Numerik 1 Übungstunde, Gruppe 1 (Donnerstag)

Woche 04

Übungstunden werden registriert. Bitte informieren Sie den Assistenten, falls Sie nicht in der Aufzeichnung erscheinen möchten.
Streaming (öffentlich) und Aufzeichnungen (Anmeldung nötig):

https://seminarlive.mnf.uzh.ch/semlive/index.php?id=event_detail&module=mat801&semester=fs23&group=1

Notizen (diese Datei) und Python scripts:
https://user.math.uzh.ch/florian/stable/page/2023Mat801/notes.php

We focus here on the function $f(x) = x^{\frac{3}{2}}$ on $I = [0, 1]$. Now consider an equidistant set of interpolation points: $x_i, 0 \le i \le n$ and the continuous linear spline interpolation $p(f, \Theta_n) \in \mathcal{S}_{\mathcal{G}}^{0,1}$. Let τ_i be given as $\tau_i := [x_{i-1}, x_i], 1 \le i \le n$ and denote the length of $\tau_i, 1 \le i \le n$ as h.

(a) Determine the maximal $m \in \mathbb{N}$, such that $f(x) \in C^m(I)$.

(b) Estimate $\|f - p(f, \Theta_n)\|_{\max,\tau_i}, \forall i = 2, \ldots, n$ by using `Satz 2.20` in the script.

(c) Compute $\|f - p(f, \Theta_n)\|_{\max,\tau_1}$ explicitly for the first interval τ_1.

(c) $f(x) = x^{3/2} \quad C^m([0,1])$

$\tau_1 = [0, x_1]$

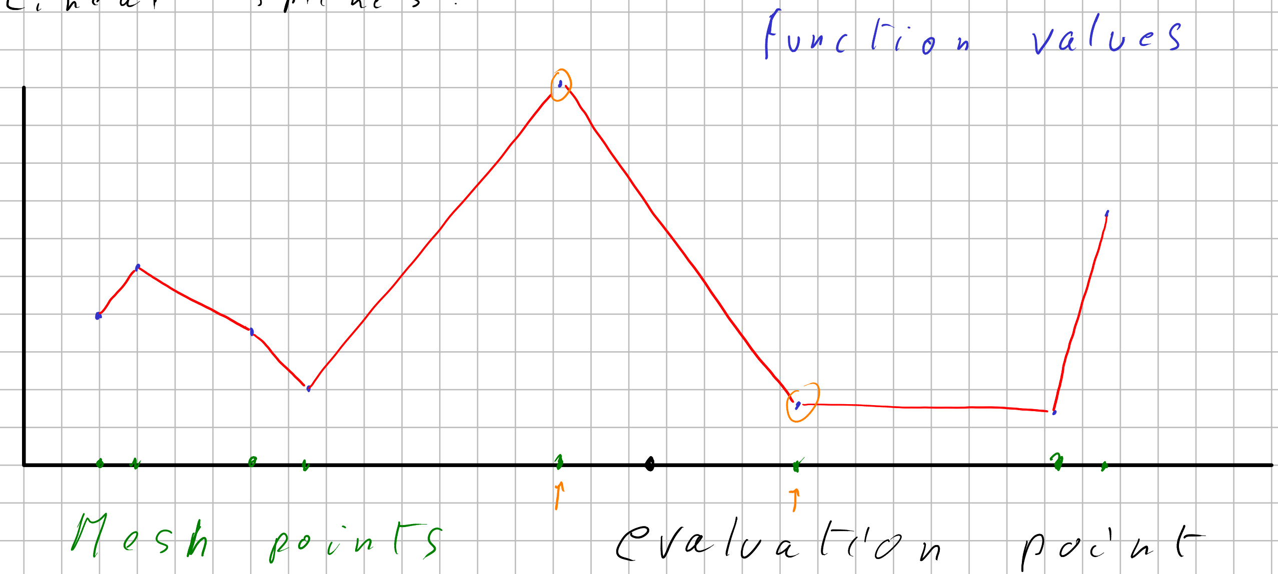
$$p(f, \Theta_n)\big|_{\tau_1}(x) = \frac{f(x_1)}{x_1} x = \frac{x_1^{3/2}}{x_1} x = \sqrt{x_1}\, x$$

$$\| f|_{\tau_1} - p(f,\Theta_n)|_{\tau_1} \|_{max} = \max_{x \in [0, x_1]} |\underbrace{x^{3/2} - \sqrt{x_1}\, x}_{g(x)}|$$

Compute the maximum of $g(x)$

(d) $\| h \|_{max} = \max_{1 \le i \le N} \{ \| h|_{\tau_i} \|_{max} \}$ (follows from the definition)

Linear splines:



To define a loop:

- start condition
- The operations performed in the loop
- stop condition(s)

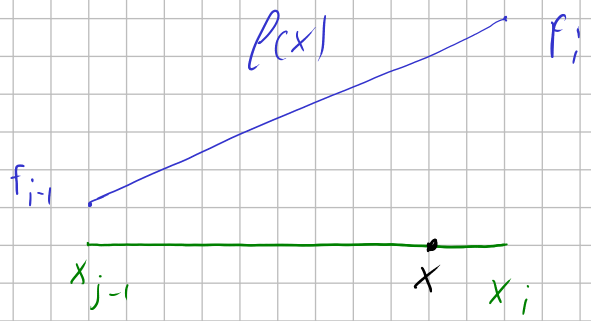
Loop over intervals (x_{j-1}, x_j): $(0 \le i \le N)$

start from $j = 1$
advance j
stop if $x < x_j$, $x > x_{j-1}$
or $x > x_N$

$j = 1 \to$ if $x < x_1$, $x \geq x_0$ then exit

$j = 2 \to x < x_2$, $\underline{x \geq x_1}$ ↳ always true

- Compute the linear spline in $I_j = [x_{j-1}, x_j]$
We have the values F_{j-1}, F_j



$$l(x) = F_i \frac{x - x_{i-1}}{x_i - x_{i-1}} + F_{i-1} \frac{x - x_i}{x_{i-1} - x_i}$$

$$= \frac{(x - x_i) F_{i-1} - (x - x_{i-1}) F_i}{x_{i-1} - x_i}$$

Find the correct interval with the bisection algorithm:

Mesh

point $=: x$

start: [1, len(mesh)]
(a, b)

In the loop, whe shrink the interval of indices, s.t. point is still inside *

stop when $b = a$

* We divide $[a, b] \cap \mathbb{Z}$ into two similarly sized sets

$$m = \left\lfloor \frac{a+b}{2} \right\rfloor$$

$$\lfloor x \rfloor := \max\{z \in \mathbb{Z},\ z \leq x\}$$

$$\lceil x \rceil := \min\{z \in \mathbb{Z},\ z \geq x\}$$

$[a, m]$, $[m+1, b]$

recall: we want in output i s.t. point $\in [x_{i-1}, x_i]$

$\to [x_{a-1}, x_m]$ $[x_m, x_b]$

if $x < x_m$ then
$b \leftarrow m$ $\to [a, m]$
else $a \leftarrow m+1 \to [m+1, b]$

Ex: start with $x_i = i$, $0 \leq i \leq 10$ $x \in 3.1$
$a = 1$, $b = 10$, $m = 5$
$[0, 5][5, 10]$
$\to a = 1$, $b = m$ $(=5)$

How long does it take?

How many iterations do we need?

(How long does an iteration take → $O(1)$)

iteration	a	b	indices
start (0)	1	N	N
1	1	$\left\lfloor \frac{1+N}{2} \right\rfloor$	$\approx N/2$
	$\left\lfloor \frac{1+N}{2} \right\rfloor + 1$	N	$\approx N/2$
2	---	---	$\approx N/4$
k	---	---	$\approx N/2^k$

For some k $\quad N \cdot 2^{-k} \approx 1$

$$k \approx \log_2 N$$

Bisection time for N intervals:

$$O(\log N)$$

Linear search: $\quad O(N) \quad (\approx N/2)$

So e.g. $N = 2^{15}$ → Bisection ≈ 15

→ Linear ≈ 16000