Numerik 1 Übungstunde, Gruppe 1 (Donnerstag)

Woche 03

Übungstunden werden registriert. Bitte informieren Sie den Assistenten, falls Sie nicht in der Aufzeichnung erscheinen möchten.
Streaming (öffentlich) und Aufzeichnungen (Anmeldung nötig):

https://seminarlive.mnf.uzh.ch/semlive/index.php?id=event_detail&module=mat801&semester=fs23&group=1

Notizen (diese Datei) und Python scripts:
https://user.math.uzh.ch/florian/stable/page/2023Mat801/notes.php

Let $I := [a, b]$ for $a < b$ be a closed interval. Consider a set of ordered nodes

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

for some $n \in \mathbb{N}$. Let τ_i for $1 \leq i \leq n$ be given as $\tau_i := [x_{i-1}, x_i]$ and denote the interval partition of I by $\mathcal{G} := \{\tau_i \mid 1 \leq i \leq n\}$. Prove that for all $k, m \in \mathbb{N}_0$ with $k \geq m$ the following equality holds:

$$\mathcal{S}_{\mathcal{G}}^{k,m} = \mathbb{P}_m(I).$$

- $\mathbb{P}_m(I) \subseteq S_{\mathcal{G}}^{k,m}$ (Trivial)

$\mathbb{P}_m(I) \subseteq C^\infty(I) \subseteq C^k(I)$

Let $p \in \mathbb{P}_m(I)$, then $p|_{\tau_i} \in \mathbb{P}_m(\tau_i)$

$S_{\mathcal{G}}^{k,m} := \{f \in C^k(I),\ f|_\tau \in \mathbb{P}_m(\tau)\ \forall \tau \in \mathcal{G}\}$

Hence $p \in S_{\mathcal{G}}^{k,m}$

- $S_{\mathcal{G}}^{k,m} \subseteq \mathbb{P}_m(I) \quad \forall k \geq m$

Note that $S_{\mathcal{G}}^{k,m} \subseteq S_{\mathcal{G}}^{m,m} \quad \forall k \geq m$

So we can prove $S_{\mathcal{G}}^{m,m} \subseteq \mathbb{P}_m(I)$

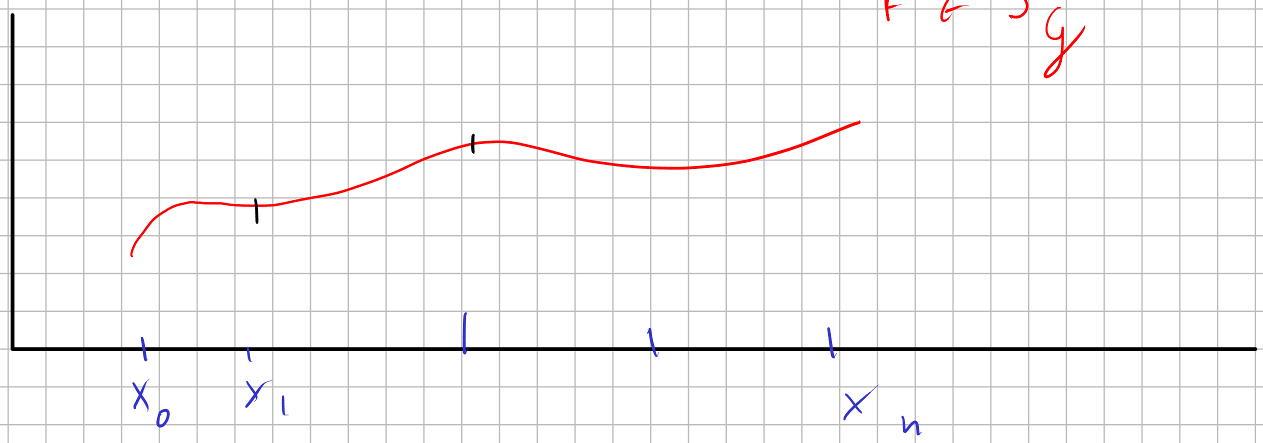
Various ways

- "Official" solution

We write $f|_{\tau_i}$ in the monomial basis.

Then we can use the same coefficient,
but the monomial basis on I instead of Z_i
To define a polynomial on I
And: that polynomial (or the coefficients)
do not depend on I
So the polynomial p we obtained is the original spline function.

$f \in S_g^{m,n}$



Take $f|_{Z_1}$ this is a polynomial
We can extend it to $\hat{f} \in \mathbb{P}(I)$

$\hat{f} = f$

Consider the Taylor expansion of $\hat{f}$
in x_1

$$\hat{f}(x) = \sum_{j=0}^{m} \hat{f}^{(j)}(x_1)\,(x - x_1)^j$$

Consider Z_2, $f|_{Z_2}$ extended to $g \in \mathbb{P}(I)$
and do the same

$$g(x) = \sum_{i=0}^{m} g^{(i)}(x_1)\,(x - x_1)^i$$

$f^{(j)}$ exists and is continuous $\forall j \leq m$
↳ spline

Then $F^{(i)}(x_1) = \hat{F}^{(i)}(x_1) = g^{(i)}(x_1) \; \forall i \leq m$

Hence $\hat{F} = g$

So $F|_{Z_1 \cup Z_2}$ is a polynomial

Homework 3, ex 1

Hint: you can assume the grid points are ordered, and you can assume (but you don't have to) that you will evaluate the spline on an ordered set of points.

Hint: think about how to store a spline function in the computer (usually you don't return a python function but the list of coefficient in some standard basis)

Hint: given $x \in [a, b]$, you need to determine i s.t. $x \in Z_i$

It might be convenient to make this a function

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spline evaluation:

define a function with inputs:

- grid points Θ
- either-or
  - $\to \hat{F}$ function to interpolate
  - $\to \hat{F}(x)$ for x in Θ

- a vector t where to evaluate the interpolant

We assume t is also ordered and all points of t are in $[a, b]$

start with $k=0$, $j=1$

j is the index for θ

k is the index for t

while True. → loop that never stops
we stop it manually!

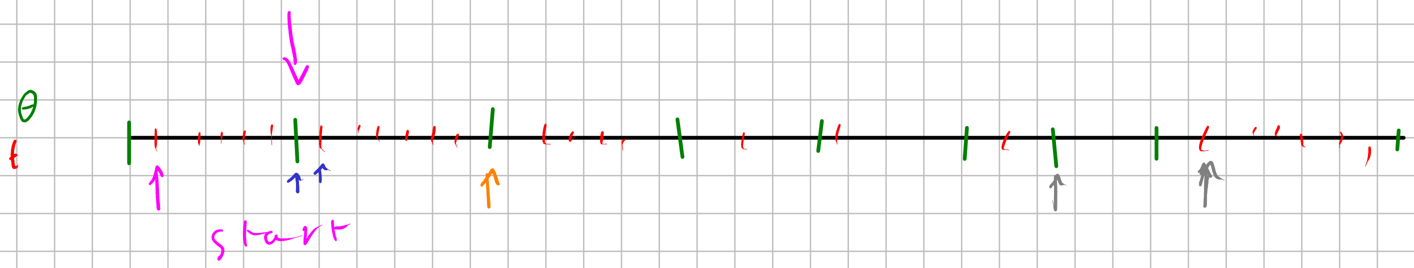
check if $t_k \leq \theta_j$

then evaluate the spline in t_k using z_j
and $k = k+1$

otherwise $j = j+1$

if $k \geq$ length (t) or $\underbrace{j \geq \text{length}(\theta)}$

this will never happen
because of our assumption



if • ≤ • all good do what you have to do
and go to the next red point

otherwise •

then increase • to • and do nothing else.

Consider a single interval $I_i = [x_{i-1}, x_i]$
for evaluating the spline in
$x \in I_i$

you need the coefficients of the polynomial
in that interval
for Newton/Lagrange or similar
the case degree 0 is special
because for any other case you need at least
two endpoints