Qualitative and Quantitative Evaluation of Covariance Tapering in Large Kriging Problems

Tapering and Kriging

Introduce a sparseness structure in the covariance via tapering to gain computational advantages in large kriging problems constraint to maintaining asymptotic optimality.

In collaboration with Marc Genton and Doug Nychka.

Motivation

Monthly aggregated precipitation in April 1948

Best Linear Unbiased Predictor

Suppose a spatial process of the form

\[ Z(x) = m(x)^T \beta + \varepsilon(x), \quad E(\varepsilon) = 0, \quad \text{Cov}(\varepsilon) = C \]

with observations \( Z = (Z(x_1), \ldots, Z(x_n))^T, \quad x_i \in D \subset \mathbb{R}^d. \)

The kriging estimator (BLUP) is

\[ \hat{Z}(x_0) = \lambda^T Z \]

where

\[ \lambda = C^{-1}(I - M(M^T C^{-1} M)^{-1}(M^T C^{-1} c - m(x_0))) \]

with \( c = \text{Cov}(Z(x_0), Z(x_i)), \quad M = (m(x_1), \ldots, m(x_n))^T. \)

Motivation

Precipitation anomaly in April 1948

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\[ \hat{Z}(x_0) = \lambda^T Z \]

where

\[ \lambda = C^{-1}(I - M(M^T C^{-1} M)^{-1}(M^T C^{-1} c - m(x_0))) = C^{-1} c \]

with \( c = \text{Cov}(Z(x_0), Z(x_i)), \quad \text{and} \quad M = (m(x_1), \ldots, m(x_n))^T. \)
Motivation

Precipitation anomaly along 40° latitude

Matérn Covariogram

We need a broad, flexible class of covariograms to describe spatial processes.

We recommend the Matérn class given by

\[ c(h) \propto (|\alpha|h)^{\nu} \psi_{|\alpha|h} \]

and with spectral density

\[ f(\omega) \propto \frac{1}{(\alpha^2 + |\omega|^2)^{\nu+d/2}} \]

Differentiability at the origin of the covariogram is related to the tail behavior of the spectrum, i.e. the smoothness of the process.

The process is \( m \) times mean squared differentiable iff \( m < \nu \).

Matérn Covariogram

\( \nu = 0.0 \) (white noise)

effective range is 0.2

\( \nu = 0.5 \)

effective range is 0.2

\( \nu = 1.0 \)

effective range is 0.2

\( \nu = 1.5 \)

effective range is 0.2
Misspecified Covariances

In a series of (Annals) papers, Stein gives asymptotic results for misspecified covariances.

Suppose the true spectrum \( f_0 \). If we krig with the misspecified covariance characterized by \( f_1 \) then under the conditions

\[
\frac{f_1(\omega)}{f_0(\omega)} \to 1 \quad \text{as} \quad |\omega| \to \infty
\]

we have

\[
\frac{\text{MSE pseudo-BLUP}(f_1)}{\text{MSE BLUP}(f_0)} \to 1
\]

\[
\frac{\text{MSE pseudo-BLUP}(f_1)}{\text{MSE BLUP}(f_0)} \to 1
\]

Tapered Covariances

Tapering, i.e. multiplying a covariance function \( c(\cdot) \) with \( k(\cdot) \), is a form of misspecification if

\[
\frac{\mathcal{F}(c(\|h\|)k(\|h\|))}{\mathcal{F}(c(\|h\|))} \to 1 \quad \text{as} \quad |\omega| \to \infty
\]

Which taper satisfies this condition?

The answer lies in the Principal Irregular Term.

- has to be as differentiable at the origin as the original covariogram
- has to be more differentiable throughout the domain than at the origin
- has to be inflated (factor obtained via PIT)

Examples

- Exponential
- Spheric
- Exp * Spheric
Examples

- Exponential
- Spheric
- Exp * Spheric

- Matern (1.5)
- Wu
- Matern * Wu

Covariance vs. Lag

Numerical Study

MSE for prediction along baseline 1000 MC samples
exp. covariogram (range=5.75) with Wu tapering

Recap

To summarize we have discussed:

- the BLUP, requiring solutions of linear systems
  \( \Rightarrow \) one point \( \mathbf{C}^{-1} \mathbf{Z} \), field \( \mathbf{C} \)
- optimal covariances, decay exponentially
  \( \Rightarrow \) no zeros in \( \mathbf{C} \), \( \mathbf{c} \) or \( \mathbf{C} \)
- that tapering preserves optimality,
  \( \Rightarrow \) covariance matrices are sparse

Is there a computational gain?

Computational Efficiency

- Matlab: standard
- Matlab: sparse, bw=20
- R: standard
- R: sparse, bw=20
- R: sparse, bw=10

It would be coffee time...