Correlated Errors in Geophysical Applications

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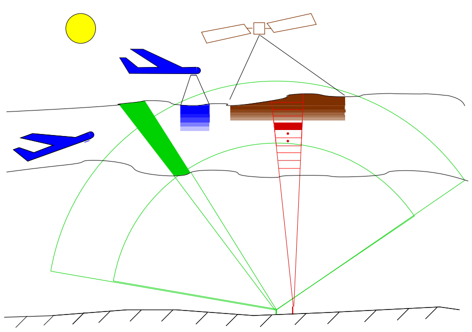
Outline

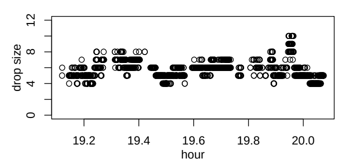
Example 1

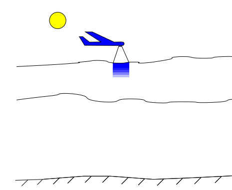
Inverse Problem: drop size retrievals



Example 1

Inverse Problem: drop size retrievals





Example 2

Ensemble Kalman filter: Weather forecasting

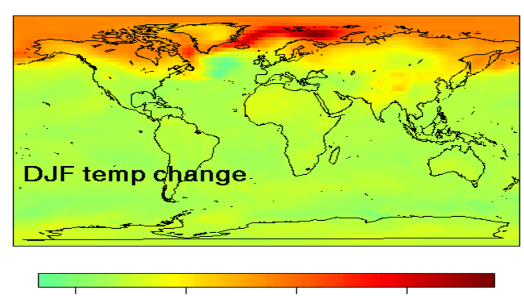
A numerical model is used to make short-range forecasts, with new observations contributing to data history as they become available.

- Surface (towers, ships)
- Altitude (planes, ballons)
- Radar
- Satellites

Example 3

Optimal prediction: climate projection

Preciptiation or temperature fields are decomposed into a large scale trend and a small scale variation.



Statistical Model

Common notation of the three examples:

$$y_i = F(m) + \varepsilon_i$$
$$\mathbf{y} = F(\mathbf{x}) + \boldsymbol{\varepsilon}$$
$$Y(\mathbf{x}_i) = f(\mathbf{x}_i) + \varepsilon(\mathbf{x}_i)$$

Assumptions on the error process:

$$\mathrm{E}(\boldsymbol{\varepsilon}) = \mathbf{0} \quad \text{and} \quad \mathrm{Cov}(\boldsymbol{\varepsilon}) = \boldsymbol{\Sigma}$$

$$\boldsymbol{\varepsilon} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}) \quad \text{or} \quad \boldsymbol{\varepsilon} \sim \mathcal{G}, \quad \mathcal{G} \text{ symmetric}$$

$\rightsquigarrow$ Characterize the "error" process ε

Method of Moments

Objective:

a parametric description of $\boldsymbol{\Sigma}$ with $\Sigma_{ij} = \mathrm{Cov}\big(\varepsilon(\mathbf{x}_i), \varepsilon(\mathbf{x}_j)\big)$.

Without distributional assumptions, analysis is usually based on the variogram:

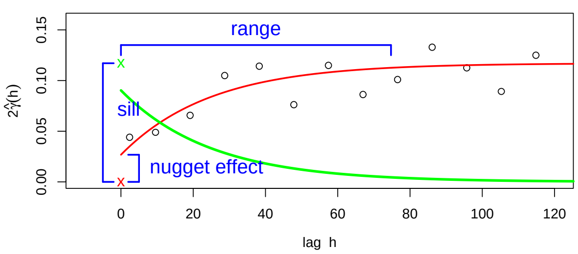
$$2\gamma(\mathbf{x}_i - \mathbf{x}_j; \boldsymbol{\theta}) = \mathrm{Var}\big(\varepsilon(\mathbf{x}_i) - \varepsilon(\mathbf{x}_j)\big)$$

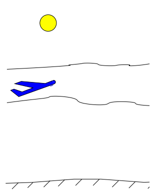
With sample measurement errors a sample variogram is obtained.

Variogram Fitting

We fit a parametric model $2\gamma(\cdot; \boldsymbol{\theta})$:

$$\widehat{\theta}_{\mathrm{MoM}} \quad \text{minimizes} \quad \ell\big(2\widehat{\gamma}(\mathbf{h}) - 2\gamma(\mathbf{h}; \boldsymbol{\theta})\big)$$



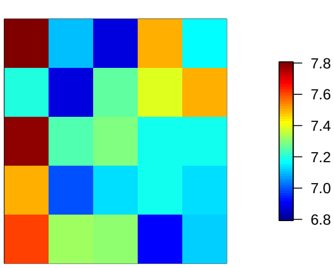


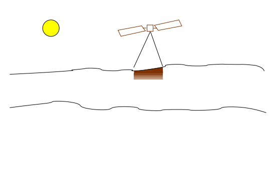
(Dis)Advantages

- \+ classical geostatistical approach
- \+ no distributional assumptions on the error required
- \+ highly robust versions exist
- − difficult to describe uncertainty
- − two-step procedure with many "hidden parameters"

Test of Independence

Do we have spatial dependence?

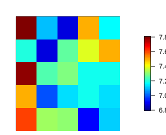


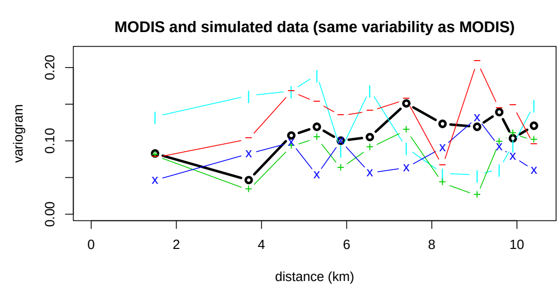


Test of Independence

Variogram of white noise $\varepsilon \sim \mathcal{N}(0, \sigma^2)$ is

$$2\gamma(\mathbf{x}_i - \mathbf{x}_j; \boldsymbol{\theta}) = \mathrm{Var}\big(\varepsilon(\mathbf{x}_i) - \varepsilon(\mathbf{x}_j)\big) \equiv 2\sigma^2$$





Maximum Likelihood

We assume a distribution for our model:

$$\mathbf{y} = F(\mathbf{x}) + \varepsilon \qquad \varepsilon \sim \mathcal{N}\big(\mathbf{0}, \Sigma(\boldsymbol{\theta})\big)$$

$$p(\mathbf{y};\boldsymbol{\theta}) \propto \big|\Sigma(\boldsymbol{\theta})\big|^{-1/2} \exp\Big(-\frac{1}{2}\big(\mathbf{y} - F(\mathbf{x})\big)^T \Sigma(\boldsymbol{\theta})^{-1}\big(\mathbf{y} - F(\mathbf{x})\big)\Big)$$

$\widehat{\theta}_{\mathsf{MLE}}$ maximizes $\log p(\mathbf{y};\boldsymbol{\theta})$

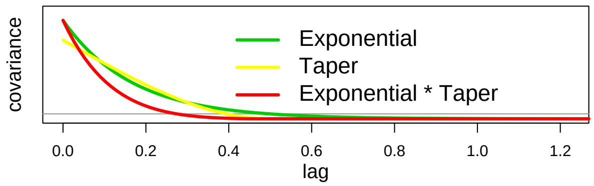
(Dis)Advantages

+ one-step procedure
+ straight-forward inference on parameters
+ can be extended with parameterized large scale structures
− distributional assumptions on the error required
− computationally expensive

Large Systems

Maximum Likelihood is computationally expensive.

(Motivated from a prediction point of view) we approximate $\boldsymbol{\Sigma}$.



Large Systems

Maximum Likelihood is computationally expensive.

(Motivated from a prediction point of view) we approximate $\boldsymbol{\Sigma}$.

Let $\mathbf{T}$ be a "sparse" positive definite matrix.

Base likelihood on $\tilde{\boldsymbol{\Sigma}} = \boldsymbol{\Sigma} \circ \mathbf{T}$ and use sparse matrix techniques.

Consistency and optimality are preserved.

Summary

Characterisation of correlated processes (spatial processes)

Geostatistics traditionally deals with correlated processes

Efficient methods for describing the process, but . . .

Enough gaps for further research . . .