponentially close to each other (Lemma 3.2), there exists $z_1 \in [x, y']$ with

$$d(z, z_1) = \mathbf{O}_{A,B}(e^{-d(x,y)/2}).$$

Similarly, since $[y', x]$ and $[y', x']$ get exponentially close to each other, there exists $z' \in [y', x']$ with

$$d(z_1, z') = \mathbf{O}(e^{-d(x,y)/2}).$$

□

The following lemma plays a major role in the proofs of Lemma 4.2 and Lemma 4.3.

Lemma 3.8. *Fix $R > 0$, and let x, y, z , and ζ be such that $d(x, y) \leq R$ and $z \in [x, \zeta]$. Then*

$$\beta(\zeta, x, y) = d(z, x) - d(z, y) + \mathbf{O}_R(e^{-d(x,z)}).$$

Proof. Without loss of generality assume $\beta(\zeta, y, x) > 0$. If $y' \in [y, \zeta]$ is such that $d(y, y') = \beta(\zeta, y, x)$, then $\beta(\zeta, x, y') = 0$. We have

$$d(x, y') \leq d(x, y) + d(y, y') \leq R + \beta(\zeta, y, x) \leq R + d(x, y) \leq 2R.$$

Let $T := d(x, z)$, and choose $y'_T \in [y', \zeta]$ such that $d(y', y'_T) = T$. Lemma 3.2 implies $d(z, y'_T) = \mathbf{O}_R(e^{-T})$, so we have

$$d(z, x) - d(z, y) = d(z, x) - d(y'_T, y) + \mathbf{O}_R(e^{-T}) = \beta(\zeta, y, x) + \mathbf{O}_R(e^{-T}).$$

□

Lemma 3.9. *The Busemann cocycle $\beta(\cdot, \cdot, \cdot)$ is $\frac{1}{2}$ -Hölder in its first variable and 1-Lipschitz in its second and third variables.*

Proof. For $\zeta \in \partial_\infty \widetilde{M}$ and $x, x', y \in \widetilde{M}$, we have

$$|\beta(\zeta, x', y) - \beta(\zeta, x, y)| = |\beta(\zeta, x', x)| \leq d(x, x').$$

This proves that β is 1-Lipschitz in its second variable. In a similar way, β is 1-Lipschitz in its third variable.

To prove Hölder-continuity in the first variable, let $R > 0$ be arbitrary, and fix $x, y \in \mathcal{B}_R(o)$. Let $\zeta, \eta \in \partial_\infty \widetilde{M}$ and let T be such that $d_x(\zeta, \eta) = e^{-T}$. Choose $x_\zeta, x'_\zeta \in [x, \zeta]$ and $x_\eta, x'_\eta \in [x, \eta]$ such that $d(x, x_\zeta) = d(x, x_\eta) = T$, and $d(x, x'_\zeta) = d(x, x'_\eta) = T/2$. Since $d(x_\zeta, x_\eta) \asymp 1$, Lemma 3.2 implies $d(x'_\zeta, x'_\eta) = \mathbf{O}(e^{-T/2})$. By Lemma 3.8 we have

$$\begin{aligned} \beta(\zeta, x, y) &= d(x'_\zeta, x) - d(x'_\zeta, y) + \mathbf{O}_R(e^{-T/2}); \\ \beta(\eta, x, y) &= d(x'_\eta, x) - d(x'_\eta, y) + \mathbf{O}_R(e^{-T/2}). \end{aligned}$$

Thus,

$$\begin{aligned} |\beta(\zeta, x, y) - \beta(\eta, x, y)| &\leq |d(x'_\zeta, x) - d(x'_\eta, x)| + |d(x'_\zeta, y) - d(x'_\eta, y)| + \mathbf{O}_R(e^{-T/2}) \\ &\leq 2d(x'_\zeta, x'_\eta) + \mathbf{O}_R(e^{-T/2}) = \mathbf{O}_R(e^{-T/2}) = \mathbf{O}_R(d_o(\zeta, \eta)^{\frac{1}{2}}), \end{aligned}$$

where to obtain the last equality we used the fact that $d_x(\zeta, \eta) \asymp_R d_o(\zeta, \eta)$. □

The following lemma is only used in the proof of Lemma 3.11, in which we prove the Hölder-continuity of a function that appears in the proof of Lemma 4.2.

Lemma 3.10. *Fix $A > 0$, and let G be a geodesic in \widetilde{M} and x a point in \widetilde{M} such that $d(x, G) \leq A$. Set $h := P_G(x)$ and assume $h' \in G$ is such that $d(h, h') \leq 1$. Then we have*

$$d(x, h') - d(x, h) \succ_A d(h, h')^2.$$

Proof. Let $\Delta(\bar{x}, \bar{h}, \bar{h}')$ be a triangle in the Euclidean plane such that $d(\bar{x}, \bar{h}) = d(x, h)$, $d(\bar{h}', \bar{h}) = d(h', h)$, and the angle that $[\bar{h}, \bar{x}]$ and $[\bar{h}, \bar{h}']$ make at \bar{h} is equal to $\frac{\pi}{2}$. By comparison, it is enough to prove the above inequality for $\Delta(\bar{x}, \bar{h}, \bar{h}')$, which is an exercise in Euclidean geometry. \square

Lemma 3.11. *The function $P_o: \partial_\infty \widetilde{M} \times \partial_\infty \widetilde{M} \rightarrow \widetilde{M}$ defined by $P_o(\zeta_1, \zeta_2) = P_{(\zeta_1, \zeta_2)}(o)$ is $\frac{1}{2}$ -Hölder continuous.*

Proof. For $R > 0$, define the compact set $\mathbf{K}_R := \{(\zeta_1, \zeta_2) : d(o, (\zeta_1, \zeta_2)) \leq R\}$, and let $\boldsymbol{\zeta} := (\zeta_1, \zeta_2)$ and $\boldsymbol{\eta} := (\eta_1, \eta_2)$ be elements in \mathbf{K}_R . Define $T > 0$ to be such that

$$e^{-T} = d(\boldsymbol{\zeta}, \boldsymbol{\eta}) = \max\{d_o(\zeta_1, \eta_1), d_o(\zeta_2, \eta_2)\}.$$

Without loss of generality we can assume that T is large enough. By the definition of d_o , if x_1 and y_1 are points at distance T from o on $[o, \zeta_1)$ and $[o, \eta_1)$ respectively, we have $d(x_1, y_1) \prec 1$. Letting $h_\zeta := P_o(\boldsymbol{\zeta})$, since $d(o, h_\zeta) \leq R$ and T is large enough, Lemma 3.2 Part (iv) gives $x'_1 \in [h_\zeta, \zeta_1)$ with $d(x_1, x'_1) \leq 1$. Letting $h_\eta := P_o(\boldsymbol{\eta})$, a similar argument gives $y'_1 \in [h_\eta, \eta_1)$ with $d(y_1, y'_1) \leq 1$. Triangle inequality then implies

$$d(x'_1, y'_1) \leq d(x'_1, x_1) + d(x_1, y_1) + d(y_1, y'_1) \prec 1.$$

The same argument for ζ_2, η_2 replacing ζ_1, η_1 gives $x'_2 \in [h_\zeta, \zeta_2)$ and $y'_2 \in [h_\eta, \eta_2)$ with $d(x'_2, y'_2) \prec 1$. Since $d(x'_1, h_\zeta)$ and $d(h_\zeta, x'_2)$ are both $T + \mathbf{O}(R)$, Lemma 3.7 applied to the quadruple (x'_1, y'_1, x'_2, y'_2) gives a point $h'_\zeta \in (\eta_1, \eta_2)$ with $d(h_\zeta, h'_\zeta) = \mathbf{O}(e^{-T})$. Hence we have

$$d(o, h_\eta) = \inf\{d(o, h) : h \in (\eta_1, \eta_2)\} \leq d(o, h'_\zeta) \leq d(o, h_\zeta) + d(h_\zeta, h'_\zeta) \leq d(o, h_\zeta) + ce^{-T}.$$

for some constant c . In a similar fashion, Lemma 3.7 gives a point $h'_\eta \in (\zeta_1, \zeta_2)$ with $d(h_\eta, h'_\eta) = \mathbf{O}(e^{-T})$, thus we have

$$d(o, h'_\eta) \leq d(o, h_\eta) + d(h_\eta, h'_\eta) \leq d(o, h_\eta) + c'e^{-T} \leq d(o, h_\zeta) + c''e^{-T},$$

for some constants c' and c'' . Since T is large enough, the above gives $d(o, h'_\eta) - d(o, h_\zeta) < 1$. Thus by Lemma 3.10,

$$d(h_\zeta, h'_\eta)^2 \prec_R d(o, h'_\eta) - d(o, h_\zeta),$$

which implies $d(h_\zeta, h'_\eta) = \mathbf{O}_R(e^{-T/2})$. By triangle inequality

$$d(h_\zeta, h_\eta) \leq d(h_\zeta, h'_\eta) + d(h'_\eta, h_\eta) = \mathbf{O}(e^{-\frac{T}{2}}).$$

This concludes the proof. \square

Lemma 3.12. *Let $\gamma \in \Gamma$ and denote the axis of γ by L . Then there exists a constant $c = c(\gamma)$ such that for every $\zeta \in \partial_\infty \widetilde{M} \setminus \text{Fix}(\gamma)$, the geodesic $(\zeta, \gamma\zeta)$ passes through the ball of radius c centered at the midpoint of $[P_L(\zeta), \gamma.P_L(\zeta)]$.*

Proof. Define $f: \partial_\infty \widetilde{M} \rightarrow \mathbb{R}$ by setting $f(\zeta)$ to be the distance between the midpoint of $[P_L(\zeta), \gamma.P_L(\zeta)]$ and the geodesic $(\zeta, \gamma\zeta)$. Fixing $x_0 \in L$, since f is continuous, it attains its maximum c on the compact set $P_L^{-1}[x_0, \gamma x_0]$. Since f is γ -invariant, c is the desired constant. \square

4. REDUCTION TO ADJUSTED COUNTS

The goal of this section is to prove Proposition 4.1, which is directly used in the proof of Theorem A, given right after Theorem 6.1.

Proposition 4.1. *Let $x, y \in \widetilde{M}$, $\text{id} \neq \gamma \in \Gamma$, and denote the axis of γ by L . Then there are Hölder-continuous functions $F_1: \partial^1 L \rightarrow \mathbb{R}$ and $F_2: S(x) \rightarrow \mathbb{R}$ such that F_1 is γ -invariant, and for every $g \in \Gamma$ we have*

$$d(gx, \gamma.gy) = 2d(gx, L) + F_1(v_1(g)) + F_2(g^{-1}.v_2(g)) + \mathbf{O}_{x,y}(e^{-d(gx, L)/2}),$$

where $v_1(g)$ and $v_2(g)$ are tangents to $[gx, P_L(gx)]$ at $P_L(gx)$ and gx respectively.

Proof. The proposition follows from Lemma 4.2 and Lemma 4.3, proved below. \square

We fix x, y, γ, L to be as in Proposition 4.1, and for the rest of this section we do not show the dependence of the implied constants in $\mathbf{O}(\cdot)$ on these fixed parameters; for example, instead of $A = \mathbf{O}_\gamma(B)$ we write $A = \mathbf{O}(B)$.

Lemma 4.2. *Define the function $F_1: \partial^1 L \rightarrow \mathbb{R}$ by*

$$F_1(v) := \beta(v^+, P_{(v^+, \gamma v^+)} o, \pi(v)) + \beta(\gamma v^+, P_{(v^+, \gamma v^+)} o, \gamma \pi(v)). \quad (3)$$

Then F_1 is γ -invariant and Hölder-continuous, and for every $g \in \Gamma$,

$$d(gx, \gamma gx) - 2d(gx, L) = F_1(v_1(g)) + \mathbf{O}(e^{-d(gx, L)}). \quad (4)$$

Proof. The map $v \mapsto v^+$ is Hölder-continuous by Corollary 3.6, $P_{(\cdot, \cdot)} o$ and $\beta(\cdot, \cdot, \cdot)$ are Hölder-continuous by Lemma 3.11 and Lemma 3.9, and $\pi(\cdot)$ is 1-Lipschitz (see Section 2.3. of [PPS15]). One can check that $\gamma: \partial_\infty \widetilde{M} \rightarrow \partial_\infty \widetilde{M}$ is also Hölder-continuous (in fact, it is Lipschitz), thus F_1 , being a combination of Hölder-continuous functions, is Hölder-continuous itself. If we change g with $\gamma^i g$ for some $i \in \mathbb{Z}$, all the terms on the left and right hand sides of (4) remain the same, hence, without loss of generality, we can assume $d(o, P_L(gx)) = \mathbf{O}(1)$. We can moreover assume that $d(gx, L)$ is large enough. We fix such an element g for the rest of the proof.

Let $x_1 := gx$, $x_2 := \gamma gx$, $h := P_L(x_1)$, and $\zeta := (v_1(g))^+$. Letting m denote the midpoint of $[h, \gamma h]$, by Lemma 3.12 there is a point $z \in (\zeta, \gamma \zeta)$ with $d(z, m) = \mathbf{O}(1)$, thus, by triangle inequality, $d(h, z) = \mathbf{O}(1)$. Since $d(gx, L)$ is large enough and $(\zeta, z]$ and $(\zeta, h]$ converge (see Lemma 3.2), we can find $x'_1 \in (\zeta, z]$ with $d(x_1, x'_1) < 1$. The same argument applied to $(\gamma \zeta, z]$ and $(\gamma \zeta, \gamma h]$ implies the existence of $x'_2 \in (\gamma \zeta, z]$ with $d(x_2, x'_2) < 1$. By triangle inequality we have

$$|d(x'_1, z) - d(x_1, h)| \leq d(x'_1, x_1) + d(h, z) = \mathbf{O}(1),$$

thus $d(x'_1, z) = d(x_1, h) + \mathbf{O}(1)$. Similarly, we obtain $d(x'_2, z) = d(x_2, \gamma h) + \mathbf{O}(1) = d(x_1, h) + \mathbf{O}(1)$, hence $d(x'_1, z) = d(x'_2, z) + \mathbf{O}(1)$. This means that z is close to the midpoint of $[x'_1, x'_2]$. Since $[x_1, x_2]$ and $[x'_1, x'_2]$ have close endpoints, by Lemma 3.7 they get exponentially close in the middle. More precisely, there exists $z' \in [x_1, x_2]$ with $d(z, z') = \mathbf{O}(e^{-d(gx, L)})$. We have

$$\begin{aligned} d(gx, \gamma gx) - 2d(gx, L) &= d(x_1, x_2) - d(x_1, h) - d(x_2, \gamma h) \\ &= d(x_1, z') + d(z', x_2) - d(x_1, h) - d(x_2, \gamma h) \\ &= (d(x_1, z') - d(x_1, h)) + (d(x_2, z') - d(x_2, \gamma h)) \\ &= (d(x_1, z) - d(x_1, h)) + (d(x_2, z) - d(x_2, \gamma h)) + \mathbf{O}(e^{-d(gx, L)}) \\ &= \beta(\zeta, z, h) + \beta(\gamma \zeta, z, \gamma h) + \mathbf{O}(e^{-d(gx, L)}), \end{aligned}$$

where the last equality is by Lemma 3.8. Since $\beta(\zeta, z, h) + \beta(\gamma \zeta, \gamma h, z)$ remains the same if z is replaced by an arbitrary $z'' \in (\zeta, \gamma \zeta)$, (4) follows.

For $t > 0$, we can repeat the above argument for x_1 and x_2 replaced by $\pi(v_t)$ and $g.\pi(v_t)$ respectively, to obtain

$$d(\pi(v_t), g.\pi(v_t)) - 2t = F_1(v) + \mathbf{O}(e^{-t}),$$

thus

$$F_1(v) = \lim_{t \rightarrow \infty} d(\pi(v_t), g.\pi(v_t)) - 2t. \quad (5)$$

It is clear from the above formula that F_1 is γ -invariant. \square

Lemma 4.3. *Define the function $F_2: S(x) \rightarrow \mathbb{R}$ by*

$$F_2(v) := \beta(v^+, y, x).$$

Then F_2 is Hölder-continuous, and for every $g \in \Gamma$,

$$d(gx, \gamma gy) - d(gx, \gamma gx) = F_2(g^{-1}.v_2(g)) + \mathbf{O}(e^{-d(gx, L)/2}). \quad (6)$$

Proof. The map $v \mapsto v^+$ is Hölder-continuous by Corollary 3.6 and $\beta(\cdot, \cdot, \cdot)$ is Hölder-continuous by Lemma 3.9, thus F_2 is Hölder-continuous. Fix $g \in \Gamma$ and let x_1, x_2, h, z' be as in Lemma 4.2. Letting $\bar{x}_1 := (\gamma g)^{-1}x_1$, we have

$$d(gx, \gamma gy) - d(gx, \gamma gx) = d(x_1, \gamma gy) - d(x_1, x_2) = d(\bar{x}_1, y) - d(\bar{x}_1, x).$$

Set $\bar{h} := (\gamma g)^{-1}\gamma h$ and $\bar{z}' := (\gamma g)^{-1}z'$, let v and w be tangents to $[x, \bar{h}]$ and $[x, \bar{z}']$ at x respectively, and note that $v = g^{-1}.v_2(g)$. Since $d(\bar{x}_1, x) = d(x_1, x_2) = 2d(gx, L) + \mathbf{O}(1)$, Lemma 3.8 implies

$$d(\bar{x}_1, y) - d(\bar{x}_1, x) = \beta(w^+, y, x) + \mathbf{O}(e^{-2d(gx, L)}).$$

Since $d(\bar{h}, \bar{z}') = d(\gamma h, z') = \mathbf{O}(1)$ and $d(x, \bar{h}) = d(gx, L)$, part (i) of Lemma 3.2 implies $d_x(v^+, w^+) = \mathbf{O}(e^{-d(gx, L)})$. Thus, by Lemma 3.9,

$$\beta(w^+, y, x) = \beta(v^+, y, x) + \mathbf{O}(e^{-d(gx, L)/2}),$$

which concludes the proof. \square

5. THE TEST FUNCTIONS

The goal of this section is to set the stage for Section 6, in which we obtain asymptotics for adjusted counts with a power saving error term. (See Theorem 6.1.) This section and the next closely follow sections 2-4 of [PP17], and Theorem 6.1 is a generalization of Theorem 3 of that paper (when the locally convex sets D^\pm in the statement of that theorem are replaced by $\Pi(x)$ and $\Pi(L)$). Since we make frequent references to [PP17], it is useful to summarize the main differences between our exposition and theirs.

- (i) Instead of a *thickening* (or a *dynamical neighbourhood*) of $\partial^1 L$ (resp. $S(x)$), we first flow $\partial^1 L$ (resp. $S(x)$) by a Hölder-continuous function, and then consider the dynamical thickening of this latter set. (Compare (10) and Equation (16) of [PP17].)
- (ii) Instead of the Hamenstädt distance on the strongly stable foliation, we use the distance induced by $d_{\widetilde{M}}$. This is essential to establish the RHC property. (See Remark 5.1.)
- (iii) Instead of flowing the dynamical neighbourhoods mentioned in (i) by $\frac{t}{2}$, we flow the neighbourhood corresponding to $\partial^1 L$ by time t and keep the neighbourhood corresponding to $S(x)$ fixed. This keeps our presentation closer to the original paper of [Rob03].

As mentioned in the introduction, our main contributions are as follows.

- (a) We verify the RHC condition under the assumptions of Theorem A. (See Section 7.)
- (b) We provide the details of a ‘smoothing argument’ which is only sketched in [PP17, PP14]. (See Appendix A.)

Both of the above are necessary to obtain a power saving error term. In fact, (a) is used in the proof of Lemma 6.4, and (b) is used in the proof of Lemma 6.6.

For a subset A of a topological space X , we denote the characteristic function of A by $\mathbb{1}(A)$. For a compactly supported function $\varphi: T^1\widetilde{M} \rightarrow \mathbb{R}$, define the function $\bar{\varphi}: T^1M \rightarrow \mathbb{R}$ by

$$\bar{\varphi}(u) := \sum_{u' \in \Pi^{-1}(u)} \varphi(u'), \tag{7}$$

and note that

$$\int_{T^1M} \bar{\varphi} = \int_{T^1\widetilde{M}} \varphi.$$

Fix a point $y_0 \in L$, and note that the set $\Delta_\gamma := \{u \in T^1\widetilde{M} : P_L(u^+) \in [y_0, \gamma.y_0]\}$ is a fundamental domain for the action of γ on $T^1\widetilde{M} \setminus \{u : u^+ \in \text{Fix}(\gamma)\}$. For a γ -invariant function $\psi: T^1\widetilde{M} \rightarrow \mathbb{R}$ such that $\psi \times \mathbb{1}(\Delta_\gamma)$ is compactly supported, define the function $\bar{\psi}_\gamma: T^1M \rightarrow \mathbb{R}$ by

$$\bar{\psi}_\gamma(u) := \sum_{u' \in \Pi^{-1}(u) \cap \Delta_\gamma} \psi(u'), \tag{8}$$

and note that

$$\int_{T^1 M} \bar{\psi}_\gamma = \int_{T^1 \widetilde{M}} \psi \times \mathbb{1}(\Delta_\gamma). \quad (9)$$

A *Patterson-Sullivan density* for Γ is a family $(\mu_x)_{x \in \widetilde{M}}$ of finite measures such that for every $x \in \widetilde{M}$ and $\gamma \in \Gamma$ we have $\gamma_* \mu_x = \mu_{\gamma x}$; and for all $x, y \in \widetilde{M}$ and (almost) all $\zeta \in \partial_\infty \widetilde{M}$, we have

$$\frac{d\mu_x}{d\mu_y}(\zeta) = e^{-\delta\beta(\zeta, x, y)}.$$

Such a family exists and is unique upto a multiplicative constant. We pick the family that makes m_{BM} , defined just above Proposition 5.2, a probability measure. To simplify the notation, from now on we set $\beta_o(\zeta, x) := \beta(\zeta, x, o)$, for $x \in \widetilde{M}$ and $\zeta \in \partial_\infty \widetilde{M}$.

Fix $u \in T^1 \widetilde{M}$ for the rest of this paragraph. Define the *strongly stable manifold* of u by

$$W^{\text{ss}}(u) := \{v \in T^1 \widetilde{M} : v^+ = u^+, \text{ and } \beta(u^+, \pi(u), \pi(v)) = 0\}.$$

Using the homeomorphism $v \mapsto v^-$ from $W^{\text{ss}}(u)$ to $\partial_\infty \widetilde{M} - \{u^+\}$, define the measure μ_u^{ss} on $W^{\text{ss}}(u)$ by

$$d\mu_u^{\text{ss}}(v) := e^{-\delta\beta_o(v^-, \pi(v))} d\mu_o(v^-).$$

More precisely, denoting the pushforward of μ_o to $W^{\text{ss}}(u)$ under the above-mentioned homeomorphism by μ'_o , we have

$$d\mu_u^{\text{ss}}(v) = e^{-\delta\beta_o(v^-, \pi(v))} d\mu'_o(v).$$

Define the *stable manifold* of u by $W^s(u) := \{v \in T^1 \widetilde{M} : v^+ = u^+\}$. Using the homeomorphism $(v, t) \mapsto w = \mathcal{G}_t(v)$ from $W^{\text{ss}}(u) \times \mathbb{R}$ to $W^s(u)$, define the measure μ_u^s on $W^s(u)$ by

$$d\mu_u^s(w) := e^{-\delta t} d\mu_u^{\text{ss}}(v) dt.$$

Note that μ_u^{ss} (resp. μ_u^s) only depends on $W^{\text{ss}}(u)$ (resp. $W^s(u)$). Given $r > 0$, define the strongly stable ball of radius r centered at u by

$$B^{\text{ss}}(u, r) := \{v \in W^{\text{ss}}(u) : d_{\widetilde{M}}(\pi(u), \pi(v)) \leq r\},$$

and let $\mu_{\text{B}}^{\text{ss}}(u, r) := \mu_u^{\text{ss}}(B^{\text{ss}}(u, r))$. For $\tau > 0$, define

$$\mathbf{B}_\tau^s(u, r) := \bigcup_{t \in [-\tau, \tau]} \mathcal{G}_t(B^{\text{ss}}(u, r)),$$

and for a subset $S \subset T^1 \widetilde{M}$, define $\mathbf{B}_\tau^s(S, r)$ to be the union of $\mathbf{B}_\tau^s(u, r)$ for all $u \in S$.

Remark 5.1. In [PP17], the strongly stable balls are defined using the Hamenstädt distance instead of $d_{\widetilde{M}}$. In that paper (and in many other references, ex. [PPS15, PP14, BAPP19]) the Hamenstädt distance between $v, v' \in W^{\text{ss}}(u)$ is defined by

$$d_u^{\text{Ham}}(v, v') := \lim_{t \rightarrow +\infty} e^{\frac{1}{2}d(\pi(v_{-t}), \pi(v'_{-t})) - t}.$$

In the original paper of Hamenstädt [Ham89], however, for a fixed positive real number R , a different distance $d_u^{\text{Ham}, R}$ is defined as follows. (This distance is denoted by $\eta_{u, R}$ in [Ham89]). For $t \in \mathbb{R}$, let d_t denote the distance on $W^{\text{ss}}(u_t)$ induced by restriction of the Riemannian structure of \widetilde{M} to its submanifold $\pi(W^{\text{ss}}(u_t))$. For $v, v' \in W^{\text{ss}}(u)$, let T be such that $d_{-T}(v_{-T}, v'_{-T}) = R$, and set

$$d_u^{\text{Ham}, R}(v, v') := e^{-T}.$$

The distances d^{Ham} and $d^{\text{Ham}, R}$ are the same upto a multiplicative constant that only depends on \mathfrak{b} and R , and the equality $d_{u_t}^{\text{Ham}}(v_t, v'_t) = e^{-t} d_u^{\text{Ham}}(v, v')$ also holds for $d^{\text{Ham}, R}$.

To verify the RHC condition in Section 7, we need the boundary of $B^{\text{ss}}(u, r)$ to be C^1 , thus we can use $d_u^{\text{Ham}, R}$ (but not d_u^{Ham}) instead of $d_{\widetilde{M}}$ in the definition of $B^{\text{ss}}(u, r)$. Finally, it is worth mentioning

that the strongly stable balls considered in [Rob03] are different from both ours and [PP17]. Compare $\mathbf{B}_{\frac{r}{2}}^s(u, r)$ with $K^+(\pi(u), r, u^+)$ defined in Chapter 4 of [Rob03].

For the following definitions let D be either a point $x \in \widetilde{M}$, or equal to L . In the former case define $\partial^1 D := S(x)$, and in the latter, set $\partial^1 D$ to be the unite normal bundle of L . Let F be a real-valued function on $\partial^1 D$, and define

$$\mathcal{G}^F : \partial^1 D \rightarrow T^1 \widetilde{M} \quad \text{by} \quad \mathcal{G}^F(u) = \mathcal{G}_{F(u)}(u).$$

For $u \in T^1 \widetilde{M}$, set $p_D(u) := P_D^1(u^+)$, and for $\tau, r > 0$ define the test function $\phi_D^F(r, \tau) : T^1 \widetilde{M} \rightarrow \mathbb{R}$ by

$$\phi_D^F(r, \tau)(u) := \frac{1}{2\tau \mu_{\mathbb{B}}^{\text{ss}}(\mathcal{G}^F(p_D u), r)} \times \mathbb{1}(\mathbf{B}_{\tau}^s(\mathcal{G}^F(\partial^1 D), r))(u). \quad (10)$$

$\mathbf{B}_{\tau}^s(\mathcal{G}^F(\partial^1 D), r)$ is called a thickening, or a dynamical neighbourhood of $\mathcal{G}^F(\partial^1 D)$. By using (7) or (8), depending on whether D is a point or is equal to L , $\phi_D^F(r, \tau)$ descends to a function from $T^1 M$ to \mathbb{R} , which we denote by $\bar{\phi}_D^F(r, \tau)$.

We define the *skinning measure* σ_D on $\partial^1 D$ by

$$d\sigma_D(v) := e^{-\delta\beta_o(v^+, \pi(v))} d\mu_o(v^+). \quad (11)$$

If $D = x$ is a point, then $d\sigma_D(v) = d\mu_x(v^+)$, and for $D = L$, σ_D is γ -invariant ([PP14] Proposition 4(ii)). For a real valued function F_1 defined on $\partial^1 L$, define

$$\sigma_{\gamma}(F_1) := \int_{\partial^1 [y_0, \gamma \cdot y_0]} e^{\delta F_1(u)} d\sigma_L(u) \quad \text{for some } y_0 \in L, \quad (12)$$

and for a real valued function F_2 defined on $S(x)$, set

$$\sigma_x(F_2) := \int_{S(x)} e^{\delta F_2(u)} d\sigma_x(u). \quad (13)$$

For $u \in T^1 \widetilde{M}$, let $\zeta := u^+$, $\eta := u^-$, and $t := d(P_{(\eta, \zeta)} o, \pi(u))$. The map $u \mapsto (\zeta, \eta, t)$ gives a homeomorphism between $T^1 \widetilde{M}$ and $\partial_{\infty} \widetilde{M} \times \partial_{\infty} \widetilde{M} \setminus \{(\zeta, \zeta) : \zeta \in \partial_{\infty} \widetilde{M}\}$. This is called the *Hopf parametrization*. Note that the parameter t above is the *signed distance* between $P_{(\eta, \zeta)} o$ and $\pi(u)$, and the sign is chosen in a way that \mathcal{G}_T in Hopf coordinates is given by $(\zeta, \eta, t) \mapsto (\zeta, \eta, t + T)$. We can write the *Bowen-Margulis measure* in these coordinates as

$$d\widetilde{m}_{\text{BM}} = e^{-\delta d_o(\zeta, \eta)} d\mu_o(\zeta) d\mu_o(\eta) dt,$$

where $d_o(\zeta, \eta) := \beta_o(\zeta, y) + \beta_o(\eta, y)$ for some (and hence all) $y \in (\zeta, \eta)$. Since $\widetilde{m}_{\text{BM}}$ is Γ -invariant, it descends to a measure on $T^1 M$, which we denote by m_{BM} .

Proposition 5.2. *Let D be either a point $x \in \widetilde{M}$, or equal to L , and let F be a real valued function on $\partial^1 D$. Then we have*

$$\int_{T^1 M} \bar{\phi}_D^F(r, \tau)(u) dm_{\text{BM}}(u) = \sigma_{\bullet}(F),$$

where $\bullet = \gamma$ (resp. $\bullet = x$) when $D = L$ (resp. $D = x$).

Proof. We prove the proposition for $D = L$. The proof is similar for the case where D is a point. Let $U_L := \{u \in T^1 \widetilde{M} : u^+ \notin \partial_{\infty} L\}$. Then $p_L : U_L \rightarrow \partial^1 L$ is a fibration, and by Proposition 8 of [PP14], under this fibration, the Bowen-Margulis measure on U_L disintegrates over the skinning measure σ_L on $\partial^1 L$, with the conditional measures $e^{\delta\beta(v^+, \pi(v), \pi(w))} d\mu_v^s(w)$ on $p_L^{-1}(v) = W^s(v)$ for $v \in \partial^1 L$.

Fix a point $y_0 \in L$, and note that $\Delta_\gamma := \{u \in T^1\widetilde{M} : P_L(u^+) \in [y_0, \gamma.y_0]\}$ is a fundamental domain for the action of γ on U_L . By (9) we have

$$\begin{aligned} \int_{T^1M} \bar{\phi}_L^F(r, \tau)(u) dm_{\text{BM}}(u) &= \int_{\Delta_\gamma} \phi_L^F(r, \tau)(u) d\tilde{m}_{\text{BM}}(u) \\ &= \int_{\partial^1[y_0, \gamma.y_0]} \frac{1}{2\tau\mu_{\mathbb{B}}^{\text{ss}}(\mathcal{G}^F(v), r)} \int_{\mathbf{B}_\tau^{\text{s}}(\mathcal{G}^F(v), r)} e^{\delta\beta(v^+, \pi(v), \pi(w))} d\mu_v^{\text{s}}(w) d\sigma_L(v). \end{aligned} \quad (14)$$

Using the homeomorphism $W^{\text{ss}}(\mathcal{G}^F(v)) \times \mathbb{R} \rightarrow W^{\text{s}}(v)$ that sends (w', t) to $\mathcal{G}_t(w')$, we have

$$d\mu_v^{\text{s}}(w) = e^{-\delta t} d\mu_{\mathcal{G}^F(v)}^{\text{ss}}(w') dt = e^{-\delta\beta(v^+, \pi(w'), \pi(w))} d\mu_{\mathcal{G}^F(v)}^{\text{ss}}(w') dt.$$

Hence we can compute

$$\int_{\mathbf{B}_\tau^{\text{s}}(\mathcal{G}^F(v), r)} e^{\delta\beta(v^+, \pi(v), \pi(w))} d\mu_v^{\text{s}}(w) = \int_{\mathbf{B}^{\text{ss}}(\mathcal{G}^F(v), r) \times [-\tau, \tau]} e^{\delta\beta(v^+, \pi(v), \pi(w)) + \delta\beta(v^+, \pi(w), \pi(w'))} d\mu_{\mathcal{G}^F(v)}^{\text{ss}}(w') dt.$$

Since

$$\beta(v^+, \pi(v), \pi(w)) + \beta(v^+, \pi(w), \pi(w')) = \beta(v^+, \pi(v), \pi(w')) = \beta(v^+, \pi(v), \pi(\mathcal{G}^F(v))) = F(v),$$

the above integral is equal to $2\tau e^{\delta F(v)} \mu_{\mathbb{B}}^{\text{ss}}(\mathcal{G}^F(v), r)$. Plugging this into Equation (14) concludes the proof of the proposition. \square

6. ADJUSTED COUNTS

Let $\gamma \in \Gamma$, and let $h: \Gamma \rightarrow \mathbb{R}$ be a γ -invariant function, that is, $h(\gamma g) = h(g)$ for all $g \in \Gamma$. For such a function, define

$$\mathcal{B}_\gamma^h(T) := \{\langle \gamma \rangle \cdot g \in \langle \gamma \rangle \backslash \Gamma : h(g) \leq T\}$$

to be the set of $\langle \gamma \rangle$ -orbits on which the value of h is at most T . For the following theorem, recall that $-\mathfrak{b}^2$ is the lower bound on the curvature of M .

Theorem 6.1. *Assume M is two-dimensional, or M is of dimension at least three and $\mathfrak{b} \leq \frac{n-1}{n-2}$. Then for every $\alpha, \kappa > 0$, there exists a constant κ' , only depending on α, κ , and the geometry of M , such that the following holds. Let $x \in \widetilde{M}$, $\text{id} \neq \gamma \in \Gamma$, and L denote the axis of γ . Let*

$$F_1: \partial^1 L \rightarrow \mathbb{R}, \quad \text{and} \quad F_2: S(x) \rightarrow \mathbb{R}$$

be α -Hölder functions, and assume that F_1 is γ -invariant. Let the γ -invariant function $h: \Gamma \rightarrow \mathbb{R}$ satisfy

$$h(g) = d(gx, L) - F_1(P_L^1(gx)) - F_2(g^{-1}.P_{gx}^1(L)) + \mathbf{O}(e^{-\kappa d(gx, L)}) \quad (15)$$

for $g \in \Gamma$. Then for $T > 0$,

$$\#\mathcal{B}_\gamma^h(T) = (1 + \mathbf{O}_{F_1, F_2}(e^{-\kappa' T})) \frac{e^{\delta T}}{\delta} \sigma_\gamma(F_1) \sigma_x(F_2),$$

where $\sigma_\gamma(F_1)$ and $\sigma_x(F_2)$ are as in (12) and (13)

Assuming the above theorem, we give a proof of Theorem A.

Proof of Theorem A. We say that an element $g \in \Gamma$ is primitive if $g = \hat{g}^k$ (for some $\hat{g} \in \Gamma$) implies $k = \pm 1$. Let the primitive element $\hat{\gamma}$ be such that $\gamma = \hat{\gamma}^k$ for some $k \in \mathbb{N}$. Since

$$d(x, g^{-1}\gamma g.y) = d(gx, \gamma g.y),$$

and the centralizer of γ is $\langle \hat{\gamma} \rangle$, the map $g \mapsto g^{-1}\gamma g.y$ gives a one-to-one correspondance between the sets

$$\{\langle \hat{\gamma} \rangle \cdot g : d(gx, \gamma g.y) \leq T\}, \quad \text{and} \quad B_T(x) \cap \text{Conj}_\gamma \cdot y.$$

Denoting the axis of γ by L , by Proposition 4.1 there exists a γ -invariant function F_1 on $\partial^1 L$ and a function F_2 on $S(x)$, both Hölder-continuous, such that for every $g \in \Gamma$ we have

$$d(gx, \gamma g.y) = 2d(gx, L) + F_1(P_L^1(gx)) + F_2(g^{-1}.P_{gx}^1(L)) + \mathbf{O}_{x,y,\gamma}(e^{-d(gx,L)/2}).$$

Note that by (5), F_1 is $\hat{\gamma}$ -invariant, thus, applying Theorem 6.1 to $\hat{\gamma}$, $-F_1$, $-F_2$ and $h(g) := d(gx, \gamma g.y)/2$, the desired result follows for

$$\sigma := \frac{1}{\delta} \sigma_{\hat{\gamma}}\left(\frac{-F_1}{2}\right) \sigma_x\left(\frac{-F_2}{2}\right).$$

□

For the rest of this section, we fix $\alpha_1 := \alpha$, $\kappa_1 := \kappa$, x , γ , L , F_1 , F_2 to be as in Theorem 6.1, and denote $\sigma_{\gamma}(F_1)$ (resp. $\sigma_x(F_2)$) by $\sigma(F_1)$ (resp. $\sigma(F_2)$). We also fix the function h to be as in this theorem, and denote $\mathcal{B}_{\gamma}^h(T)$ by $\mathcal{H}(T)$. Using the action of deck transformations on $T^1\widetilde{M}$, we can define F_2 on $\cup_{g \in \Gamma} S(gx)$, so from now on we assume that F_2 is defined on this larger domain. For $g \in \Gamma$, let $v_1(g)$ and $v_2(g)$ be tangents to $[gx, P_L(gx)]$ at $P_L(gx)$ and gx respectively. With these definitions, the assumption (15) of Theorem 6.1 can be written as

$$h(g) = d(gx, L) - F_1(v_1(g)) - F_2(v_2(g)) + \mathbf{O}(e^{-\kappa_1 d(gx,L)}), \quad (16)$$

and the statement of this theorem can be written as

$$\#\mathcal{H}(T) = (1 + \mathbf{O}(e^{-\kappa'T})) \frac{e^{\delta T}}{\delta} \sigma(F_1) \sigma(F_2) \quad \text{for some } \kappa' > 0.$$

Note that $h(g)$ differs from $d(gx, L)$ by a bounded amount which is determined, upto an exponentially small error, by the manner that the perpendicular from gx to L departs from gx and enters L . We say that $h(g)$ is the distance between gx and L , adjusted by *adjustment functions* F_1 and F_2 , or simply the *adjusted height* of gx .

Let X be a topological space, $\varphi: X \rightarrow \mathbb{R}$ an arbitrary function, and $G: X \rightarrow X$ a homeomorphism. Define $G \cdot \varphi: X \rightarrow \mathbb{R}$ by $G \cdot \varphi := \varphi \circ G^{-1}$. Note that if $H: X \rightarrow X$ is another homeomorphism, then $(G \circ H) \cdot \varphi = G \cdot (H \cdot \varphi)$. Since \mathcal{G}_t and ι are homeomorphisms from $T^1\widetilde{M}$ (resp. T^1M) to itself, $\mathcal{G}_t \cdot \varphi$ and $\iota \cdot \varphi$ can be defined for a function $\varphi: T^1\widetilde{M} \rightarrow \mathbb{R}$ (resp. $\varphi: T^1M \rightarrow \mathbb{R}$). Associating to $g \in \Gamma$ its deck transformation, $g \cdot \varphi$ can be defined for a function $\varphi: T^1\widetilde{M} \rightarrow \mathbb{R}$. Let φ and ψ be arbitrary functions from $T^1\widetilde{M}$ to \mathbb{R} , and assume that φ is γ -invariant. Let $\bar{\varphi}_{\gamma}$ and $\bar{\psi}$ be defined by equations (8) and (7) respectively. Denoting $\langle \gamma \rangle \cdot g$ by $[g]$, one can check that

$$\int_{T^1M} \bar{\varphi}_{\gamma} \times \bar{\psi} dm_{\text{BM}} = \sum_{[g] \in \langle \gamma \rangle \backslash \Gamma} \int_{T^1\widetilde{M}} \varphi \times g \cdot \psi d\tilde{m}_{\text{BM}}. \quad (17)$$

Fix a radius R for the rest of this section, and using equation (10), define the test functions ϕ_{τ}^i for $i = 1, 2$ by

$$\phi_{\tau}^1 := \phi_L^{F_1}(R, \tau), \quad \text{and} \quad \phi_{\tau}^2 := \phi_x^{F_2}(R, \tau).$$

Plugging $\varphi := \mathcal{G}_t \cdot \phi_{\tau}^1$ and $\psi := \iota \cdot \phi_{\tau}^2$ in (17) we obtain

$$\int_{T^1M} \mathcal{G}_t \cdot \bar{\phi}_{\tau}^1 \times \iota \cdot \bar{\phi}_{\tau}^2 dm_{\text{BM}} = \sum_{[g] \in \langle \gamma \rangle \backslash \Gamma} \int_{T^1\widetilde{M}} j_{\tau}(g, t) d\tilde{m}_{\text{BM}} = \sum_{[g] \in \langle \gamma \rangle \backslash \Gamma} J_{\tau}(g, t), \quad (18)$$

where $j_{\tau}(g, t): T^1\widetilde{M} \rightarrow \mathbb{R}$ and $J_{\tau}(g, t) \in \mathbb{R}$ are defined by

$$j_{\tau}(g, t) := \mathcal{G}_t \cdot \phi_{\tau}^1 \times (\iota \cdot g \cdot \phi_{\tau}^2), \quad \text{and} \quad J_{\tau}(g, t) := \int_{T^1\widetilde{M}} j_{\tau}(g, t) d\tilde{m}_{\text{BM}}.$$

For $0 < T_1 < T_2$, multiplying left and right hand sides of (18) by $e^{\delta t}$ and integrating over $T_1 \leq t \leq T_2$, we obtain

$$I_\tau(T_1, T_2) := \int_{T_1}^{T_2} e^{\delta t} \int_{T^1 M} \mathcal{G}_t \cdot \bar{\phi}_\tau^1 \times \iota \cdot \bar{\phi}_\tau^2 dm_{\text{BM}} dt = \sum_{[g] \in \langle \gamma \rangle \backslash \Gamma} \int_{T_1}^{T_2} e^{\delta t} J_\tau(g, t) dt \quad (19)$$

To prove Theorem 6.1, in the first step we show that the right hand side of the above, for $T_1 := \frac{T}{2}$ and $T_2 := T$, approximates the number of γ -orbits of adjusted height at most T . In the next step, we compute the left hand side of (19), using the mixing property of geodesic flow. For the first step we need Lemma 6.4, and for the second step we need Lemma 6.6. Assuming these lemmas, the details of the proof is given in Section 6.3. This method is similar to the one used in [PP17, Rob03].

6.1. $J_\tau^\infty(g)$ is exponentially close to 1. Recall the notation introduced in Section 5, and recall that we fixed a radius $R > 0$ throughout the paper. The following is an adaptation of Lemma 7 of [PP17] to our setting. (See Item (iii) at the beginning of Section 5.)

Lemma 6.2. *There are constants κ_2 and κ_3 such that the following holds. Let $\tau > 0$, and let $w_1 \in \mathbf{B}_\tau^s(\mathcal{G}^{F_1}(\partial^1 L), R)$ be such that $w_2 := \iota(\mathcal{G}_t(w_1)) \in \mathbf{B}_\tau^s(\mathcal{G}^{F_2}(S(gx)), R)$. Setting $v'_1 := p_L(w_1)$ and $v'_2 := p_{gx}(w_2)$, we have*

$$d(v_i(g), v'_i) = \mathbf{O}(e^{-\kappa_2 d(gx, L)}), \quad \text{for } i = 1, 2; \quad (20)$$

$$d(gx, L) = t + F_1(v_1(g)) + F_2(v_2(g)) + \mathbf{O}(\tau + e^{-\kappa_3 d(gx, L)}). \quad (21)$$

Proof. Let $t' := t + F_1(v'_1) + F_2(v'_2)$ and $w' := \mathcal{G}_{\frac{t'}{2} - F(v'_1)} w_1$. Then for some $r = \mathbf{O}(R)$ we have

$$\iota(\mathcal{G}_{\frac{t'}{2}} w') \in \mathbf{B}_\tau^s(S(x), r), \quad \text{and} \quad \mathcal{G}_{-\frac{t'}{2}} w' \in \mathbf{B}_\tau^s(\partial^1 L, r).$$

Let $h := P_L(gx)$ be the foot of perpendicular from gx to L . By the third item in [PP17] Lemma 7, there exists $y \in [gx, h]$ with $d(\pi(w'), y) = \mathbf{O}(e^{-\frac{t'}{2}})$. By the same item (of the same lemma) we have $d(\pi(w'), y') = \mathbf{O}(\tau + e^{-\frac{t'}{2}})$ for $y' := \mathcal{G}_{\frac{t'}{2}} v'_1$. Triangle inequality then implies $d(y, y') = \mathbf{O}(\tau + e^{-\frac{t'}{2}}) = \mathbf{O}(1)$. Since $d(y', \pi(v'_1)) = \frac{1}{2}d(gx, L) + \mathbf{O}(1)$, Proposition 3.3 implies $d(v_1(g), v'_1) = \mathbf{O}(e^{-\kappa_2 d(gx, L)})$ for $\kappa_2 := \frac{1}{2}$. This proves (20) for $i = 1$. The proof for $i = 2$ is analogous.

The first item of [PP17] Lemma 7 implies that

$$d(gx, L) = t' + \mathbf{O}(\tau + e^{-\frac{t'}{2}}) = t + F_1(v'_1) + F_2(v'_2) + \mathbf{O}(\tau + e^{-\frac{t'}{2}})$$

Since $d(v_i(g), v'_i) = \mathbf{O}(e^{-\kappa_2 d(gx, L)})$ and F_i is α_1 -Hölder we have $F_i(v'_i) = F_i(v_i(g)) + \mathbf{O}(e^{-\kappa d(gx, L)})$ for $\kappa := \kappa_2 \alpha_1$. Since $t = d(gx, L) + \mathbf{O}(1)$, we have $\mathbf{O}(\tau + e^{-\frac{t'}{2}}) = \mathbf{O}(\tau + e^{-\frac{d(gx, L)}{2}})$. Equation (21) now follows for $\kappa_3 := \min\{\frac{1}{2}, \kappa\}$. \square

Corollary 6.3. *There exists a constant κ_4 such that the following holds. Let $\tau > 0$, $g \in \Gamma$, and $t > 0$ be such that $j_\tau(g, t) \neq 0$. Then we have*

$$h(g) = t + \mathbf{O}(\tau + e^{-\kappa_4 d(gx, L)}).$$

Proof. Let u be such that $j_\tau(g, t)(u) \neq 0$, and set $w_1 := \mathcal{G}_{-t} u$ and $w_2 := \iota(u)$. Then w_1 and w_2 satisfy the assumptions of Lemma 6.2, hence Equation (21) implies

$$d(gx, L) - F_1(v_1(g)) - F_2(v_2(g)) = t + \mathbf{O}(\tau + e^{-\kappa_3 d(gx, L)}).$$

This, combined with (16), implies the lemma for $\kappa_4 := \min\{\kappa_1, \kappa_3\}$. \square

Define

$$J_\tau^\infty(g) := \int_0^\infty e^{\delta t} J_\tau(g, t) dt.$$

The following lemma is proved in Section 3.2. of [PP17] (see Equation (23) of that paper), when F_1 and F_2 are both zero functions and with a slightly different definition for $B^{\text{ss}}(u, r)$. (See Remark 5.1.) The proof is completely analogous in our case, hence we only provide a sketch of the proof.

Lemma 6.4. *Assume M is two-dimensional, or M is of dimension at least three and $\mathfrak{b} \leq \frac{n-1}{n-2}$. Then there exists a constant κ_5 such that for all $g \in \Gamma$ and $\tau \geq e^{-\kappa_5 d(gx, L)}$ we have*

$$J_\tau^\infty(g) = 1 + \mathbf{O}(\tau).$$

Proof. We fix g throughout the proof. We also fix $\tau \geq e^{-\hat{\kappa}d(gx, L)}$ for some small enough $\hat{\kappa}$ to be optimized in the course of the proof. We say that two vectors $u, v \in T^1\widetilde{M}$ are exponentially close if $d(u, v) = \mathbf{O}(e^{-\kappa d(gx, L)})$ for some constant κ . We make a similar definition for real numbers.

If $J_\tau(g, t) \neq 0$ for some t , then by Corollary 6.3 (and imposing $\hat{\kappa} \leq \kappa_4$) we have $t = h(g) + \mathbf{O}(\tau)$, therefore

$$J_\tau^\infty(g) = (1 + \mathbf{O}(\tau)) e^{\delta h(g)} \int_0^\infty J_\tau(g, t) dt = \int_{T^1\widetilde{M} \times \mathbb{R}} j(g, t)(u) d\widetilde{m}_{\text{BM}} dt.$$

For $D = L$ or D a point in \widetilde{M} define $p_D^F(u) := \mathcal{G}^F(p_D(u))$. Define $v_i^F := \mathcal{G}^{F_i}(v_i(g))$ for $i = 1, 2$. If $j_\tau(g, t)(u) \neq 0$ for some u , then by Lemma 6.2 $v_1(g)$ and $p_L u$ are exponentially close, hence by Lemma 3.1, $v_1^F := \mathcal{G}^F(v_1(g))$ and $p_L^F(u)$ are exponentially close. by Theorem 7.7, M has the RHC property, thus Lemma A.5 implies $\mu_B^{\text{ss}}(v_1^F, R)$ and $\mu_B^{\text{ss}}(p_L^F(u), R)$ are exponentially close. By a similar argument, $\mu_B^{\text{ss}}(v_2^F, R)$ and $\mu_B^{\text{ss}}(p_{gx}^F(\iota(u)), R)$ are also exponentially close. As a result, there exists κ such that for all $u \in \text{Supp}(j_\tau(g, t))$ we have

$$j_\tau(g, t)(u) = \frac{1}{4\tau^2 \mu_B^{\text{ss}}(p_L^F(u), R) \mu_B^{\text{ss}}(p_{gx}^F(\iota(u)), R)} = (1 + \mathbf{O}(e^{-\kappa d(gx, L)})) \frac{1}{4\tau^2 \mu_B^{\text{ss}}(v_1^F, R) \mu_B^{\text{ss}}(v_2^F, R)}.$$

Let S_g denote the support of $j_\tau(g, \cdot)(\cdot)$, that is,

$$S_g := \{(u, t) : j_\tau(g, t)(u) \neq 0\}.$$

We have

$$J_\tau^\infty(g) = (1 + \mathbf{O}(e^{-\kappa d(gx, L)} + \tau)) e^{\delta h(g)} \frac{1}{4\tau^2 \mu_B^{\text{ss}}(v_1^F, R) \mu_B^{\text{ss}}(v_2^F, R)} \int_{T^1\widetilde{M} \times \mathbb{R}} \mathbb{1}_{S_g}(u, t) d\widetilde{m}_{\text{BM}}(u). dt. \quad (22)$$

Imposing $\hat{\kappa} \leq \kappa$, we can replace the term $\mathbf{O}(e^{-\kappa d(gx, L)} + \tau)$ in the above equation by $\mathbf{O}(\tau)$.

For $u \in T^1\widetilde{M}$ and for $i = 1, 2$ define $u'_i \in W^{\text{ss}}(v_i^F)$ such that $(u'_1)^- = u^-$ and $(u'_2)^- = u^+$, and let $s := d(P_{(u^-, u^+)o}, \pi(u))$ be the time parameter in the Hopf parametrization u (see Section 5). The map $u \mapsto (u'_1, u'_2, s)$ is a homeomorphism from $T^1\widetilde{M} \setminus \{u : u^- = (v_1^F)^+ \text{ or } u^+ = (v_2^F)^+\}$ to $W^{\text{ss}}(v_1^F) \times W^{\text{ss}}(v_2^F) \times \mathbb{R}$. Note that if $d(gx, L)$ is large enough and $j_\tau(g, t)(u) \neq 0$, then $u \in T^1\widetilde{M} \setminus \{u : u^- = (v_1^F)^+, \text{ or } u^+ = (v_2^F)^+\}$. We identify u with (u'_1, u'_2, s) for the rest of this proof.

Define $\mathbf{B}_g \subset W^{\text{ss}}(v_1^F) \times W^{\text{ss}}(v_2^F)$ by

$$\mathbf{B}_g := \{(u'_1, u'_2) : j_\tau(g, t)(u'_1, u'_2, s) \neq 0 \text{ for some } t, s\}.$$

By the properties of $H^{\bullet\bullet}$ introduced in Lemma A.5, there exists κ' such that for $\epsilon := e^{-\kappa' d(gx, L)}$ we have

$$B^{\text{ss}}(v_1^F, r - \epsilon) \times B^{\text{ss}}(v_2^F, r - \epsilon) \subset \mathbf{B}_g \subset B^{\text{ss}}(v_1^F, r + \epsilon) \times B^{\text{ss}}(v_2^F, r + \epsilon). \quad (23)$$

For $(u'_1, u'_2) \in \mathbf{B}_g$, set

$$P_{(u'_1, u'_2)} := \{(t, s) : j_\tau(g, t)(u'_1, u'_2, s) \neq 0\}.$$

Let P_τ denote the rhombus $\{(t, s) : |s| \leq \tau \text{ and } |t - s| \leq \tau\}$. It can be proved that $P_{(u'_1, u'_2)}$ is always a translation of P_τ . More precisely, there are functions \bar{t} and \bar{s} from \mathbf{B}_g to \mathbb{R} such that

$$P_{(u'_1, u'_2)} = (\bar{t}(u'_1, u'_2), \bar{s}(u'_1, u'_2)) + P_\tau \text{ for } (u'_1, u'_2) \in \mathbf{B}_g. \quad (24)$$

Arguing as in Lemma 10 of [PP17], for $\hat{\kappa}$ small enough we have

$$d\tilde{m}_{\text{BM}}(u) = (1 + \mathbf{O}(\tau)) e^{-\delta h(g)} d\mu_{v_1^F}^{\text{ss}}(u'_1) d\mu_{v_2^F}^{\text{ss}}(u'_2) ds,$$

therefore

$$\begin{aligned} \int_{T^1 \tilde{M} \times \mathbb{R}} \mathbb{1}_{S_g}(u, t) d\tilde{m}_{\text{BM}}(u) dt &= (1 + \mathbf{O}(\tau)) e^{-\delta h(g)} \int_{W^{\text{ss}}(v_1^F) \times W^{\text{ss}}(v_2^F) \times \mathbb{R} \times \mathbb{R}} \mathbb{1}_{S_g}(u'_1, u'_2, s, t) \times \\ &\quad d\mu_{v_1^F}^{\text{ss}}(u'_1) d\mu_{v_2^F}^{\text{ss}}(u'_2) ds dt \\ &= 4\tau^2 e^{-\delta h(g)} \int_{W^{\text{ss}}(v_1^F) \times W^{\text{ss}}(v_2^F)} \mathbb{1}_{\mathbf{B}_g}(u'_1, u'_2) d\mu_{v_1^F}^{\text{ss}}(u'_1) d\mu_{v_2^F}^{\text{ss}}(u'_2) \text{ (by (24))} \\ &= 4\tau^2 e^{-\delta h(g)} (1 + \mathbf{O}(\tau)) \mu_{\mathbf{B}}^{\text{ss}}(v_1^F, R) \mu_{\mathbf{B}}^{\text{ss}}(v_2^F, R). \end{aligned}$$

For the last equality, we used (23) and Theorem 7.7, and assumed $\hat{\kappa}$ is small enough. Plugging the above into (22) concludes the proof for $\kappa_5 := \hat{\kappa}$. \square

6.2. The exponential decay of correlations for test functions. For a real valued α -Hölder function φ on a metric space (X, d) , define

$$\|\varphi\|_\alpha := \max \left\{ \|\varphi\|_\infty, \sup_{x \neq y \in X} \frac{|\varphi(x) - \varphi(y)|}{d(x, y)^\alpha} \right\}.$$

We say that the geodesic flow \mathcal{G}_t (on M) is *exponentially mixing for Hölder regularity α* (for Bowen-Margulis measure), if there are constants $\kappa = \kappa(\alpha)$ and $c = c(\alpha)$ such that for all α -Hölder functions $\varphi_1, \varphi_2: T^1 M \rightarrow \mathbb{R}$ and $t > 0$ we have

$$\int_{T^1 M} \mathcal{G}_t \cdot \varphi_1 \times \varphi_2 dm_{\text{BM}} = \int_{T^1 M} \varphi_1 dm_{\text{BM}} \int_{T^1 M} \varphi_2 dm_{\text{BM}} + \mathbf{O}(ce^{-\kappa t}) \|\varphi_1\|_\alpha \|\varphi_2\|_\alpha. \quad (25)$$

If \mathcal{G}_t is exponentially mixing for some Hölder regularity α_0 , (or for C^{n_0} functions equipped with Sobolev norm of order n_0), then it is exponentially mixing for all Hölder regularity α (and for C^n functions equipped with Sobolev norm of order n , for all positive integers n). See Lemma B.1. of [DKRH21] for a proof. If M is two-dimensional, or if M is of dimension at least three and $\frac{1}{9}$ -pinched (that is, $\mathbf{b} \leq 3$), the geodesic flow is exponentially mixing. (See [Dol98] and [GLP13].)

The functions $\bar{\phi}_\tau^i$, introduced at the beginning of this section, are not Hölder-continuous; in fact, they are not even continuous. However, an equation similar to (25) holds for these functions. (See Lemma 6.6.) The reason is that $\bar{\phi}_\tau^i$ can be well-approximated by Hölder-continuous functions, with some control over the Hölder constants. Such functions are constructed in Appendix A, and a similar construction is sketched in Section 6 of [PP14]. We use the following elementary lemma in the proof of Lemma 6.6.

Lemma 6.5. *Let (X, μ) be a measured space and $\varphi, \psi, \varphi', \psi'$ be measurable functions with finite L^1 and L^∞ norms. Then we have*

$$|\langle \varphi, \psi \rangle - \langle \varphi', \psi' \rangle| \leq \|\varphi - \varphi'\|_{L^1} \|\psi\|_{L^\infty} + \|\varphi'\|_{L^\infty} \|\psi' - \psi\|_{L^1},$$

where the inner product $\langle \varphi, \psi \rangle$ is defined to be $\int_X \varphi \psi d\mu$.

Lemma 6.6. *Assume M is two-dimensional, or M is of dimension at least three and $\mathbf{b} \leq 3$. Then there exists a constant κ_6 such that for all $t > 0$ and $\tau \geq e^{-\kappa_6 t}$ we have*

$$\int_{T^1 M} \mathcal{G}_t \cdot \bar{\phi}_\tau^1 \times \iota \cdot \bar{\phi}_\tau^2 dm_{\text{BM}} = (1 + \mathbf{O}(\tau)) \sigma_\gamma(F_1) \sigma_x(F_2).$$

Proof. To simplify the notation, we will drop dm_{BM} when we integrate over T^1M . Let α_6 , α_7 , and n_1 be the constants given by Lemma A.3. Then, by this lemma, for every $\epsilon < \tau^{n_1}$ there are functions $\bar{\chi}_i(\tau, \epsilon) := \bar{\chi}_{F_i}(R, \tau, \epsilon)$, for $i = 1, 2$, such that

- (i) $0 \leq \bar{\chi}_i(\tau, \epsilon) \leq \bar{\phi}_\tau^i$.
- (ii) $\bar{\chi}_i(\tau, \epsilon)$ is α_6 -Hölder with α_6 -norm of order $\frac{1}{\tau\epsilon}$.
- (iii) $\|\bar{\phi}_\tau^i - \bar{\chi}_i(\tau, \epsilon)\|_{L^1} = \mathbf{O}(\frac{\epsilon^{\alpha_7}}{\tau})$.

Since ι is an isometry with respect to Sasaki metric, $\iota \cdot \bar{\chi}_2(\tau, \epsilon)$ is $(\alpha_6, \mathbf{O}(\frac{1}{\tau\epsilon}))$ -Hölder. Writing (25) for $\alpha := \alpha_6$, we obtain $\kappa' = \kappa'(\alpha_6)$ and $c = c(\alpha_6)$ such that for all $t > 0$ we have

$$\begin{aligned} \int_{T^1M} \mathcal{G}_t \cdot \bar{\chi}_1(\tau, \epsilon) \times \iota \cdot \bar{\chi}_2(\tau, \epsilon) &= \int_{T^1M} \bar{\chi}_1(\tau, \epsilon) \times \int_{T^1M} \iota \cdot \bar{\chi}_2(\tau, \epsilon) \\ &\quad + \mathbf{O}(ce^{-\kappa't}) \|\bar{\chi}_1(\tau, \epsilon)\|_{\alpha_6} \|\iota \cdot \bar{\chi}_2(\tau, \epsilon)\|_{\alpha_6}. \end{aligned}$$

Since $\iota: T^1M \rightarrow T^1M$ is measure preserving, $\int_{T^1M} \iota \cdot \bar{\chi}_2(\tau, \epsilon) = \int_{T^1M} \bar{\chi}_2(\tau, \epsilon)$, thus

$$\int_{T^1M} \mathcal{G}_t \cdot \bar{\chi}_1(\tau, \epsilon) \times \iota \cdot \bar{\chi}_2(\tau, \epsilon) = \int_{T^1M} \bar{\chi}_1(\tau, \epsilon) \times \int_{T^1M} \bar{\chi}_2(\tau, \epsilon) + \mathbf{O}(\frac{e^{-\kappa't}}{\tau^2\epsilon^2}). \quad (26)$$

Item (iii) above implies that

$$\int_{T^1M} \bar{\chi}_i(\tau, \epsilon) = \int_{T^1M} \bar{\phi}_i(\tau) + \mathbf{O}(\frac{\epsilon^{\alpha_7}}{\tau}) = (1 + \mathbf{O}(\frac{\epsilon^{\alpha_7}}{\tau})) \sigma(F_i). \quad (27)$$

After using Lemma 6.5, Item (iii) above, coupled with the fact that both $\|\bar{\phi}_\tau^i\|_{L^\infty}$ and $\|\bar{\chi}_i(\tau, \epsilon)\|_{L^\infty}$ are of order $\frac{1}{\tau}$, implies

$$\int_{T^1M} \mathcal{G}_t \cdot \bar{\chi}_1(\tau, \epsilon) \times \iota \cdot \bar{\chi}_2(\tau, \epsilon) = \int_{T^1M} \mathcal{G}_t \cdot \bar{\phi}_\tau^1 \times \iota \cdot \bar{\phi}_\tau^2 + \mathbf{O}(\frac{\epsilon^{\alpha_7}}{\tau^2}).$$

Plugging the above equation and (27) into (26), we get

$$\int_{T^1M} \mathcal{G}_t \cdot \bar{\phi}_\tau^1 \times \iota \cdot \bar{\phi}_\tau^2 = (1 + \mathbf{O}(\frac{e^{-\kappa't}}{\tau^2\epsilon^2} + \frac{\epsilon^{\alpha_7}}{\tau} + \frac{\epsilon^{\alpha_7}}{\tau^2})) \sigma(F_1)\sigma(F_2).$$

Let $n := \max\{n_1, \frac{3}{\alpha_7}\}$. Setting $\epsilon := \tau^n$ in the above equation gives

$$\int_{T^1M} \mathcal{G}_t \cdot \bar{\phi}_\tau^1 \times \iota \cdot \bar{\phi}_\tau^2 = (1 + \mathbf{O}(\frac{e^{-\kappa't}}{\tau^{2n+2}} + \tau)) \sigma(F_1)\sigma(F_2).$$

Let κ_6 be a small enough constant such that $\kappa_6(2n+3) < \kappa'$. Then we have

$$\frac{e^{-\kappa't}}{\tau^{2n+2}} < \tau,$$

hence we can replace the term

$$\mathbf{O}(\frac{e^{-\kappa't}}{\tau^{2n+2}} + \tau)$$

in the last Equation by $\mathbf{O}(\tau)$. This concludes the proof of the lemma. \square

6.3. Proof of Theorem 6.1. For $0 < T_1 < T_2$, recall the definition of $I_\tau(T_1, T_2)$ given in (19).

Lemma 6.7. *Assume M is two-dimensional, or M is of dimension at least three and $\mathfrak{b} \leq 3$. Then there exists a constant κ_7 such that the following holds. Let $a > 0$, then there exists a constant $c = c(a)$ such that for all $T \geq 0$, $\tau \geq e^{-\kappa_7 T}$, and real number a' with $|a'| \leq a$ we have*

$$I_\tau(\frac{T}{2} + a', T) = (1 + \mathbf{O}(c\tau)) \frac{\sigma(F_1)\sigma(F_2)}{\delta} e^{\delta T}.$$

Proof. Let κ_6 be the constant given by Lemma 6.6. Choose $\kappa_7 < \min\{\frac{\kappa_6}{2}, \frac{\delta}{2}\}$ and, without loss of generality, assume T is large enough. Since $\kappa_7 < \frac{\kappa_6}{2}$, for $t \geq \frac{T}{2} + a'$ we have $\tau \geq e^{-\kappa_6 t}$, hence Lemma 6.6 implies that for such t ,

$$\begin{aligned} \int_{T^1 M} \mathcal{G}_t \cdot \bar{\phi}_\tau^1 \times \iota \cdot \bar{\phi}_\tau^2 dm_{\text{BM}} &= (1 + \mathbf{O}(\tau)) \sigma(F_1) \sigma(F_2) \\ \implies I_\tau\left(\frac{T}{2} + a', T\right) &= (1 + \mathbf{O}(\tau)) \sigma(F_1) \sigma(F_2) \int_{\frac{T}{2} + a'}^T e^{\delta t} dt. \end{aligned}$$

Letting $c' := e^{\delta a}$, we get

$$\int_{\frac{T}{2} + a'}^T e^{\delta t} dt = \frac{e^{\delta T}}{\delta} (1 + \mathbf{O}(c' e^{-\frac{\delta}{2} T})),$$

thus

$$I_\tau\left(\frac{T}{2} + a', T\right) = (1 + \mathbf{O}(\tau + c' e^{-\frac{\delta}{2} T})) \frac{\sigma(F_1) \sigma(F_2)}{\delta} e^{\delta T}.$$

Since $\kappa_7 < \frac{\delta}{2}$, the lemma follows. \square

Recall the definitions of the adjusted height function $h(g)$ and counting sets $\mathcal{H}(T)$ given at the beginning of this section. The following upper bound on $\#\mathcal{H}(T)$ is the last ingredient of the proof of Theorem 6.1.

Lemma 6.8. *For $T \geq 0$, we have $\#\mathcal{H}(T) = \mathbf{O}(e^{\delta T})$.*

Proof. Recall that the origin o belongs to the axis of γ , denoted by L . Define

$$\mathcal{H}_\Delta(T) := \{gx : [g] \in \mathcal{H}(T), \text{ and } P_L(gx) \in [o, \gamma o)\},$$

and note that we have $\#\mathcal{H}(T) = \#\mathcal{H}_\Delta(T)$. Given $gx \in \mathcal{H}_\Delta(T)$ with large enough distance from L , by Equation (16) we have $d(gx, L) \leq T + M_1 + M_2 + 1$, where M_i is the supremum of $|F_i|$ for $i = 1, 2$. By triangle inequality we have,

$$d(o, gx) \leq d(o, p_L(gx)) + d(gx, L) \leq T + c$$

for $c := d(o, \gamma o) + M_1 + M_2 + 1$. Denoting the ball of radius R centered at o in \widetilde{M} by $\mathcal{B}_R(o)$, the above inequality implies that $\mathcal{H}_\Delta(T) \subset \Gamma \cdot x \cap \mathcal{B}_{T+c}(o)$. Since $\#(\Gamma \cdot x \cap \mathcal{B}_t(o)) = \mathbf{O}(e^{\delta t})$, the result follows. \square

Proof of Theorem 6.1. Choose κ' to be strictly less than $\min\{\frac{\kappa_4}{2}, \frac{\kappa_5}{2}, \frac{\delta}{2}\}$, fix T to be large enough, and let $\tau := e^{-\kappa' T}$. For $0 < T_1 < T_2$, set

$$\mathcal{H}(T_1, T_2) := \{[g] : T_1 \leq h([g]) \leq T_2\}.$$

Since $\kappa' < \frac{\kappa_5}{2}$, by Lemma 6.4 there exists a constant c such that for every $[g] \in \mathcal{H}(\frac{T}{2}, T)$ we have

$$1 - c\tau \leq J_\tau^\infty(g) \leq 1 + c\tau.$$

For an element $g \in \Gamma$, let $t_{\max}(g)$ and $t_{\min}(g)$ denote the supremum and infimum of the set $\{t \geq 0 : j_\tau(g, t) \neq 0\}$ respectively. Since $\kappa' \leq \frac{\kappa_4}{2}$, by Corollary 6.3 there exists a constant c' such that for all $[g] \in \mathcal{H}(\frac{T}{2}, T)$ we have

$$t_{\max}(g) \leq T + c'\tau, \text{ and } t_{\min}(g) \geq \frac{T}{2} - c'\tau.$$

Hence, for such $[g]$, the term

$$J_\tau^\infty(g) = \int_0^\infty e^{\delta t} J_\tau(g, t) dt = \int_{t_{\min}(g)}^{t_{\max}(g)} e^{\delta t} J_\tau(g, t) dt = \int_{\frac{T}{2} - c'\tau}^{T + c'\tau} e^{\delta t} J_\tau(g, t) dt$$

contributes to the sum (19) written for $I_\tau(\frac{T}{2} - c'\tau, T + c'\tau)$. As a result,

$$I_\tau(\frac{T}{2} - c'\tau, T + c'\tau) \geq \sum_{[g] \in \mathcal{H}(\frac{T}{2}, T)} J_\tau^\infty(g) \geq (1 - c\tau) \#\mathcal{H}(\frac{T}{2}, T).$$

By Lemma 6.7, we obtain c'' such that

$$\#\mathcal{H}(\frac{T}{2}, T) \leq (1 + c''\tau) \frac{\sigma(F_1)\sigma(F_2)}{\delta} e^{\delta T}.$$

In a similar way we can find constants $c^{(3)}$ and $c^{(4)}$ such that

$$\begin{aligned} I_\tau(\frac{T}{2} + c^{(3)}\tau, T - c^{(3)}\tau) &\leq \sum_{[g] \in \mathcal{H}(\frac{T}{2}, T)} J_\tau^\infty(g) \leq (1 + c\tau) \#\mathcal{H}(\frac{T}{2}, T) \\ \implies (1 - c^{(4)}\tau) \frac{\sigma(F_1)\sigma(F_2)}{\delta} e^{\delta T} &\leq \#\mathcal{H}(\frac{T}{2}, T). \end{aligned}$$

By Lemma 6.8,

$$\#\mathcal{H}(T) = \#\mathcal{H}(\frac{T}{2}) + \#\mathcal{H}(\frac{T}{2}, T) = \#\mathcal{H}(\frac{T}{2}, T) + \mathbf{O}(e^{\frac{\delta}{2}T}),$$

hence the proposition follows from the above upper and lower bounds on $\#\mathcal{H}(\frac{T}{2}, T)$. \square

7. RADIUS HÖLDER-CONTINUITY OF BOWEN-MARGULIS MEASURE

Definitions of RC and RHC. Radius Hölder-continuity of Bowen-Margulis measures of strongly stable (unstable) balls, referred to as RHC throughout the text (see below for the precise definition), was first introduced in [PP17] as a technical assumption to obtain exponentially small error terms. However, in that paper, this condition was only verified for compact manifolds of constant negative curvature, in which the measure of a strongly stable ball is a smooth function of its radius. The goal of this section is to prove the RHC condition under more relaxed assumptions. (See Theorem 7.7.) We adopt the notation introduced in Section 5, and for $v \in T^1\widetilde{M}$, define the distance d_v^{ss} on $W^{\text{ss}}(v)$ by

$$d_v^{\text{ss}}(u, u') := d_{\widetilde{M}}(\pi(u), \pi(u')).$$

Recall the definition of strongly stable ball of radius $r > 0$ centered at v ,

$$B^{\text{ss}}(v, r) = \{u \in W^{\text{ss}}(v) : d_v^{\text{ss}}(v, u) \leq r\},$$

and for a set $S \subset W^{\text{ss}}(v)$, define

$$B^{\text{ss}}(S, r) := \bigcup_{u \in S} B^{\text{ss}}(u, r).$$

Define the strongly stable sphere of radius r centered at v by

$$S^{\text{ss}}(v, r) := \{u \in W^{\text{ss}}(v) : d_v^{\text{ss}}(v, u) = r\}.$$

Proposition 7.1. *The following are equivalent.*

- (i) $\mu_v^{\text{ss}}(S^{\text{ss}}(v, r)) = 0$ for all $v \in T^1\widetilde{M}$ and $r > 0$.
- (ii) For every $v \in T^1\widetilde{M}$, $\mu_v^{\text{ss}}(B^{\text{ss}}(v, r))$ is a continuous function of $r > 0$.
- (iii) $(v, r) \mapsto \mu^{\text{ss}}(B^{\text{ss}}(v, r))$ is a continuous function on $T^1\widetilde{M} \times \mathbb{R}^{>0}$.

Proof. (i) \iff (ii) and (iii) \implies (ii) are obvious. (ii) \implies (iii) follows from Lemma 1.16 of [Rob03]. \square

Definition 7.2. If the items in Proposition 7.1 are satisfied, we say that the strongly stable Bowen-Margulis measure of balls is radius-continuous, or in short, the RC condition is satisfied.

For constants $\alpha, c > 0$ and $v \in T^1\widetilde{M}$ and $r > 0$, we say that $S^{\text{ss}}(v, r)$ is (α, c) Hölder-thin, or equivalently, $B^{\text{ss}}(v, r)$ has (α, c) Hölder-thin boundary, if for every $\epsilon \leq 1$ we have

$$\mu_v^{\text{ss}}(B^{\text{ss}}(S^{\text{ss}}(v, r), \epsilon)) \leq c\epsilon^\alpha.$$

Proposition 7.3. *The following are equivalent.*

- (i) *There exists a constant $\alpha > 0$ such that the following holds. For every $r_0 > 0$ there exists a constant $c = c(r_0)$ such that $S^{\text{ss}}(v, r)$ is (α, c) Hölder-thin for all $v \in T^1\widetilde{M}$ and $0 \leq r \leq r_0$.*
- (ii) *There exists a constant $\alpha' > 0$ such that the following holds. For every $r_0 > 0$ there exists a constant $c' = c'(r_0)$ such that for every $v \in T^1\widetilde{M}$, $\mu_v^{\text{ss}}(B^{\text{ss}}(v, r))$ is an (α', c') Hölder-continuous function of $0 \leq r \leq r_0$.*
- (iii) *$(v, r) \mapsto \mu_v^{\text{ss}}(B^{\text{ss}}(v, r))$ is a Hölder-continuous function on $T^1\widetilde{M} \times \mathbb{R}^{\geq 0}$.*

Proof. (ii) \implies (i) follows since

$$B^{\text{ss}}(S^{\text{ss}}(v, r), \epsilon) \subset B^{\text{ss}}(v, r + \epsilon) \setminus \overset{\circ}{B}^{\text{ss}}(v, r - \epsilon),$$

for all $0 \leq r, v \in T^1\widetilde{M}$, and $\epsilon \leq 1$. (i) \implies (ii) follows since there exists a constant $c_0 = c_0(r_0) > 0$ such that

$$B^{\text{ss}}(v, r + \epsilon) \setminus B^{\text{ss}}(v, r - \epsilon) \subset B^{\text{ss}}(S^{\text{ss}}(v, r), c_0\epsilon),$$

for all $0 \leq r \leq r_0, v \in T^1\widetilde{M}$, and $\epsilon \leq 1$. (iii) \implies (i) is obvious, and (ii) \implies (iii) is proved in Lemma A.5. \square

Definition 7.4. If the items in Proposition 7.3 are satisfied, we say that the strongly stable Bowen-Margulis measure of balls is radius Hölder-continuous, or in short, the RHC condition is satisfied.

Comments on RC and RHC. For this discussion, let M be a manifold of variable negative curvature which is not necessarily compact. Fix $x, y \in \widetilde{M}$, and let $\mathcal{B}(T)$ denote the set of Γ -orbits of y that lie in the ball of radius T centered at x . If M is 2-dimensional and geometrically finite, the RC condition is satisfied since μ_o has no atoms, and RHC is satisfied by an argument similar to that of Theorem 7.7. Hence, for the rest of this discussion, we may assume that M is of dimension at least three.

If M is compact, asymptotics for $\#\mathcal{B}(T)$ as T goes to infinity was obtained by Margulis [Mar04] by considering a partition of $V := S(x)$ into finitely many sets V_i , where each V_i satisfies conditions (1)-(4) of Lemma 7.3 of that paper. (See Theorem 8 and the proof of Theorem 6 of that paper.) Condition (4) of that lemma is that the boundary of V_i has measure zero (where the measure on $S(x)$ is the pullback of μ_x under the homeomorphism from $S(x)$ to $\partial_\infty\widetilde{M}$ given by $u \mapsto u^+$). Choosing a partition that satisfies conditions (1)-(3) of that lemma and perturbing it slightly, we can always obtain a partition that also satisfies the fourth condition.

If M is geometrically finite manifold of variable negative curvature (and under the additional assumption that μ_o is finite), the asymptotics for $\#\mathcal{B}(T)$ was obtained by Roblin [Rob03]. As mentioned earlier, our proof for Theorem 6.1 resembles the proof of Theorem 4.1.1 of [Rob03]. In particular, our definition of a dynamical neighbourhood, given in Section 5, is similar to [Rob03] (see the definition of $K(z, r)$ at page 58 of [Rob03], also see Remark 5.1), and in fact, Roblin faces a similar problem regarding the boundary of the strongly stable balls. Since [Rob03] is not concerned with error terms, it suffices that the RC condition is satisfied. If M is geometrically finite and of constant negative curvature, the RC condition is proved in [Rud82, Lemma 1] and [Rob00, Proposition 3.1], but we could not understand either of these proofs. If M is compact and of constant negative curvature, the RC condition is trivial, but we could not verify this condition for compact manifolds of variable negative curvature. In page 81 of [Rob03], Roblin mentions that “la condition technique exigeant que le bord des boules instables soit négligeable est acquise en courbure constante (voir [Rob00], proposition 3.1), tandis qu’autrement il est toujours possible de l’éluder à l’aide de fonctions plateaux.” We do not know the exact method Roblin had in mind while writing those words.

Inspecting the proof of [Rob03, Theorem 4.1.1], we realize that it is not sensitive to the shape of the strongly stable sets $B^{\text{ss}}(u, r)$, and it goes through as long as they have measure zero boundary. This leads us to believe that (as in [Mar04]) fixing a partition $\bar{\mathcal{P}} = \{\bar{V}_i\}$ of $\partial_\infty \widetilde{M}$, where each \bar{V}_i has measure zero boundary (with respect to the measure μ_o , and hence μ_z for all $z \in \widetilde{M}$), and then defining the sets $B^{\text{ss}}(u, \bar{\mathcal{P}}) \subset W^{\text{ss}}(u)$ using this partition, one can circumvent the RC problem in Roblin. Assuming that the Bowen-Margulis measure is finite, such a partition can always be obtained by perturbing a given partition. To tackle the RHC problem using the same idea, we need for the boundary of V_i to be Hölder-thin. However, we do not know how to construct such partitions in general, even when M is compact.

Proof of RHC under additional assumptions. We use the following two theorems in the proof of Theorem 7.7.

Theorem 7.5. [HIH77, Theorem 2.4] *Let $v \in T^1 \widetilde{M}$ and $u \in S(\pi(v))$ be perpendicular to v . Let $J(t)$ denote the Jacobi field along v_t such that $J(0) = u$ and the norm of $J(t)$ is bounded as t goes to $-\infty$. Then for $t \geq 0$ we have*

$$\|J(t)\| \leq e^{bt} \|u\|.$$

The following theorem is well-known. See [OS84] for a discussion, and a stronger version.

Theorem 7.6. *Let M be an n -dimensional compact manifold with curvature bounded above by -1 , and let δ denote the topological entropy of geodesic flow on the unit tangent bundle of M . Then we have*

$$\delta \geq n - 1,$$

and the equality holds if and only if the curvature is constantly equal to -1 .

For the following theorem, recall that \mathfrak{b} is such that the curvature of M is bounded below by $-\mathfrak{b}^2$.

Theorem 7.7. *Assume M is two-dimensional, or M is of dimension $n \geq 3$ and $\mathfrak{b} \leq \frac{n-1}{n-2}$. Then there exists a constant $\alpha_2 > 0$ such that the following holds. For every $r_0 > 0$, there exists a constant $c = c(r_0)$ such that for every $v \in T^1 \widetilde{M}$, $\mu_B^{\text{ss}}(v, r)$ is an (α_2, c) -Hölder function of $0 \leq r \leq r_0$.*

For the proof of this theorem, it is more natural to work with the strongly unstable foliation. For $v \in T^1 \widetilde{M}$, define

$$W^{\text{uu}}(v) := \{u \in T^1 \widetilde{M} : u^- = v^-, \text{ and } \beta(v^-, \pi(v), \pi(u)) = 0\}.$$

For $r > 0$, d_v^{uu} , $S^{\text{uu}}(v, r)$, $B^{\text{uu}}(v, r)$, μ_v^{uu} , and $\mu_B^{\text{uu}}(v, r)$ can be defined as in the strongly stable case. The RHC (or RC) condition can also be written in terms of the strongly unstable foliation, and one can directly check that RHC (or RC) holds for the strongly stable foliation if and only if it holds for the strongly unstable foliation.

Proof. By the above discussion, it is enough to verify item (i) of Proposition 7.3 for the strongly unstable foliation. To this end, fix $r_0 > 0$, $0 \leq r \leq r_0$, $v \in T^1 \widetilde{M}$, and $\epsilon \ll 1$, and let $\varphi_i: U_i \rightarrow V_i \subset S^{\text{uu}}(v, r)$, for i belonging to some finite set I , be charts on $S^{\text{uu}}(v, r)$. Choose compact sets $K_i \subset U_i$ such that $\{\varphi_i(K_i)\}_{i \in I}$ covers $S^{\text{uu}}(v, r)$, and for each $i \in I$, cover K_i by $\mathbf{O}((\frac{1}{\epsilon})^{n-2})$ distinct cubes of the form

$$C := [k_1 \epsilon, (k_1 + 1) \epsilon] \times \cdots \times [k_{n-2} \epsilon, (k_{n-2} + 1) \epsilon], \quad \text{where } k_j \in \mathbb{Z} \text{ for } 1 \leq j \leq n - 2,$$

and denote the collection of these cubes by \mathcal{K}_i . Let \mathcal{A} denote the collection of $B^{\text{uu}}(\varphi_i(C), \epsilon)$ for $i \in I$ and $C \in \mathcal{K}_i$. Then \mathcal{A} covers $B^{\text{uu}}(S^{\text{uu}}(v, r), \epsilon)$; $\#\mathcal{A} = \mathbf{O}((\frac{1}{\epsilon})^{n-2})$, and $\text{diam}_v^{\text{uu}}(A) = \mathbf{O}(\epsilon)$ for all $A \in \mathcal{A}$, where $\text{diam}_v^{\text{uu}}$ denotes the diameter with respect to the metric d_v^{uu} .

Let \hat{d}_v^{uu} denote the distance function on $W^{\text{uu}}(v)$ induced by restriction of the Riemannian structure of \widetilde{M} to its submanifold $\pi(W^{\text{uu}}(v))$. On $B^{\text{uu}}(v, r_0 + 1)$, the metrics \hat{d}_v^{uu} and d_v^{uu} coincide upto a multiplicative constant that only depends on r_0 . Hence, for every $A \in \mathcal{A}$, we have $\widehat{\text{diam}}_v^{\text{uu}}(A) = \mathbf{O}(\epsilon)$, where $\widehat{\text{diam}}_v^{\text{uu}}$ denotes the diameter with respect to the metric \hat{d}_v^{uu} . Let T be such that $e^{-\mathfrak{b}T} = \epsilon$. Then

by Theorem 7.5 we have $\widehat{\text{diam}}_v^{\text{uu}}(\mathcal{G}_T(A)) = \mathbf{O}(1)$, thus, the compactness of M implies $\mu_{\mathcal{G}_T(v)}^{\text{uu}}(\mathcal{G}_T(A)) = \mathbf{O}(1)$. Since the homeomorphism $u \mapsto u_T$ from $W^{\text{uu}}(v)$ to $W^{\text{uu}}(v_T)$ sends μ_v^{uu} to $e^{-\delta T} \mu_{v_T}^{\text{uu}}$, we have $\mu_v^{\text{uu}}(A) = \mathbf{O}(e^{-\delta T})$. Since $A \in \mathcal{A}$ was arbitrary, we obtain

$$\mu_v^{\text{uu}}(\mathbf{B}^{\text{uu}}(S^{\text{uu}}(v, r), \epsilon)) \prec (\#\mathcal{A})e^{-\delta T} = \mathbf{O}(e^{(\mathfrak{b}(n-2)-\delta)T}). \quad (28)$$

If $n = 2$, then the above implies

$$\mu_v^{\text{uu}}(\mathbf{B}^{\text{uu}}(S^{\text{uu}}(v, r), \epsilon)) = \mathbf{O}(\epsilon^\alpha)$$

for $\alpha := \frac{\delta}{\mathfrak{b}}$. This proves the theorem when M is two-dimensional. If $n \geq 3$, by Theorem 7.6, either M is of constant curvature -1 , or otherwise $\delta > n - 1$. In the former case the theorem follows directly, so we can assume we are in the latter case. If $K \leq \frac{n-1}{n-2}$, then $\kappa := \delta - \mathfrak{b}(n-2) > 0$, thus by (28) and the choice of T we have

$$\mu_v^{\text{uu}}(\mathbf{B}^{\text{uu}}(S^{\text{uu}}(v, r), \epsilon)) = \mathbf{O}(\epsilon^\alpha)$$

for $\alpha := \frac{\kappa}{\mathfrak{b}}$. Since the implicit constant in the above \mathbf{O} can be chosen to only depend on r_0 , the theorem is proved. \square

APPENDIX A. A SMOOTHING ARGUMENT

In this section we prove the smoothing lemma (Lemma A.3) which is used in the proof of Lemma 6.6. We adopt the notation introduced in Section 5, and assume that D is either a point $x \in \widetilde{M}$, or equal to L_γ , the axis an element $\text{id} \neq \gamma \in \Gamma$. The function $F: \partial^1 D \rightarrow \mathbb{R}$ is assumed to be α_3 -Hölder, and if $D = L$ we assume that F is γ -invariant. Given $u \in T^1 \widetilde{M}$, recall that we defined $p_D^F(u)$ to be $\mathcal{G}^F(p_D(u))$. Let $p_F^{\text{ss}}(u)$ be the unique element of $W^{\text{ss}}(p_D^F(u))$ such that $\mathcal{G}_t(p_F^{\text{ss}}(u)) = u$ for some t , and set $\tau_F^{\text{ss}}(u) := t$, that is, $\tau_F^{\text{ss}}(u)$ is such that

$$\mathcal{G}_{\tau_F^{\text{ss}}(u)}(p_F^{\text{ss}}(u)) = u. \quad (29)$$

Finally, define $r_F^{\text{ss}}(u) := d_{p_F^{\text{ss}}(u)}^{\text{ss}}(p_D^F(u), p_F^{\text{ss}}(u)) = d_{\widetilde{M}}(\pi(p_D^F(u)), \pi(p_F^{\text{ss}}(u)))$.

Lemma A.1. *There exists a constant $\alpha_4 > 0$ such that $\tau_F^{\text{ss}}(u)$ and $r_F^{\text{ss}}(u)$ are both α_4 -Hölder.*

Proof. Given $u \in T^1 \widetilde{M}$, by the definition of $p_F^{\text{ss}}(u)$ we have

$$\beta_o(u^+, \pi(p_D^F(u))) = \beta_o(u^+, \pi(p_F^{\text{ss}}(u))).$$

By the defining equation for τ_F^{ss} (Equation (29)) we have

$$\beta_o(u^+, \pi(p_F^{\text{ss}}(u))) = \beta_o(u^+, \pi(u)) + \tau_F^{\text{ss}}(u).$$

As a result,

$$\tau_F^{\text{ss}}(u) = \beta_o(u^+, \pi(p_D^F(u))) - \beta_o(u^+, \pi(u)).$$

Since the right hand side of the above equation is a combination of Hölder-continuous functions by the results in Section 3, τ_F^{ss} is Hölder-continuous as well. Plugging $p^{\text{ss}}(u) = \mathcal{G}_{-\tau_F^{\text{ss}}(u)}(u)$ into the defining equation for r_F^{ss} , we have

$$r_F^{\text{ss}}(u) = d_{\widetilde{M}}(\pi(p_D^F(u)), \pi(\mathcal{G}_{-\tau_F^{\text{ss}}(u)}(u))).$$

As before, all the functions appearing on the right hand side are Hölder-continuous, hence r_F^{ss} is Hölder-continuous as well. \square

Let (X, d) be a metric space. For $\epsilon > 0$ and a subset $A \subset X$, define the function $L_\epsilon(A): X \rightarrow \mathbb{R}$ by

$$L_\epsilon(A)(x) := 1 - \min\left\{\frac{1}{\epsilon}d(x, A), 1\right\}.$$

Note that $L_\epsilon(A)$ is $\frac{1}{\epsilon}$ -Lipschitz and it is constantly equal to 1 on A . Denoting the r -neighbourhood of a set $S \subset X$ by $\mathcal{B}_r(S)$, one can readily check that $\text{Supp}(L_\epsilon(A)) = \overline{\mathcal{B}_\epsilon(A)}$.

For $r, \tau > 0$ set

$$\mathbf{B}_F(r, \tau) := \{u \in T^1\widetilde{M} : r_F^{\text{ss}}(u) \leq r, \text{ and } |\tau_F^{\text{ss}}(u)| \leq \tau\}, \quad (30)$$

and note that this set coincides with $\mathbf{B}_\tau^s(\mathcal{G}^F(\partial^1 D), r)$, defined in Section 5.

Lemma A.2. *There are constants $c_1, \alpha_5 > 0$ such that if r, τ , and ϵ are all of order 1, then we have*

$$\mathcal{B}_\epsilon(\mathbf{B}_F(r, \tau)) \subset \mathbf{B}_F(r + c_1\epsilon^{\alpha_5}, \tau + c_1\epsilon^{\alpha_5}).$$

Proof. Let $v \in \mathcal{B}_\epsilon(\mathbf{B}_F(r, \tau))$. This means that there exists $u \in \mathbf{B}_F(r, \tau)$ with $d(u, v) < \epsilon$. Since $u \in \mathbf{B}_F(r, \tau)$, by (30) we have $r_F^{\text{ss}}(u) \leq r$ and $|\tau_F^{\text{ss}}(u)| \leq \tau$. By Lemma A.1, for $\alpha_5 := \alpha_4$ and $c_1 = \mathbf{O}(1)$ we have

$$|r_F^{\text{ss}}(u) - r_F^{\text{ss}}(v)| \leq c_1 d(u, v)^{\alpha_5} \leq c_1 \epsilon^{\alpha_5} \implies r_F^{\text{ss}}(v) \leq r + c_1 \epsilon^{\alpha_5}.$$

In a similar way, $|\tau_F^{\text{ss}}(v)| \leq \tau + c_1 \epsilon^{\alpha_5}$. The lemma now follows from (30). \square

For $r, \tau > 0$, define the function $\varphi_F(r, \tau): T^1\widetilde{M} \rightarrow \mathbb{R}$ by

$$\varphi_F(r, \tau)(u) := \frac{1}{2\tau \mu_{\mathbf{B}}^{\text{ss}}(p_D^F(u), r)},$$

and recall from Section 5 that $\phi_D^F(r, \tau)(u) = \varphi_F(r, \tau)(u) \times \mathbb{1}(\mathbf{B}_F(r, \tau))$. Let $n_1 := \frac{2}{\alpha_5}$, and for $\tau \ll r < 1$ and $\epsilon < \tau^{n_1}$ define

$$\chi_F(r, \tau, \epsilon) := \phi_D^F(r, \tau) \times L_\epsilon(\mathbf{B}_F(r - c_1\epsilon^{\alpha_5}, \tau - c_1\epsilon^{\alpha_5})).$$

If $D = L$, then $\chi_F(r, \tau, \epsilon)$ is γ -invariant. Hence, using (7) or (8), depending on whether D is a point or $D = L$, $\chi_F(r, \tau, \epsilon)$ descends to a function on T^1M , which we denote by $\bar{\chi}_F(r, \tau, \epsilon)$.

Lemma A.3. (the smoothing lemma) *There are constants α_6 and α_7 , only depending on the Hölder exponent of F , such that for all $\tau \ll r < 1$ and $\epsilon < \tau^{n_1}$, we have*

- (1) $0 \leq \bar{\chi}_F(r, \tau, \epsilon) \leq \bar{\phi}_D^F(r, \tau)$.
- (2) $\bar{\chi}_F(r, \tau, \epsilon)$ is α_6 -Hölder with α_6 -norm of order $\frac{1}{\tau\epsilon}$.
- (3) $\|\bar{\phi}_D^F(r, \tau) - \bar{\chi}_F(r, \tau, \epsilon)\|_{L^1} = \mathbf{O}(\frac{\epsilon^{\alpha_7}}{\tau})$, where $\|\cdot\|_{L^1}$ denotes the L^1 -norm.

The first item of the above lemma follows directly from the definition of $\bar{\chi}_F(r, \tau, \epsilon)$. The second item follows from Lemma A.4, and the third item is proved at the end of this section.

Lemma A.4. *There exists α_6 such that for all $\tau \ll r < 1$ and $\epsilon < \tau^{n_1}$, $\chi(r, \tau, \epsilon)$ is $(\alpha_6, \mathbf{O}(\frac{1}{\tau\epsilon}))$ -Hölder.*

Proof. By Lemma A.2, the support of $L_\epsilon(\mathbf{B}_F(r - c_1\epsilon^{\alpha_5}, \tau - c_1\epsilon^{\alpha_5}))$ is a subset of $\mathbf{B}_F(r, \tau)$, hence we have

$$\chi_F(r, \tau, \epsilon)(u) = \varphi_F(r, \tau)(u) \times L_\epsilon(\mathbf{B}_F(r - c_1\epsilon^{\alpha_5}, \tau - c_1\epsilon^{\alpha_5})). \quad (31)$$

The function $u \mapsto \mu_{\mathbf{B}}^{\text{ss}}(p_D^F(u), r)$ is a combination of Hölder-continuous functions (the fact that $\mu_{\mathbf{B}}^{\text{ss}}(\cdot, \cdot)$ is Hölder-continuous is proved in Lemma A.5), hence it is α -Hölder for some constant $\alpha > 0$. This makes $\varphi_F(r, \tau)$ a $(\alpha, \mathbf{O}(\frac{1}{\tau}))$ -Hölder function. As mentioned at the beginning of the section, $L_\epsilon(A)$ is $\frac{1}{\epsilon}$ -Lipschitz for every subset A of a general topological space X , hence $L_\epsilon(\mathbf{B}_F(r - c_1\epsilon^{\alpha_5}, \tau - c_1\epsilon^{\alpha_5}))$ is $\frac{1}{\epsilon}$ -Lipschitz. It follows that $\chi_F(r, \tau, \epsilon)$ is $(\alpha_6, \mathbf{O}(\frac{1}{\tau\epsilon}))$ -Hölder for $\alpha_6 := \alpha$. \square

Lemma A.5. *The function $\mu_{\mathbf{B}}^{\text{ss}}(u, r): T^1\widetilde{M} \times [0, \infty) \rightarrow \mathbb{R}$ is Hölder-continuous.*

Proof. Fixing $u \in T^1\widetilde{M}$, $\mu_{\mathbf{B}}^{\text{ss}}(u, \cdot)$ is Hölder-continuous by Theorem 7.7, hence we only need to show that $\mu_{\mathbf{B}}^{\text{ss}}(\cdot, r)$ is Hölder-continuous for a fixed r . This is essentially proved in Lemma 11 of [PP17], thus we only give an outline of the proof.

Fix r , and let $u, v \in T^1\widetilde{M}$ be close by. Define the function $H^{uv}: \mathbf{B}^{\text{ss}}(u, r) \rightarrow \mathbf{W}^{\text{ss}}(v)$, $u' \mapsto v'$ in a way that $(u')^- = (v')^-$. This function satisfies the following.

- (1) There exists α such that $d(\pi(u'), \pi(H^{uv}u')) = \mathbf{O}(d(u, v)^\alpha)$ for $u' \in \mathbf{B}^{\text{ss}}(u, r)$.

(2) For $u' \in \mathbf{B}^{\text{ss}}(u, r)$,

$$d\mu_v^{\text{ss}}(v') = (1 + \mathbf{O}(d(u, v)^\alpha)) d\mu_u^{\text{ss}}(u').$$

By the first item and triangle inequality, if we choose ϵ to be a large multiple of $d(u, v)^\alpha$, then $H^{uv}(\mathbf{B}^{\text{ss}}(u, r - \epsilon)) \subset \mathbf{B}^{\text{ss}}(v, r)$, thus

$$\begin{aligned} \mu_{\mathbf{B}}^{\text{ss}}(v, r) &\geq \mu_v^{\text{ss}}(H^{uv}(\mathbf{B}^{\text{ss}}(u, r - \epsilon))) = (1 + \mathbf{O}(d(u, v)^\alpha)) \mu_{\mathbf{B}}^{\text{ss}}(u, r - \epsilon) && \text{(by item (2))} \\ &= (1 + \mathbf{O}(d(u, v)^{\alpha\alpha_2})) \mu_{\mathbf{B}}^{\text{ss}}(u, r). && \text{(by Theorem 7.7)} \end{aligned}$$

Therefore, letting $\alpha' := \alpha\alpha_2$, there exists a constant c such that

$$\mu_{\mathbf{B}}^{\text{ss}}(v, r) \geq \mu_{\mathbf{B}}^{\text{ss}}(u, r) - cd(u, v)^{\alpha'}.$$

Changing the role of u and v in the above argument, we deduce that $\mu_{\mathbf{B}}^{\text{ss}}(\cdot, r)$ is α' -Hölder. \square

Lemma A.6. *There are constants c_2, α_7 such that for $\tau \ll r < 1$ and $\epsilon < \tau^{n_1}$ we have*

$$(1 - c_2 \frac{\epsilon^{\alpha_7}}{\tau}) \phi_D^F(r, \tau) \leq \chi_F(r + c_1 \epsilon^{\alpha_5}, \tau + c_1 \epsilon^{\alpha_5}, \epsilon). \quad (32)$$

Proof. Let $r' := r + c_1 \epsilon^{\alpha_5}$ and $\tau' := \tau + c_1 \epsilon^{\alpha_5}$. Since $\text{Supp}(\phi_D^F(r, \tau)) = \mathbf{B}_F(r, \tau)$ and $\chi(r', \tau', \epsilon)$ is equal to $\varphi_F(r', \tau')$ on $\mathbf{B}_F(r, \tau)$, it is enough to show that for $u \in \mathbf{B}_F(r, \tau)$ we have

$$(1 - \mathbf{O}(c_1(\frac{\epsilon^{\alpha_5}}{\tau} + \epsilon^{\alpha_2\alpha_5}))) \varphi_F(r, \tau) \leq \varphi_F(r', \tau'),$$

where α_2 is the constant given by Theorem 7.7. (then we take $\alpha_7 := \alpha_2\alpha_5$ and $c_2 := \mathbf{O}(c_1)$.) This follows by writing the defining equation for $\varphi_F(\cdot, \cdot)$ on both sides of the above and using the equalities

$$\frac{\tau'}{\tau} = 1 + \mathbf{O}(c_1 \frac{\epsilon^{\alpha_5}}{\tau}), \quad \text{and} \quad \frac{\mu_{\mathbf{B}}^{\text{ss}}(p_D^F u, r')}{\mu_{\mathbf{B}}^{\text{ss}}(p_D^F, r)} = 1 + \mathbf{O}(c_1 \epsilon^{\alpha_2\alpha_5}).$$

\square

Proof of item (3) of Lemma A.3. Let $r' := r - c_1 \epsilon^{\alpha_5}$ and $\tau' := \tau - c_1 \epsilon^{\alpha_5}$, and recall that by Proposition 5.2,

$$\sigma(F) = \int_{T^1 M} \bar{\phi}_D^F(r, \tau) dm_{\text{BM}} = \int_{T^1 M} \bar{\phi}_D^F(r', \tau') dm_{\text{BM}}, \quad (33)$$

where $\sigma(F) := \sigma_\gamma(F)$ (resp. $\sigma(F) := \sigma_x(F)$) when $D = L_\gamma$ (resp. $D = x$). By Lemma A.6,

$$(1 - c_2 \frac{\epsilon^{\alpha_7}}{\tau}) \bar{\phi}_D^F(r', \tau') \leq \bar{\chi}_F(r, \tau, \epsilon).$$

By integrating and using (33) we obtain

$$\sigma(F) - \int_{T^1 M} \bar{\chi}_F(r, \tau, \epsilon) = \mathbf{O}(c_2 \frac{\epsilon^{\alpha_7}}{\tau}).$$

Since $0 \leq \bar{\chi}_F(r, \tau, \epsilon) \leq \bar{\phi}_D^F(r, \tau)$, we have

$$\|\bar{\phi}_D^F(r, \tau) - \bar{\chi}_F(r, \tau, \epsilon)\|_{L^1} = \int_{T^1 M} \bar{\phi}_D^F(r, \tau) - \int_{T^1 M} \bar{\chi}_F(r, \tau, \epsilon) = \sigma(F) - \int_{T^1 M} \bar{\chi}_F(r, \tau, \epsilon) = \mathbf{O}(\frac{\epsilon^{\alpha_7}}{\tau}).$$

\square

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