Numerik 1 Übungstunde, Gruppe 1 (Donnerstag)

Woche 06

Übungstunden werden registriert. Bitte informieren Sie den Assistenten, falls Sie nicht in der Aufzeichnung erscheinen möchten.
Streaming (öffentlich) und Aufzeichnungen (Anmeldung nötig):

https://seminarlive.mnf.uzh.ch/semlive/index.php?id=event_detail&module=mat801&semester=fs23&group=1

Notizen (diese Datei) und Python scripts:
https://user.math.uzh.ch/florian/stable/page/2023Mat801/notes.php

(a) Explain how the quantities in the `FFT` algorithm (§5.4, pp. 45-56) correspond to the quantities of the abstract hierarchical algorithm as in Definition 1 and 2., i.e give a full description of $\mathcal{N}(\mathcal{T}_{\mathrm{FFT}})$, $\mathcal{E}(\mathcal{T}_{\mathrm{FFT}})$, $\mathcal{L}(\mathcal{T}_{\mathrm{FFT}})$ and $\mathcal{N}_m(\mathcal{T}_{\mathrm{FFT}})$ for all $0 \le m \le \log_2(n_{\mathrm{FFT}}) \in \mathbb{N}$.

2) Organisation in Baumstruktur.
Die Nachfolgerstruktur

$$\begin{array}{ccccc} & & \mathcal{W} & & \\ \nearrow & \uparrow & \cdots & \uparrow & \nwarrow \\ \mathcal{W}_0 & \mathcal{W}_1 & \cdots & \mathcal{W}_{q-2} & \mathcal{W}_{q-1} \end{array} \tag{5.11}$$

lässt sich als Baum organisieren. Das globale Problem $\mathcal{W}$ ist die n-te Stufe (Wurzel des Bau-

Definition 1: A tree $\mathcal{T}$ is given by a set of nodes $\mathcal{N}(\mathcal{T})$ and a set of directed edges $\mathcal{E}(\mathcal{T}) \subseteq \{(v,v') \mid v,v' \in \mathcal{N}(\mathcal{T})\}$. Nodes $v \in \mathcal{N}$ can have successors, meaning there exists a node v' and a directed edge $(v,v') \in \mathcal{E}(\mathcal{T})$ and the set of all successors of v is denoted by $\sigma(v)$. We say a vertex $\tilde{v}$ is the predecessor of a vertex v if $v \in \sigma(\tilde{v})$. A tree $\mathcal{T}$ must satisfy the following properties:

(a) All nodes $v \in \mathcal{N}(\mathcal{T})$ have at most one predecessor.

(b) There exists exactly one node $v_0 \in \mathcal{N}(\mathcal{T})$ which has no predecessor. v_0 is called the root.

(c) There exist nodes $v \in \mathcal{N}$ which have no successor. These nodes are called leafs. The set of leafs is denoted by $\mathcal{L}(\mathcal{T})$.

For $m \in \mathbb{N}_0$, the set of all nodes with depth m is denoted by $\mathcal{N}_m(\mathcal{T})$.

$\mathcal{T}_{FFT} = ?$

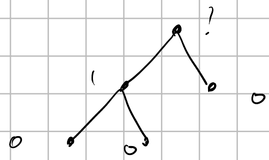
$W \to FFT(\underbrace{0 \dots\dots N}_{\text{set of indices } =: W})$

$W_{0,0} = W = \{0 \dots N\}$

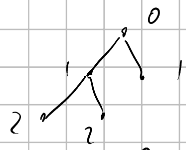
$W_{1,0} := \{0, 2, \dots 2\lfloor \frac{N}{2} \rfloor\}$

$W_{1,1} := \{1, 3, \dots\dots\}$

Problem: if depth(v) = 0 for all leaves what happens with



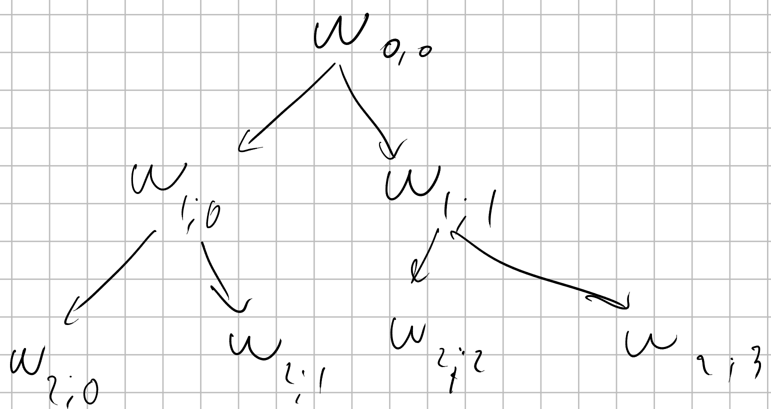
→ Solution: depth of root is 0!



if N is not a power of 2 this may happen

$$Z_{FFT} = \Big(\{ W_{j,k},\ 0 \le j \le \operatorname{depth}(Z_{FFT}),\ 0 \le k \le 2^j - 1 \},\ \{ (W_{j,k} ; W_{j+1,\ell}) \text{ with } \ell = 2k \text{ or } \ell = 2k+1 \} \Big)$$

if balanced!



Leaves: $W_{k,h}$ with $k =$, $h =$

Nodes of depth m: $W_{k,h}$ with $k =$, $h =$

homework!

b) $\sigma(v) :=$ successors of v.

Assume $\exists\, q \in \mathbb{N} \setminus \{0, 1\}$ s.t.

$$2 \le \#(\sigma(v)) \le q$$

$n := \text{depth}(w) \quad w \in \mathcal{L}(\mathcal{T})$

Give a lower and upper bound for $\#(\mathcal{N}(\mathcal{T}))$

$$\#(\mathcal{N}(\mathcal{T})) = \sum_{i=0}^{n} \#(\mathcal{N}_i(\mathcal{T}))$$

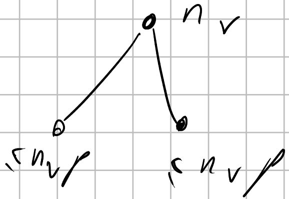
$$? \leq \frac{\#\mathcal{N}_i(\mathcal{T})}{\#\mathcal{N}_{i+1}(\mathcal{T})} \leq ?$$

c) For every node v, $n(v) =: n_v \qquad n: \mathcal{N}(\mathcal{T}) \to \mathbb{R}_{\geq 0}$

- Assume $n(v) \geq 1$ for each node
- "

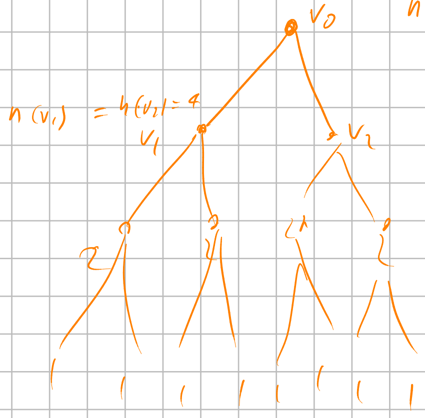
$$\underbrace{\sum_{v \in \mathcal{N}_i(\mathcal{T})} n(v)}_{\text{does not depend on } i} = n(v_0) = \sum_{v \in \underbrace{\mathcal{N}_0(\mathcal{T})}_{=\{v_0\}}} n(v)$$

- " $\quad n_z \leq p\, n_v \quad \forall z \in \sigma(v), \ \forall v \in \mathcal{N}(\mathcal{T}) \setminus \mathcal{L}(\mathcal{T})$



e.g. $p = \frac{1}{2} \qquad n_{v_0} = 8$

$q = 2$ (binary tree, in our setting exactly two successors per node)



$n(v_0) = 8$

15 nodes

at most $n(v_0)/2$ per child, but

$$\sum_{v \in \sigma(v_0)} n(v) = n(v_0)$$

$\downarrow$

$$\sum_{i \in \{1,2\}} n(v_i) \leq \sum_{i \in \{1,2\}} \frac{1}{2} n(v_0) = n(v_0)$$

$\downarrow$

then this is $=$

Now for generic q, p $\rightarrow$ more difficult

- Function $ops : \mathcal{N} \rightarrow \mathbb{R}$

  $\exists\, C_{init}$ s.t. $ops(v) \leq C_{init} \quad \forall v \in \mathcal{L}(\mathcal{T})$

  $ops(w) \leq C\, n_w \quad \forall w \in \mathcal{N}(\mathcal{T}) \setminus \mathcal{L}(\mathcal{T})$

- Actual question: $\sum_{v \in \mathcal{N}} ops(v) \leq\ ?$

  ! depends on depth, p, q