Geometric Aspects of Field Theories on Manifolds with Low-Codimensional Strata

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Abstract

In this work, we analyze the geometric structure of field theories on manifolds with boundary and manifolds with corners with codimension-1 and codimension-2 strata. We focus on gravity in the coframe formalism coupled with scalar, gauge, and spinor fields, including their mutual interactions, where the vielbein and the connection form are treated as dynamical fields.

We show that the action functional in the bulk induces a pre-symplectic structure on the boundary, which can be reduced to a symplectic manifold via symplectic reduction. Non-dynamical field equations, when restricted to the boundary, are treated as constraints—local functionals—on the space of boundary fields. The constraint algebra is also studied for the aforementioned interactions.

No further assumptions on the boundary metric are made; in particular, we consider the possibility of a null-boundary where the induced metric is degenerate. However, when the boundary metric is non-degenerate, the vanishing locus of the constraint functionals defines a coisotropic submanifold, and its Marsden–Weinstein reduction yields the reduced phase space of the theory.

Finally, we study the codimension-2 structure of gravity, showing that the resulting geometric setup is that of a Courant algebroid. In this context, the Dorfman/Courant algebraic structure of vector fields and 1-forms—the ones induced from the codimension-1 boundary—over the space of fields on the codimension-2 corner is shown to be isotropic and involutive.

Self Plagiarism Statement

Chapters 4 to 6 present original research, previously published in the paper works mentioned below. Moreover, some sections of the Introduction incorporate material from those publications and draw on results from the literature, as referenced in the relevant sections. On top of that, Chapter 6 and the related work "Gravitational Structure on Manifolds with Codimension-2 Strata" originate from discussions with Giovanni Canepa and from unpublished notes by Francesco Bonechi titled "On Field-Theory-Induced Dirac Structures on Corners."

– A. S. Cattaneo, F. Fila–Robattino and M. Tecchiolli, "Gravitational structure on manifolds with codimension-2 corners," work-in-progress paper.

– A. S. Cattaneo, F. Fila–Robattino, V. Huang and M. Tecchiolli, "Gravity Coupled with Scalar, SU(n), and Spinor Fields on Manifolds with Null-Boundary," *Adv. Theor. Math. Phys.*, January 2025.

– G. Canepa, A. S. Cattaneo, F. Fila–Robattino and M. Tecchiolli, "Boundary structure of the standard model coupled to gravity," *Annales Henri Poicaré*, Semptember 2024.

– G. Canepa, A. S. Cattaneo and M. Tecchiolli, "Gravitational Constraints on a Lightlike boundary," *Annales Henri Poicaré*, March 2021.

– M. Tecchiolli, "On the Mathematics of Coframe Formalism and Einstein Cartan Theory–A Brief Review," arXiv:2008.08314, September 2019.

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Chapter 1

Introduction

Given a globally hyperbolic space-time and a Cauchy surface Σ within it, the reduced phase space of a theory characterizes the set of admissible initial conditions on Σ. That is, not all possible fields on the Cauchy surface evolve in time into a solution of the field equations on space-time; only those satisfying certain conditions do. The reduced phase space corresponds to this subset quotiented by the minimal relations that are needed to restore a symplectic structure.

In order to build the reduced phase space, the boundary structure is recovered by applying the method developed by Kijowski and Tulczijew (KT) in [KT79]. This approach defines the reduced phase space as a reduction—that is, a quotient—of the space of boundary fields, rather than following the framework introduced by Dirac in [Dir58]. The KT method offers several mathematical advantages, such as a more transparent formulation of constraints and natural compatibility with the BV-BFV formalism, as detailed in [CMR14] (for gravity, see [CS19], [CCS21a], and [CCS21b]). Moreover, the KT method works on more general space-times than hyperbolic ones.

The KT procedure goes as follows. One begins by varying the action of the theory, from which one can isolate the Euler–Lagrange equations and a boundary term arising from integrations by parts. This boundary term can then be reinterpreted as a 1-form on the space of pre-boundary fields. Taking the variation of this 1-form yields a closed 2-form. If this 2-form is non-degenerate, it defines the symplectic form on the space given by the restriction of the fields to the boundary. If instead the 2-form is degenerate but has a regular kernel, the geometric phase space is obtained as the quotient by this kernel.

Next, the Euler–Lagrange equations are restricted to the boundary, allowing one to distinguish between evolution equations—those involving derivatives normal to the boundary—and constraints. One then seeks structural conditions on the geometric phase space such that these constraints can be expressed in terms of the reduced boundary variables (i.e., after the aforementioned reduction). Finally, the reduced phase space is defined as the further reduction of the geometric phase space that restores the symplectic structure.

In Chapters 2 to 4, we recall the fundamental geometric framework that serves as the basis for our analysis, along with the main results for gravity in the coframe formulation on manifolds with boundary. We then move on to analyzing the boundary structure of gravity coupled to scalar, $\mathrm{SU}(n)$, and spinor fields (Chapter 5, Sections 5.1 to 5.3), also allowing for the case in which the boundary is null—that is, where the boundary metric is degenerate. Our analysis builds upon the results of [CCF22], where the geometric structures of gravity coupled to scalar, $\mathrm{SU}(n)$, and spinor fields were studied under the assumption of a non-degenerate boundary metric. The first goal here is thus to extend those results to the most general case, including degenerate boundaries. From a different perspective, this work generalizes the analysis carried out in [CCT21], which addressed the degenerate boundary structure of the Palatini–Cartan theory by incorporating gravity coupled to matter and gauge fields.

The second step consists in analyzing their mutual interactions. In Chapter 5, Sections 5.4 to 5.6, we carry out the boundary constraint analysis for gravity coupled to Yang–Mills–spinor, Yang–Mills-Higgs, and Yukawa fields. Also in this case, the study of the constraint algebra is performed without imposing any additional assumptions on the boundary metric, thus allowing for the possibility of a null-boundary. As such, the results displayed in these first chapters provide a stepping stone toward a formulation of the Standard Model on manifolds with boundary, presented in [Can+23], where also the BV-BFV approach is carried out.

Finally, in Chapter 6, we analyze the geometric structure of gravity on manifolds with codimension-2 corners. The problem of gravity on manifolds with corners has been addressed in [CC24], where the goal was to characterize the Poisson structure—defined up to homotopy—that arises on codimension-2 corners in four-dimensional gravity within the coframe formalism. This analysis was carried out by means of the BFV formalism, which replaces a (possibly singular) symplectic quotient with a cohomological resolution. Notice that, in the framework of the BFV formalism, the space of boundary fields is extended to a superspace endowed with a symplectic form and a Hamiltonian vector field whose square vanishes.

In this work, we do not attempt to extend the boundary theory within a BFV setting. Instead, we observe that, in the presence of a corner, the variation of the constraints naturally gives rise to a linear structure of the form $T \oplus T^*$ on the space of fields at the corner. From this starting point, our analysis shows that the structure inherited from the codimension-1 stratum (the boundary) forms a subspace of a Courant algebroid, which is both closed under the Dorfman bracket and isotropic.

Chapter 2

The geometric framework

2.1 Coframe formalism

A modern approach to the theory of gravity is by means of the *coframe formalism*. The general set-up of the theory will consist of:

- An N-dimensional smooth oriented[1] pseudo-riemannian manifold M with boundary Σ;
- A principal $\mathrm{GL}(N, \mathbb{R})$-bundle LM called the *frame bundle*, which can be reduced to a principal $\mathrm{SO}(N-1,1)$-bundle P;
- An associated vector bundle $\mathcal{V} := P \times_{\rho} V$ called the *Minkowski bundle*, where V is an N-dimensional real pseudo-riemannian vector space with reference metric $\eta = \mathrm{diag}(1, ..., -1)$ and $\rho\colon \mathrm{SO}(N-1,1) \to \mathrm{Aut}(V)$ is the fundamental representation of $\mathrm{SO}(N-1,1)$.

In the following, we will build the aforementioned geometrical set-up for a theory of gravity in the coframe formalism. We start by giving some definitions.[2]

Definition 2.1.1 (Smooth fiber bundle)**.** A *fiber bundle* is a structure (E, M, π, F), where E, M, and F are topological spaces,[3] and $\pi\colon E \to M$ is a smooth surjective function such that for every $x \in M$, there exists an open neighborhood $U \subseteq M$ of x and a diffeomorphism

$$\varphi\colon \pi^{-1}(U) \to U \times F$$

[1] Orientability is not necessary (see, e.g., [CCS21a, Section 2.1]), but we assume it here for simplicity of notations.

[2] We refer to [Tec19b] and references therein

[3] We will also assume M to be connected.

such that the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi} & U \times F \\ \pi \downarrow & & \downarrow p_1 \\ U & = & U \end{array}$$

where $p_1 \colon U \times F \to U$ is the projection onto the first factor. Because of this last property, a fiber bundle is said to be *locally trivial* and the object (U, φ) is called a *local trivialization*.

Definition 2.1.2 (Principal G-bundle)**.** Let M be a smooth manifold and G be a Lie group. A principal G-bundle P is a fiber bundle $\pi \colon P \to M$ together with a smooth right action $\mathcal{R} \colon G \times P \to P$ such that $\mathcal{R}$ acts freely and transitively on the fibers[4] of P and such that $\pi(\mathcal{R}_g(p)) = \pi(p)$ for all $g \in G$ and $p \in P$. Moreover, for a local trivialization (U, φ), it holds

$$\varphi(\mathcal{R}_g(p)) = \varphi(p)g = (u, g')g = (u, g'g). \tag{2.1}$$

Remark 2.1.3. Notice that the requirement for the right action of the principal bundle to be smooth is, in general, not required for the definition of such a bundle. It is our intention, nonetheless, to strengthen this definition for later purposes.

Here, we recall the similarity with a differentiable manifold. For a manifold, when we change charts, we have an induced diffeomorphism between the neighborhoods of the two charts, given by the composition of the two maps.

Thus, having two charts (U_i, ϕ_i) and (U_j, ϕ_j), we define the following:

$$\phi_j \circ \phi_i^{-1} \colon \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j). \tag{2.2}$$

At a level up, we have an analogous thing when we change trivialization. Of course, here, we have one more element: the element of fiber.

Taking two local trivializations (U_i, φ_i) and (U_j, φ_j) and given a smooth left action $\mathcal{T} \colon G \to \text{Diffeo}(F)$ of G on F, we then have

$$(\varphi_j \circ \varphi_i^{-1})(x, f) = \Big(x, \mathcal{T}(g_{ij}(x))(f)\Big) \qquad \forall x \in U_i \cap U_j, f \in F, \tag{2.3}$$

where the maps $g_{ij} \colon U_i \cap U_j \to G$ are called the *transition functions* for this change of trivialization and G is called the structure group.

Such functions obey the following transition functions conditions for all $x \in U_i \cap U_j$:

- $g_{ii}(x) = id$

[4] The space $\pi^{-1}(x)$ for $x \in M$ is called the fiber over x.

– $g_{ij}(x) = (g_{ji}(x))^{-1}$

– $g_{ij}(x) = g_{ik}(x)g_{kj}(x)$ for all $x \in U_i \cap U_k \cap U_j$.

The last condition is called the *cocycle condition.*

Theorem 2.1.4 (Fiber bundle construction theorem[5])**.** *Let M be a smooth manifold, F be a space, and G be a Lie group with faithful smooth left action $\mathcal{T}\colon G \to$ Diffeo(F) of G on F.*

Given an open cover $\{U_i\}$ of M and a set of smooth maps,

$$t_{ij}\colon U_i \cap U_j \to G \tag{2.4}$$

defined on each nonempty overlap, satisfying the transition function conditions.

Then, there exists a fiber bundle $\pi\colon E \to M$ such that

– *$\pi^{-1}(x) \simeq F$ for all $x \in M$*

– *its structure group is G, and*

– *it is trivializable over $\{U_i\}$ with transition functions given by t_{ij}.*

It is clear now that having E as a fiber bundle over M with fibers isomorphic to F and F' as a space equipped with the smooth action $\mathcal{T}'$ of G, implies the possibility of building a bundle E' *associated* to E, which shares the same structure group and the same transition functions g_{ij}. By the fiber bundle construction theorem, we have a new bundle E' over M with fibers isomorphic to F'. This bundle is called the *associated bundle* to E.

Depending on the nature of the associated bundle,[6] we have the following two definitions.

Definition 2.1.5 (Associated principal G-bundle)**.** Let $\pi\colon E \to M$ be a fiber bundle over a smooth manifold M, G be a Lie group, F' be a topological space, and $\mathcal{R}$ be a smooth right action of G on F'. Let also E' be the associated bundle to E with fibers isomorphic to F'.

If F' is the principal homogeneous space[7] for $\mathcal{R}$, namely $\mathcal{R}$ acts freely and transitively on F', then E' is called the principal G-bundle associated to E.

[5] A proof of the theorem can be found in [Sha97].

[6] We will be dealing with two particular types of associated bundles: a principal bundle associated to a vector bundle and a vector bundle associated to a principal bundle.

[7] The space where the orbits of G span all the space.

Definition 2.1.6 (Associated bundle to a principal G-bundle)**.** Let P be a principal G-bundle over M, F' be a space, and $\rho\colon \to \mathrm{Diffeo}(F')$ be a smooth effective left action of the group G on F'. We then have an induced right action of the group G over $P \times F'$ given by

$$(p, f') * g = (\mathcal{R}_g(p), \rho(g^{-1})(f')). \tag{2.5}$$

We define the associated bundle E to the principal bundle P, as an equivalence relation:

$$E \coloneqq P \times_\rho F' = \frac{P \times F'}{\sim}, \tag{2.6}$$

where $(p, f') \sim (\mathcal{R}_g(p), \rho(g^{-1})(f'))$, $p \in P$, and $f' \in F'$ with projection $\pi_\rho\colon E \to M$ s.t. $\pi_\rho([p, f']) = \pi(p) = x \in M$.

Therefore, $\pi_\rho\colon E \to M$ is a fiber bundle over M with $\pi_\rho^{-1}(x) \simeq F'$ for all $x \in M$.

Remark 2.1.7. The new bundle, given by the latter definition, is what we expected from a general associated bundle: a bundle with the same base space, different fibers, and the same structure group.

The idea would be therefore to take a principal G-bundle P as an associated bundle to TM, and we build a vector bundle associated to P with a fiber-wise metric η. We shall call this associated bundle $\mathcal{V}$.

First of all, we display the principal G-bundle as the principal G-bundle associated to TM.

Definition 2.1.8 (Orthonormal coframe)**.** Let (M, g) be a pseudo-Riemannian N-dimensional smooth manifold and (V, η) be an N-dimensional vector space with Minkowskian metric η.

A coframe at $x \in M$ is the linear isometry.

$${}_xe \coloneqq \Big\{ {}_xe\colon T_xM \to V \Big|\ {}_xe^*\eta \coloneqq \eta_{ab}\, {}_xe^a\, {}_xe^b = g \Big\}, \tag{2.7}$$

equivalently ${}_xe^a$ forms an ordered orthonormal basis in T_x^*M.

On the other hand, an orthonormal frame is defined as the dual of a coframe.

Remark 2.1.9. Locally, coframes can be identified with local covector fields. A necessary and sufficient condition for identifying them with global covector fields (namely a coframe for each point of the manifold) is to have a *parallelizable* manifold, namely a trivial tangent bundle.

Definition 2.1.10 (Orthonormal coframe bundle)**.** Let (M, g) be a smooth N-dimensional manifold with pseudo-riemannian metric g and T^*M be its cotangent bundle (real vector bundle of rank N). Then, we call the coframe bundle LM^* the principal G-bundle where the fiber at $x \in M$ is the set of all orthonormal coframes at x and where the group $G = \mathrm{O}(N-1, 1)$ acts freely and transitively on them.

The dual bundle of this is the orthonormal frame bundle, and it is denoted by LM, made up of orthonormal frames (dual of orthonormal coframes).

Remark 2.1.11.

- The orthonormal frame bundle is an associated principal G-bundle to TM.
- We will usually require the orientability of M (namely, the first Stiefel–Whitney class to be vanishing). This induces a further $\mathrm{SO}(N-1,1)$ reduction of the frame bundle.
- In contrast to the case of a Riemannian manifold, we stress that the bundle $\mathcal{V}$ is not canonically isomorphic to TM. This is equivalent to saying that there exists no canonical soldering form $\theta : TM \to \mathcal{V}$, as is the case in the Riemannian setting. The obstruction comes from the topological structure of the group $\mathrm{O}(N-1,1)$. In fact, $\mathrm{O}(N-1,1)$ has four connected components, and $\mathrm{SO}(N-1,1)$ has two, only one of which — the time-oriented component — is connected to the identity. This component is usually denoted by $\mathrm{SO}^+(N-1,1)$. As a consequence, a partition of unity cannot be used to canonically define a global soldering form, since the local 1-forms may belong to different connected components of $\mathrm{SO}(N-1,1)$. To overcome this obstruction, one must require an additional topological property of the spacetime: time-orientability.[8]
- If the manifold is parallelizable, we have the bundle isomorphism $e\colon TM \to \mathcal{V}$, which is given by the identity map over M and ${}_xe\colon T_xM \to V$ $\forall x \in M$. It can be regarded as a $\mathcal{V}$-valued 1-form $e \in \Omega^1(M,\mathcal{V})$. We can identify e with an element of $\Omega^1(M,V)$, thus with global sections of the cotangent bundle such that, at each point in M, the corresponding covectors ${}_xe^a$ obey $\eta_{ab}\, {}_xe^a\, {}_xe^b = g$.

Then, we define the vielbein via a reduction of the frame bundle.

Definition 2.1.12 (Vielbein)**.** We define the *vielbein* $\tilde{e}\colon P \to LM$ as the principal bundle isomorphism such that the following diagram commutes

$$\begin{array}{ccc} P & \xrightarrow{\ \tilde{e}\ } & LM \\ {\scriptstyle p'}\big\downarrow & & \big\downarrow{\scriptstyle \pi'} \\ \mathcal{V} & \underset{e}{\rightleftarrows} & TM \end{array}$$

[8]We have the presence of a Lorentzian metric, which induces a reduction of the structure group to $\mathrm{O}(1,N-1)$. Then, assuming that the manifold M is orientable, one can further reduce the structure group to $\mathrm{SO}(1,N-1)$ in order to perform integration on the manifold. This reduction becomes canonical once an orientation is chosen. Potentially, one could also introduce a time-orientation, which allows for a further canonical reduction to $\mathrm{SO}^+(1,N-1)$, provided that a time-orientation is selected.

where $e\colon TM \to \mathcal{V}$ is the vector bundle isomorphism induced by $\tilde{e}\colon P \to LM$ and p', π' the corresponding associated bundle maps. This means that the vielbein consists of the elements in $\Omega^1(M, \mathcal{V})$ possessing smooth inverse. We can call this space $\widetilde{\Omega}^1(M, \mathcal{V})$.

Remark 2.1.13.

– Given $i\colon \mathrm{SO}(N-1,1) \to \mathrm{GL}(N, \mathbb{R})$ as the canonical embedding, we recall that, in order for $\tilde{e}$ to be a principal bundle isomorphism, it must be an isomorphism of fiber bundles and also satisfy the equivariance condition

$$\mathcal{R}_{i(g)} \circ \tilde{e} = \tilde{e} \circ \mathcal{R}_g \quad \text{for all} \quad g \in G. \tag{2.8}$$

This is equivalent to asking that the following diagram commutes

$$\begin{array}{ccc} P & \xrightarrow{\tilde{e}} & LM \\ {\scriptstyle \mathcal{R}_g}\downarrow & & \downarrow{\scriptstyle \mathcal{R}_{i(g)}} \\ P & \xrightarrow[\tilde{e}]{} & LM \end{array}$$

– The existence and uniqueness of the map can be guaranteed through the use of the universal property of the quotient for the bundle isomorphism $\pi' \circ \tilde{e}\colon P \to TM$. This is possible thanks to the equivariance condition of $\tilde{e}$. The isomorphism property of the map $e\colon TM \to \mathcal{V}$ is simply inherited from $\tilde{e}$ by passing to the quotient.

– Since the map $e\colon TM \to \mathcal{V}$ is an isomorphism of vector bundles, it acts like a linear isomorphism on the fibers. It means it can be written in the following way:

$$\begin{aligned} {}_x e\colon T_xM &\to V \\ v &\mapsto v^a = v^\mu e^a_\mu, \end{aligned} \tag{2.9}$$

where $v = v^\mu \partial_\mu \in T_xM$. Consider now the dual basis $\{dx^\mu\}$. We can collect the components of the isomorphism into the covector $e^a_\nu dx^\nu(\partial_\mu) = e^a_\mu$, since a covector is a linear map over the tangent space. Given that a basis of the cotangent space can be seen as a family of N covectors $e^a_\mu dx^\mu$ and also that an isomorphism sends a basis to another basis, on a chart over $U \in M$, we can identify the map $e\colon TM \to \mathcal{V}$ with a family of N covector fields or directly with a V-valued covector field in $\Omega^1(U, V)$. Therefore, if M is parallelizable, we can identify the whole map e with a V-valued covector field $e \in \Omega^1(M, V)$.

– The name coframe formalism comes from the fact that e not only defines an isomorphism, but, thanks to the fact that it is obtained from the reduction of the structure group of the frame bundle to the pseudo-orthogonal group $\mathrm{SO}(N-1,1)$, it is also a linear isometry on the fibers. In fact, the reduction to $\mathrm{SO}(N-1,1)$ means by definition that the frames of the frame bundle are orthonormal, namely we have on the fibers $g_{\mu\nu}e^\mu_a e^\nu_b = \eta_{ab}$. On the other hand, in terms of their dual basis (coframes) $\{e^a\}$, we have $g_{\mu\nu} = \eta_{ab}e^a_\mu e^b_\nu$, which can be written as

$$g = e^*\eta. \tag{2.10}$$

This means that e is a linear isometry.

In the ordinary formulation of General Relativity (as in the original Einstein's work, for instance), we have objects called Γs, which are coefficients of a linear connection ∇ and thus determined by a parallel transport of tangent vectors.

The biggest advantage of treating $\mathrm{O}(3,1)$ as an "explicit symmetry" of the theory is that we have obtained the possibility of defining a *principal connection form*, which is the same kind of entity we have in a Yang–Mills gauge theory.

If we consider a smooth fiber bundle $\pi\colon E \to M$, where fibers are smooth manifolds, we can of course take tangent spaces at points $e \in E$. Having the tangent bundle TE, we may wonder if it is possible to separate the contributions coming from M to the ones from the fibers.

This cannot be done just by stating $TE = TM \oplus TF$, unless $E = M \times F$ is the trivial bundle. Namely, we cannot split directly vector fields on M from vector fields on the fibers F.

We can formalize this idea: use our projection π for constructing a tangent map $\pi_* = d\pi\colon TE \to TM$, and consider its kernel.

Definition 2.1.14 (Vertical bundle)**.** Let M be a smooth manifold and $\pi\colon E \to M$ be a smooth fiber bundle.

We call the sub-bundle $VE = \mathrm{Ker}(\pi_*\colon TE \to TM)$ the vertical bundle.

Following this definition, we have the natural extension to the complementary bundle of the vertical bundle, which is somehow the formalization of the idea we had of a bundle that takes care of tangent vector fields on M.

Definition 2.1.15 (Ehresmann connection)**.** Let M be a smooth manifold and $\pi\colon E \to M$ be a smooth fiber bundle.

Consider a complementary bundle HE such that $TE = HE \oplus VE$. We call this smooth sub-bundle HE the horizontal bundle or Ehresmann connection.

Remark 2.1.16. Thus, vector fields will be called *vertical* or *horizontal* depending on whether they belong to $\Gamma(VE)$ or $\Gamma(HE)$, respectively.

Remark 2.1.17. In the case where the fiber bundle is a principal bundle, HE is called *principal (Ehresmann) connection*.

Remark 2.1.18. We recall the case of the linear connection ∇; it was uniquely determined by a parallel transport procedure.

Here, we have an analogy.

Definition 2.1.19 (Lift)**.** Let $\pi \colon E \to M$ be a fiber bundle, M be a smooth manifold, $x \in M$ and $e \in E$ such that $\pi(e) = x$.

Given a smooth curve $\gamma \colon \mathbb{R} \to M$ such that $\gamma(0) = x$, we define a lift of γ through e as the curve $\tilde{\gamma}$, satisfying

$$\tilde{\gamma}(0) = e \quad \text{and} \quad \pi(\tilde{\gamma}(t)) = \gamma(t) \quad \forall t. \tag{2.11}$$

If E is smooth, then a lift is horizontal if every tangent to $\tilde{\gamma}$ lies in a fiber of HE, namely

$$\dot{\tilde{\gamma}}(t) \in HE_{\tilde{\gamma}(t)} \;\; \forall t. \tag{2.12}$$

Remark 2.1.20. It can be shown that an Ehresmann connection uniquely determines a horizontal lift. Here, it is the analogy with parallel transport.

We now focus on the case where the smooth fiber bundle is a principal G-bundle with smooth action $\mathcal{R}$. Here, we need a group G, that we generally take to be a matrix Lie group. We then have the corresponding algebra $\mathfrak{g}$, a matrix vector space in the present case.

Definition 2.1.21. The action $\mathcal{R}$ defines a map $\sigma \colon \mathfrak{g} \to \Gamma(VE)$ called the *fundamental map*,[9] where at $p \in P$, for an element $\xi \in \mathfrak{g}$, it is given via the exponential map $Exp \colon \mathfrak{g} \to G$.

$$\sigma_p(\xi) = \frac{d}{dt} \mathcal{R}_{e^{t\xi}}(p)\Big|_{t=0}. \tag{2.13}$$

The map is vertical because

$$\pi_* \sigma_p(\xi) = \frac{d}{dt} \pi(\mathcal{R}_{e^{t\xi}}(p))\Big|_{t=0} = \frac{d}{dt} \pi(p) = 0, \tag{2.14}$$

and, for this reason, the vector $\sigma_p(\xi)$ is vertical and it is called the *fundamental vector* associated to ξ.

Before proceeding, we need some Lie group theory.

Definition 2.1.22. Let G be a Lie group (a smooth manifold) with $\mathfrak{g}$ as its Lie algebra and $\forall g, h \in G$. We define:

[9] It turns out that it is an isomorphism, since $\mathcal{R}$ is regular.

– $L_g \colon G \to G$ and $R_g \colon G \to G$, such that $L_g h = gh$ and $R_g h = hg$ are the *left* and *right* actions, respectively;

– The adjoint map $\mathrm{Ad}_g \colon G \to G$ via such left and right actions is $\mathrm{Ad}_g := L_g \circ R_{g^{-1}}$, namely $\mathrm{Ad}_g h = ghg^{-1}$. It also acts on elements of the algebra $\xi \in \mathfrak{g}$ as $\mathrm{Ad}_g \colon \mathfrak{g} \to \mathfrak{g}$ via the exponential map[10]

$$\begin{aligned}\mathrm{Ad}_g \xi &= \frac{d}{dt}\Big((L_g \circ R_{g^{-1}})(e^{t\xi})\Big)\Big|_{t=0} = \frac{d}{dt}(g e^{t\xi} g^{-1})\Big|_{t=0} \\ &= g\xi g^{-1} \ \in \mathfrak{g},\end{aligned} \tag{2.15}$$

where the last two equalities hold in the present case of matrix Lie groups. This is not to be confused with the adjoint action $\mathrm{ad} \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, which is generated by the derivative of the adjoint map with $g = e^{t\chi}$ and $\chi \in \mathfrak{g}$, such that $\mathrm{ad}_\chi \xi = [\chi, \xi]$;

– The left invariant vector fields $v \in \Gamma(TG)$ as $L_{g*} \circ v = v$, namely $v(g) = L_{g*} v(e)$;

– The *Maurer–Cartan form* is the left invariant $\mathfrak{g}$-valued 1-form $\theta \in \Omega^1(G, \mathfrak{g})$ defined by its values at g.

$$\theta_g := L_{g^{-1}*} \colon T_g G \to T_e G \cong \mathfrak{g}. \tag{2.16}$$

Remark 2.1.23. Since for any left invariant vector field v, it holds $\forall g \in G$ that $\theta_g(v(g)) = v(e)$, we have that left invariant vector fields are identified by their values over the identity thanks to the Maurer–Cartan form θ. So we can state ([Fec11]) that this identification $v(e) \mapsto v$ defines an isomorphism between the space of left invariant vector fields on G and the space of vectors in $T_e G$, thus, the Lie algebra $\mathfrak{g}$. For matrix Lie groups, it holds that $\theta_g = g^{-1} dg$.

Lemma 2.1.24. *Given the group action $\mathcal{R}_g \colon P \to P$ and the associated tangent map $\mathcal{R}_{g*}\colon TP \to TP$ for an element $g \in G$, we have*

$$\mathcal{R}_{g*} \circ \sigma(\xi) = \sigma(Ad_{g^{-1}} \xi) \tag{2.17}$$

for all elements $g \in G$.

[10] We stress that the exponential map is not an isomorphism for all Lie groups; thus, the elements generated by the exponential map belong, in general, to a connected subgroup of the total group. More in general, the isomorphism is between a subset of the algebra containing 0 and a subset of the group containing the identity. Moreover, for a compact, connected, and simply connected Lie group, the algebra always generates the whole group via the exponential map. See [Fig06] for more details about this an other parts of this section.

Proof. At $p \in P$

$$\mathcal{R}_{g*}\sigma_p(\xi) = \frac{d}{dt}\Big((\mathcal{R}_g \circ \mathcal{R}_{e^{t\xi}})(p)\Big)\Big|_{t=0} = \frac{d}{dt}\Big((\mathcal{R}_g \circ \mathcal{R}_{e^{t\xi}} \circ \mathcal{R}_{g^{-1}} \circ \mathcal{R}_g)(p)\Big)\Big|_{t=0}, \tag{2.18}$$

we then use the fact that $\mathcal{R}_g \circ \mathcal{R}_{e^{t\xi}} \circ \mathcal{R}_{g^{-1}} = \mathcal{R}_{g^{-1}e^{t\xi}g} = \mathcal{R}_{\mathrm{Ad}_{g^{-1}}}e^{t\xi}$ and the identity for matrix groups $\mathrm{Ad}_g e^{t\xi} = e^{t\mathrm{Ad}_g\xi}$ to get the following:

$$\mathcal{R}_{g*}\sigma_p(\xi) = \frac{d}{dt}\Big(\mathcal{R}_{e^{t(\mathrm{Ad}_{g^{-1}}\xi)}}(\mathcal{R}_g(p))\Big)\Big|_{t=0} = \sigma_{\mathcal{R}_g(p)}(\mathrm{Ad}_{g^{-1}}\xi). \tag{2.19}$$

□

It is time to define what we were aiming to construct at the beginning of the section.

Definition 2.1.25 (Connection form)**.** Let P be a smooth principal G-bundle and $HE \subset TP$ be an Ehresmann connection.
We call the $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1(P, \mathfrak{g})$, satisfying

$$\omega(v) = \begin{cases} \xi & if \;\; v = \sigma(\xi),\; \xi \in \mathcal{C}^\infty(P, \mathfrak{g}) \\ 0 & if \;\; v \quad \text{horizontal}, \end{cases} \tag{2.20}$$

the connection 1-form. We will refer to the space of connection forms on P as $\mathcal{A}(P)$.

Remark 2.1.26. Notice that, in what follows, the terms "connection", "connection form", and "principal connection" will be used interchangeably. This abuse of terminology should not lead to confusion, as the context and notation will always make the intended meaning clear.

Theorem 2.1.27. *Given the quantities defined as above, we have*

$$\mathcal{R}_g^*\omega = Ad_{g^{-1}} \circ \omega. \tag{2.21}$$

Proof. Suppose $v = \sigma(\xi)$, since the other case left is trivial.

We can carry out some calculations on the left-hand side, and, following from Lemma 2.1.24, we have

$$\Big(\mathcal{R}_g^*\omega\Big)\Big(\sigma(\xi)\Big) = \omega\Big(\mathcal{R}_{g*} \circ \sigma(\xi)\Big) = \omega\Big(\sigma(\mathrm{Ad}_{g^{-1}}(\xi)\Big) = \mathrm{Ad}_{g^{-1}}(\xi). \tag{2.22}$$

Then, we only need to manipulate the right-hand side as

$$\mathrm{Ad}_{g^{-1}}\Big(\omega(\sigma(\xi))\Big) = \mathrm{Ad}_{g^{-1}}(\xi). \tag{2.23}$$

Both times, we used the definition of connection 1-form (i.e. Eq. (2.20)). □

Remark 2.1.28. This last theorem is called *G-equivariance*. It can be imposed instead of by assuming that HE is an Ehresmann connection, and then HE can be shown to be such an Ehresmann connection.

Definition 2.1.29 (Tensorial form)**.** Let $\rho\colon G \to \mathrm{Aut}(V)$ be a representation over a vector space V and $\alpha \in \Omega^k(P, V)$ be a vector valued differential form.

We call α a tensorial form if it is the following:

– horizontal, i.e., $\alpha(v_1, ..., v_k) = 0$ if at least one v_i is a vertical vector field, and

– equivariant, i.e., for all $g \in G$, $\mathcal{R}_g^* \alpha = \rho(g^{-1}) \circ \alpha$.

We define horizontal and equivariant forms as maps belonging to $\Omega^k_G(P, V)$.

Remark 2.1.30. The connection form ω is not, in general, horizontal; thus, it is not a tensorial form, $\omega \notin \Omega^1_G(P, \mathfrak{g})$.

Given our connection 1-form ω, we can proceed in two ways: the first consists in taking a map called the *horizontal projection* and in defining the curvature as this projection applied on the exterior derivative of ω. In this way, we naturally see that curvature measures the displacement of the commutator of two vectors from being horizontal.

We will proceed in a different way though. We will define the curvature through a *structure equation*.

Definition 2.1.31. Given $\omega \in \Omega^1(P, \mathfrak{g})$, a principal connection 1-form, the 2-form $\Omega \in \Omega^2_G(P, \mathfrak{g})$ satisfies the following:

$$\Omega = d\omega + \frac{1}{2}[\omega \wedge \omega], \tag{2.24}$$

which is called curvature 2-form. Here, $[\omega \wedge \omega]$ denotes the bilinear operation on the Lie algebra $\mathfrak{g}$ called differential Lie bracket, defined as

$$[\omega \wedge \eta](u, v) = \frac{1}{2}\Big([\omega(u), \eta(v)] - [\omega(v), \eta(u)]\Big), \tag{2.25}$$

where u and v are vector fields.

Remark 2.1.32. It the following sections, we will usually omit the wedge symbols. The bracket we will use will be of the form of the one in Definition 2.1.53, and we will generalize the meaning of the notation case by case.

Remark 2.1.33. It follows straightforwardly that, if we take two general horizontal vector fields $u, v \in \Gamma(HE)$ and we use the ordinary formula[11] for the exterior derivative of a 1-form $d\omega(u,v) = u\omega(v) - v\omega(u) - \omega([u,v])$, since $\omega(u) = \omega(v) = 0$, we get

$$\Omega(u,v) = -\omega([u,v]). \tag{2.26}$$

We see that Ω measures how the commutator of two horizontal vector fields is far from being horizontal as well.

Remark 2.1.34. $\Omega^k_G(P,V)$ is not closed under the ordinary exterior derivative. In that sense, if $\alpha \in \Omega^k_G(P,V)$, then $d\alpha \notin \Omega^{k+1}_G(P,V)$. This is what a covariant differentiation will do instead.

The idea of a covariant exterior derivative for a connection HE is, given such an Ehresmann connection HE, the one of projecting vector fields onto this horizontal bundle and then feed our ordinary exterior derivative with such horizontal vector fields. First of all, we define a map acting as a pull-back. Namely, given a map $h\colon TP \to HE$ (called the *horizontal projection*) such that, for all vertical vector fields v, we get $h \circ v := hv = 0$, we define the dual map $h^*\colon T^*P \to HE^*$ such that, for $\alpha \in \Omega^1(P,V)$ and V a vector space, we have $h^* \circ \alpha := h^*\alpha = \alpha \circ h$.

Definition 2.1.35 (d^h)**.** Let P be a principal G-bundle, V be a vector space, and $\alpha \in \Omega^k(P,V)$ be an equivariant form. We define the exterior covariant derivative d^h as a map $d^h\colon \Omega^k(P,V) \to \Omega^{k+1}_G(P,V)$ such that

$$d^h\alpha(v_0, ..., v_k) := h^* d\alpha(v_0, ..., v_k) = d\alpha(hv_0, ..., hv_k), \tag{2.27}$$

where $v_0, ..., v_k$ are vector fields.

Remark 2.1.36. Such object depends on the choice of our Ehresmann connection HE, which reflects onto the horizontal projection h; that is the reason why the index h is adopted.

Remark 2.1.37. We can make our covariant derivative depend only on ω, if we restrict it to only forms in $\Omega^k_G(P,V)$ and if we consider the representation of the algebra induced by the derivative of ρ that we denote $d\rho\colon \mathfrak{g} \to \mathrm{End}(V)$. Then, we have $d\rho \circ \omega \in \Omega^k\Big(P, \mathrm{End}(V)\Big)$.

Definition 2.1.38 (d_ω)**.** Let P be a principal G-bundle, V a vector space, $\omega \in \mathcal{A}(P)$ a connection form, and $\alpha \in \Omega^k_G(P,V)$ a tensorial form.

[11] Here, we regard $\omega(u)$ as a function $\omega(u)\colon P \to \mathfrak{g}$ belonging to the algebra of smooth functions to $\mathfrak{g}$, $\mathcal{C}^\infty(P,\mathfrak{g})$.

We define the *exterior covariant derivative* $d_\omega \colon \Omega^k_G(P, V) \to \Omega^{k+1}_G(P, V)$ as[12]

$$d_\omega \alpha \coloneqq d\alpha + \omega \wedge_{d\rho} \alpha, \tag{2.28}$$

where $\omega \wedge_{d\rho} \alpha \coloneqq d\alpha + d\rho \circ \omega \wedge \alpha$.

Remark 2.1.39.

- We observe that $d^2_\omega \alpha \neq 0$ for a general $\alpha \in \Omega^k_G(P, V)$, but it is easy to show that it holds
$$d^2_\omega \alpha = \Omega \wedge_{d\rho} \alpha. \tag{2.29}$$
Thus, for a *flat connection* such that $\Omega = 0$, we have $d^2_\omega \alpha = d^2 \alpha = 0$.

- We have observed that $\omega \notin \Omega^1_G(P, \mathfrak{g})$. Therefore, $d_\omega \omega$ is not well defined. However, we can consider $d^h \omega \in \Omega^2_G(P, \mathfrak{g})$, and this is precisely our curvature $\Omega = d\omega + \frac{1}{2}[\omega \wedge \omega]$, where the anomalous $\frac{1}{2}$ factor comes from the "non-tensoriality" of ω. As a matter of fact, there is no representation that would make the $\frac{1}{2}$ term arise if we considered $d_\omega \omega$ instead.

- The fact that d_ω is not well defined for non-tensorial forms does not mean that ω defines a less general derivative than what d^h does. As a matter of fact, HE could be defined starting from ω, as we mentioned above, since $HE = \operatorname{Ker} \omega$.

Definition 2.1.40 (Gauge field)**.** Let $P \to M$ be a principal G-bundle, G be a Lie group with $\mathfrak{g}$ as the respective Lie algebra, $\{U_\beta\}$ be an open cover of M, and $s_\beta \colon U_\beta \to P$ be a section.

We define the gauge field as the pull-back of the connection form $\omega \in \Omega^1(P, \mathfrak{g})$, i.e. as

$$A_\beta = s^*_\beta \omega \ \in \Omega^1(U_\beta, \mathfrak{g}). \tag{2.30}$$

Remark 2.1.41. Notice that, under a change of trivialization, the gauge field changes via the action of the adjoint map.

In fact, we have the following:

Lemma 2.1.42. *The restriction of ω to $\pi^{-1}(U_\beta)$ agrees with*

$$\omega_\beta = Ad_{g_\beta^{-1}} \circ \pi^* A_\beta + g^*_\beta \theta, \tag{2.31}$$

where $g_\beta \colon \pi^{-1}(U_\beta) \to G$ is the map induced by the trivialization map φ_β and where $Ad_{g_\beta^{-1}}$ is adjoint map at the group element given by $g_\beta(p)^{-1}$ at a point $p \in \pi^{-1}(U_\beta)$.

[12] For a general k-form:

$$(\omega \wedge_{d\rho} \alpha)(v_1, ..., v_{k+1}) = \frac{1}{(1+k)!} \sum_\sigma \operatorname{sign}(\sigma) d\rho\big(\omega(v_{\sigma(1)})\big)\big(\alpha(v_{\sigma(2)}, ..., v_{\sigma(k+1)}\big).$$

The proof comes from the observation that Eqs. (2.20) and (2.31) coincide in the open set $\pi^{-1}(U_\beta)$ for both a horizontal (for which they are zero) and a vertical vector field.

Thanks to this, we easily have the following result:

Theorem 2.1.43. *Let G be a matrix Lie group. Then it holds the following transformation for a gauge field:*

$$A_\beta = g_{\beta\gamma} A_\gamma g_{\beta\gamma}^{-1} - dg_{\beta\gamma} g_{\beta\gamma}^{-1}. \tag{2.32}$$

Proof. Using Eq. (2.30) and Eq. (2.31) for all $x \in U_\beta \cap U_\gamma$,

$$\begin{aligned} A_\beta &= s_\beta^* \omega \\ &= s_\beta^* \omega_\beta = s_\beta^* \omega_\gamma \\ &= s_\beta^* \Big(\mathrm{Ad}_{g_\gamma^{-1}} \circ \pi^* A_\gamma + g_\gamma^* \theta \Big) \\ &= \mathrm{Ad}_{g_{\beta\gamma}^{-1}} \circ A_\gamma + g_{\gamma\beta}^* \theta \qquad \text{(using } g_\gamma \circ s_\beta \coloneqq g_{\beta\gamma} \colon U_\beta \cap U_\gamma \to G) \\ &= \mathrm{Ad}_{g_{\beta\gamma}} \circ \Big(A_\gamma - g_{\beta\gamma}^* \theta \Big) \qquad (\mathrm{Ad}_{g_{\beta\gamma}} \circ g_{\beta\gamma}^* \theta = -g_{\gamma\beta}^* \theta), \end{aligned} \tag{2.33}$$

which reduces to the assert for matrix Lie groups. □

Remark 2.1.44.

- We observe that a local gauge transformation of the gauge field corresponds to a change of trivialization chart.

- Non-tensoriality of ω was given by the fact that it is, in general, not horizontal. For the gauge field A, we can generalize to forms on M the concept of tensoriality/non-tensoriality by noticing that a form obtained by the pull-back of a tensorial form, denoted with $t \in \Omega_G^1(P, V)$, would transform differently compared to A, namely as

$$t_\beta \coloneqq s_\beta^* t = g_{\beta\gamma} t_\gamma g_{\beta\gamma}^{-1}. \tag{2.34}$$

  The Maurer–Cartan form θ reflects the non-horizontality of ω to the gauge field, from Eq. (2.31).

- A difference of two gauge fields like $A - A'$ transforms as Eq. (2.34). In fact, the transformation rule is one of a tensorial form, since the Maurer–Cartan forms simplify.

– We notice that (iii) is a particular case of a more general one. Indeed, it is possible to show with proof in [KN69] (Chapter 5) that $\Omega^k_G(P,V) \cong \Omega^k(M, P\times_\rho V)$. This is essentially due to the fact that, thanks to the equivalence relation of the associated bundle and the gluing condition of sections on overlaps, the pull-backs by sections $s_\beta\colon U_\beta \to P$ give a one-to-one correspondence between these two spaces. Therefore, we can obtain forms with a tensorial transformation like Eq. (2.34) just by taking the pull-back of tensorial forms on P; these will be forms on M with values into the associated bundle $P\times_\rho V$.

– Our construction ensures that an object built with gauge fields $A_\beta \in \Omega^1(U_\beta, \mathfrak{g})$ (which transform on overlaps by Eq. (2.32)) will be in $\Omega^2(M, P\times_{\mathrm{Ad}} \mathfrak{g})$.

Definition 2.1.45 (Field strength)**.** Let $P \to M$ be a principal G-bundle, G be a Lie group with $\mathfrak{g}$ as the respective Lie algebra, $\{U_\beta\}$ be an open cover of M, and $s_\beta\colon U_\beta \to P$ be a section.

We define the field strength as the pull-back of the curvature form $\Omega \in \Omega^2_G(P,\mathfrak{g})$ as

$$F_\beta = s_\beta^*\Omega \ \in \Omega^2_G(U_\beta, \mathfrak{g}), \tag{2.35}$$

which, by definition of Ω, is

$$F_\beta = dA_\beta + \frac{1}{2}[A_\beta \wedge A_\beta]. \tag{2.36}$$

Remark 2.1.46. Similarly to what we have done for the gauge field, we can show[13] that the field strength transforms as

$$F_\beta = \mathrm{Ad}_{g_{\beta\gamma}} \circ F_\gamma = g_{\beta\gamma} F_\gamma g^{-1}_{\beta\gamma}, \tag{2.37}$$

where the last equality holds for matrix Lie groups with g and g^{-1} in G. This is indeed the transformation of a tensorial form, as in Eq. (2.34).

Remark 2.1.47. Since, as already pointed out, is a canonical isomorphism between $\Omega^k_G(P,V)$ and $\Omega^k(M, P\times_\rho V)$, we can relate Ω and F_β with a form[14] $F_A \in \Omega^2(M, \mathrm{ad}P)$. Namely, there is a canonical isomorphism sending $\Omega \in \Omega^2_G(P,\mathfrak{g})$ to $F_A \in \Omega^2(M,\mathrm{ad}P)$. Indeed, given the transformation law for the field strength (through the adjoint representation) in Eq. (2.37), we see that $\{F_\beta\}$ are horizontal and equivariant and, thus, form a global section belonging to $\Omega^2(M,\mathrm{ad}P)$, which is usually denoted as F_A.
The notation F_A stresses that it is obtained from gauge fields in $\Omega^1(U_\beta, \mathfrak{g})$.

[13]Using the Cartan structure equation for θ, $d\theta = -\frac{1}{2}[\theta,\theta]$.
[14]Where we have introduced the notation $\Omega^k(M, P\times_{\mathrm{Ad}}\mathfrak{g}) \coloneqq \Omega^k(M,\mathrm{ad}P)$.

In the case of a trivial bundle, it is also possible to define a global gauge field $A \in \Omega^1(M, \mathfrak{g})$.

Definition 2.1.48. The collection of gauge fields defines an exterior covariant derivative for $P \times_\rho V$-valued forms on M. We denote such a map with

$$d_A \colon \Omega^k(M, P \times_\rho V) \to \Omega^{k+1}(M, P \times_\rho V). \tag{2.38}$$

Consider $d_A \colon \Omega^k(M, P \times_\rho V) \to \Omega^{k+1}(M, P \times_\rho V)$ as the exterior covariant derivative and $F_A \in \Omega^2(M, \operatorname{ad} P)$ as the field strength.

Then, we have the following result, called the second Bianchi identity.

Theorem 2.1.49. *Given the quantities defined as above, we have*

$$d_A F_A = 0. \tag{2.39}$$

Proof. Given

$$F_A = dA + \frac{1}{2}[A \wedge A], \tag{2.40}$$

then

$$\begin{aligned} d_A F_A &= dF_A + [A \wedge F_A] \\ &= d^2 A + \frac{1}{2} d[A \wedge A] + [A \wedge dA] + \frac{1}{2}[A \wedge [A \wedge A]] \\ &= \frac{1}{2}[A \wedge [A \wedge A]] \qquad \left(d^2 A = 0 \text{ and } \frac{1}{2} d[A \wedge A] = -[A \wedge dA]\right) \\ &= 0. \qquad \text{(because of the Jacobi identity)} \end{aligned} \tag{2.41}$$

□

Theorem 2.1.50. *The inner product on V allows the identification $\mathfrak{so}(N-1,1) \cong \bigwedge^2 V$.*

Because of this theorem, we can identify $\mathfrak{so}(N-1,1)$-valued forms[15] with $\bigwedge^2 \mathcal{V}$-valued forms and we will use the following shortened notation to indicate the spaces of i-forms on M with values in the jth wedge product of $\mathcal{V}$

$$\Omega^{i,j} := \Omega^i \left(M, \textstyle\bigwedge^j \mathcal{V}\right), \tag{2.42}$$

which is generalized to all possible $i, j \in \mathbb{N}$.

[15] In the sense of a vector bundle with fibers $\mathfrak{so}(N-1,1)$

Remark 2.1.51. These spaces form indeed a graded algebra with graded product

$$\begin{aligned} \wedge\colon \Omega^{i,j} \times \Omega^{k,l} &\to \Omega^{i+k,j+l} & \text{for } i+k \leq N,\ j+l \leq N \\ (\alpha,\beta) &\mapsto \alpha \wedge \beta = (-1)^{(i+j)(k+l)} \beta \wedge \alpha. \end{aligned}$$

We will refer to an element in $\Omega^{i,j}$ also as an (i,j)-form.

Remark 2.1.52. If we consider a principal connection form on the principal $\mathrm{SO}(N-1,1)$-bundle P, namely an element $\omega \in \Omega^1(P, \bigwedge^2 V)$ (thanks to Theorem 2.1.50), we can pull it back using local sections. We will obtain a family of local connections $\omega_\alpha \in \Omega^1(U_\alpha, \bigwedge^2 \mathcal{V})$. These forms define a covariant derivative on M (see Definition 2.1.57).

The action of the Lie algebra on $\bigwedge^j \mathcal{V}$-differential forms will be denoted in the following way.

Definition 2.1.53. Let $\alpha \in \Omega^{i,j}$ and $\beta \in \Omega^{k,l}$. Then, we define the bracket

$$[\ ,\ ]\colon \Omega^{i,j} \times \Omega^{k,l} \to \Omega^{i+k,j+l-2} \tag{2.43}$$

through

$$\begin{aligned} &[\alpha,\beta]^{a_1\dots a_{j+l-2}}_{\mu_1\dots\mu_{i+k}} = \\ &= \sum_{\sigma_{i+k}} \sum_{\sigma_{j+l-2}} \mathrm{sign}(\sigma_{i+k}) \mathrm{sign}(\sigma_{j+l-2}) \alpha^{a_{\sigma(1)}\cdots a_{\sigma(j-1)} a}_{\mu_{\sigma(1)}\cdots\mu_{\sigma(i)}} \beta^{a_{\sigma(j)}\cdots a_{\sigma(j+l-2)} b}_{\mu_{\sigma(i+1)}\cdots\mu_{\sigma(i+k)}} \iota(\rho)_{ab}, \end{aligned} \tag{2.44}$$

where $\iota(\rho)$ is the contraction map that $\bigwedge^m \mathcal{V}$ inherits[16] from the representation ρ of $\mathrm{SO}(N-1,1)$. For the fundamental representation, this map is just the contraction with the η.

Remark 2.1.54. Shortly speaking, the bracket acts as a wedge product on both space-time and internal indices not contracted with the contraction map.

Remark 2.1.55. The contraction map $\iota(\rho)$ is obtained from the representation map of the algebra $\mathrm{d}\rho\colon \mathfrak{so}(N-1,1) \to \mathrm{End}(V)$ composed with the isomorphism of Theorem 2.1.50.

Remark 2.1.56. As mentioned in Remark 2.1.32, in the following sections we will retain the notation of Definition 2.1.53 while generalizing the meaning of the bracket case by case.

We now rewrite Definition 2.1.48, generalizing it to the case of an exterior product of bundles, using the notation introduced in Definition 2.1.53. This will serve as our standard notation for the exterior covariant derivative in the following sections.

---

[16]The representation ρ induces an algebra representation $\mathrm{d}\rho$ and we can translate that to $\bigwedge^2 \mathcal{V}$ thanks to Theorem 2.1.50. Then, we can easily generalize this action to $\bigwedge^m \mathcal{V}$.

Definition 2.1.57 (Reference form of the exterior covariant derivative)**.** Local connections in $\Omega^{1,2}$ define an exterior covariant derivative for $\bigwedge^j \mathcal{V}$-valued i-forms on M. We denote such a map with

$$d_\omega \colon \Omega^{i,j} \to \Omega^{i+1,j}. \tag{2.45}$$

Explicitly, it reads

$$d_\omega \alpha = d\alpha + [\omega, \alpha], \tag{2.46}$$

where $\alpha \in \Omega^{i,j}$.

Remark 2.1.58. Note that the representation of the brackets is the fundamental one. This is due to the fact that $\mathcal{V}$ is the associated bundle to P through the fundamental representation. In the case of a different associated bundle, the brackets will accordingly replaced.

Definition 2.1.59. Let $\omega \in \mathcal{A}(P)$ be a principal connection. Then, the associated local connections on M define a global 2-form $F_\omega \in \Omega^{2,2}$, which satisfies, in any arbitrary trivialization chart (U_α, s_α),

$$F_\omega|_{U_\alpha} = d\omega_\alpha + \frac{1}{2}[\omega_\alpha, \omega_\alpha], \tag{2.47}$$

with $\omega_\alpha = s_\alpha^* \omega$.

A more detailed derivation of this definition can be found in [Tec19b].

2.2 Stratified manifolds

Definition 2.2.1 (Stratified manifold[17])**.** A *stratified manifold* is a topological space X equipped with a decomposition

$$X = \bigsqcup_{\alpha \in A} S_\alpha, \tag{2.48}$$

where each S_α is a (locally closed) smooth manifold of dimension d_α, called a *stratum*, and the following conditions hold:

1. Each S_α is a smooth manifold.

[17] See [Pfl01] for a detailed treatment.

2. For any two strata S_α, S_β, if

$$S_\beta \cap \overline{S_\alpha} \neq \varnothing, \tag{2.49}$$

then $S_\beta \subset \overline{S_\alpha}$ and $\dim S_\beta < \dim S_\alpha$. This is sometimes called the *frontier condition.*

3. The topology of X is such that each closure $\overline{S_\alpha}$ is a union of strata.

Remark 2.2.2. The stratification induces a partial order on the index set A defined by

$$\beta \preceq \alpha \quad \Longleftrightarrow \quad S_\beta \subset \overline{S_\alpha}, \tag{2.50}$$

which, by the frontier condition, is well-defined and acyclic.

Example 2.2.3 (The Circular Cone)*.* Consider the quadratic cone

$$X = \{(x, y, z) \in \mathbb{R}^3 : z^2 = x^2 + y^2\}. \tag{2.51}$$

It admits a two-stratum decomposition:

$$S_2 = X \setminus \{(0, 0, 0)\}, \tag{2.52}$$
$$S_0 = \{(0, 0, 0)\}. \tag{2.53}$$

Here the apex is a zero-dimensional stratum, and the smooth part is two-dimensional.

Example 2.2.4 (Algebraic Varieties and Whitney Stratification)*.* Let $V \subset \mathbb{C}^n$ be an affine algebraic variety defined by the vanishing of polynomials $f_1, \ldots, f_m$. For each $k = 0, \ldots, n$, set

$$V_k = \Big\{ p \in V : \operatorname{rank}\Big(Df(p)\Big) = n - k \Big\}, \tag{2.54}$$

where $Df(p)$ is the Jacobian matrix of $(f_1, \ldots, f_m)$ at p. Equivalently, V_k is the locus where V is a smooth complex submanifold of complex dimension k (real dimension $2k$).

Then

$$V = \bigsqcup_{k=0}^{n} V_k \tag{2.55}$$

defines a stratification of V satisfying:

1. *Frontier condition:* if $V_j \cap \overline{V_k} \neq \emptyset$, then $j < k$ and $V_j \subset \overline{V_k}$.

2. *Whitney regularity:* for each $j < k$, the pair (V_j, V_k) satisfies Whitney's Conditions (A) and (B), ensuring that near any $p \in V_j$ the tangent spaces to V_k vary continuously and intersect T_pV_j in the expected dimension.

Example 2.2.5 (Square in $\mathbb{R}^2$). Let

$$X = [0,1]^2 \ \subset \ \mathbb{R}^2. \tag{2.56}$$

We stratify X by

$$S_0 = (0,1)^2, \tag{2.57}$$

$$S_1 = \Big(\{0\} \times (0,1)\Big) \cup \Big(\{1\} \times (0,1)\Big) \cup \Big((0,1) \times \{0\}\Big) \cup \Big((0,1) \times \{1\}\Big), \tag{2.58}$$

$$S_2 = \{(0,0), (0,1), (1,0), (1,1)\}. \tag{2.59}$$

Here S_2 consists of the four *corners* of the square—points where two boundary edges meet. The interior $(0,1)^2$ is S_0, the open edges are S_1, and the corner points are S_2.

Definition 2.2.6 (Manifold with corners[18]). A *manifold with corners* is a smooth manifold M of dimension N equipped with a stratification

$$M = \bigsqcup_{k=0}^{K} M^{[k]}, \tag{2.60}$$

for some $K \leq N$, satisfying the following:

1. Each $M^{[k]}$ is the codimension-k stratum, consisting of points locally diffeomorphic to an open subset of

$$\{x_1 = x_2 = \cdots = x_k = 0\} \subset \mathbb{R}^k \times \mathbb{R}^{N-k}, \tag{2.61}$$

i.e., modeled on $[0,\infty)^k \times \mathbb{R}^{N-k}$ with exactly k vanishing coordinates.

2. Each stratum $M^{[k]}$ is a smooth manifold of dimension $N-k$ (without boundary).

3. The stratification satisfies the frontier condition:

$$\overline{M^{[k]}} \supset \bigsqcup_{\ell > k} M^{[\ell]}. \tag{2.62}$$

4. The manifold M admits a smooth structure modeled on open subsets of $[0,\infty)^k \times \mathbb{R}^{N-k}$ for $k = 0, \ldots, K$, with smooth transition maps between overlapping charts.

[18] See [Mel] for further details.

Remark 2.2.7. In the following sections, we will restrict our attention to manifolds with corners admitting only strata of codimension at most 2, that is:

$$M = M^{[0]} \sqcup M^{[1]} \sqcup M^{[2]}. \tag{2.63}$$

Remark 2.2.8. Notice that, even though a manifold with boundary is a special case of a manifold with corners, consisting of only two strata, one of codimension-0 (the bulk) and one of codimension-1 (the boundary), the definition of manifold with corners usually refers to stratified structures with higher depth.

Chapter 3

Field theories on the boundary

3.1 Marsden–Weinstein reduction

The following notion of reduction we refer to originates from the work of Marsden and Weinstein [MW74] and is commonly known as Marsden–Weinstein reduction. The construction begins with a symplectic manifold equipped with a group action. The goal is to produce a new symplectic manifold by effectively quotienting out the group action.

We start with the fundamental ingredient of the whole construction.

Definition 3.1.1 (Symplectic manifold)**.** A *symplectic manifold* is a manifold M equipped with a closed, non-degenerate differential 2-form ϖ, called the *symplectic form*. That is, it holds:

- Closedness: $d\varpi = 0$;
- Non-degeneracy: for every point $p \in M$, the bilinear form

$$\varpi_p \colon T_pM \times T_pM \to \mathbb{R} \tag{3.1}$$

  satisfies the condition that if $\varpi_p(v, w) = 0$ for all $w \in T_pM$, then $v = 0$. This is equivalent to requiring that the map

$$\flat \colon T_pM \to T_p^*M, \quad v \mapsto \varpi_p(v, \cdot) \tag{3.2}$$

  is injective.[1]

Remark 3.1.2. In the finite dimensional case,[2] the dimension of a symplectic manifold must be even.

[1]Notice that, in the infinite dimensional case, only the injectivity of the map is required.

[2]All anti-symmetric matrices on an odd-dimensional space have zero determinant (degeneracy).

Theorem 3.1.3 (Darboux's Theorem). *Let (M, ϖ) be a symplectic manifold of dimension $2n$. Then, for every point $m \in M$, there exists a local chart around m, with coordinates $(x^1, \dots, x^n, p_1, \dots, p_n)$, such that the symplectic form ϖ takes locally the canonical form*

$$\varpi = \sum_{i=1}^{n} dp_i \wedge dx^i. \tag{3.3}$$

In other words, all symplectic manifolds are locally symplectomorphic to $(\mathbb{R}^{2n}, \varpi_0)$, where $\varpi_0 = \sum_i dp_i \wedge dx^i$ is the canonical symplectic form.

On top of a symplectic manifold, we add an action of a group encoded by the following object.

Definition 3.1.4 (Momentum map). Let (M, ϖ) be a symplectic manifold, and let G be a Lie group acting[3] on M via symplectomorphisms.[4] Denote by $\mathfrak{g}$ the Lie algebra of G, and for each $\xi \in \mathfrak{g}$, let X_ξ be the fundamental vector field on M associated to ξ. Then, a *momentum map* is a smooth map

$$J \colon M \to \mathfrak{g}^* \tag{3.4}$$

such that for every $\xi \in \mathfrak{g}$, the function $J_\xi := \langle J, \xi \rangle \colon M \to \mathbb{R}$ satisfies

$$\mathrm{d}J_\xi + \iota_{X_\xi} \varpi = 0. \tag{3.5}$$

If, in addition, J is equivariant with respect to the given action of G on M and the coadjoint action of G on $\mathfrak{g}^*$, that is,

$$J(g \cdot p) = \mathrm{Ad}^*_{g^{-1}} J(p) \quad \text{for all } g \in G,\ p \in M, \tag{3.6}$$

then J is called an *equivariant momentum map*.

Definition 3.1.5. Let $f \colon M \to N$ be a smooth map between smooth manifolds, with M of dimension m and N of dimension n. A point $y \in N$ is called a *regular value* of f if, for every $x \in f^{-1}(y)$, the following map is surjective

$$\mathrm{d}f_x \colon T_x M \to T_y N. \tag{3.7}$$

If $y \in N$ is not a regular value, it is called a *critical value*. A point $x \in M$ is called a *critical point* of f if $\mathrm{d}f_x$ is not surjective.

[3]We will always consider smooth actions.

[4]Namely, the action preserves the symplectic form.

Theorem 3.1.6 (Marsden–Weinstein). *Let (M, ϖ) be a symplectic manifold, and let G be a Lie group acting on M via symplectomorphisms. Suppose the action is free and proper, and that there exists an equivariant momentum map*

$$J\colon M \to \mathfrak{g}^*. \tag{3.8}$$

Let $\mu \in \mathfrak{g}^*$ be a regular value of J, and denote by $M_\mu := J^{-1}(\mu)/G_\mu$ the quotient of the level set by the stabilizer subgroup G_μ of μ under the coadjoint action.

Then M_μ is a smooth manifold of dimension $\dim M - 2\dim G + \dim G_\mu$, and it inherits a natural symplectic form ϖ_μ uniquely characterized by the condition

$$\iota^*\varpi = \pi^*\varpi_\mu, \tag{3.9}$$

where $\iota\colon J^{-1}(\mu) \hookrightarrow M$ is the inclusion and $\pi\colon J^{-1}(\mu) \to M_\mu$ is the canonical projection.

3.2 Symplectic reduction and the RPS

The concept of the reduced phase space (RPS) originates in physics. In gauge theories, not all field equations describe the true dynamics of the fields; some impose constraints. Simultaneously, the gauge algebra restricts the number of independent local degrees of freedom. The reduced phase space arises precisely when one aims to extract the genuine physical content of the theory by factoring out gauge-related degrees of freedom and non-dynamical relations.

Definition 3.2.1 (RPS). The *reduced phase space* (RPS) is the symplectic space of gauge-inequivalent boundary data that extend to solutions of the field equations in the bulk.

The procedure we follow to obtain the reduced phase space is known as the KT construction([KT79]). We start with a field theory on a manifold with boundary M with action $\mathcal{S}$ where we notice that the integration by parts in the variation of the action commonly gives rise to a boundary term α, which we call the Noether 1-form.

We denote the space of the fields of the theory as $\mathcal{W}$. By considering the pull-back of the fields in $\mathcal{W}$ to the boundary Σ via the natural inclusion $i\colon \Sigma \to M$, we obtain the space of pulled-back fields denoted by $\widetilde{\mathcal{W}}$. In this setting, the boundary term α can be interpreted as a 1-form on the space of pulled-back fields. Furthermore, the variational operator δ is regarded as a de Rham differential of the complex of differential forms on $\widetilde{\mathcal{W}}$.

Note that the 2-form on $\widetilde{\mathcal{W}}$ defined via

$$\tilde{\varpi} = \delta\alpha \tag{3.10}$$

is closed.

Remark 3.2.2. It is important to note that a closed 2-form does not necessarily have to be non-degenerate. A closed 2-form with a possibly degenerate kernel is called a *pre-symplectic form*, and a space endowed with such a form is called a *pre-symplectic space*.

Definition 3.2.3. We define the *space of pre-boundary fields* for a field theory as the pre-symplectic space $(\widetilde{\mathcal{W}}, \tilde{\varpi})$, where $\widetilde{\mathcal{W}}$ is the space of pulled-back fields on the boundary Σ and $\tilde{\varpi}$ is the pre-symplectic form.

In order to obtain a symplectic space, we could just quotient by the distribution given by the kernel of the pre-symplectic form.

Definition 3.2.4. We define the *geometric phase space* of a field theory as the symplectic space $(\mathcal{W}, \varpi)$ obtained as the quotient of the space of pre-boundary fields by the kernel of its pre-symplectic form[5]

$$\mathcal{W} := \frac{\widetilde{\mathcal{W}}}{\ker(\tilde{\varpi})} \tag{3.11}$$

and with symplectic form ϖ, the unique 2-form on $\mathcal{W}$ such that $p^*\varpi = \tilde{\varpi}$, where $p\colon \widetilde{\mathcal{W}} \to \mathcal{W}$ is the canonical projection.

Definition 3.2.5 (Symplectic reduction)**.** The procedure we employed in constructing the geometric phase space is referred to as *symplectic reduction.* In particular, when the reduction is carried out from a coisotropic submanifold, it is called *coisotropic reduction.*

Remark 3.2.6. Notice that the Marsden–Weinstein reduction is a particular case of coisotropic reduction, in which the characteristic distribution—i.e., the distribution arising as the kernel of the presymplectic form—is generated by the action of a Lie group through a momentum map.

As we mentioned in the beginning, in field theory, it is commonly understood that not all field equations are dynamical, and on a manifold with boundary, this is equivalent to having some field equations that are non-transverse with respect to the boundary. The resulting non-dynamical equations can be interpreted as constraints that must be satisfied by the boundary fields.
We can give these constraints the form of local functionals on $\mathcal{W}$ (or $\widetilde{\mathcal{W}}$), just by integrating the pulled-back equations on Σ. We denote this set of constraints as $\mathcal{C}$ (or $\widetilde{\mathcal{C}}$).

[5]This quotient is to be intended in the sense of distributions on the tangent bundle. Note that $\ker(\tilde{\varpi})$ is involutive, since $\tilde{\varpi}$ is closed.

The first understanding of the nature of a set of constraints on a symplectic space is due to Dirac [Dir50]. He pointed out correctly that the nature of the constraints, which he divided in first- and second-class, had important implications on the local degrees of freedom of the theory.[6] More precisely, the hamiltonian vector fields of the first-class constraints generate the algebra of the symmetry group of the theory and the ones of the second-class constraints do not. In Section 4.3, a more detailed discussion of first- and second-class constraints is presented.

If these constraints are first class, their vanishing locus—which is a coisotropic submanifold—can be possibly identified[7] with the zero level set of a momentum map associated to the gauge algebra generated by the Hamiltonian vector fields of the constraints. The Marsden–Weinstein reduction of this locus yields the *reduced phase space* (RPS). This is indeed the space of the non-gauge equivalent (thanks to the quotient) initial conditions (the fields are on the boundary) for the dynamical field equations of the theory (since we have considered the vanishing locus of the constraints).

$$
\begin{array}{l}
(\mathcal{W}, S) \\
\quad \Big\downarrow \text{Pull-back to the boundary} \\
(\widetilde{\mathcal{W}}_\Sigma, \tilde{\varpi}, \widetilde{\mathcal{C}}) \\
\quad \Big\downarrow \text{Symplectic reduction} \\
(\mathcal{W}_\Sigma, \varpi, \mathcal{C}) \\
\quad \Big\downarrow \text{Reduction (possibly Marsden-Weinstein)} \\
\text{RPS}
\end{array}
$$

Table 3.1: Construction of the RPS.

3.3 The peculiarity of the null–boundary

It is worth setting aside a short section to discuss the nature of a null-boundary. As we will see in the next sections, this kind of boundary requires additional conditions to be imposed, due to the degeneracy of the induced metric. This feature has direct consequences on the constraint algebra, since these extra conditions can be reinterpreted as new local functionals defining the constraints of the theory. In

[6] The local degrees of freedom are defined as the dimension of the reduced phase space and the dimension of a space is define as the rank of the fiber or its dimension as a C^∞-module.

[7] The gauge algebra must integrate to a Lie group. Otherwise one has to perform another coisotropic reduction.

this section, we will focus on clarifying what a null-boundary is and present a representative example to better illustrate the concept.

Definition 3.3.1 (Null-boundary)**.** Let M be a differentiable pseudo-Riemannian manifold with metric g. We say that the boundary of M, denoted as $\Sigma = \partial M$, is a *null-boundary* if the pullback of the metric g to Σ via the boundary inclusion $i\colon \Sigma \to M$ is degenerate, i.e., if $\dim(\mathrm{Ker}(\iota^* g)) = 1$, where the boundary metric is considered as a map $\iota^* g\colon T\Sigma \to T^*\Sigma$.

To give an idea about why this degeneracy arises, we can take into account the following examples.

Example 3.3.2 (Minkowski). Consider the Minkowski space-time given by $\mathbb{R}^4$ with metric $\eta = \mathrm{diag}(1,1,1,-1)$. The vielbein is given by the set of the following tetrads

$$e^a_\mu = \delta^a_\mu, \tag{3.12}$$

where $a, \mu = 1,2,3,4$.

If we consider as our space-time the submanifold given by time-like part of the Minkowski space-time which has the light-cone as its boundary, we can take the boundary vielbein with the following form[8]

$$e^a_i = \begin{cases} e^a_1 = \delta^a_1 \\ e^a_2 = \delta^a_2 \\ e^a_+ = \delta^a_3 - \delta^a_4, \end{cases} \tag{3.13}$$

where again $a = 1,2,3,4$ but $i = 1,2,+$ the boundary coordinates. It follows that the boundary metric is given by

$$\eta^\partial_{ij} = \eta_{ab} e^a_i e^b_j = \mathrm{diag}(1,1,0), \tag{3.14}$$

and it is thereby degenerate. Following from Definition 3.3.1, we have that the light-cone is indeed a null-boundary.

Example 3.3.3 (Schwarzschild). Consider in the bulk ($N = 4$) the metric

$$g = e^{2B(r)} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\varphi^2) - e^{2A(r)} dt^2. \tag{3.15}$$

After properly setting A and B as

$$e^{2B(r)} = \Big(1 - \frac{2m}{r}\Big)^{-1} \tag{3.16}$$

$$e^{2A(r)} = 1 - \frac{2m}{r}, \tag{3.17}$$

[8]Note that, as we will be doing in the subsequent sections, we denote the bulk and boundary vielbein in the same way.

with $2m$ a constant called the *Schwarzschild radius*, this turns to be metric discovered/invented by Karl Schwarzschild in 1916 ([K S16]). As well known, this metric presents a particular singularity at $r = 2m$, which spatially lies on the surface $\mathcal{S}^2$. This singularity is called *event horizon*.

We can see, by inspection, that

$$e^t = e^A dt \qquad e^r = e^B dr \tag{3.18}$$

$$e^\theta = r d\theta \qquad e^\varphi = r \sin\theta d\varphi, \tag{3.19}$$

satisfy $\eta_{ab} e^a e^b = g$, and can be taken as a choice of the vielbein.

The vanishing torsion condition completely determines the connection ω, through the equation

$$(d_\omega e)^a = de^a + \omega^a{}_b e^b = 0. \tag{3.20}$$

Taking the exterior derivative of Eq. (3.18), yields

$$de^r = 0 \tag{3.21}$$

$$de^t = \frac{dA}{dr} e^A dr dt \tag{3.22}$$

$$de^\theta = dr d\theta \tag{3.23}$$

$$de^\varphi = \sin\theta dr d\varphi + r\cos\theta d\theta d\varphi. \tag{3.24}$$

We can write $(d\theta, d\varphi, dr, dt)$ in terms of the coframes and, therefore, Eq. (3.21) becomes

$$de^r = 0 \tag{3.25}$$

$$de^t = (-\frac{dA}{dr} e^{-B}) e^t e^r \tag{3.26}$$

$$de^\theta = (r^{-1} e^{-B}) e^r e^\theta \tag{3.27}$$

$$de^\varphi = (r^{-1} e^{-B}) e^r e^\varphi + (r^{-1}\cot\theta) e^\theta e^\varphi, \tag{3.28}$$

which can be used in Eq. (3.20) in order to identify the connection form ω, for which we have the six independent components

$$\omega^t{}_r = (\frac{dA}{dr} e^{-B}) e^t \tag{3.29}$$

$$\omega^r{}_\theta = (-r^{-1} e^{-B}) e^\theta \tag{3.30}$$

$$\omega^\theta{}_\varphi = (-r^{-1}\cot\theta) e^\varphi \tag{3.31}$$

$$\omega^r{}_\varphi = (-r^{-1} e^{-B}) e^\varphi \tag{3.32}$$

$$\omega^t{}_\theta = 0 \tag{3.33}$$

$$\omega^t{}_\varphi = 0, \tag{3.34}$$

where the other components are given by the antisymmetry $\omega^i{}_j = -\omega_j{}^i$.

We observe that these coframes present a degeneracy. More precisely, we notice that that both $A(r)$ and $B(r)$ diverge when restricted to the event horizon. therefore, in order to overcome the problem of the degeneracy, consider the coframes of Eq. (3.18) under the change of coordinates, with θ and φ that do not change and $r_s = 2m$ the Schwarzschild radius with m the mass in natural units, given by (for $r > r_s$)

$$T = \Big(\frac{r}{r_s} - 1\Big)^{1/2} e^{r/2r_s} \cosh(\frac{t}{2r_s}) \tag{3.35}$$

$$X = \Big(\frac{r}{r_s} - 1\Big)^{1/2} e^{r/2r_s} \sinh(\frac{t}{2r_s}), \tag{3.36}$$

where we have set $A(r) = \log(1 - \frac{r_s}{r})$ and $B(r) = -\log(1 - \frac{r_s}{r})$. Moreover, notice that r becomes a function of T and X.

This leads to the following coframes (Kruskal)

$$e^T = 2\frac{r_s^{\frac{3}{2}}}{\sqrt{r}} e^{-r/2r_s} dT \tag{3.37}$$

$$e^X = 2\frac{r_s^{\frac{3}{2}}}{\sqrt{r}} e^{-r/2r_s} dR \tag{3.38}$$

$$e^\theta = r d\theta \tag{3.39}$$

$$e^\varphi = r \sin\theta d\varphi. \tag{3.40}$$

We take the components of Eq. (3.37) and restrict them to the $S^2 \times \mathbb{R}$ hypersurface with fixed r and coordinates $(+, 1, 2)$. We have

$$e_1^a = r\delta_1^a \tag{3.41}$$

$$e_2^a = r \sin\theta \delta_2^a \tag{3.42}$$

$$e_+^a = 2\frac{r_s^{\frac{3}{2}}}{\sqrt{r}} e^{-r/2r_s} (\delta_3^a + \delta_4^a), \tag{3.43}$$

with $a = 1, 2, 3, 4$. These coframes are no longer degenerate in the present case for $r = 2m$. On the other hand, the pulled back metric[9]

$$g^\partial = e^*\eta = \mathrm{diag}(0, r^2, r^2 \sin^2\theta) \tag{3.44}$$

turns out to be degenerate. Thus, these coframes define a null-hypersurface.

[9]Here, e is the vielbein given by e_1, e_2, e_+.

Chapter 4

Codimension-1 structure of gravity

When it comes to pure gravity, it is worth dedicating an entire chapter only to that. In fact, given that gravity will be present throughout all chapters of this thesis, we will develop the underlying geometrical framework that will serve as the foundation for the entirety of our discussion.

In the following chapter, we will recall the results of [CCS21a], where the non-degenerate boundary structure is presented, and [CCT21], where the degenerate one is treated instead.

4.1 The Palatini-Cartan theory

General Relativity is traditionally formulated ([Ein16]) using a linear (or Koszul) affine connection ∇ on the tangent bundle TM. This leads to Einstein's field equations, expressed in terms of the Ricci tensor $R_{\mu\nu}$, which depends on the Christoffel symbols Γ and, consequently, on the metric $g_{\mu\nu}$. The Ricci scalar R and the metric $g_{\mu\nu}$ further appear in the Einstein-Hilbert action functional:

$$S = \int R\sqrt{-g}\, d^4x. \tag{4.1}$$

Since the connection is assumed to be Levi-Civita—i.e., the unique metric-compatible and torsion-free affine connection—the Christoffel symbols are symmetric by definition.

Alternatively, one can consider the same action (4.1), but treat the connection ∇ as an independent variable, so that $S = S[g, \nabla]$. This approach allows Γ to represent the coefficients of a generic affine connection, which may, in general, be non-symmetric. If we impose metric compatibility, the variational principle ensures the torsion-free condition (and vice versa). This formulation is known as the Palatini formalism ([Pal19]).

By means of the coframe formalism we developed in Chapter 2, we can cast the theory into a different setting. Namely, we are able to express the affine and the metric structure of the theory through a vielbein e and a principal connection ω, letting general relativity resemble, from a geometrical perspective, an $\mathrm{SU}(N)$ gauge theory. This approach is known as the coframe or Palatini–Cartan formulation.

Then, we define the theory as follows:

Definition 4.1.1. The classical Palatini–Cartan theory is the assignment of the pair $(\mathcal{F}, \mathcal{S})_M$ to every pseudo riemannian N-dimensional manifold and vector space V with reference metric[1] η with space of fields

$$\mathcal{F} = \widetilde{\Omega}^{1,1} \times \mathcal{A}(P) \ni (e, \omega) \tag{4.2}$$

and action functional

$$\mathcal{S} = \int_M \frac{1}{(N-2)!} e^{N-2} F_\omega + \frac{1}{N!} \Lambda e^N, \tag{4.3}$$

where $\Lambda \in \mathbb{R}$ and the powers in e are in terms of the wedge product.

Remark 4.1.2. In Eq. (4.3), we have omitted both the wedge product and the trace. The trace operator is the map $\mathrm{Tr}\colon \bigwedge^N V \to \mathbb{R}$ such that, given a basis $\{u_i\}_{i=1,\dots,N}$ of V, it holds that $\mathrm{Tr}[u_{i_1} \wedge \cdots \wedge u_{i_n}] = \varepsilon_{i_1 \dots i_n}$, thus the trace works as a choice of the orientation on M, which must be compatible with the $\mathrm{SO}(N-1,1)$ reduction.

Remark 4.1.3. In the subsequent sections, we will avoid reiterating a similar definition for each distinct case. Rather, we will provide the space of fields on M, and it is important to bear in mind that the definitions of the upcoming theories will be straightforward generalizations of Definition 4.1.1.

The fields of the theory are therefore the coframe $e \in \widetilde{\Omega}^{1,1}$ and the pull-back of the principal connection $\omega \in \mathcal{A}(P)$ via sections of P.

The Euler–Lagrange equations coming from the action principle $\delta\mathcal{S} = 0$ are, respectively for the variations in e and ω theory read

$$\frac{1}{(N-3)!} e^{N-3} F_\omega - \frac{1}{(N-1)!} \Lambda e^{N-1} = 0 \tag{4.4}$$

$$e^{N-3} d_\omega e = 0, \tag{4.5}$$

[1] Note that any particular choice of the Lorentzian structure on V is immaterial, since a change in V would just isomorphically reflect to the space of fields without changing $\mathcal{S}_{PC}$.

which, in $N = 4$, reduce to

$$eF_\omega - \frac{1}{3!}\Lambda e^3 = 0 \tag{4.6}$$

$$ed_\omega e = 0. \tag{4.7}$$

By injectivity of the map $e \wedge \cdot$ on $\Omega^{2,1}$ (see [CCT21]), Eq. (4.7) is equivalent to[2]

$$d_\omega e = 0, \tag{4.8}$$

which is the torsion-free condition. Therefore, this is the equation that identifies the Levi-Civita connection for the metric (2.10).

4.2 Boundary structure of gravity

In this section, we will assume that our stratified structure, with respect of Definition 2.2.6, has only two strata, i.e. we will work on a manifold with boundary. Moreover, from now on, we will take $N = 4$. Notice that the geometrical method implemented to the study of the boundary structure of the theory is the KT one, based on the symplectic reduction defined in Definition 3.2.5, and described in the previous sections.

Remark 4.2.1. Given $i\colon \ :\Sigma \to M$ the boundary inclusion. We will use the following definition for bundle valued differential forms on the boundary

$$\Omega_\Sigma^{i,j} \coloneqq \Omega^i(\Sigma, \textstyle\bigwedge^j i^*\mathcal{V}). \tag{4.9}$$

We notice that the integration by parts in the variation of the action $\mathcal{S}$ gives rise to a boundary term

$$\alpha = \int_\Sigma \frac{1}{2} e^2 \delta\omega, \tag{4.10}$$

which we call the Noether 1-form.

Remark 4.2.2. Notice that we denote fields on the boundary with the exact same notation of the ones in the bulk.

Remark 4.2.3. By considering the pull-back of the fields in $\mathcal{F}$ to the boundary Σ via the natural inclusion $i\colon \Sigma \to M$, we obtain the space of pulled-back fields denoted by $\widetilde{\mathcal{F}}_\Sigma$. In this setting, the boundary term α defined in (4.10) can be interpreted as a 1-form on the space of pulled-back fields. Furthermore, the variational operator δ is regarded as a de Rham differential of the complex of differential forms on $\widetilde{\mathcal{F}}_\Sigma$.

[2] This generalizes to a generic N.

Lemma 4.2.4. *The 2-form on $\widetilde{\mathcal{F}}_\Sigma$ defined via*

$$\tilde{\varpi} = \delta\alpha = \int_\Sigma e\delta e\delta\omega \tag{4.11}$$

is closed and degenerate.

Proof. The proof of the closeness follows immediately from the definition of the graded algebra structure. The degeneracy comes from noticing that a vector $\mathbb{X} \in \mathrm{Ker}(\tilde{\varpi})$ acts as a shift $\omega \mapsto \omega + v$ with $v \in \Omega_\Sigma^{1,2}$ such that $ev = 0$. In particular, given

$$\mathbb{X} = \mathbb{X}_e \frac{\delta}{\delta e} + \mathbb{X}_\omega \frac{\delta}{\delta\omega}, \tag{4.12}$$

we get

$$\tilde{\varpi}(\mathbb{X}) = \int_\Sigma e(\mathbb{X}_\omega \delta e + \mathbb{X}_e \delta\omega) = 0. \tag{4.13}$$

Since we already mentioned that $e \wedge \cdot$ is injective on $\Omega_\Sigma^{1,1}$ but not on $\Omega_\Sigma^{1,2}$, it follows that $\mathbb{X}_e = 0$ and $e\mathbb{X}_\omega = 0$, namely $\mathbb{X}$ acts on ω as $\omega \mapsto \omega + v$ with $v \in \Omega_\Sigma^{1,2}$ such that $ev = 0$. □

Definition 4.2.5. We define the *space of pre-boundary fields* for the Palatini–Cartan theory as the pre-symplectic space $(\widetilde{\mathcal{F}}_\Sigma, \tilde{\varpi})$, where $\widetilde{\mathcal{F}}_\Sigma$ is the space of pulled-back fields (e, ω) on the boundary Σ and $\tilde{\varpi} = \int_\Sigma e\delta e\delta\omega$ is the pre-symplectic form.

As we pointed out in Section 3.1, in order to obtain a symplectic space, we could just quotient by the distribution given by the kernel of the pre-symplectic form. We need first to define some maps.

Definition 4.2.6. Let $e \in \widetilde{\Omega}_\Sigma^{1,1}$ and $e^k \in \Omega_\Sigma^{k,k}$ be the wedge product of k elements e. Then, we define the following maps:

$$\begin{aligned} W_k^{\Sigma,(i,j)} \colon \Omega_\Sigma^{i,j} &\longrightarrow \Omega_\Sigma^{i+k,j+k} \\ \alpha &\longmapsto e^k \wedge \alpha \end{aligned} \tag{4.14}$$

$$\begin{aligned} \varrho^{(i,j)} \colon \Omega_\Sigma^{i,j} &\longrightarrow \Omega_\Sigma^{i+1,j-1} \\ \alpha &\longmapsto [e, \alpha] \end{aligned} \tag{4.15}$$

$$\begin{aligned} \tilde{\varrho}^{(i,j)} \colon \Omega_\Sigma^{i,j} &\longrightarrow \Omega_\Sigma^{i+1,j-1} \\ \alpha &\longmapsto [\tilde{e}, \alpha], \end{aligned} \tag{4.16}$$

with $\tilde{e} \in \widetilde{\Omega}_\Sigma^{1,1}$ being a degenerate vielbein, namely $\tilde{e}^*\eta = 0$.

Theorem 4.2.7. *The geometric phase space for the Palatini–Cartan theory is the symplectic manifold* $(\mathcal{F}_\Sigma, \varpi)$ *obtained by a symplectic reduction of the space of pre-boundary fields* $\widetilde{\mathcal{F}}_\Sigma$*. The symplectic form* ϖ *is the unique* 2*-form on* $\mathcal{F}_\Sigma$ *such that* $p^*\varpi = \tilde{\varpi}$*, where* $p\colon \widetilde{\mathcal{F}}_\Sigma \to \mathcal{F}_\Sigma$ *is the canonical projection, and it is given by*

$$\varpi = \int_\Sigma e\delta e\delta[\omega], \tag{4.17}$$

where

$$\omega' \sim \omega \quad \iff \quad \omega' - \omega \in \mathrm{Ker} W_1^{\Sigma,(1,2)}. \tag{4.18}$$

We refer to this equivalence class as $\mathcal{A}_{red}(\Sigma)$*.*

Proof. It follows from Lemma 4.2.4. □

Remark 4.2.8. As mentioned in Chapter 3, we interpret the field equations containing non-transversal terms to the boundary as local functionals (constraints) on the space of pre-boundary fields of the theory.
The local functionals coming from the transversal components to the boundary are given by

$$L_c = \int_\Sigma c e d_\omega e \tag{4.19}$$

$$P_\xi = \int_\Sigma \iota_\xi e e F_\omega + \iota_\xi(\omega - \omega_0) e d_\omega e \tag{4.20}$$

$$H_\lambda = \int_\Sigma \lambda e_n \left(e F_\omega + \frac{1}{3!}\Lambda e^3 \right) \tag{4.21}$$

where $c \in \Omega_\Sigma^{0,2}[1]$, $\xi \in \mathfrak{X}[1](\Sigma)$ and $\lambda \in \Omega_\Sigma^{0,0}[1]$ are (odd) Lagrange multipliers and the notation [1] denotes that the fields are shifted in parity and are treated as odd variables.[3]
However, these functionals are not the only ones forming the constraints set of the theory. In Section 3.3, we mention that, for a case of a null-boundary, the pull-back of the metric to the boundary is in general degenerate. This fact has a few implications. In particular, the degeneracy of the metric has an impact on the constraints of the theory. Indeed, one has to encompass the consequences of this degeneracy into a new constraint if one wants to take into account all possible pseudo-Riemannian geometric structures of the boundary.

[3]The symbol [1] indicates indeed a shift in the parity. Note that this does not mean necessarily that their total parity is odd. Moreover, this formulation is natural for the BV–BFV formalism; while it lies beyond the scope of the present work, it may prove useful for future developments.

Remark 4.2.9. On the boundary Σ, the injectivity property of the map $e \wedge \cdot$ acting on boundary $(2,1)$-forms is lost ([CCS21a]). This property guaranteed the equivalence of $d_\omega e = 0$ and $ed_\omega e = 0$ in the bulk. This situation is indeed problematic. In fact, in the bulk we have two perfectly equivalent conditions, namely two equivalent ways of writing one of the field equations. If we take the equation $ed_\omega e = 0$ as a constraint functional of the theory on the boundary, we need the equivalence with $d_\omega e = 0$ to hold in order to make sense of the field equations on the boundary themselves. In other words, since in the bulk $ed_\omega e = 0$ must give rise to the same solution space of $d_\omega e = 0$, if the solutions space of these two equations on the boundary were to differ, then the theory on the boundary would be ill-defined. I.e., the pull-back to the boundary of the solutions obtained from the field equations in the bulk would be different from the boundary fields obtained from the solutions of the fields equations on the boundary. It means that one has to impose some additional conditions in order to maintain this equivalence on the boundary[4]

Definition 4.2.10. Let J be a complement[5] in $\Omega_\Sigma^{2,1}$ of the space $\mathrm{Im}\, \varrho^{(1,2)}|_{\mathrm{Ker} W_1^{\Sigma,(1,2)}}$. Then, we define the following subspaces:

$$\mathcal{T} \coloneqq \mathrm{Ker} W_1^{\Sigma(2,1)} \cap J \subset \Omega_\Sigma^{2,1} \tag{4.22}$$

$$\mathcal{S} \coloneqq \mathrm{Ker} W_1^{\Sigma,(1,3)} \cap \mathrm{Ker} \tilde{\varrho}^{(1,3)} \subset \Omega_\Sigma^{1,3} \tag{4.23}$$

$$\mathcal{K} \coloneqq \mathrm{Ker} W_1^{\Sigma,(1,2)} \cap \mathrm{Ker} \varrho^{(1,2)} \subset \Omega_\Sigma^{1,2}. \tag{4.24}$$

Definition 4.2.11 (Completion vielbein)**.** We define the completion vielbein as a fixed non-vanishing section $e_n \in \Omega_\Sigma^{0,1}$ such that, for $e \in \widetilde{\Omega}_\Sigma^{1,1}$, $\{e(v_1), e(v_2), e(v_3), e_n\}$ is a basis of $i^*\mathcal{V}$, where $\{v_1, v_2, v_3\}$ is a basis of $T\Sigma$.

Remark 4.2.12. Notice that fixing a completion vielbein e_n selects an open subset for $e \in \widetilde{\Omega}_\Sigma^{1,1}$ on which the defining conditions for e_n hold.

Remark 4.2.13. Notice in particular that, in any neighborhood of e of the space of boundary fields, we are allowed to pick e_n independently of the dynamics of the vielbein e. In other words, we can state that e_n is constant in the field e. This trivially implies that e_n has no variation along e.

[4]Notice that one could theoretically consider the pull-back to the boundary of the equation $d_\omega e = 0$, instead of $ed_\omega e = 0$, and take that as a constraint. In fact, the solution space of $d_\omega e = 0$ contains that one of $ed_\omega e = 0$. Notice that the constraint arising from $d_\omega e = 0$ would be no longer basic with respect to the quotient map defining the geometric phase space. Therefore, one has to split it into an invariant and a non-invariant part, leading to a situation perfectly equivalent to the one discussed above.

[5]To obtain an explicit expression for the complement, one can follow these steps. Start by selecting an arbitrary Riemannian metric on the boundary manifold Σ and extend it to the space $\Omega^{2,1}$. Then, the orthogonal complement of the image of the map $\varrho^{(1,2)}|\mathrm{Ker} W_1^{\Sigma,(1,2)}$ in $\Omega_\Sigma^{2,1}$ can be identified as the space J, with respect to the chosen Riemannian metric.

Lemma 4.2.14 (Corollary of Lemma A.0.1)**.** *Given $\alpha \in \Omega_\Sigma^{2,1}$, $e_n \in \Omega_\Sigma^{0,1}$ a completion vielbein and $\mathcal{T}$ as defined in Definition 4.2.10, we have that*

$$\alpha = 0$$

if and only if

$$\begin{cases} \alpha \in \mathrm{Ker} W_1^{\Sigma,(2,1)} \\ e_n(\alpha - p_\mathcal{T}\alpha) \in \mathrm{Im}\, W_1^{\Sigma,(1,1)} \\ p_\mathcal{T}\alpha = 0, \end{cases} \tag{4.25}$$

where $p_\mathcal{T}$ is the projector onto $\mathcal{T}$. We call the second and third conditions in (4.25) *respectively the structural and the degeneracy constraints.*

The next lemma provides a formulation of the degeneracy constraint in terms of an integral functional.

Lemma 4.2.15 ([CCT21])**.** *Let $\alpha \in \Omega_\Sigma^{2,1}$. Then, we have the following equivalence*

$$p_\mathcal{T}\alpha = 0 \quad \Longleftrightarrow \quad \int_\Sigma \tau\alpha = 0 \quad \forall \tau \in \mathcal{S}. \tag{4.26}$$

Remark 4.2.16*.* As long as we do not specify any α, these two lemmas remain purely geometrical and do not depend on the properties of the field equations. We will then be able to use these results for the interactive theories where the equivalence condition on the boundary will differ from $d_\omega e = 0$ and $e d_\omega e = 0$ (since the field equations will be different themselves). Therefore, in general, we need to specify α for each different theory. In particular, for the Palatini–Cartan theory, $\alpha = d_\omega e$ and the structural reads

$$e_n(d_\omega e - p_\mathcal{T} d_\omega e) \in \mathrm{Im}\, W_1^{\Sigma,(1,1)}, \tag{4.27}$$

together with the additional constraint

$$R_\tau = \int_\Sigma \tau d_\omega e. \tag{4.28}$$

Remark 4.2.17*.* It is important to emphasize that, in the non-degenerate case (i.e. if the boundary has no null-components), Eq. (4.25) is trivially equivalent to the structural constraint

$$e_n\alpha \in \mathrm{Im}\, W_1^{\Sigma,(1,1)}. \tag{4.29}$$

In fact, in the non-degenerate case, one can easily see that $p_\mathcal{T}\alpha = 0$ holds trivially, which also implies that the additional constraint R_τ is identically vanishing.

Remark 4.2.18. To study the reduced phase space of the theory, we make use of representatives for the equivalence classes defined in (4.18). In the non-degenerate case, these representatives are uniquely determined by the structural constraint itself. In other words, ensuring the equivalence of $d_\omega e = 0$ and $ed_\omega e = 0$ on the boundary, is enough to determine uniquely the representatives of the equivalence classes defined in (4.18). However, in the degenerate case, the structural constraint and the degenerate constraint (or its integral form R_τ), despite the fact that they indeed ensure on the boundary the equivalence mentioned above, are not sufficient to uniquely assign a representative to each equivalence class. Therefore, it is necessary to seek an alternative way to guarantee the unambiguous determination of these representatives.

We can accomplish this through the following lemma.

Lemma 4.2.19 ([CCT21])**.** *Given $\omega \in \Omega_\Sigma^{1,2}$, $e_n \in \Omega_\Sigma^{0,1}$ a completion vielbein and $\mathcal{K}$ as defined in Definition 4.2.10, then the conditions*

$$\begin{cases} e_n(d_\omega e - p_{\mathcal{T}}(d_\omega e)) \in \operatorname{Im} W_1^{\Sigma,(1,1)} \\ p_{\mathcal{K}}\omega = 0 \end{cases} \tag{4.30}$$

uniquely define a representative of the equivalence class $[\omega] \in \mathcal{A}_{red}(\Sigma)$.

Remark 4.2.20. In [CCT21], it has been proved that the analysis is independent of the choice of the representative of the equivalence class (4.18). In more rigorous terms, for each choice of the representatives there is a canonical symplectomorphism between the symplectic space defined by representatives and the geometric phase space of the theory.

Remark 4.2.21. It is important to highlight that, in the non-degenerate case, the subspaces $\mathcal{T}$, $\mathcal{S}$, and $\mathcal{K}$ of Definition 4.2.10 are trivial. It follows that the projectors $p_{\mathcal{K}}$ and $p_{\mathcal{T}}$ are also trivial. Once again, this means that, in the non-degenerate theory, the structural constraint alone serves the purpose of establishing the equivalence between $d_\omega e = 0$ and $ed_\omega e = 0$ on the boundary, as well as uniquely determining the representatives of the equivalence classes defined in Eq. (4.18).

We have seen that, on a null-boundary, we need both the structural and the degeneracy constraints together with the additional equation $p_{\mathcal{K}}\omega = 0$ in order to both guarantee the equivalence between $d_\omega e = 0$ and $ed_\omega e = 0$ on the boundary and uniquely fix the representative of the equivalence class $[\omega] \in \mathcal{A}(i^*P)_{red}$.
More specifically, the role of the structural constraint together with the integral constraint R_τ is the one of ensuring the aforementioned equivalence condition, whereas, the structural constraint together with $p_{\mathcal{K}}\omega = 0$ will uniquely fix the representatives.

From Lemma 4.2.19, it naturally follows the following theorem.

Theorem 4.2.22. *The geometric phase space of the Palatini–Cartan theory is symplectomorphic to the space* $(\mathcal{F}^\partial, \varpi^\partial)$*, where*[6]

$$\mathcal{F}^\partial \subset \widetilde{\Omega}^{1,1}_\Sigma \times \Omega^{1,2}_\Sigma \tag{4.31}$$

with $(e, \omega) \in \mathcal{F}^\partial$ *satisfying*

$$\begin{cases} e_n(d_\omega e - p_{\mathcal{T}}(d_\omega e)) \in \operatorname{Im} W_1^{\Sigma,(1,1)} \\ p_{\mathcal{K}}\omega = 0, \end{cases} \tag{4.32}$$

as defined in Lemma 4.2.19.
The corresponding symplectic form on $\mathcal{F}^\partial$ *is given by*

$$\varpi^\partial = \int_\Sigma e\delta e\delta\omega. \tag{4.33}$$

Remark 4.2.23. Because of convenience, from this to the following sections, we will always work within spaces defined by representatives. We will equivalently call them the geometric phase spaces of the theories.

Remark 4.2.24. Note that the space of fields $\mathcal{F}^\partial$ described in Theorem 4.2.22 can equally be formulated as a general fiber bundle. All results in this paper carry over verbatim under this substitution.

We now display the constraints of the theory. The vanishing locus $\widetilde{\mathcal{C}}$ of these functionals will then be the submanifold upon which we will build the reduced phase space of the theory.

Definition 4.2.25. Let $c \in \Omega^{0,2}_\Sigma[1]$, $\xi \in \mathfrak{X}(\Sigma)[1]$, $\lambda \in C^\infty(\Sigma)[1]$ and $\tau \in \mathcal{S}[1]$. Then, we define the following functionals

$$L_c = \int_\Sigma c e d_\omega e \tag{4.34}$$

$$P_\xi = \int_\Sigma \frac{1}{2}\iota_\xi(e^2)F_\omega + \iota_\xi(\omega - \omega_0)e d_\omega e \tag{4.35}$$

$$H_\lambda = \int_\Sigma \lambda e_n\Big(eF_\omega + \frac{\Lambda}{3!}e^3\Big) \tag{4.36}$$

$$R_\tau = \int_\Sigma \tau d_\omega e. \tag{4.37}$$

We refer to these as the constraints of the Palatini–Cartan theory.

[6]Notice that we can either choose $\mathcal{A}(\iota^*\mathcal{P})$ or the space of local connections on the boundary $\Omega^{1,2}_\Sigma$.

Remark 4.2.26. Although these constraints follow from the field equations, here we adopt them as a definition. Our derivation relies on several geometric constructions and includes an arbitrary form of the constraints chosen for subsequent computations. Therefore, it is convenient to cast them explicitly as a definition.

Remark 4.2.27. Notice that the structure of the constraints is invariant with respect to the quotient map of Theorem 4.2.7. This is a fundamental point to make sense of the whole construction.

We are now able to determine the algebra of the constraints of the theory. This differs from the one of the non-degenerate theory, since the new constraint R_τ changes the nature of the Poisson brackets, which become second class.

Theorem 4.2.28. *The Poisson brackets of the constraints of Definition 4.2.25 read*

$$\begin{aligned}
&\{L_c, L_c\} = -\frac{1}{2}L_{[c,c]} && \{P_\xi, P_\xi\} = \frac{1}{2}P_{[\xi,\xi]} - \frac{1}{2}L_{\iota_\xi\iota_\xi F_{\omega_0}}\\
&\{L_c, P_\xi\} = L_{\mathcal{L}_\xi^{\omega_0}c} && \{H_\lambda, H_\lambda\} \approx 0\\
&\{L_c, R_\tau\} = -R_{p_\mathcal{S}[c,\tau]} && \{P_\xi, R_\tau\} = R_{p_\mathcal{S}\mathcal{L}_\xi^{\omega_0}\tau}.\\
&\{R_\tau, H_\lambda\} \approx G_{\lambda\tau} && \{R_\tau, R_\tau\} \approx F_{\tau\tau}\\
&\{L_c, H_\lambda\} = -P_{X^{(a)}} + L_{X^{(a)}(\omega-\omega_0)_a} - H_{X^{(n)}}\\
&\{P_\xi, H_\lambda\} = P_{Y^{(a)}} - L_{Y^{(a)}(\omega-\omega_0)_a} + H_{Y^{(n)}}
\end{aligned}$$

with $X = [c, \lambda e_n]$ *and* $Y = \mathcal{L}_\xi^{\omega_0}(\lambda e_n)$ *and where the superscripts* (a) *and* (n) *describe their components with respect to* e_a, e_n. *Furthermore* $F_{\tau\tau}$ *and* $G_{\lambda\tau}$ *are functionals of* e, ω, τ *and* λ *that are not proportional to any other constraint.*

Remark 4.2.29. The symbol $\approx$ indicates the identity on the zero locus of the constraints. In particular, this means that those brackets written with this symbol are not a linear combination of the constraints themselves. On the other hand, all the brackets written with a $=$ vanish on the zero locus, for example $\{L_c, L_c\} \approx 0$.

Corollary 4.2.30. *If the boundary metric* i^*g *is non-degenerate, then the functionals in Definition 4.2.25 define a coisotropic submanifold.*

The first step in each of the following sections will be the one of finding the correct set of equations as a choice for the structural constraint. Then, we will find the relations for uniquely fixing the representatives. Once established the correct geometrical set-up, we will proceed by computing the algebra of the constraints of the theory at hand.

4.3 Digression: First and second class constraints

Definition 4.3.1. Consider a symplectic manifold $\mathcal{W}$ and a set of smooth maps $\phi_i \in C^\infty(\mathcal{W})$ defined on it. Let $C_{ij} = \{\phi_i, \phi_j\}$ represent the matrix of Poisson brackets associated with these maps. The count of second-class maps in the set corresponds to the rank of the matrix C_{ij} evaluated at the zero locus defined by the ϕ_is.[7] In particular, if $C_{ij} \approx 0$, we categorize all the maps as first-class.

Proposition 4.3.2. *Let $\mathcal{W}$ be a symplectic manifold and let $\psi_i, \phi_j \in C^\infty(\mathcal{W})$, where $i = 1 \dots n$ and $j = 1 \dots m$. Moreover, denote with $C_{jj'}, B_{ij}$ and $D_{ii'}$ the matrices representing, respectively, the Poisson brackets $\{\phi_j, \phi_{j'}\}$, $\{\psi_i, \phi_j\}$ and $\{\psi_i, \psi_{i'}\}$, with $i, i' = 1 \dots n$ and $j, j' = 1 \dots m$. Then, if D is invertible and $C = -B^T D^{-1} B$, the number of second-class constraints is n, i.e. the rank of the matrix D.*

Proof. See [CCT21]. □

Theorem 4.3.3 ([CCT21])**.** *The constraints L_c, P_ξ, H_λ and R_τ do not form a first class system. In particular, R_τ is a second class constraint, whereas the others are first class.*

We can now determine the degrees of freedom of the reduced phase space. Let r denote the number of degrees of freedom in the reduced phase space, p the number of degrees of freedom in the geometric phase space, f the number of first class constraints, and s the number of second class constraints. The relationship among them is given by[8]

$$r = p - 2f - s. \tag{4.38}$$

For all the possible couplings, it follows that we obtain the same result of the Palatini–Cartan theory, i.e.,

$$r = 2. \tag{4.39}$$

Remark 4.3.4. We notice that in the specific case where the boundary metric is non-degenerate, we would obtain $r = 4$. This reflects the existence of the constraint R_τ, which has been proven giving rise to a second class system. We recall that such a constraint was implied by the geometry of the theory. In particular, together with some additional condition, it ensured the possibility of uniquely fixing a representative of the equivalence class of ω. In fact, on a general (possibly

[7] We assume the rank to be constant on the zero locus.

[8] The proof of this formula is contained in [HT92].

null-) boundary, the space $\mathcal{T}$ defined in Definition 4.2.10 is non-trivial.
In physics, it is well-known that GR carries four local degrees of freedom.[9] However, the constraint analysis of the degenerate theory sheds light of the fact that these local degrees of freedom, in the case of manifolds with a null-boundary, are reduced to only two. This fact has important implications regarding the study of black-holes, since the event horizon is a null-hypersurface.

[9] Notice that sometimes the literature reports only two degrees of freedom. This is simply a consequence of considering the dimension of the phase space or just the one of the base manifold.

Chapter 5

Codimension-1 structure of field theories

5.1 Scalar field

In the canonical formalism, the Palatini–Cartan theory coupled with a scalar field maintains the very same geometrical background of the previous sections with the addition of two new fields, the scalar field ϕ and its conjugate momentum (upon equation of motion) Π.

We must define the building blocks of our scalar Palatini–Cartan theory, starting with the space of fields on M, which reads

$$\mathcal{F}_\phi = \widetilde{\Omega}^{1,1} \times \mathcal{A}(P) \times C^\infty(M) \times \Omega^{0,1} \ni (e, \omega, \phi, \Pi), \tag{5.1}$$

and the action functional

$$\mathcal{S}_s = \mathcal{S} + \mathcal{S}_\phi = \mathcal{S} + \int_M \frac{1}{6} e^3 \Pi d\phi + \frac{1}{48} e^4 (\Pi, \Pi), \tag{5.2}$$

where the brackets indicates the inner product of the Minkowski bundle. It follows that the Euler-Lagrange equations of the theory are given by

$$e d_\omega e = 0 \tag{5.3}$$

$$e F_\omega + \frac{\Lambda}{6} e^3 + \frac{1}{2} e^2 \Pi d\phi + \frac{1}{12} e^3 (\Pi, \Pi) = 0 \tag{5.4}$$

$$d(e^3 \Pi) = 0 \tag{5.5}$$

$$e^3 (d\phi - (e, \Pi)) = 0. \tag{5.6}$$

The variation of the action leads to the following Noether 1-form on the space of pre-boundary fields

$$\tilde{\alpha}_\phi = \int_\Sigma \frac{1}{2} e^2 \delta\omega + \frac{1}{6} e^3 \Pi \delta\phi, \tag{5.7}$$

which gives rise to the following pre-symplectic form

$$\tilde{\varpi}_\phi = \delta\tilde{\alpha} = \int_\Sigma e\delta e\delta\omega + \frac{1}{6}\delta(e^3\Pi)\delta\phi. \tag{5.8}$$

Similarly to the previous section, we can define the space of pre-boundary fields $\tilde{\mathcal{F}}^\phi_\Sigma$, as in Definition 4.2.5 for the Palatini–Cartan theory, by pulling back the fields to the boundary Σ. Also in this case, we will write the fields on the boundary with the same letters as for those in the bulk.

We are now able to define the geometric phase space of the theory via a reduction through the kernel of the pre-symplectic form.

Theorem 5.1.1 ([CCF22])**.** *The geometric phase space for the scalar Palatini–Cartan theory is the symplectic manifold $(\mathcal{F}^\phi_\Sigma, \varpi_\phi)$ given by the following equivalence relations on the space of pre-boundary fields $\tilde{\mathcal{F}}^\phi_\Sigma$*

$$\omega' \sim \omega \quad \iff \quad \omega' - \omega \in \mathrm{Ker} W_1^{\Sigma,(1,2)} \tag{5.9}$$

$$\Pi' \sim \Pi \quad \iff \quad \Pi' - \Pi \in \mathrm{Ker} W_3^{\Sigma,(0,1)} \tag{5.10}$$

and the symplectic form

$$\varpi_\phi = \int_\Sigma e\delta e\delta[\omega] + \frac{1}{6}\delta(e^3[\Pi])\delta\phi. \tag{5.11}$$

We refer to these equivalence classes as $\mathcal{A}(\Sigma)_{red}$ and $\Omega^{0,1}_{\Sigma,red}$.

We notice that the first field equation (5.4) does not couple with the scalar field. Therefore, since this purely geometrical term is equivalent to the one of the Palatini–Cartan theory (namely $\alpha = d_\omega e$), the structural and degeneracy constraints possess the same form of the free theory. In fact, as we said, they serve the purpose of maintaining the equivalence between $ed_\omega e = 0$ and $d_\omega e = 0$ on the boundary. We recall here the aforementioned constraints, which thus read

$$\begin{cases} e_n(d_\omega e - p_\mathcal{T} d_\omega e) \in \mathrm{Im}\, W_1^{\Sigma,(1,1)} \\ p_\mathcal{T} d_\omega e = 0. \end{cases} \tag{5.12}$$

Similarly to the Palatini–Cartan theory, we focus on fixing the representative of the equivalence classes defined in Theorem 5.1.1. The purely gravitational part remains the same, since it follows uniquely from the kernel of the piece of (5.8) equal to the pre-symplectic form of the free Palatini–Cartan theory. In other words, since in the present case the equivalence class $[\omega]$ is defined in the same way of the Palatini–Cartan theory, as well as the structural constraint, it follows that Lemma 4.2.19 applies verbatim to the scalar field theory.
Although, we are left to fix the representative of the equivalence class for Π. For this purpose, we give the following lemma.

Lemma 5.1.2. *Let* $\phi \in C^\infty(\Sigma)$*,* $\Pi \in \Omega_\Sigma^{0,1}$ *and* $e_n \in \Omega_\Sigma^{0,1}$ *as in Lemma 4.2.14. Then, the conditions*

$$\begin{cases} d\phi - (e, \Pi) = 0 \\ p_W \Pi = 0, \end{cases} \tag{5.13}$$

with[1] $W = e(\mathrm{Ker}(i^*g))$*, uniquely define a representative of the equivalence class* $[\Pi] \in \Omega^{0,1}_{\Sigma,red}$.

Proof. We first notice that, if we consider a vector field along the degenerate direction (if any), namely $X \in \mathrm{Ker}(i^*g)$, and we take the contraction of the field equations with it, we obtain the condition

$$\iota_X d\phi = 0. \tag{5.14}$$

What happens is that the degeneracy in the boundary metric decouples ϕ and Π along the degenerate direction[2] and this is precisely why we need, compared to the non-degenerate case, an extra condition in order to fix the representative of the equivalence class of Π.

We can decompose the field $\Pi \in \Omega_\Sigma^{0,1}$ in the following way[3]

$$\Pi = \pi^n e_n + \pi^a e_a, \tag{5.15}$$

with $a = 1, 2, 3$. Then, we notice that, thanks to definition of the wedge product, $e^3 e_a = 0$ for every a and therefore $\pi = \pi^a e_a \in \mathrm{Ker} W_3^{\Sigma,(0,1)}$. This means that fixing a certain π^n uniquely defines an equivalence class $[\Pi] \in \Omega^{0,1}_{\Sigma,red}$ and vice versa. We are thus left to show that the conditions (5.13) fix also uniquely $\pi = \pi^a e_a$, as a function of π^n. Now, we recall that $\dim(\mathrm{Ker}(i^*g)) = 1$ and e is injective and, therefore, we have that $W \subset e(T\Sigma)$ is a 1-dimensional subspace. Furthermore, for any open neighbourhood of $e(T\Sigma)$, without loss of generality, we can assume that the basis given by the vielbein $\{e_1, e_2, e_3\}$ is such that, say, e_3 spans W. From Eq. (5.15), it follows that the condition

$$p_W \Pi = 0 \tag{5.16}$$

[1] Here, we regard the boundary metric as a map $i^*g \colon T\Sigma \to T^*\Sigma$ and therefore we have that $\mathrm{Ker}(i^*g) = \{\xi \in \mathfrak{X}(\Sigma) \mid \iota_\xi(i^*g) = 0\} \subset \mathfrak{X}(\Sigma)$.

[2] This gives a condition on the derivative of ϕ. More specifically, the degeneracy of the boundary metric complicates the selection of the components of the fields in the orthogonal direction to the boundary. This implies that we could potentially have some spurious components of the field ϕ generating the diffeomorphisms along the orthogonal direction. Therefore, we can interpret the condition of Eq. (5.14) as a geometrical constraint that selects the only component of the symmetry transformations orthogonal to the boundary which are actually generated by the Hamiltonian vector field h^ϕ_λ of Eq. (5.41). In other words, one could say that Eq. (5.14) selects the "physically meaningful" components of the derivative of the field ϕ.

[3] We take the basis of $i^*\mathcal{V}$ given by the vielbein and the completion e_n. Notice that, as a section of $i^*\mathcal{V}$, e_n will have components along the vielbein in general.

implies

$$\pi^n e_n^3 + \pi^3 = 0. \tag{5.17}$$

Moreover, with such a choice of basis, we can write the exterior derivative of the scalar field as

$$d\phi = \partial_i \phi dx^i = e_1^a \partial_a \phi dx^1 + e_2^a \partial_a \phi dx^2, \tag{5.18}$$

where we implemented the condition $\iota_X d\phi = 0$, which reads $e_3^a \partial_a \phi = 0$. Lastly, we can write the field equations implementing Eq. (5.18), obtaining

$$\begin{aligned} d\phi - (e, \Pi) &= \partial_i \phi dx^i - (e_i^a dx^i e_a, \pi^b e_b + \pi^n e_n) &(5.19)\\ &= e_i^a \partial_a \phi dx^i - e_i^a \pi^b g_{ab} dx^i - e_i^a \pi^n g_{an} dx^i &(5.20)\\ &= e_i^a (\partial_a \phi - \pi^b g_{ab} - \pi^n g_{an}) dx^i = 0, &(5.21) \end{aligned}$$

where g is the metric and $b = 1, 2$. Since the restricted inner product is non-degenerate, we have

$$\partial_a \phi - \pi^b g_{ab} - \pi^n g_{an} = 0 \tag{5.22}$$

and thus we deduce the following equation for π^b (with $b = 1, 2$)

$$\pi^b = g^{ab}(\partial_a \phi - \pi^n g_{an}). \tag{5.23}$$

It follows that Eqs. (5.17) and (5.23) completely fix the components of π in terms of π^n. Hence, since fixing π^n is equivalent to fixing a representative for $[\Pi]$ and vice versa, we have that, given an equivalence class (or equivalently a π^n), the conditions (5.13) fix uniquely the representative of $[\Pi]$. On the other hand, given a representative, the conditions (5.13) fix unambiguously a π^n and therefore an equivalence class $[\Pi]$. □

Similarly to the previous section, we obtain the following result.

Theorem 5.1.3. *The geometric phase space of the scalar Palatini–Cartan theory is symplectomorphic to the space* $(\mathcal{F}_\phi^\partial, \varpi_\phi^\partial)$, *where*

$$\mathcal{F}_\phi^\partial \subset \widetilde{\Omega}_\Sigma^{1,1} \times \Omega_\Sigma^{1,2} \times C^\infty(\Sigma) \times \Omega_\Sigma^{0,1} \tag{5.24}$$

with $(e, \omega, \phi, \Pi) \in \mathcal{F}_\phi^\partial$ *satisfying*

$$\begin{cases} e_n(d_\omega e - p_{\mathcal{T}}(d_\omega e)) \in \operatorname{Im} W_1^{\Sigma,(1,1)} \\ p_{\mathcal{K}} \omega = 0 \\ d\phi - (e, \Pi) = 0 \\ p_W \Pi = 0, \end{cases} \tag{5.25}$$

as defined in Lemma 4.2.19 and Lemma 5.1.2, and where the corresponding symplectic form on $\mathcal{F}^\partial_\phi$ is given by

$$\varpi^\partial_\phi = \int_\Sigma e\delta e\delta\omega + \frac{1}{6}\delta(e^3\Pi)\delta\phi. \tag{5.26}$$

Definition 5.1.4. Let $c \in \Omega^{0,2}_\Sigma[1]$, $\xi \in \mathfrak{X}(\Sigma)[1]$, $\lambda \in C^\infty(\Sigma)[1]$ and $\tau \in \mathcal{S}[1]$. Then, we define the following functionals

$$L_c = \int_\Sigma c e d_\omega e \tag{5.27}$$

$$P^\phi_\xi = \int_\Sigma \frac{1}{2}\iota_\xi(e^2)F_\omega + \frac{1}{3!}\iota_\xi(e^3\Pi)d\phi + \iota_\xi(\omega-\omega_0)ed_\omega e \tag{5.28}$$

$$H^\phi_\lambda = \int_\Sigma \lambda e_n\Big(eF_\omega + \frac{\Lambda}{3!}e^3 + \frac{1}{2}e^2\Pi d\phi + \frac{1}{2\cdot 3!}e^3(\Pi,\Pi)\Big) \tag{5.29}$$

$$R_\tau = \int_\Sigma \tau d_\omega e. \tag{5.30}$$

We refer to these as the constraints of the scalar Palatini–Cartan theory.

In the following theorem, we give the form of the Poisson brackets determining the constraint algebra of the theory.

Theorem 5.1.5. *The Poisson brackets of the constraints of Definition 5.1.4 read*

$$\begin{aligned}
&\{L_c, L_c\}_\phi = -\frac{1}{2}L_{[c,c]} && \{P^\phi_\xi, P^\phi_\xi\}_\phi = \frac{1}{2}P^\phi_{[\xi,\xi]} - \frac{1}{2}L_{\iota_\xi\iota_\xi F_{\omega_0}}\\
&\{L_c, P^\phi_\xi\}_\phi = L_{\mathcal{L}^{\omega_0}_\xi c} && \{H^\phi_\lambda, H^\phi_\lambda\}_\phi \approx 0\\
&\{L_c, R_\tau\}_\phi = -R_{p_\mathcal{S}[c,\tau]} && \{P^\phi_\xi, R_\tau\}_\phi = R_{p_\mathcal{S}\mathcal{L}^{\omega_0}_\xi \tau}\\
&\{R_\tau, H^\phi_\lambda\}_\phi \approx G_{\lambda\tau} && \{R_\tau, R_\tau\}_\phi \approx F_{\tau\tau}
\end{aligned} \tag{5.31}$$

$$\{L_c, H^\phi_\lambda\}_\phi = -P^\phi_{X^{(a)}} + L_{X^{(a)}(\omega-\omega_0)_a} - H^\phi_{X^{(n)}} \tag{5.32}$$

$$\{P^\phi_\xi, H^\phi_\lambda\}_\phi = P^\phi_{Y^{(a)}} - L_{Y^{(a)}(\omega-\omega_0)_a} + H^\phi_{Y^{(n)}}, \tag{5.33}$$

with $X = [c, \lambda e_n]$ and $Y = \mathcal{L}^{\omega_0}_\xi(\lambda e_n)$ and where the superscripts (a) and (n) describe their components with respect to e_a, e_n. Furthermore, $F_{\tau\tau}$ and $G_{\lambda\tau}$ are functionals of e, ω, τ and λ which are not proportional to any other constraint.[4]

Proof. First, we introduce the following notation[5]

$$P^\phi_\xi = P_\xi + p^\phi_\xi \qquad\qquad H^\phi_\lambda = H_\lambda + h^\phi_\lambda, \tag{5.34}$$

[4] They are properly defined in [CCT21] (proof of Theorem 30).
[5] With P_ξ and H_λ of Definition 4.2.25.

in order to simplify the computations.

In accordance with the results from [CCT21] and [CCF22], we possess knowledge of the some of the brackets as follows

$$\begin{aligned} &\{L_c, L_c\} = -\frac{1}{2}L_{[c,c]} && \{L_c, P_\xi^\phi\}_\phi = L_{\mathcal{L}_\xi^{\omega_0} c} \\ &\{P_\xi^\phi, P_\xi^\phi\}_\phi = \frac{1}{2}P^\phi_{[\xi,\xi]} - \frac{1}{2}L_{\iota_\xi \iota_\xi F_{\omega_0}} && \{L_c, R_\tau\} = -R_{p_{\mathcal{S}}[c,\tau]} \\ &\{P_\xi, R_\tau\} = R_{p_{\mathcal{S}}\mathcal{L}_\xi^{\omega_0}\tau} && \{R_\tau, H_\lambda\} \approx G_{\lambda\tau} \\ &\{R_\tau, R_\tau\} \approx F_{\tau\tau} && \{H_\lambda^\phi, H_\lambda^\phi\}_\phi \approx 0 \end{aligned} \tag{5.35}$$

$$\{L_c, H_\lambda^\phi\}_\phi = -P^\phi_{X^{(a)}} + L_{X^{(a)}(\omega-\omega_0)_a} - H^\phi_{X^{(n)}} \tag{5.36}$$

$$\{P_\xi^\phi, H_\lambda^\phi\}_\phi = P^\phi_{Y^{(a)}} - L_{Y^{(a)}(\omega-\omega_0)_a} + H^\phi_{Y^{(n)}} \tag{5.37}$$

with F and G non-identically-vanishing functional of τ and λ defined in [CCT21] (Theorem 30) and $X = [c, \lambda e_n] \in \Omega_\Sigma^{0,1}$ divided into a tangential component $X^{(a)} = [c, \lambda e_n]^{(a)}$ and a normal component $X^{(n)} = [c, \lambda e_n]^{(n)}$. We are therefore left to compute the brackets $\{R_\tau, h_\lambda^\phi\}_\phi$ and $\{R_\tau, p_\xi^\phi\}_\phi$. Also, we can recall the known results from [CCT21] and [CCT21] for what concerns all Hamiltonian vector fields. In particular, for p_ξ^ϕ, we have

$$\mathbb{p}_e^\phi = 0 \qquad \mathbb{p}_\omega^\phi = 0 \tag{5.38}$$

$$\mathbb{p}_\rho^\phi = -\mathcal{L}_\xi^{\omega_0}\rho \qquad \mathbb{p}_\phi^\phi = -\xi(\phi), \tag{5.39}$$

whereas, for h_λ^ϕ, we have

$$\mathbb{h}_e^\phi = 0 \qquad \mathbb{h}_\omega^\phi = \lambda e_n\Big(\Pi d\phi + \frac{e}{4}(\Pi,\Pi)\Big) - \frac{\lambda}{2}e\Pi(\Pi, e_n) \tag{5.40}$$

$$\mathbb{h}_\rho^\phi = \frac{1}{2}d_\omega(\lambda e_n e^2 \Pi) \qquad \mathbb{h}_\phi^\phi = -\lambda(e_n, \Pi), \tag{5.41}$$

where we have defined a new field $\rho := \frac{1}{3!}e^3\Pi \in \Omega_\Sigma^{3,4}$.

Next, it is helpful to write explicitly the variation[6] of R_τ, which reads[7]

$$\delta R_\tau = \int_\Sigma \delta\tau d_\omega e - \tau[\delta\omega, e] + \tau d_\omega \delta e \tag{5.42}$$

$$= \int_\Sigma (g(\tau,\omega,e) + d_\omega\tau)\delta e + [\tau, e]\delta\omega, \tag{5.43}$$

[6]We compute the Hamiltonian vector fields in the following manner. Let $\mathbb{X}$ be the Hamiltonian vector field of the functional F for the symplectic form ϖ, then it holds $\iota_{\mathbb{X}}\varpi - \delta F = 0$, where δF is the functional derivative of F.

[7]Since τ is defined on $\mathcal{S}$ and the latter is defined making use of e, it follows that τ has a non-trivial variation along e.

where we have introduced the formal expression $g = g(\tau, e, \omega)$ which encodes the dependence of τ on e (see [CCT21] Theorem 30 for further details). It follows that the Hamiltonian vector fields are

$$e\mathbb{R}_e = [\tau, e] \qquad e\mathbb{R}_\omega = g(\tau, \omega, e) + d_\omega \tau \tag{5.44}$$
$$\mathbb{R}_\rho = 0 \qquad \mathbb{R}_\phi = 0. \tag{5.45}$$

Now that we possess all Hamiltonian vector fields, we are ready to compute the Poisson brackets of the remaining constraints. First, we notice that

$$\{R_\tau, p_\xi^\phi\}_\phi = 0 \tag{5.46}$$

since p_ξ^ϕ has trivial Hamiltonian vector fields along e or ω. Then, we compute

$$\{R_\tau, h_\lambda^\phi\}_\phi = \int_\Sigma -\lambda e_n \Pi d\phi[\tau, e] - \frac{e}{4}\lambda e_n(\Pi, \Pi)[\tau, e] + \frac{\lambda}{2} e\Pi(\Pi, e_n)[\tau, e]. \tag{5.47}$$

Here, the last two terms are zero thanks to $e[\tau, e] = 0$ (following from $e\tau = 0$ in the definition of $\mathcal{S}$). For the term bracket, we have

$$\begin{aligned}
\int_\Sigma -\lambda e_n \Pi d\phi[\tau, e] &= \int_\Sigma -\lambda \Pi d\phi \tau[e_n, e] && (5.48)\\
&= \int_\Sigma -\lambda \Pi d\phi \tau(e_n, e) && (5.49)\\
&\overset{(5.13)}{=} \int_\Sigma -\lambda \Pi(e, \Pi)\tau(e_n, e) && (5.50)\\
&= \int_\Sigma -\lambda \Pi(e, \Pi) e_n[\tau, e] && (5.51)\\
&= 0, && (5.52)
\end{aligned}$$

where we implemented the Leibniz identity for the squared brackets, the definition of $\tau \in \mathcal{S}$ and Proposition A.0.6, thanks to the fact that there are no derivatives in the integral.[8]

Finally, in order to complete the proof, we can simply exploit the linearity of the Poisson brackets and recall the definition of the split introduced in Eq. (5.34) together with the known results mentioned above. □

Corollary 5.1.6. *If the boundary metric i^*g is non-degenerate, then the functionals in Definition 5.1.4 define a coisotropic submanifold.*

[8] Roughly speaking, we can "diagonalize" the vielbein.

5.2 Yang–Mills field

In this section, we will examine the case of an SU(n)-gauge-field,[9] namely a principal connection A of a principal SU(n)-bundle over M denoted with P (see [Tec19b] Section 5). It follows that the space of gauge fields is locally modelled on $\Omega^1(M, \mathfrak{su}(n))$, via the pull-backs along the sections of G. In the Standard Model of particle physics, this kind of field is responsible for the mediation of a variety of interactions, in particular, the Electroweak and the Strong interaction. Moreover, similarly to what we did in the previous section, we associate to the gauge field[10] A an independent field $B \in \Gamma\Big(\bigwedge^2 \mathcal{V} \otimes \mathfrak{su}(n)\Big)$.

Hence, the Yang–Mills Palatini–Cartan theory is defined by the following space of fields

$$\mathcal{F}_A = \widetilde{\Omega}^{1,1} \times \mathcal{A}(P) \times \mathcal{A}(G) \times \Gamma\Big(\textstyle\bigwedge^2 \mathcal{V} \otimes \mathfrak{su}(n)\Big) \ni (e, \omega, A, B), \tag{5.53}$$

and the action functional

$$\mathcal{S}_{YM} = \mathcal{S} + \mathcal{S}_A = \mathcal{S} + \int_M \frac{1}{4} e^2 \mathrm{Tr}(BF_A) + \frac{1}{48} e^4 \mathrm{Tr}(B, B), \tag{5.54}$$

where $\Omega^2(M, \mathfrak{su}(n)) \ni F_A = dA + \frac{1}{2}[A, A]$ is the field strength, $(\ ,\ )$ is the canonical pairing in $\bigwedge^2 \mathcal{V}$ and $\mathrm{Tr}\colon \mathfrak{su}(n) \to \mathbb{R}$ is the trace over the algebra.

The Euler-Lagrange equations are as follows

$$d_\omega e = 0 \tag{5.55}$$

$$e(F_\omega + \mathrm{Tr}(BF_A)) + \frac{e^3}{6}(\Lambda + \frac{1}{2}\mathrm{Tr}(B, B)) = 0 \tag{5.56}$$

$$e^2\Big(F_A + \frac{1}{2}(e^2, B)\Big) = 0 \tag{5.57}$$

$$d_A(e^2 B) = 0, \tag{5.58}$$

whereas the Noether 1-form becomes

$$\tilde{\alpha}_{YM} = \int_\Sigma \frac{e^2}{2}\delta\omega + \frac{e^2}{2}\mathrm{Tr}(B\delta A). \tag{5.59}$$

It follows that the pre-symplectic form of the theory is

$$\tilde{\varpi}_{YM} = \varpi + \varpi_A = \delta\tilde{\alpha}_{YM} = \int_\Sigma e\delta e\delta\omega + \mathrm{Tr}(eB\delta e\delta A) + \frac{1}{2}\mathrm{Tr}(e^2\delta B\delta A). \tag{5.60}$$

This is a 2-form over the space of pre-boundary fields obtained as the pull-back of bulk fields along $i\colon \Sigma \to M$ and denoted in this case as $\widetilde{\mathcal{F}}^A_\Sigma$. Notice that, also in this case, we refer to boundary fields with the same notation of bulk fields.

[9] All the considerations below work with a general Lie algebra $\mathfrak{g}$.

[10] Note that we refer to both A and its pull-back as the gauge field.

Theorem 5.2.1. *The geometric phase space for the Yang–Mills Palatini–Cartan theory is the symplectic manifold $(\mathcal{F}^A_\Sigma, \varpi_A)$ given by the following equivalence relations on the space of pre-boundary fields $\widetilde{\mathcal{F}}^A_\Sigma$*

$$\omega' \sim \omega \quad \Longleftrightarrow \quad \omega' - \omega \in \mathrm{Ker} W_1^{\Sigma,(1,2)} \tag{5.61}$$

$$B' \sim B \quad \Longleftrightarrow \quad B' - B \in \mathrm{Ker} W_2^{\Sigma,(0,2*)}, \tag{5.62}$$

where $2*$ indicates that the $\bigwedge^2 \mathcal{V}$-algebra is tensored with $\mathfrak{su}(n)$, and the symplectic form

$$\varpi_{YM} = \int_\Sigma e\delta e\delta[\omega] + \frac{1}{2}\mathrm{Tr}(\delta(e^2[B])\delta A). \tag{5.63}$$

We refer to these equivalence classes as $\mathcal{A}(\Sigma)_{red}$ and $\Gamma\Big(\bigwedge^2 i^*\mathcal{V} \otimes \mathfrak{su}(n)\Big)_{red}$.

Remark 5.2.2. In the context of the Yang–Mills Palatini–Cartan theory, we can indeed establish unique representatives for these equivalence classes. Subsequently, we can proceed to formulate the constraints in a manner analogous to the approach we previously employed in the preceding section. The representative for $[\omega] \in \mathcal{A}(\Sigma)_{red}$ is already uniquely fixed thanks to equivalent considerations to the ones articulated in the previous sections. Therefore, the structural and the degeneracy constraints for the Yang–Mills Palatini–Cartan theory read

$$\begin{cases} e_n(d_\omega e - p_\mathcal{T} d_\omega e) \in \mathrm{Im}\, W_1^{\Sigma,(1,1)} \\ p_\mathcal{T} d_\omega e = 0. \end{cases} \tag{5.64}$$

We are therefore left with the problem of the representative for $[B] \in \Gamma\Big(\bigwedge^2 i^*\mathcal{V} \otimes \mathfrak{su}(n)\Big)_{red}$, which is determined by the following lemma.

Lemma 5.2.3. *Given $A \in \mathcal{A}(i^*G)$, $B \in \Gamma\Big(\bigwedge^2 i^*\mathcal{V} \otimes \mathfrak{su}(n)\Big)$ and $e_n \in \Omega^{0,1}_\Sigma$ as in Lemma 4.2.14, the conditions*

$$\begin{cases} F_A + \frac{1}{2}(e^2, B) = 0 \\ p_{\Omega_e^{0,1*} \wedge W} B = 0, \end{cases} \tag{5.65}$$

with $\Omega_e^{i,j*} := \Omega^i\Big(\Sigma, \bigwedge^j e(T\Sigma) \otimes \mathfrak{su}(n)\Big)$ where $W = e(\mathrm{Ker}(i^*g))$, uniquely define a representative of the equivalence class $[B] \in \Gamma\Big(\bigwedge^2 i^*\mathcal{V} \otimes \mathfrak{su}(n)\Big)_{red}$.

Proof. We can decompose an element[11] $B \in \Gamma\Big(\bigwedge^2 i^*\mathcal{V} \otimes \mathfrak{su}(n)\Big)$ as[12]

$$B = b^{an} e_a e_n + \frac{1}{2} b^{ab} e_a e_b, \tag{5.66}$$

with $b^{an}, b^{ab} \in \Gamma(\mathfrak{su}(n))$ and $a, b = 1, 2, 3$. We notice that $b = b^{ab} e_a e_b \in \mathrm{Ker}(W_2^{\Sigma,(0,2*)})$, since $e^2 e_a e_b = 0$ for all $a, b = 1, 2, 3$. This directly implies that the components b^{an} are already uniquely determined by the equivalence class $[B]$ and vice versa. Now, as we did in the proof of Lemma 5.1.2, we observe that $\dim(\mathrm{Ker}(i^*g)) = 1$ and e is injective and, therefore, we have that $W \subset e(T\Sigma)$ is a 1-dimensional subspace. Hence, for any open neighbourhood of $e(T\Sigma)$, without loss of generality, we can take as a basis of $e(T\Sigma)$ the one given by the $\{e_1, e_2, e_3\}$ such that e_3 spans W. Then, since a basis of $\Omega_e^{0,1*} \wedge W$ is given by $\{e_1e_3, e_2e_3\} \otimes \mathfrak{su}(n)$, we have that, similarly to the scalar case, we first notice that the field equations imply the condition $\iota_X F_A = 0$, with $X \in \mathrm{Ker}(i^*g)$. Moreover, the condition $p_{\Omega_e^{0,1*}\wedge W} B = 0$ implies that

$$2b^{[1n}e_n^{3]} + b^{13} = 2b^{[2n}e_n^{3]} + b^{23} = 0, \tag{5.67}$$

where the square brackets in the indices denote the anti-symmetrization.

Next, consider the condition $\iota_X F_A = 0$. Then, we can write

$$F_A = \frac{1}{2} F_{ab} e_i^a e_j^b dx^i dx^j = \frac{1}{2} F_{12} dx^1 dx^2. \tag{5.68}$$

Furthermore, similarly to the preceding case, we can write

$$\begin{aligned}
&2F_A + (e^2, B) = && (5.69)\\
&= F_{ij} dx^i dx^j + (\frac{1}{2} e_i^a e_j^b dx^i dx^j e_a e_b, b^{cd} e_c e_d + b^{cn} e_c e_n) && (5.70)\\
&= F_{ab} e_i^a e_j^b dx^i dx^j + b^{cd} e_i^a e_j^b g_{ac} g_{bd} dx^i dx^j + b^{cn} e_i^a e_j^b g_{ac} g_{bn} dx^i dx^j && (5.71)\\
&= e_i^a e_j^b (F_{ab} + b^{cd} g_{ac} g_{bd} + b^{cn} g_{ac} g_{bn}) dx^i dx^j = 0. && (5.72)
\end{aligned}$$

We observe that, since the restricted inner product is non-degenerate, we have

$$F_{ab} + b^{cd} g_{ac} g_{bd} + b^{cn} g_{ac} g_{bn} = 0 \tag{5.73}$$

and, given $a, b, c, d \neq 3$, we can use the inverse metric to write

$$b^{cd} = -(g^{ac} g^{bd} F_{ab} + g^{bd} g_{bn} b^{cn}). \tag{5.74}$$

[11] We can consider the basis for $e(T\Sigma)$ given by the vielbein. See the proof of Lemma 5.1.2 for more details.

[12] Apart from the wedge product, in order to lighten the notation, we also omit the tensor product.

This result together with Eq. (5.67) fixes uniquely the elements b^{ab} in terms of b^{an} (with $a, b = 1, 2, 3$). The completion of the proof follows from analogous considerations to the ones of the scalar case in the previous section. □

Also in this case, we get the following result.

Theorem 5.2.4. *The geometric phase space of the Yang–Mills Palatini–Cartan theory is symplectomorphic to the space* $(\mathcal{F}^{\partial}_{YM}, \varpi^{\partial}_{YM})$, *where*

$$\mathcal{F}^{\partial}_{YM} \subset \widetilde{\Omega}^{1,1}_{\Sigma} \times \Omega^{1,2}_{\Sigma} \times A(i^*G) \times \Gamma\Big(\textstyle\bigwedge^2 i^*\mathcal{V} \otimes \mathfrak{su}(n)\Big) \tag{5.75}$$

with $(e, \omega, A, B) \in \mathcal{F}^{\partial}_{YM}$ *satisfying*

$$\begin{cases} e_n(d_\omega e - p_{\mathcal{T}}(d_\omega e)) \in \operatorname{Im} W_1^{\Sigma,(1,1)} \\ p_{\mathcal{K}}\omega = 0 \\ F_A + \frac{1}{2}(e^2, B) = 0 \\ p_{\Omega_e^{0,1*}\wedge W} B = 0, \end{cases} \tag{5.76}$$

as defined in Lemma 4.2.19 and Lemma 5.2.3, and where the corresponding symplectic form on $\mathcal{F}^{\partial}_{YM}$ *is given by*

$$\varpi^{\partial}_{YM} = \varpi^{\partial} + \varpi^{\partial}_{A} = \int_{\Sigma} e\delta e\delta\omega + \frac{1}{2}\mathrm{Tr}(\delta(e^2B)\delta A). \tag{5.77}$$

Definition 5.2.5. Let $c \in \Omega^{0,2}_{\Sigma}[1]$, $\mu \in C^\infty(\Sigma, \mathfrak{g})[1]$, $\xi \in \mathfrak{X}(\Sigma)[1]$, $\lambda \in C^\infty(\Sigma)[1]$ and $\tau \in \mathcal{S}[1]$. Moreover, let $\rho = e^2B \in \Omega^{2,4*}_{\Sigma}$. Then, we define the following functionals

$$L_c = \int_{\Sigma} ced_\omega e \tag{5.78}$$

$$M_\mu = \int_{\Sigma} \mathrm{Tr}(\mu d_A \rho) \tag{5.79}$$

$$P^A_\xi = \int_{\Sigma} \frac{1}{2}\iota_\xi e^2 F_\omega + \iota_\xi(\omega - \omega_0)ed_\omega e + \frac{1}{2}\mathrm{Tr}(\iota_\xi \rho F_A) \tag{5.80}$$

$$+ \mathrm{Tr}\Big(\iota_\xi(A - A_0)d_A\rho\Big) \tag{5.81}$$

$$H^A_\lambda = \int_{\Sigma} \lambda e_n\Big(eF_\omega + \frac{\Lambda}{3!}e^3 + e\mathrm{Tr}(BF_A) + \frac{1}{2\cdot 3!}e^3\mathrm{Tr}(B, B)\Big) \tag{5.82}$$

$$R_\tau = \int_{\Sigma} \tau d_\omega e. \tag{5.83}$$

We refer to these as the constraints of the Yang–Mills Palatini–Cartan theory.

Theorem 5.2.6. *The Poisson brackets of the constraints of Definition 5.2.5 read*

$$\begin{aligned}
&\{L_c, L_c\}_{YM} = -\frac{1}{2}L_{[c,c]} && \{M_\mu, M_\mu\}_{YM} = -\frac{1}{2}M_{[\mu,\mu]}\\
&\{L_c, P_\xi^A\}_{YM} = L_{\mathcal{L}_\xi^{\omega_0} c} && \{H_\lambda^A, H_\lambda^A\}_{YM} \approx 0\\
&\{L_c, M_\mu\}_{YM} = 0 && \{P_\xi, M_\mu\}_{YM} = M_{\mathcal{L}_\xi^{A_0}\mu}\\
&\{H_\lambda^A, M_\mu\}_{YM} = 0 && \{R_\tau, M_\mu\}_{YM} = 0\\
&\{L_c, R_\tau\}_{YM} = -R_{p_{\mathcal{S}}[c,\tau]} && \{P_\xi^A, R_\tau\}_{YM} = R_{p_{\mathcal{S}}\mathcal{L}_\xi^{\omega_0}\tau}\\
&\{R_\tau, H_\lambda^A\}_{YM} \approx G_{\lambda\tau} + K_{\lambda\tau}^A && \{R_\tau, R_\tau\}_{YM} \approx F_{\tau\tau}
\end{aligned} \tag{5.84}$$

$$\{P_\xi^A, P_\xi^A\}_{YM} = \frac{1}{2}P_{[\xi,\xi]}^A - \frac{1}{2}L_{\iota_\xi\iota_\xi F_{\omega_0}} - \frac{1}{2}M_{\iota_\xi\iota_\xi F_{\omega_0}} \tag{5.85}$$

$$\{L_c, H_\lambda^A\}_{YM} = -P_{X^{(a)}}^A + L_{X^{(a)}(\omega-\omega_0)_a} - H_{X^{(n)}}^A \tag{5.86}$$

$$\{P_\xi^A, H_\lambda^A\}_{YM} = P_{Y^{(a)}}^A - L_{Y^{(a)}(\omega-\omega_0)_a} + H_{Y^{(n)}}^A, \tag{5.87}$$

with $X = [c, \lambda e_n]$ *and* $Y = \mathcal{L}_\xi^{\omega_0}(\lambda e_n)$ *and where the superscripts* (a) *and* (n) *describe their components with respect to* e_a, e_n*. Furthermore,* $F_{\tau\tau}$*,* $G_{\lambda\tau}$ *and* $K_{\lambda\tau}^A$ *are functional of* e, ω, A, B, τ *and* λ *defined in the proof*[13] *which are not proportional to any other constraint.*

Proof. Similarly to the proof of Theorem 5.1.5, we will introduce now a split in some of the constraints. In this case, we have

$$P_\xi^A = P_\xi + p_\xi^A \qquad\qquad H_\lambda^A = H_\lambda + h_\lambda^A. \tag{5.88}$$

Moreover, from [CCT21] and [CCF22], we have knowledge of the following brackets

$$\begin{aligned}
&\{L_c, L_c\} = -\frac{1}{2}L_{[c,c]} && \{L_c, P_\xi^A\}_{YM} = L_{\mathcal{L}_\xi^{\omega_0} c}\\
&\{P_\xi^A, P_\xi^A\}_{YM} = \frac{1}{2}P_{[\xi,\xi]}^A - \frac{1}{2}L_{\iota_\xi\iota_\xi F_{\omega_0}} && \{L_c, R_\tau\} = -R_{p_{\mathcal{S}}[c,\tau]}\\
&\{P_\xi, R_\tau\} = R_{p_{\mathcal{S}}\mathcal{L}_\xi^{\omega_0}\tau} && \{R_\tau, H_\lambda\} \approx G_{\lambda\tau}\\
&\{R_\tau, R_\tau\} \approx F_{\tau\tau} && \{H_\lambda^A, H_\lambda^A\}_{YM} \approx 0\\
&\{M_\mu, M_\mu\}_{YM} = -\frac{1}{2}M_{[\mu,\mu]} && \{L_c, M_\mu\}_{YM} = 0\\
&\{P_\xi^A, M_\mu\}_{YM} = M_{\mathcal{L}_\xi^{A_0}\mu} && \{M_\mu, H_\lambda^A\}_{YM} = 0
\end{aligned} \tag{5.89}$$

$$\{L_c, H_\lambda^A\}_{YM} = -P_{X^{(a)}}^A + L_{X^{(a)}(\omega-\omega_0)_a} - H_{X^{(n)}}^A \tag{5.90}$$

$$\{P_\xi, H_\lambda^A\}_{YM} = P_{Y^{(a)}}^A - L_{Y^{(a)}(\omega-\omega_0)_a} + H_{Y^{(n)}}^A, \tag{5.91}$$

[13] F and G are properly defined in [CCT21] (proof of Theorem 30).

with F and G non-identically-vanishing functional of τ and λ defined in [CCT21] (Theorem 30), $X = [c, \lambda e_n]$ and $Y = \mathcal{L}_\xi^{\omega_0}(\lambda e_n)$. We are thus left with computing the remaining brackets.

Equivalently to the scalar case, the Hamiltonian vector fields for R_τ are given by

$$e\mathbb{R}_e = [\tau, e] \qquad e\mathbb{R}_\omega = g(\tau, \omega, e) + d_\omega \tau \tag{5.92}$$

$$\mathbb{R}_{YM} = 0 \qquad \mathbb{R}_\rho = 0, \tag{5.93}$$

since in does not possess any variation along the gauge fields. We consider now the variation

$$\delta M_\mu = \int_\Sigma \mathrm{Tr}(\mu \delta(d_A \rho)) = \int_\Sigma \mathrm{Tr}(-\mu([\delta A, \rho] + d_A(\delta\rho)) \tag{5.94}$$

$$= \int_\Sigma \mathrm{Tr}\Big([\mu, \rho]\delta A + d_A \mu \, \delta\rho\Big), \tag{5.95}$$

and therefore we obtain the following Hamiltonian vector fields

$$\mathbb{M}_e = 0 \qquad \mathbb{M}_\omega = 0 \tag{5.96}$$

$$\mathbb{M}_\rho = [\mu, \rho] \qquad \mathbb{M}_{YM} = d_A \mu. \tag{5.97}$$

From [CCF22], for p_ξ^A, we have

$$\mathbb{p}_e^A = 0 \qquad \mathbb{p}_\omega^A = 0 \tag{5.98}$$

$$\mathbb{p}_\rho^A = -\mathcal{L}_\xi^{A_0} \rho \qquad \mathbb{p}_{YM}^A = -\mathcal{L}_\xi^{A_0}(A - A_0) - \iota_\xi F_{A_0}, \tag{5.99}$$

whereas, for h_λ^A, the Hamiltonian vector fields read

$$\mathbb{h}_e^A = 0 \qquad \mathbb{h}_\rho^A = d_A(\lambda e_n e B) \tag{5.100}$$

$$\mathbb{h}_{YM}^A = \lambda(B, e_n e) \quad e\mathbb{h}_\omega^A = \mathrm{Tr}\Big(\lambda e_n B F_A + \lambda e_n \frac{e^2}{4}(B, B) - \lambda e B(B, e_n e)\Big). \tag{5.101}$$

Now, we are left with computing the Poisson brackets of the constraints for $\{R_\tau, h_\lambda^A\}_{YM}$, $\{R_\tau, p_\xi^A\}_{YM}$ and $\{R_\tau, M_\mu\}_{YM}$. We start with noticing that

$$\{R_\tau, p_\xi^A\}_{YM} = \{R_\tau, M_\mu\}_{YM} = 0 \tag{5.102}$$

since both p_ξ^A and M_μ have vanishing Hamiltonian vector fields along e and ω. Then, we are left with computing

$$\{R_\tau, h_\lambda^A\}_{YM} = \int_\Sigma \mathrm{Tr}\Big(\lambda e_n B F_A W_1^{-1}[\tau, e] + \lambda e_n \frac{e}{4}(B, B)[\tau, e] \tag{5.103}$$

$$- \lambda B(B, e_n e)[\tau, e]\Big), \tag{5.104}$$

where the second term is zero because of $e[\tau, e] = 0$ and the first and third terms in general do not vanish. In fact, we have

$$\{R_\tau, h_\lambda^A\}_{YM} = \tag{5.105}$$

$$= \int_\Sigma \mathrm{Tr}\Big(\lambda e_n B F_A W_1^{-1}[\tau, e] - \lambda B(B, e_n e)[\tau, e]\Big) \tag{5.106}$$

$$= \int_\Sigma \mathrm{Tr}\Big(\frac{\lambda e_n}{2} B(B, e^2) - \lambda B(B, e_n e)e\Big) W_1^{-1}[\tau, e] \tag{5.107}$$

$$= \int_\Sigma \mathrm{Tr}\bigg(\lambda B\Big(\frac{e_n}{2}(B, e^2) - (B, e_n e)e\Big)\bigg) W_1^{-1}[\tau, e] \tag{5.108}$$

$$\approx\colon K_{\lambda\tau}^A, \tag{5.109}$$

where $W_1^{-1}\colon \Omega_\Sigma^{2,2} \to \Omega_\Sigma^{1,1}$ indicates the inverse of the map $W_1^{\Sigma,(1,1)}$ and the symbol $\approx\colon$ means that we are defining the quantity $K_{\lambda\tau}^A$ on the constraint submanifold. Then, thanks to Corollary 12 of [CCT21], we can write the explicit form of $K_{\lambda\tau}^A$ by means of the independent components $\mathcal{X}$ and $\mathcal{Y}$ of τ, defined in Proposition 8 of[CCT21]. Hence, we define $K_{\lambda\tau}^A$ as

$$K_{\lambda\tau}^A = \int_\Sigma \mathrm{Tr}\bigg(\lambda\Big(\frac{1}{2}\Big(\sum_{\mu=1}^{2} \mathcal{Y}_\mu \mathcal{C}_\mu^\mu - \sum_{\mu_1 \neq \mu_2 = 1}^{2} \mathcal{X}_{\mu_1}^{\mu_2} \mathcal{C}_{\mu_2}^{\mu_1}\Big) \tag{5.110}$$

$$- \Big(\sum_{\mu=1}^{2} \mathcal{Y}_\mu \mathcal{D}_\mu^\mu - \sum_{\mu_1 \neq \mu_2 = 1}^{2} \mathcal{X}_{\mu_1}^{\mu_2} \mathcal{D}_{\mu_2}^{\mu_1}\Big)\Big)\bigg), \tag{5.111}$$

where $\mathcal{C}_\sigma^\rho := (B^{\rho 3} - B^{\rho 4})(B, e^2)_{3\sigma}$ and $\mathcal{D}_\sigma^\rho := (B^{\rho 3} - B^{\rho 4})(B, e_n e)_\sigma$.

Therefore, thanks to the linearity of the Poisson brackets together with the known results, this completes the proof. □

Corollary 5.2.7. *If the boundary metric i^*g is non-degenerate, then the functionals in Definition 5.2.5 define a coisotropic submanifold.*

5.3 Spinor field

The concept of a spinor field is central in mathematical physics. The idea of a spinor field is funded on the definition a particular subalgebra of the tensor algebra over a vector space, called the *Clifford algebra.* In the following, we will recall the basic and fundamental results about the structure of these algebras in order to be able to write the Palatini–Cartan theory coupled with a Dirac spinor and then we will proceed to compute the algebra of its constraints.

Definition 5.3.1. Let, V be a vector space over $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and $g\colon V \times V \to \mathbb{K}$ be a symmetric bilinear form.[14] Moreover, let I_g be the two sided ideal in the tensor graded algebra $T(V)$ of V generated by

$$\{v \otimes v + g(v, v)1, v \in V\}, \tag{5.112}$$

where $1 \in T(V)$ is the unit element. Then, we define the Clifford algebra $\mathrm{Cl}(V, g)$ as the filtered algebra given by the quotient

$$\mathrm{Cl}(V, g) := \frac{T(V)}{I_g}. \tag{5.113}$$

Remark 5.3.2. The general definition of a Clifford algebra is given by means of a universal property in the category of unital associative algebras. One can recover Definition 5.3.1 by building a functor between the category of vector spaces endowed with a symmetric bilinear form and the category of unital associative algebras. Then, the universal property guarantees that morphisms extend uniquely to Clifford algebras homomorphisms.

In the following, we will state some results which are well-known facts in the literature. They will serve as a basis in order to build the theory of spin coframes, which can be regarded as a sort of generalization of the vielbein and the coframe formalism. We refer to [Wer19], [Fat18] and references therein for the proofs of these results as well as more details.

Definition 5.3.3. Let V be a quadratic vector space on $\mathbb{R}$ and let (p, q) be the signature of g. Moreover, let $\mathrm{Cl}^+(V, g) := \mathrm{Cl}^0 \oplus \mathrm{Cl}^2 \oplus \mathrm{Cl}^4 \oplus ...$ be the subalgebra defined by the even grading. We define the group $\mathrm{Pin}_{p,q} \subset \mathrm{Cl}(V, g)$ as the subgroup of the group of units in $\mathrm{Cl}(V, g)$ generated by $v \in V$ such that $|g(v, v)| = 1$.
Then, we defined the group $\mathrm{Spin}_{p,q}$ as the subgroup of $\mathrm{Pin}_{p,q}$ given by

$$\mathrm{Spin}_{p,q} := \mathrm{Pin}_{p,q} \cap \mathrm{Cl}^+(V, g). \tag{5.114}$$

Proposition 5.3.4. *Let V be a quadratic vector space on $\mathbb{R}$ and let (p, q) be the signature of g. Moreover, let $\rho\colon \mathrm{Spin}_{p,q} \to \mathrm{GL}\big(\mathfrak{spin}_{p,q}\big)$ be the adjoint representation. Then, we have the following:*

- $\mathfrak{spin}_{p,q} \subset \mathrm{Cl}(V, g)$;
- *The map ρ acts as $\mathrm{SO}(p, q)$ on V[15] (or, for its complexification, as $\mathrm{SO}(n) \times \mathrm{U}(1)$ with $n = p + q$);*

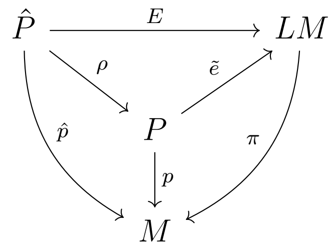
[14] We call this space a *quadratic vector space*.

[15] Here, we regard V as a first grade subspace of the Clifford algebra.

– *The map* ρ *defines a covering map*[16] $\rho\colon \mathrm{Spin}_{p,q} \to \mathrm{SO}(p,q)$.

Furthermore, the group $\mathrm{Spin}_{p,q}$ *is simply connected and it is the universal cover of* $\mathrm{SO}(p,q)$. *Therefore, in particular,* $\mathrm{Spin}_{3,1} \cong \mathrm{SL}(2,\mathbb{C})$.

Definition 5.3.5. Let $\hat{P}$ be a principal $\mathrm{Spin}_{p,q}$-bundle on M and LM the frame bundle. Then, we define the *spin map* $E\colon \hat{P} \to LM$ as the principal bundle morphism such that the following diagram commutes



where $\rho\colon \hat{P} \to P$ denotes the bundle morphism induced by the covering map of Proposition 5.3.4 and $\tilde{e}$ the vielbein of Definition 2.1.12.

The following result will be a particular example of the broader spectrum of the classification of Clifford algebras. In a nutshell, they exhibit a 2-periodicity in the complex case and a 8-periodicity in the real case.

Theorem 5.3.6. *Let* V *be a* 4*-dimensional quadratic vector space on* $\mathbb{K}$ *and, in particular, if* $\mathbb{K} = \mathbb{R}$*, let* $(p,q) = (3,1)$. *Furthermore, let* $\mathrm{M}_{4\times4}(\mathbb{K})_{Cl}$ *denote the algebra of* 4×4 *matrices on* $\mathbb{K}$ *endowed with the Clifford structure. Then, we have the following isomorphism*

$$\lambda\colon \mathrm{Cl}(V,g) \to \mathrm{M}_{4\times4}(\mathbb{K})_{Cl}. \tag{5.115}$$

Remark 5.3.7. If we consider the complexification of the algebra $\mathfrak{spin}_{3,1}^{\mathbb{C}}$, as a consequence of Theorem 5.3.6, the adjoint representation $\rho\colon \mathrm{Spin}_{3,1} \to \mathrm{GL}\left(\mathfrak{spin}_{3,1}^{\mathbb{C}}\right)$ can be regarded as acting on $\mathrm{M}_{4\times4}(\mathbb{C})_{Cl}$, since $\mathfrak{spin}_{p,q} \subset \mathrm{Cl}(V,g)$. Moreover, we know by Proposition 5.3.4 that ρ acts as $\mathrm{SO}(3,1)$ on V. Hence, this statement takes the form

$$\rho_S(\gamma^a) = S\gamma^a S^{-1} = \Lambda^a_b \gamma^b, \tag{5.116}$$

where $S \in \lambda\left(\mathrm{Spin}_{3,1}\right)$ and $\Lambda \in \mathrm{SO}(3,1)$ is the matrix associated to S under the covering map with $a, b = 1,2,3,4$. In other words, the complexified algebra of the

[16] By abuse of notation, we denote the covering map and the adjoint representation in the same manner.

spin group, where the adjoint representation acts, can be expressed in terms of γ-matrices, which can be also labeled according to a basis of V, i.e. $\gamma = \gamma^a v_a \in V \otimes \mathrm{M}_{4\times 4}(\mathbb{C})_{Cl}$, such that the Clifford relation reads

$$\{\gamma^a, \gamma^b\} = -2\eta^{ab} 1_{4\times 4}, \tag{5.117}$$

where the brackets denote the anti-commutators.
Furthermore, if we denote with $f\colon \mathrm{SO}(3,1) \to \mathrm{Aut}(V)$ the fundamental representation of $\mathrm{SO}(3,1)$, by composition with the adjoint representation of $\mathrm{Spin}_{3,1}$, we can construct the Minkowski bundle as the associated vector bundle to $\hat{P}$ under the composition, i.e.

$$\mathcal{V} := \hat{P} \times_{f\circ\rho} V. \tag{5.118}$$

Note that the isomorphism of Theorem 5.3.6 defines a representation of the complexified group $\mathrm{Spin}^{\mathbb{C}}_{3,1}$ on $\mathbb{C}^4$. This representation is called the γ-representation and it corresponds to the representation $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ of $\mathrm{SL}(2,\mathbb{C})$ (thanks to the group isomorphism $\mathrm{Spin}_{3,1} \cong \mathrm{SL}(2,\mathbb{C})$). This fact allows to have the following definition.

Definition 5.3.8. Let $\gamma\colon \mathrm{Spin}^{\mathbb{C}}_{3,1} \to \mathrm{Aut}(\mathbb{C}^4)$ be the γ-representation of the spin group. Then, we define the *spinor bundle* as the associated vector bundle to $\hat{P}$ under γ, namely

$$S := \hat{P} \times_\gamma \mathbb{C}^4. \tag{5.119}$$

We define a *spinor field*[17] as a section of the odd-bundle ΠS, where Π indicates the parity reversal operation.[18]

Remark 5.3.9. Notice that, in this context, we can regard the γ-matrices as elements $\gamma \in \Gamma\big(\mathcal{V} \otimes \mathrm{End}(\Pi S)\big)$. Note also that, by construction, the parity of a spinor field $\psi \in \Gamma(\Pi S)$ is given by $|\psi| = 1$.

Proposition 5.3.10. *Given a real vector space V and the isomorphism $\mathfrak{so}(3,1) \cong \bigwedge^2 V$, we have the following algebra isomorphism*

$$\mathrm{d}\rho\colon \mathfrak{spin}_{3,1} \to \bigwedge^2 V, \tag{5.120}$$

[17]In our case, we will only deal with Dirac spinors. Therefore, the term "spinor" refers uniquely to a Dirac one. In a more general setting, we must slightly generalize our definition in order to include other spin structures.

[18]Parity inversion is fundamental since we want spinors to be Grassmannian/odd quantities.

which is given by

$$\mathrm{d}\rho^{-1}(v \wedge w) = -\frac{1}{4}[\tilde{v}, \tilde{w}], \tag{5.121}$$

where $v, w \in V$, $\tilde{v}, \tilde{w} \in \mathfrak{spin}_{3,1}$ *and* $\rho\colon \mathrm{Spin}_{3,1} \to \mathrm{GL}\Big(\mathfrak{spin}_{3,1}\Big)$ *is the adjoint representation.*

If we consider the complexified Lie algebra $\mathfrak{spin}_{3,1}^{\mathbb{C}}$ and the isomorphism of Proposition 5.3.10, we can build a covariant derivative for spinor fields in terms of local connections in $\Omega^{1,2}$. Explicitly, it reads

$$d_\omega \psi = d\psi + [\omega, \psi] = d\psi - \frac{1}{4}\omega^{ab}\gamma_a\gamma_b\psi. \tag{5.122}$$

We define the covariant derivative for the conjugate of ψ such that $d_\omega\overline{\psi} = \overline{d_\omega \psi}$. Therefore, we have

$$d_\omega \overline{\psi} = d\overline{\psi} + [\omega, \overline{\psi}] = d\overline{\psi} - \frac{1}{4}\omega^{ab}\overline{\psi}\gamma_a\gamma_b. \tag{5.123}$$

By Remark 5.3.9, we can extend the definition of the covariant derivative also to the γ-matrices. It follows the upcoming lemma.

Lemma 5.3.11 ([CCF22])**.** *Let* $\gamma \in \Gamma\Big(\mathcal{V} \otimes \mathrm{End}(\Pi S)\Big)$. *Then, it holds*

$$d_\omega \gamma = 0. \tag{5.124}$$

The space of fields of the Spinor Palatini–Cartan theory is given by[19]

$$\mathcal{F}_\psi = \widetilde{\Omega}^{1,1} \times \mathcal{A}(P) \times \Gamma(\Pi S) \times \Gamma(\Pi \bar{S}) \ni (e, \omega, \psi, \overline{\psi}), \tag{5.125}$$

whereas the action functional reads

$$\mathcal{S}_S = \mathcal{S} + \mathcal{S}_\psi = \mathcal{S} + \int_M \frac{i}{12} e^3(\overline{\psi}\gamma d_\omega \psi - d_\omega\overline{\psi}\gamma\psi). \tag{5.126}$$

It follows that the field equations are the following Euler-Lagrange equations for the action $\mathcal{S}_S$

$$eF_\omega + \frac{i}{4}e^2(\overline{\psi}\gamma d_\omega\psi - d_\omega\overline{\psi}\gamma\psi) = 0 \tag{5.127}$$

$$ed_\omega e + \frac{i}{6}(\overline{\psi}\gamma[e^3, \psi] - [e^3, \overline{\psi}]\gamma\psi) = 0 \tag{5.128}$$

$$\frac{e^3}{6}\gamma d_\omega \psi - \frac{1}{12}d_\omega e^3 \gamma\psi = 0 \tag{5.129}$$

$$\frac{e^3}{6}d_\omega\overline{\psi}\gamma + \frac{1}{12}d_\omega e^3\overline{\psi}\gamma = 0, \tag{5.130}$$

[19]Where $\bar{S}$ is simply given by the conjugate representation $\bar{\gamma}$.

where we define, for $X \in \Gamma(\mathcal{V})$ and $\alpha \in \Omega_\Sigma^{r,k}$, the contraction

$$\iota_X \alpha := \frac{\eta_{ab}}{(k-1)!} X^a \alpha^{b i_2 \cdots i_k} v_{i_2} \wedge \cdots \wedge v_{i_k} \tag{5.131}$$

and consequently, for $\chi \in \Omega_\Sigma^{i,j}$, the brackets

$$\begin{cases} [\chi, \psi] & := \frac{1}{4(j-1)} \iota_\gamma \iota_\gamma \chi \psi \\ [\chi, \overline{\psi}] & := -\frac{(-1)^{|\chi||\psi|}}{4(j-1)} \overline{\psi} \iota_\gamma \iota_\gamma \chi, \end{cases} \tag{5.132}$$

where $|\chi|$ is the parity of χ and $|\psi|$ the parity of ψ.

Similar to the preceding sections, the space of pre-boundary fields $\widetilde{\mathcal{F}}_\Sigma^\psi$, as defined in Definition 4.2.5 for the Palatini–Cartan theory, can be established by pulling back the fields to the boundary Σ. Furthermore, we will keep denoting the fields on the boundary in the same way as those in the bulk.

The Noether 1-form becomes

$$\tilde{\alpha}_\psi = \int_\Sigma \frac{e^2}{2} \delta\omega + i \frac{e^3}{12} \left(\overline{\psi} \gamma \delta \psi - \delta \overline{\psi} \gamma \psi \right) \tag{5.133}$$

and therefore the pre-symplectic form of the theory is

$$\tilde{\varpi}_\psi = \int_\Sigma e \delta e \delta \omega + i \frac{e^2}{4} \left(\overline{\psi} \gamma \delta \psi - \delta \overline{\psi} \gamma \psi \right) \delta e + i \frac{e^3}{3!} \delta \overline{\psi} \gamma \delta \psi. \tag{5.134}$$

As outlined in [CCF22], we can now define the geometric phase space of the theory through a reduction using the kernel of the pre-symplectic form.

Theorem 5.3.12. *The geometric phase space for the Spinor Palatini–Cartan theory is the symplectic manifold* $(\mathcal{F}_\Sigma^\psi, \varpi_\psi)$ *given by the following equivalence relations on the space of pre-boundary fields* $\widetilde{\mathcal{F}}_\Sigma^\psi$

$$\omega' \sim \omega \quad \Longleftrightarrow \quad \omega' - \omega \in \mathrm{Ker} W_1^{\Sigma,(1,2)} \tag{5.135}$$

and the symplectic form

$$\varpi_\psi = \int_\Sigma e \delta e \delta [\omega] + i \frac{e^2}{4} \left(\overline{\psi} \gamma \delta \psi - \delta \overline{\psi} \gamma \psi \right) \delta e + i \frac{e^3}{3!} \delta \overline{\psi} \gamma \delta \psi. \tag{5.136}$$

We denote this equivalence class as $\mathcal{A}(\Sigma)_{red}$.

Remark 5.3.13. Likewise the preceding cases, we notice that the equivalence class of ω, defining the geometric phase space, remains equal to the Palatini–Cartan theory. In fact, similarly to the previous couplings, this can be seen as a consequence of the fact that the symplectic form does not have any other piece along ω, but the one equal to the Palatini–Cartan case.

Remark 5.3.14. In the case at hand, the field equations see a substantial difference. Namely, the Levi-Civita (or torsion-free) condition $ed_\omega e = 0$ no longer holds. Indeed, the Lagrangian of the theory couples the connection with the spinor. Therefore, the structural and the degeneracy constraints take the form

$$\begin{cases} e_n(a_\psi - p_\mathcal{T} a_\psi) \in \operatorname{Im} W_1^{\Sigma,(1,1)} \\ p_\mathcal{T} a_\psi = 0. \end{cases} \tag{5.137}$$

with

$$a_\psi := d_\omega e + \frac{i}{4}(\overline{\psi}\gamma[e^2,\psi] - [e^2,\overline{\psi}]\gamma\psi). \tag{5.138}$$

Theorem 5.3.15. *The geometric phase space of the spinor Palatini–Cartan theory is symplectomorphic to the space* $(\mathcal{F}^\partial_\psi, \varpi^\partial_\psi)$*, where*

$$\mathcal{F}^\partial_\psi \subset \widetilde{\Omega}^{1,1}_\Sigma \times \Omega^{1,2}_\Sigma \times \Gamma(\Pi i^* S) \times \Gamma(\Pi i^* \bar{S}) \tag{5.139}$$

with $(e,\omega,\psi,\overline{\psi}) \in \mathcal{F}^\partial_\psi$ *satisfying*

$$\begin{cases} e_n(a_\psi - p_\mathcal{T} a_\psi) \in \operatorname{Im} W_1^{\Sigma,(1,1)} \\ p_\mathcal{K} \omega = 0, \end{cases} \tag{5.140}$$

as defined in Lemma 4.2.19 and Remark 5.3.14, and where the corresponding symplectic form on $\mathcal{F}^\partial_\psi$ *is given by*

$$\varpi^\partial_\psi = \int_\Sigma e\delta e\delta\omega + i\frac{e^2}{4}\Big(\overline{\psi}\gamma\delta\psi - \delta\overline{\psi}\gamma\psi\Big)\delta e + i\frac{e^3}{3!}\delta\overline{\psi}\gamma\delta\psi. \tag{5.141}$$

The following proposition will ensure that, although the form of the degeneracy constraint is sensibly different from the preceding cases, the form of the functional R^ψ_τ will coincide with the one of the Palatini–Cartan theory.

Proposition 5.3.16. *Let* $\tau \in \mathcal{S}$*. Then, we have the following identity*

$$\tau(\overline{\psi}\gamma[e^2,\psi] - [e^2,\overline{\psi}]\gamma\psi) = 0. \tag{5.142}$$

Proof. The proof comes by applying twice Lemma A.0.9. Therefore, by means of Proposition A.0.5, we have

$$\tau(\overline{\psi}\gamma[e^2,\psi] - [e^2,\overline{\psi}]\gamma\psi) = e_n\beta(\overline{\psi}\gamma[e^2,\psi] - [e^2,\overline{\psi}]\gamma\psi) \tag{5.143}$$

$$= e_n e^2(\overline{\psi}\gamma[\beta,\psi] - [\beta,\overline{\psi}]\gamma\psi) \tag{5.144}$$

$$= e\beta(\overline{\psi}\gamma[e_n e,\psi] - [e_n e,\overline{\psi}]\gamma\psi) \tag{5.145}$$

$$= 0, \tag{5.146}$$

since $\beta \in \operatorname{Ker} W_1^{\Sigma,(1,2)}$. □

We are now able to properly give the constraints of the theory.

Definition 5.3.17. Let $c \in \Omega_{\Sigma}^{0,2}[1]$, $\xi \in \mathfrak{X}(\Sigma)[1]$, $\lambda \in C^{\infty}(\Sigma)[1]$ and $\tau \in \mathcal{S}[1]$. Then, we define the following functionals

$$L_c^{\psi} = \int_{\Sigma} c e d_{\omega} e - i\frac{e^3}{2\cdot 3!}\Big([c,\overline{\psi}]\gamma\psi - \overline{\psi}\gamma[c,\psi]\Big) \tag{5.147}$$

$$P_{\xi}^{\psi} = \int_{\Sigma} \frac{1}{2}\iota_{\xi}(e^2)F_{\omega} + \iota_{\xi}(\omega-\omega_0)e d_{\omega}e - i\frac{e^3}{2\cdot 3!}\Big(\overline{\psi}\gamma \mathrm{L}_{\xi}^{\omega_0}(\psi) - \mathrm{L}_{\xi}^{\omega_0}(\overline{\psi})\gamma\psi\Big) \tag{5.148}$$

$$H_{\lambda}^{\psi} = \int_{\Sigma} \lambda e_n\Big(eF_{\omega} + \frac{\Lambda}{3!}e^3 + i\frac{e^2}{4}\Big(\overline{\psi}\gamma d_{\omega}\psi - d_{\omega}\overline{\psi}\gamma\psi\Big)\Big) \tag{5.149}$$

$$R_{\tau} = \int_{\Sigma} \tau d_{\omega} e. \tag{5.150}$$

We refer to these as the constraints of the spinor Palatini–Cartan theory.

Theorem 5.3.18. *The Poisson brackets of the constraints of Definition 5.3.17 read*

$$\begin{aligned}
&\{L_c^{\psi}, L_c^{\psi}\}_{\psi} = -\frac{1}{2}L_{[c,c]}^{\psi} && \{P_{\xi}^{\psi}, P_{\xi}^{\psi}\}_{\psi} = \frac{1}{2}P_{[\xi,\xi]}^{\psi} - \frac{1}{2}L_{\iota_{\xi}\iota_{\xi}F_{\omega_0}}^{\psi}\\
&\{L_c^{\psi}, P_{\xi}^{\psi}\}_{\psi} = L_{\mathcal{L}_{\xi}^{\omega_0}c}^{\psi} && \{H_{\lambda}^{\psi}, H_{\lambda}^{\psi}\}_{\psi} \approx 0\\
&\{L_c^{\psi}, R_{\tau}\}_{\psi} = -R_{p_{\mathcal{S}}[c,\tau]} && \{R_{\tau}^{\psi}, P_{\xi}^{\psi}\}_{\psi} = R_{p_{\mathcal{S}}\mathcal{L}_{\xi}^{\omega_0}\tau}\\
&\{R_{\tau}, H_{\lambda}^{\psi}\}_{\psi} \approx G_{\lambda\tau} + K_{\lambda\tau}^{\psi} && \{R_{\tau}, R_{\tau}\}_{\psi} \approx F_{\tau\tau}
\end{aligned} \tag{5.151}$$

$$\{L_c^{\psi}, H_{\lambda}^{\psi}\}_{\psi} = -P_{X^{(a)}}^{\psi} + L_{X^{(a)}(\omega-\omega_0)_a}^{\psi} - H_{X^{(n)}}^{\psi} \tag{5.152}$$

$$\{P_{\xi}^{\psi}, H_{\lambda}^{\psi}\}_{\psi} = P_{Y^{(a)}}^{\psi} - L_{Y^{(a)}(\omega-\omega_0)_a}^{\psi} + H_{Y^{(n)}}^{\psi}, \tag{5.153}$$

with $X = [c, \lambda e_n]$, $Y = \mathcal{L}_{\xi}^{\omega_0}(\lambda e_n)$ *and where the superscripts* (a) *and* (n) *describe their components with respect to* e_a, e_n. *Furthermore,* $F_{\tau\tau}$, $G_{\lambda\tau}$ *and* $K_{\lambda\tau}^{\psi}$ *are functionals of* $e, \omega, \psi, \overline{\psi}, \tau$ *and* λ *defined in the proof which are not proportional to any other constraint.*

Proof. First, we notice that the contraction of the symplectic form with a vector field $\mathbb{X} \in \mathfrak{X}(\mathcal{F}_{\Sigma}^{\psi})$ is given by

$$\iota_{\mathbb{X}}\varpi_{\psi}^{\partial} = \int_{\Sigma} e\mathbb{X}_e\delta\omega + \Big[e\mathbb{X}_{\omega} + \frac{i}{4}e^2(\overline{\psi}\gamma\mathbb{X}_{\psi} - \mathbb{X}_{\overline{\psi}}\gamma\psi)\Big]\delta e \tag{5.154}$$

$$+ i\delta\overline{\psi}\Big(-\frac{e^2}{4}\gamma\psi\mathbb{X}_e + \frac{e^3}{3!}\gamma\mathbb{X}_{\psi}\Big) + i\Big(\frac{e^2}{4}\overline{\psi}\gamma\mathbb{X}_e + \frac{e^3}{3!}\mathbb{X}_{\overline{\psi}}\gamma\Big)\delta\psi. \tag{5.155}$$

Then, we start giving the Hamiltonian vector fields of the constraints. For L_c^ψ and P_ξ^ψ, from [CCF22], we have

$$\mathbb{L}_e^\psi = [c, e] \qquad \mathbb{L}_\psi^\psi = [c, \psi] \tag{5.156}$$
$$\mathbb{L}_\omega^\psi = d_\omega c \qquad \mathbb{L}_{\overline{\psi}}^\psi = [c, \overline{\psi}] \tag{5.157}$$
$$\mathbb{P}_e^\psi = -\mathcal{L}_\xi^{\omega_0} e \qquad \mathbb{P}_\psi^\psi = -\mathcal{L}_\xi^{\omega_0}(\psi) \tag{5.158}$$
$$\mathbb{P}_\omega^\psi = -\mathcal{L}_\xi^{\omega_0}(\omega - \omega_0) - \iota_\xi F_{\omega_0} \qquad \mathbb{P}_{\overline{\psi}}^\psi = -\mathcal{L}_\xi^{\omega_0}(\overline{\psi}). \tag{5.159}$$

Whereas, for H_λ^ψ, we have

$$\mathbb{H}_e^\psi = d_\omega(\lambda e_n) + \lambda\sigma + \frac{i}{4}\lambda\overline{\psi}\Big(\iota_\gamma\iota_\gamma e_n e\gamma - \gamma\iota_\gamma\iota_\gamma e_n e\Big)\psi \tag{5.160}$$
$$e\mathbb{H}_\omega^\psi = \lambda e_n\Big(F_\omega + \frac{\Lambda}{2}e^2\Big) - i\frac{\lambda e_n}{4}e(\overline{\psi}\gamma d_\omega\psi - d_\omega\overline{\psi}\gamma\psi) \tag{5.161}$$
$$\frac{e^3}{3!}\gamma\mathbb{H}_\psi^\psi = \frac{\lambda e_n}{2}e^2\gamma d_\omega\psi - \frac{\lambda e_n}{4}ed_\omega e\gamma\psi \tag{5.162}$$
$$+\frac{i}{64}\lambda e\Big[\overline{\psi}\Big(\iota_\gamma\iota_\gamma(e_n e^2)\gamma - \gamma\iota_\gamma\iota_\gamma(e_n e^2)\Big)\psi\Big]\gamma\psi \tag{5.163}$$
$$\frac{e^3}{3!}\mathbb{H}_{\overline{\psi}}^\psi\gamma = \frac{\lambda e_n}{2}e^2 d_\omega\overline{\psi}\gamma + \frac{\lambda e_n}{4}ed_\omega e\overline{\psi}\gamma \tag{5.164}$$
$$-\frac{i}{64}\lambda e\overline{\psi}\gamma\Big[\overline{\psi}\Big(\iota_\gamma\iota_\gamma(e_n e^2)\gamma - \gamma\iota_\gamma\iota_\gamma(e_n e^2)\Big)\psi\Big], \tag{5.165}$$

where $\sigma \in \Omega_\Sigma^{1,1}$. Lastly, the Hamiltonian vector fields of R_τ, are given by

$$e\mathbb{R}_e = [\tau, e] \tag{5.166}$$
$$e\mathbb{R}_\omega = \frac{\delta\tau}{\delta e}d_\omega e + d_\omega\tau \tag{5.167}$$
$$\mathbb{R}_\psi = \mathbb{R}_{\overline{\psi}} = 0, \tag{5.168}$$

since they coincide with the ones of the Palatini–Cartan theory of Definition 4.2.25. Notice that, instead of using the function $g = g(\tau, e, \omega)$, we preferred expressing the variation of τ with respect to e by means of the functional derivative $\frac{\delta\tau}{\delta e}$. However, we have the relation

$$g(\tau, e, \omega) = \frac{\delta\tau}{\delta e}d_\omega e. \tag{5.169}$$

Now, we are ready to compute the Poisson brackets of the constraints. From

[CCF22], we have already knowledge of the following Poisson brackets

$$\{P_\xi^\psi, P_\xi^\psi\}_\psi = \frac{1}{2}P^\psi_{[\xi,\xi]} - \frac{1}{2}L^\psi_{\iota_\xi\iota_\xi F_{\omega_0}} \qquad \{H_\lambda^\psi, H_\lambda^\psi\}_\psi = 0 \tag{5.170}$$

$$\{L_c^\psi, P_\xi^\psi\}_\psi = L^\psi_{\mathcal{L}_\xi^{\omega_0} c} \qquad \{L_c^\psi, L_c^\psi\}_\psi = -\frac{1}{2}L^\psi_{[c,c]} \tag{5.171}$$

$$\{L_c^\psi, H_\lambda^\psi\}_\psi = -P^\psi_{X^{(a)}} + L^\psi_{X^{(a)}(\omega-\omega_0)_a} - H^\psi_{X^{(n)}} \tag{5.172}$$

$$\{P_\xi^\psi, H_\lambda^\psi\}_\psi = P^\psi_{Y^{(a)}} - L^\psi_{Y^{(a)}(\omega-\omega_0)_a} + H^\psi_{Y^{(n)}}, \tag{5.173}$$

with $X = [c, \lambda e_n]$ and $Y = \mathcal{L}_\xi^{\omega_0}(\lambda e_n)$ as above. Therefore, we are left with computing the remaining constraints. First, we notice that

$$\{R_\tau, L_c^\psi\}_\psi = \{R_\tau, L_c\} = -R_{p_\mathcal{S}[c,\tau]} = -R_{p_\mathcal{S}[c,\tau]}. \tag{5.174}$$

Similarly, we can also compute the bracket

$$\{R_\tau, P_\xi^\psi\}_\psi = \{R_\tau, P_\xi\} = R_{p_\mathcal{S}\mathcal{L}_\xi^{\omega_0}\tau} = R_{p_\mathcal{S}\mathcal{L}_\xi^{\omega_0}\tau}. \tag{5.175}$$

Now, we move on to compute the bracket $\{R_\tau, H_\lambda^\psi\}_\psi$, i.e.

$$\{R_\tau, H_\lambda^\psi\}_\psi = \int_\Sigma \Big(e_n \frac{\delta\beta}{\delta e} d_\omega e + d_\omega(e_n\beta)\Big)\Big(d_\omega(\lambda e_n) + \lambda\sigma \tag{5.176}$$

$$- i\lambda(\overline{\psi}\gamma[e_n e, \psi] - [e_n e, \overline{\psi}]\gamma\psi)\Big) \tag{5.177}$$

$$+ W_1^{-1}[e_n\beta, e]\Big(\lambda e_n(F_\omega + \frac{\Lambda}{2}e^2) \tag{5.178}$$

$$- \frac{i}{4}\lambda e_n e(\overline{\psi}\gamma d_\omega\psi - d_\omega\overline{\psi}\gamma\psi)\Big) \tag{5.179}$$

$$\approx \int_\Sigma -i\lambda\beta d_\omega e_n(\overline{\psi}\gamma[e_n e, \psi] - [e_n e, \overline{\psi}]\gamma\psi) \tag{5.180}$$

$$- \frac{i}{4}[e_n\beta, e]\lambda e_n(\overline{\psi}\gamma d_\omega\psi - d_\omega\overline{\psi}\gamma\psi) \tag{5.181}$$

$$+ G_{\lambda\tau}, \tag{5.182}$$

where, in the last passage, we used Lemma A.0.9 and the fact that $e_n^2 = 0$. Moreover, the quantity $G_{\lambda\tau}$ and the map W_1^{-1} are defined respectively in Theorem 30 of [CCT21] and in the proof of Theorem 5.2.6. Now, we can notice that, thanks to Lemma A.0.9, we can write

$$\lambda\beta d_\omega e_n(\overline{\psi}\gamma[e_n e, \psi] - [e_n e, \overline{\psi}]\gamma\psi) = \tag{5.183}$$

$$= \lambda e_n e d_\omega e_n(\overline{\psi}\gamma[\beta, \psi] - [\beta, \overline{\psi}]\gamma\psi) \tag{5.184}$$

$$= \lambda e\beta(\overline{\psi}\gamma[e_n d_\omega e_n, \psi] - [e_n d_\omega e_n, \overline{\psi}]\gamma\psi) \tag{5.185}$$

$$= 0, \tag{5.186}$$

obtaining

$$\{R_\tau, H_\lambda^\psi\}_\psi \approx G_{\lambda\tau} - \int_\Sigma \frac{i}{4}[e_n\beta, e]\lambda e_n(\overline{\psi}\gamma d_\omega\psi - d_\omega\overline{\psi}\gamma\psi). \tag{5.187}$$

Finally, we can write the integral as

$$\{R_\tau, H_\lambda^\psi\}_\psi \approx G_{\lambda\tau} - \int_\Sigma \frac{i}{4}\lambda\tau[e_n, \hat{e}](\overline{\psi}\gamma d_\omega\psi - d_\omega\overline{\psi}\gamma\psi), \tag{5.188}$$

where we implemented again Proposition A.0.5 and also the relation[20]

$$e_n[\tau, e] = \tau[e_n, \hat{e}] \tag{5.189}$$

with $\hat{e}$ defined as $\hat{e} := e - \tilde{e}$ (see Eq. (4.16)). More specifically, using the definition of the independent components of τ, as we did in the proof of Theorem 5.2.6, we have

$$\{R_\tau, H_\lambda^\psi\}_\psi \approx G_{\lambda\tau} + K_{\lambda\tau}^\psi, \tag{5.190}$$

with

$$K_{\lambda\tau}^\psi := -\int_\Sigma i\lambda\Big(\sum_{\mu=1}^{2}\mathcal{Y}_\mu\Big(\hat{g}_n d_\omega J_\psi\Big)_\mu^{3\mu} + \sum_{\mu_1\neq\mu_2=1}^{2}\mathcal{X}_{\mu_1}^{\mu_2}\Big(\hat{g}_n d_\omega J_\psi\Big)_{3\mu_2}^{\mu_1}\Big), \tag{5.191}$$

where $\hat{g}_n := [e_n, \hat{e}] \in \Omega_\Sigma^{1,0}$ and $d_\omega J_\psi := d_\omega(\overline{\psi}\gamma\psi) \in \Omega_\Sigma^{1,1}$.

Finally, since $\{R_\tau, R_\tau\}_\psi = \{R_\tau, R_\tau\}$, this final result completes the proof. □

Corollary 5.3.19. *If the boundary metric i^*g is non-degenerate, then the functionals in Definition 5.3.17 define a coisotropic submanifold.*

5.4 Yang–Mills-spinor

Thus far, we have focused on the fundamental fields of the theory, which provide the basic building blocks for the interactions to come. In the following sections, we will begin to consider such interactions, starting with the Yang–Mills-spinor. The space of fields of the spinor Palatini–Cartan theory is given by

$$\mathcal{F}_{YMS} = \mathcal{F}_{YM} \times \Gamma(\Pi S_{\mathrm{SU}(N)}) \times \Gamma(\Pi \bar{S}_{\mathrm{SU}(N)}), \tag{5.192}$$

[20] It simply comes from the definition of $\mathcal{S}$.

where the index SU(N) indicates that we take into account the internal gauge group,[21] and $\mathcal{F}_{YM}$ is defined in Eq. (5.53). Whereas the action functional reads

$$S_{YMS} = S + S_\psi + S_A + S_{A,\psi}, \tag{5.193}$$

where

$$S_{A,\psi} = \int_M \frac{e^{N-1}}{2(N-1)!}\big(\overline{\psi}\gamma[A,\psi] - [A,\overline{\psi}]\gamma\psi\big), \tag{5.194}$$

with $\overline{\psi}\gamma[A,\psi] = ig_i\overline{\psi}_I\gamma A^I_J\psi^J$ and g_i a constant.

Since the interaction term does not contain derivatives, the boundary structure is just the direct sum of the Yang–Mills and spinor structures. In particular, the geometric phase space is given by the following theorem.

Theorem 5.4.1. *The geometric phase space of the Yang–Mills-spinor Palatini–Cartan theory is symplectomorphic to the space $(\mathcal{F}^\partial_{YMS}, \varpi^\partial_{YMS})$, where*

$$\mathcal{F}^\partial_{YMS} \subset \mathcal{F}^\partial_{YM} \times \Gamma(\Pi i^* S_{\mathrm{SU}(N)}) \times \Gamma(\Pi i^* \bar{S}_{\mathrm{SU}(N)}) \tag{5.195}$$

with $\mathcal{F}^\partial_{YM}$ defined in Theorem 5.2.4 and $(e,\omega,A,B,\psi,\overline{\psi}) \in \mathcal{F}^\partial_{YMS}$ satisfying

$$\begin{cases} e_n(a_\psi - p_{\mathcal{T}}a_\psi) \in \operatorname{Im} W_1^{\Sigma,(1,1)} \\ p_{\mathcal{K}}\omega = 0 \\ F_A + \frac{1}{2}(e^2, B) = 0 \\ p_{\Omega_e^{0,1*}\wedge W} B = 0, \end{cases} \tag{5.196}$$

as defined in Lemma 4.2.19, Lemma 5.2.3, Lemma 4.2.19 and Remark 5.3.14, and where the corresponding symplectic form on $\mathcal{F}^\partial_{YMS}$ is given by

$$\varpi_{YMS} = \varpi + \varpi_A + \varpi_\psi. \tag{5.197}$$

[21] One takes the tensor product of the two associated vector bundles—to the principal SO(3, 1)- and SU(N)-bundle—together with the tensor product of the spinorial representation of SO(3, 1) on $\mathbb{C}^4$ and the fundamental representation of SU(N) on $\mathbb{C}^N$, respectively. This construction is equivalent to considering the principal bundle with structure group given by the direct product SO(3, 1)× SU(N), and then forming the associated vector bundle via the tensor product representation on $\mathbb{C}^4 \otimes \mathbb{C}^N$.

Definition 5.4.2. Let $c \in \Omega_\Sigma^{0,2}[1]$, $\mu \in C^\infty(\Sigma, \mathfrak{g})[1]$, $\xi \in \mathfrak{X}(\Sigma)[1]$, $\lambda \in C^\infty(\Sigma)[1]$, and $\tau \in \mathcal{S}[1]$, and the other functionals be as in Definition 5.2.5 and Definition 5.3.17. Then, we define the following functionals

$$L_c^{A,\psi} = L_c + l_c^\psi \tag{5.198}$$
$$P_\xi^{A,\psi} = P_\xi + p_\xi^A + p_\xi^\psi + p_\xi^{A,\psi} \tag{5.199}$$
$$H_\lambda^{A,\psi} = H_\lambda + h_\lambda^A + h_\lambda^\psi + h_\lambda^{A,\psi} \tag{5.200}$$
$$M_\mu^{A,\psi} = M_\mu^A + m_\mu^{A,\psi} \tag{5.201}$$
$$R_\tau^{A,\psi} = R_\tau, \tag{5.202}$$

where

$$l_c^\psi = L_c^\psi - L_c \tag{5.203}$$
$$p_\xi^\psi = P_\xi^\psi - P_\xi \tag{5.204}$$
$$h_\lambda^\psi = H_\lambda^\psi - H_\lambda \tag{5.205}$$
$$p_\xi^A = P_\xi^A - P_\xi \tag{5.206}$$
$$h_\lambda^A = H_\lambda^A - H_\lambda \tag{5.207}$$

and

$$p_\xi^{A,\psi} = \int_\Sigma -i\frac{e^3}{2\cdot 3!}\Big(\overline{\psi}\gamma[\iota_\xi A_0, \psi] - [\iota_\xi A_0, \overline{\psi}]\gamma\psi\Big) \tag{5.208}$$
$$h_\lambda^{A,\psi} = \int_\Sigma -\lambda e_n\Big[i\frac{e^2}{4}\Big(\overline{\psi}\gamma[A,\psi] - [A,\overline{\psi}]\gamma\psi\Big)\Big] \tag{5.209}$$
$$m_\mu^{A,\psi} = \int_\Sigma -i\frac{e^3}{2\cdot 3!}\Big([\mu,\overline{\psi}]\gamma\psi - \overline{\psi}\gamma[\mu,\psi]\Big). \tag{5.210}$$

We refer to these as the constraints of the Yang–Mills-spinor Palatini–Cartan theory.

Theorem 5.4.3. *The Poisson brackets of the constraints of Definition 5.4.2 read*

$$\{L_c^{A,\psi}, L_c^{A,\psi}\}_{YMS} = -\frac{1}{2}L_{[c,c]}^{A,\psi} \qquad \{P_\xi^{A,\psi}, P_\xi^{A,\psi}\}_{YMS} = \frac{1}{2}P_{[\xi,\xi]}^{A,\psi} - \frac{1}{2}L_{\iota_\xi\iota_\xi F_{\omega_0}}^{A,\psi}$$
$$\{L_c^{A,\psi}, P_\xi^{A,\psi}\}_{YMS} = L_{\mathcal{L}_\xi^{\omega_0}c}^{A,\psi} \qquad \{H_\lambda^{A,\psi}, H_\lambda^{A,\psi}\}_{YMS} \approx 0$$
$$\{L_c^{A,\psi}, R_\tau\}_{YMS} = -R_{p_\mathcal{S}[c,\tau]} \qquad \{R_\tau^{A,\psi}, P_\xi^{A,\psi}\}_{YMS} = R_{p_\mathcal{S}\mathcal{L}_\xi^{\omega_0}\tau}$$
$$\{R_\tau, H_\lambda^{A,\psi}\}_{YMS} \approx G_{\lambda\tau} + K_{\lambda\tau}^\psi \qquad \{R_\tau, R_\tau\}_{YMS} \approx F_{\tau\tau}$$
$$\{L_c^{A,\psi}, H_\lambda^{A,\psi}\}_{YMS} = -P_{X^{(a)}}^{A,\psi} + L_{X^{(a)}(\omega-\omega_0)_a}^{A,\psi} - H_{X^{(n)}}^{A,\psi}$$
$$\{P_\xi^{A,\psi}, H_\lambda^{A,\psi}\}_{YMS} = P_{Y^{(a)}}^{A,\psi} - L_{Y^{(a)}(\omega-\omega_0)_a}^{A,\psi} + H_{Y^{(n)}}^{A,\psi},$$

with $X = [c, \lambda e_n]$, $Y = \mathcal{L}_\xi^{\omega_0}(\lambda e_n)$ and where the superscripts (a) and (n) describe their components with respect to e_a, e_n. Furthermore, $F_{\tau\tau}$, $G_{\lambda\tau}$ and $K_{\lambda\tau}^{\psi}$ are functionals of $e, \omega, \psi, \overline{\psi}, \tau$ and λ defined in Theorem 5.3.18 which are not proportional to any other constraint.

Proof. Notice that the proof of each bracket relies on the result presented in Appendix B. The first step is to compute the Hamiltonian vector fields of the constraints. The expressions for the Hamiltonian vector fields have been presented in the former sections. Hence the only components that we have to compute through (**??**) are $\mathbb{l}^{A,\psi}$, $\mathbb{p}^{A,\psi}$, $\mathbb{h}^{A,\psi}$ and $\mathbb{m}^{A,\psi}$. Let us start from $\mathbb{l}^{A,\psi}$. It must satisfy

$$\iota_{\mathbb{l}^{A,\psi}}(\varpi + \varpi_A + \varpi_\psi) + \iota_{\mathbb{l}^\psi}\varpi_A + \iota_{\mathbb{l}^A}\varpi_\psi = 0. \tag{5.211}$$

Since $\iota_{\mathbb{l}^\psi}\varpi_A = \iota_{\mathbb{l}^A}\varpi_\psi = 0$ we conclude $\mathbb{l}^{A,\psi} = 0$. Similarly we have

$$\iota_{\mathbb{p}^{A,\psi}}(\varpi + \varpi_A + \varpi_\psi) + \iota_{\mathbb{p}^\psi}\varpi_A + \iota_{\mathbb{p}^A}\varpi_\psi = \delta p_\xi^{A,\psi}. \tag{5.212}$$

Since $\iota_{\mathbb{p}^\psi}\varpi_A = \iota_{\mathbb{p}^A}\varpi_\psi = 0$, the computation is exactly the same as for p_ξ^ψ with A_0 instead of ω_0. Hence we get

$$\mathbb{p}_e^{A,\psi} = 0 \qquad \mathbb{p}_\omega^{A,\psi} = \mathbb{V}_{p^{A,\psi}} \tag{5.213}$$
$$\mathbb{p}_A^{A,\psi} = 0 \qquad \mathbb{p}_\rho^{A,\psi} = 0 \tag{5.214}$$
$$\mathbb{p}_\psi^{A,\psi} = -[\iota_\xi A_0, \psi] \qquad \mathbb{p}_{\overline{\psi}}^{A,\psi} = -[\iota_\xi A_0, \overline{\psi}]. \tag{5.215}$$

For $\mathbb{h}^{A,\psi}$, we have $\iota_{\mathbb{h}^\psi}\varpi_A = \iota_{\mathbb{h}^A}\varpi_\psi = 0$ and

$$\delta h_\lambda^{A,\psi} = \int_\Sigma -\lambda e_n i e \delta e\Big(\overline{\psi}\gamma[A,\psi]\Big) - \lambda e_n\Big[i\frac{e^2}{2}\Big(\delta\overline{\psi}\gamma[A,\psi] \tag{5.216}$$
$$+ \overline{\psi}\gamma[\delta A, \psi] - \overline{\psi}\gamma[A, \delta\psi]\Big)\Big]. \tag{5.217}$$

Hence, we get:

$$\mathbb{h}_e^{A,\psi} = 0 \qquad \mathbb{h}_\omega^{A,\psi} = -\frac{i}{2}\lambda e_n \overline{\psi}\gamma[A,\psi] + \mathbb{V}_{h^{A,\psi}} \tag{5.218}$$
$$\mathbb{h}_A^{A,\psi} = 0 \qquad (\mathbb{h}_\rho^{A,\psi})_I^J = -\frac{1}{2}g_i \lambda e_n e^2 \overline{\psi}_I \gamma \psi^J \tag{5.219}$$
$$\frac{e^3}{3!}\gamma \mathbb{h}_\psi^{A,\psi} = \frac{\lambda e_n e^2}{2}\gamma[A,\psi] \qquad \frac{e^3}{3!}\mathbb{h}_{\overline{\psi}}^{A,\psi}\gamma = \frac{\lambda e_n e^2}{2}[A,\overline{\psi}]\gamma. \tag{5.220}$$

As for $p_\xi^{A,\psi}$, the Hamiltonian vector field of $m_\mu^{A,\psi}$ can be obtained by noticing that it is equal to that of l_c^ψ by substituting c with μ. The result is[22]

$$\mathbb{m}_e^{A,\psi} = 0 \qquad \mathbb{m}_\omega^{A,\psi} = \mathbb{V}_{m^{A,\psi}} \tag{5.221}$$
$$\mathbb{m}_A^{A,\psi} = 0 \qquad \mathbb{m}_\rho^{A,\psi} = 0 \tag{5.222}$$
$$\mathbb{m}_\psi^{A,\psi} = [\mu,\psi] \qquad \mathbb{m}_{\overline{\psi}}^{A,\psi} = [\mu,\overline{\psi}]. \tag{5.223}$$

We can now compute the constraints using Theorem B. Before beginning the actual computation we note that for all constraints

$$\iota_{\mathbb{X}+\mathbb{x}^\psi}\iota_{\mathbb{Y}+\mathbb{y}^\psi}\varpi_A + \iota_{\mathbb{X}+\mathbb{x}^A}\iota_{\mathbb{Y}+\mathbb{y}^A}\varpi_\psi = 0, \tag{5.224}$$
$$\iota_{\mathbb{X}}\iota_{\mathbb{Y}}\varpi_A = 0, \tag{5.225}$$
$$\iota_{\mathbb{X}}\iota_{\mathbb{Y}}\varpi_\psi = 0. \tag{5.226}$$

Furthermore, it is also possible to note that

$$\iota_{\mathbb{x}^\psi}\iota_{\mathbb{y}^A}\varpi_{YMS} = 0 \tag{5.227}$$
$$\iota_{\mathbb{x}^A}\iota_{\mathbb{y}^\psi}\varpi_{YMS} = 0 \tag{5.228}$$

for all brackets except $\{H_\lambda^{A,\psi}, H_\lambda^{A,\psi}\}$. In particular,

$$\iota_{\mathbb{h}^\psi}\iota_{\mathbb{h}^A}\varpi_{YMS} = \iota_{\mathbb{h}^\psi}\iota_{\mathbb{h}^A}\varpi = \int_\Sigma e\mathbb{h}_e^\psi\mathbb{h}_\omega^A = 0, \tag{5.229}$$

since $\mathbb{h}_e^\psi \sim \lambda$, $\mathbb{h}_\omega^\psi \sim \lambda$ and $\lambda^2 = 0$. Hence, we conclude that we have

$$\{X^{A,\psi}, Y^{A,\psi}\}_{YMS} = \{X + x^A, Y + y^A\}_A + \{X + x^\psi, Y + y^\psi\}_\psi - \{X, Y\} \tag{5.230}$$
$$+ \iota_{\mathbb{y}^{A,\psi}}\delta(X + x^A + x^\psi + x^{A,\psi}) \tag{5.231}$$
$$+ \iota_{\mathbb{x}^{A,\psi}}\delta(Y + y^A + y^\psi + y^{A,\psi}) \tag{5.232}$$
$$- \iota_{\mathbb{x}^{A,\psi}}\iota_{\mathbb{y}^{A,\psi}}\varpi_{YMS}. \tag{5.233}$$

Using this formula, we can compute the following brackets

$$\{M_\mu^{A,\psi}, M_\mu^{A,\psi}\}_{YMS} = \{M_\mu^A, M_\mu^A\}_A + 2\iota_{\mathbb{m}^{A,\psi}}\delta(M_\mu^A + m_\mu^{A,\psi}) \tag{5.234}$$
$$- \iota_{\mathbb{m}^{A,\psi}}\iota_{\mathbb{m}^{A,\psi}}\varpi_{YMS} \tag{5.235}$$
$$= \frac{1}{2}M_{[\mu,\mu]}^A + 2\int_\Sigma i\frac{e^3}{2\cdot 3!}\Big([\mu,[\mu,\overline{\psi}]]\gamma\psi - \overline{\psi}\gamma[\mu,[\mu,\psi]] \tag{5.236}$$
$$+ [\mu,\overline{\psi}]\gamma[\mu,\psi]\Big) + 2\int_\Sigma i\frac{e^3}{2\cdot 3!}[\mu,\overline{\psi}]\gamma[\mu,\psi] \tag{5.237}$$
$$- 2\int_\Sigma i\frac{e^3}{3!}[\mu,\overline{\psi}]\gamma[\mu,\psi] \tag{5.238}$$
$$= \frac{1}{2}M_{[\mu,\mu]}^A + \frac{1}{2}m_{[\mu,\mu]}^{A,\psi} = \frac{1}{2}M_{[\mu,\mu]}^{A,\psi}, \tag{5.239}$$

[22] Where we refer to $\mathbb{V}_{m^{A,\psi}}$ as an object in the kernel of e.

where we omit the terms that are zero.

Since $\mathbb{l}^{A,\psi} = 0$, we get

$$
\begin{aligned}
\{M_\mu^{A,\psi}, L_c^{A,\psi}\}_{YMS} &= \{M_\mu^A, L_c\}_A + \iota_{\mathbb{m}^{A,\psi}}\delta(L_c + l_c^\psi) && (5.240)\\
&= \int_\Sigma -i\frac{e^3}{2\cdot 3!}\Big(- [c,[\mu,\overline{\psi}]]\gamma\psi - [c,\overline{\psi}]\gamma[\mu,\psi] && (5.241)\\
&\qquad - [\mu,\overline{\psi}]\gamma[c,\psi] + \overline{\psi}\gamma[c,[\mu,\psi]]\Big) && (5.242)\\
&= 0, && (5.243)
\end{aligned}
$$

where we used that $[c,[\mu,\overline{\psi}]] = [\mu,[c,\overline{\psi}]]$ and the two identities

$$
\begin{aligned}
[\mu, \overline{\psi}\gamma\psi] &= 0 && (5.244)\\
[\mu,\overline{\psi}]\gamma\psi - \overline{\psi}\gamma[\mu,\psi] &= 2[\mu,\overline{\psi}]\gamma\psi. && (5.245)
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
\{M_\mu^{A,\psi}, P_\xi^{A,\psi}\}_{YMS} &= \{M_\mu^A, P_\xi + p_\xi^A\}_A + \iota_{\mathbb{p}^{A,\psi}}\delta(M_\mu^A + m_\mu^{A,\psi}) && (5.246)\\
&\quad + \iota_{\mathbb{m}^{A,\psi}}\delta(P_\xi + p_\xi^A + p_\xi^\psi + p_\xi^{A,\psi}) && (5.247)\\
&\quad - \iota_{\mathbb{m}^{A,\psi}}\iota_{\mathbb{p}^{A,\psi}}\varpi_{YMS} && (5.248)\\
&= M^A_{\mathrm{L}_\xi^{A_0}\mu} - \int_\Sigma i\frac{e^3}{3!}\Big([\mu,\overline{\psi}]\gamma[\iota_\xi A_0,\psi] + [\iota_\xi A_0,\overline{\psi}]\gamma[\mu,\psi]\Big) && (5.249)\\
&\quad - \int_\Sigma i\frac{e^3}{2\cdot 3!}\Big([\mu,\overline{\psi}]\gamma\mathrm{L}_\xi^{\omega_0+A_0}(\psi) - \overline{\psi}\gamma\mathrm{L}_\xi^{\omega_0+A_0}([\mu,\psi]) && (5.250)\\
&\qquad + \mathrm{L}_\xi^{\omega_0+A_0}([\mu,\overline{\psi}])\gamma\psi\Big) && (5.251)\\
&\quad + \int_\Sigma i\frac{e^3}{2\cdot 3!}\Big(\mathrm{L}_\xi^{\omega_0+A_0}(\overline{\psi})\gamma[\mu,\psi] + 2\Big([\mu,\overline{\psi}]\gamma[\iota_\xi A_0,\psi] && (5.252)\\
&\qquad + [\iota_\xi A_0,\overline{\psi}]\gamma[\mu,\psi]\Big)\Big) && (5.253)\\
&= M^A_{\mathrm{L}_\xi^{A_0}\mu} - \int_\Sigma i\frac{e^3}{2\cdot 3!}\Big(- \overline{\psi}\gamma[\mathrm{L}_\xi^{A_0}\mu,\psi] + [\mathrm{L}_\xi^{A_0}\mu,\overline{\psi}]\gamma\psi\Big) && (5.254)\\
&= M^A_{\mathrm{L}_\xi^{A_0}\mu} + m^{A,\psi}_{\mathrm{L}_\xi^{A_0}\mu} = M^{A,\psi}_{\mathrm{L}_\xi^{A_0}\mu}, && (5.255)
\end{aligned}
$$

where we used $\mathrm{L}_\xi^{\omega_0+A_0}\mu = \mathrm{L}_\xi^{A_0}\mu$ and $[\mu,\overline{\psi}]\gamma\mathrm{L}_\xi^{\omega_0+A_0}(\psi) = -\overline{\psi}\gamma[\mu,\mathrm{L}_\xi^{\omega_0+A_0}(\psi)]$.

Similarly, we get

$$
\begin{aligned}
\{M_\mu^{A,\psi}, H_\lambda^{A,\psi}\}_{YMS} &= \{M_\mu^A, H_\lambda^A\}_A + \iota_{\mathbb{h}^{A,\psi}}\delta(M_\mu^A + m_\mu^{A,\psi}) &(5.256)\\
&+ \iota_{\mathbb{m}^{A,\psi}}\delta(H_\lambda + h_\lambda^A + h_\lambda^\psi + h_\lambda^{A,\psi}) &(5.257)\\
&- \iota_{\mathbb{m}^{A,\psi}}\iota_{\mathbb{h}^{A,\psi}}\varpi_{YMS} &(5.258)\\
&= -\int_\Sigma \frac{i\lambda e_n e^2}{2}\Big(\overline{\psi}\gamma[d_A\mu,\psi][\mu,\overline{\psi}]\gamma[A,\psi] - [A,\overline{\psi}]\gamma[\mu,\psi] &(5.259)\\
&\quad + [\mu,\overline{\psi}]\gamma d_\omega\psi - \overline{\psi}\gamma d_\omega([\mu,\psi]) + d_\omega([\mu,\overline{\psi}])\gamma\psi &(5.260)\\
&\quad + d_\omega(\overline{\psi})\gamma[\mu,\psi]\Big)[\mu,\overline{\psi}]\gamma[A,\psi] - \overline{\psi}\gamma[A,[\mu,\psi]] &(5.261)\\
&\quad + [A,[\mu,\overline{\psi}]]\gamma\psi + [A,\overline{\psi}]\gamma[\mu,\psi] &(5.262)\\
&\quad - [\mu,\overline{\psi}]\gamma[A,\psi] - [A,\overline{\psi}]\gamma[\mu,\psi]\Big) &(5.263)\\
&= -\int_\Sigma \frac{i\lambda e_n e^2}{2}\Big(\overline{\psi}\gamma[d_A\mu,\psi] + [\mu,\overline{\psi}]\gamma d_{\omega+A}\psi &(5.264)\\
&\quad - \overline{\psi}\gamma d_{\omega+A}[\mu,\psi] + d_{\omega+A}[\mu,\overline{\psi}]\gamma\psi &(5.265)\\
&\quad + d_{\omega+A}\overline{\psi}\gamma[\mu,\psi]\Big) &(5.266)\\
&= -\int_\Sigma \frac{i\lambda e_n e^2}{2}\overline{\psi}\gamma[d_A\mu,\psi] + \int_\Sigma \frac{i\lambda e_n e^2}{2}\overline{\psi}\gamma[d_{A+\omega}\mu,\psi] &(5.267)\\
&= 0, &(5.268)
\end{aligned}
$$

where we used $d_{A+\omega}\mu = d_A\mu$.

Using again that $\mathbb{l}^{A,\psi} = 0$, we obtain

$$
\begin{aligned}
\{L_c^{A,\psi}, L_c^{A,\psi}\}_{YMS} &= \{L_c, L_c\}_A + \{L_c + l_c^\psi, L_c + l_c^\psi\}_\psi - \{L_c, L_c\} &(5.269)\\
&= \frac{1}{2}(L_{[c,c]} + l_{[c,c]}^\psi) = \frac{1}{2}L_{[c,c]}^{A,\psi}. &(5.270)
\end{aligned}
$$

Similarly,

$$\begin{aligned}
\{L_c^{A,\psi}, P_\xi^{A,\psi}\}_{YMS} &= \{L_c, P_\xi + p_\xi^A\}_A + \{L_c + l_c^\psi, P_\xi + p_\xi^\psi\}_\psi && (5.271)\\
&\quad - \{L_c, P_\xi\} + \iota_{\mathbb{p}^{A,\psi}}\delta(L_c + l_c^\psi) && (5.272)\\
&= L^A_{\mathrm{L}_\xi^{\omega_0}c} + L^\psi_{\mathrm{L}_\xi^{\omega_0}c} - L_{\mathrm{L}_\xi^{\omega_0}c} && (5.273)\\
&\quad + \int_\Sigma i\frac{e^3}{2\cdot 3!}\Big([c,[\iota_\xi A_0,\overline{\psi}]]\gamma\psi + [c,\overline{\psi}]\gamma[\iota_\xi A_0,\psi]\Big) && (5.274)\\
&\quad + \int_\Sigma i\frac{e^3}{2\cdot 3!}\Big(-[\iota_\xi A_0,\overline{\psi}]\gamma[c,\psi] - \overline{\psi}\gamma[c,[\iota_\xi A_0,\psi]]\Big) && (5.275)\\
&= L^{A,\psi}_{\mathrm{L}_\xi^{\omega_0}c} + \int_\Sigma i\frac{e^3}{2\cdot 3!}\Big(-[\iota_\xi A_0,[c,\overline{\psi}]]\gamma\psi + [c,\overline{\psi}]\gamma[\iota_\xi A_0,\psi]\Big) && (5.276)\\
&\quad + \int_\Sigma i\frac{e^3}{2\cdot 3!}\Big(-[\iota_\xi A_0,\overline{\psi}]\gamma[c,\psi] + \overline{\psi}\gamma[\iota_\xi A_0,[c,\psi]]\Big) && (5.277)\\
&= L^{A,\psi}_{\mathrm{L}_\xi^{\omega_0}c}, && (5.278)
\end{aligned}$$

where we used $[c,[\iota_\xi A_0,\overline{\psi}]] = -[\iota_\xi A_0,[c,\overline{\psi}]]$ and $[\iota_\xi A_0,[c,\overline{\psi}]]\gamma\psi = [c,\overline{\psi}]\gamma[\iota_\xi A_0,\psi]$.

We also have

$$\{L_c^{A,\psi}, H_\lambda^{A,\psi}\}_{YMS} = \{L_c, H_\lambda + h_\lambda^A\}_A + \{L_c + l_c^\psi, H_\lambda + h_\lambda^\psi\}_\psi \tag{5.279}$$

$$- \{L_c, H_\lambda\} + \iota_{\mathbb{h}^{A,\psi}}\delta(L_c + l_c^\psi) \tag{5.280}$$

$$= -P^A_{X^{(\nu)}} + L_{X^{(\nu)}(\omega-\omega_0)_\nu} - H^A_{X^{(n)}} \tag{5.281}$$

$$+ M^A_{X^{(\nu)}(A-A_0)_\nu} - P^\psi_{X^{(\nu)}} + L^\psi_{X^{(\nu)}(\omega-\omega_0)_\nu} \tag{5.282}$$

$$- H^\psi_{X^{(n)}} - P_{X^{(\nu)}} + L_{X^{(\nu)}(\omega-\omega_0)_\nu} - H_{X^{(n)}} \tag{5.283}$$

$$- \int_\Sigma \frac{i}{2}[\lambda e_n \overline{\psi}\gamma[A,\psi], e^2] \tag{5.284}$$

$$- \int_\Sigma \frac{i}{4}\Big(- [c, \lambda e_n e^2[A,\overline{\psi}]]\gamma\psi + [c,\overline{\psi}]\gamma\lambda e_n e^2[A,\psi]\Big) \tag{5.285}$$

$$+ \int_\Sigma \frac{i}{4}\Big(- \lambda e_n e^2[A,\overline{\psi}]\gamma[c,\psi] + \overline{\psi}\gamma[c, \lambda e_n e^2[A,\psi]]\Big) \tag{5.286}$$

$$= -P^A_{X^{(\nu)}} + L_{X^{(\nu)}(\omega-\omega_0)_\nu} - H^A_{X^{(n)}} \tag{5.287}$$

$$+ M^A_{X^{(\nu)}(A-A_0)_\nu} - P^\psi_{X^{(\nu)}} + L^\psi_{X^{(\nu)}(\omega-\omega_0)_\nu} \tag{5.288}$$

$$- H^\psi_{X^{(n)}} - P_{X^{(\nu)}} + L_{X^{(\nu)}(\omega-\omega_0)_\nu} - H_{X^{(n)}} \tag{5.289}$$

$$- \int_\Sigma [c, \lambda e_n]\Big[i\frac{e^2}{4}\Big(\overline{\psi}\gamma[A,\psi] - [A,\overline{\psi}]\gamma\psi\Big)\Big] \tag{5.290}$$

$$= -P^A_{X^{(\nu)}} + L_{X^{(\nu)}(\omega-\omega_0)_\nu} - H^A_{X^{(n)}} \tag{5.291}$$

$$+ M^A_{X^{(\nu)}(A-A_0)_\nu} - P^\psi_{X^{(\nu)}} + L^\psi_{X^{(\nu)}(\omega-\omega_0)_\nu} \tag{5.292}$$

$$- H^\psi_{X^{(n)}} - P_{X^{(\nu)}} + L_{X^{(\nu)}(\omega-\omega_0)_\nu} - H_{X^{(n)}} \tag{5.293}$$

$$- p^{A,\psi}_{X^{(\nu)}} - h^{A,\psi}_{X^{(n)}} + m^{A,\psi}_{X^{(\nu)}(A-A_0)_\nu} \tag{5.294}$$

$$= -P^{A,\psi}_{X^{(\nu)}} + L^{A,\psi}_{X^{(\nu)}(\omega-\omega_0)_\nu} - H^{A,\psi}_{X^{(n)}} + M^{A,\psi}_{X^{(\nu)}(A-A_0)_\nu}, \tag{5.295}$$

where, in the second last passage, we used that $[c, \lambda e_n] = X = X^{(\nu)}e_\nu + X^{(n)}e_n$ and that

$$-p^{A,\psi}_{X^{(\nu)}} + m^{A,\psi}_{X^{(\nu)}(A-A_0)_\nu} = - \int_\Sigma [c, \lambda e_n]^{(\nu)} e_\nu \Big[i\frac{e^2}{4}\Big(\overline{\psi}\gamma[A,\psi] - [A,\overline{\psi}]\gamma\psi\Big)\Big]. \tag{5.296}$$

Let us now consider

$$\{P_\xi^{A,\psi}, P_\xi^{A,\psi}\}_{YMS} = \{P_\xi + p_\xi^A, P_\xi + p_\xi^A\}_A + \{P_\xi + p_\xi^\psi, P_\xi + p_\xi^\psi\}_\psi - \{P_\xi, P_\xi\} \tag{5.297}$$

$$+ 2\iota_{\mathbb{p}^{A,\psi}}\delta(P_\xi + p_\xi^A + p_\xi^\psi + p_\xi^{A,\psi}) - \iota_{\mathbb{p}^{A,\psi}}\iota_{\mathbb{p}^{A,\psi}}\varpi_{YMS}. \tag{5.298}$$

Then, we have

$$\{P_\xi + p_\xi^A, P_\xi + p_\xi^A\}_A + \{P_\xi + p_\xi^\psi, P_\xi + p_\xi^\psi\}_\psi - \{P_\xi, P_\xi\} \tag{5.299}$$

$$= \frac{1}{2}\Big(P^A_{[\xi,\xi]} - L_{\iota_\xi\iota_\xi F_{\omega_0}} - M^A_{\iota_\xi\iota_\xi F_{A_0}} + P^\psi_{[\xi,\xi]} - L^\psi_{\iota_\xi\iota_\xi F_{\omega_0}} - P_{[\xi,\xi]} + L_{\iota_\xi\iota_\xi F_{\omega_0}}\Big) \tag{5.300}$$

and

$$
\begin{aligned}
&2\iota_{\mathbb{p}^{A,\psi}}\delta(P_\xi + p^A_\xi + p^\psi_\xi + p^{A,\psi}_\xi) - \iota_{\mathbb{p}^{A,\psi}}\iota_{\mathbb{p}^{A,\psi}}\varpi_{YMS} && (5.301)\\
&= -\int_\Sigma i\frac{e^3}{2\cdot 3!}\Big([\iota_\xi A_0, \overline{\psi}]\gamma \mathrm{L}^{\omega_0+A_0}_\xi(\psi) + \overline{\psi}\gamma\mathrm{L}^{\omega_0+A_0}_\xi([\iota_\xi A_0, \psi])\Big) && (5.302)\\
&\quad -\int_\Sigma i\frac{e^3}{2\cdot 3!}\Big(-\mathrm{L}^{\omega_0+A_0}_\xi(\overline{\psi})\gamma[\iota_\xi A_0, \psi] - \mathrm{L}^{\omega_0+A_0}_\xi([\iota_\xi A_0, \overline{\psi}])\gamma\psi\Big) && (5.303)\\
&= -\int_\Sigma i\frac{e^3}{2\cdot 3!}\Big(-\overline{\psi}\gamma[\iota_\xi A_0, \mathrm{L}^{\omega_0+A_0}_\xi(\psi)] + \overline{\psi}\gamma\mathrm{L}^{\omega_0+A_0}_\xi([\iota_\xi A_0, \psi])\Big) && (5.304)\\
&\quad -\int_\Sigma i\frac{e^3}{2\cdot 3!}\Big(-[\iota_\xi A_0, \mathrm{L}^{\omega_0+A_0}_\xi(\overline{\psi})]\gamma\psi - \mathrm{L}^{\omega_0+A_0}_\xi([\iota_\xi A_0, \overline{\psi}])\gamma\psi\Big) && (5.305)\\
&= \int_\Sigma -i\frac{e^3}{2\cdot 3!}\Big(-\overline{\psi}\gamma[\mathrm{L}^{\omega_0+A_0}_\xi(\iota_\xi A_0), \psi] - [\mathrm{L}^{\omega_0+A_0}_\xi(\iota_\xi A_0), \overline{\psi}]\gamma\psi\Big) && (5.306)\\
&= \int_\Sigma i\frac{e^3}{4\cdot 3!}\Big(\overline{\psi}\gamma\Big[\iota_{[\xi,\xi]}A_0 + \iota_\xi\iota_\xi F_{A_0}, \psi\Big] + \Big[\iota_{[\xi,\xi]}A_0 + \iota_\xi\iota_\xi F_{A_0}, \overline{\psi}\Big]\gamma\psi\Big) && (5.307)\\
&= \frac{1}{2}p^{A,\psi}_{[\xi,\xi]} - m^{A,\psi}_{\iota_\xi\iota_\xi F_{A_0}}, && (5.308)
\end{aligned}
$$

where we used that $\mathrm{L}^{\omega_0+A_0}_\xi\iota_\xi A_0 = \mathrm{L}^{A_0}_\xi\iota_\xi A_0 = \frac{1}{2}\iota_{[\xi,\xi]}A_0 + \frac{1}{2}\iota_\xi\iota_\xi F_{A_0}$.

Thus, we get

$$
\{P^{A,\psi}_\xi, P^{A,\psi}_\xi\}_{YMS} = \frac{1}{2}\Big(P^{A,\psi}_{[\xi,\xi]} - L^{A,\psi}_{\iota_\xi\iota_\xi F_{\omega_0}} - M^{A,\psi}_{\iota_\xi\iota_\xi F_{A_0}}\Big). \qquad (5.309)
$$

We then analyze

$$\{P_\xi^{A,\psi}, H_\lambda^{A,\psi}\}_{YMS} = \{P_\xi + p_\xi^A, H_\lambda + h_\lambda^A\}_A + \{P_\xi + p_\xi^\psi, H_\lambda + h_\lambda^\psi\}_\psi - \tag{5.310}$$

$$\{P_\xi, H_\lambda\}\iota_{\mathbb{h}^{A,\psi}}\delta(P_\xi + p_\xi^A + p_\xi^\psi + p_\xi^{A,\psi}) \tag{5.311}$$

$$+ \iota_{\mathbb{p}^{A,\psi}}\delta(H_\lambda + h_\lambda^A + h_\lambda^\psi + h_\lambda^{A,\psi}) - \iota_{\mathbb{p}^{A,\psi}}\iota_{\mathbb{h}^{A,\psi}}\varpi_{YMS} \tag{5.312}$$

$$= P^A_{Y^{(\nu)}} - L_{Y^{(\nu)}(\omega-\omega_0)_\nu} \tag{5.313}$$

$$+ H^A_{Y^{(n)}} - M^A_{Y^{(\nu)}(A-A_0)_\nu} + p^\psi_{Y^{(\nu)}} - l^\psi_{Y^{(\nu)}(\omega-\omega_0)_\nu} \tag{5.314}$$

$$+ h^\psi_{Y^{(n)}} - \int_\Sigma \frac{i\lambda e_n \mathrm{L}_\xi^{\omega_0} e^2}{4}\Big(\overline{\psi}\gamma[A,\psi] - [A,\overline{\psi}])\gamma\psi\Big) \tag{5.315}$$

$$- \int_\Sigma \frac{i\lambda e_n e^2}{2}\Big(\overline{\psi}\gamma[\iota_\xi F_{\omega_0} + \mathrm{L}_\xi^{A_0}(A - A_0), \psi] \tag{5.316}$$

$$- [\iota_\xi F_{\omega_0} + \mathrm{L}_\xi^{A_0}(A - A_0), \overline{\psi}]\gamma\psi\Big) \tag{5.317}$$

$$- \int_\Sigma \frac{i\lambda e_n e^2}{2}\Big([A,\overline{\psi}]\gamma \mathrm{L}_\xi^{\omega_0+A_0}\psi - \mathrm{L}_\xi^{\omega_0+A_0}\overline{\psi}\gamma[A,\psi]\Big) \tag{5.318}$$

$$- \int_\Sigma \frac{i\lambda e_n e^2}{4}\Big([\iota_\xi A_0, \overline{\psi}]\gamma d_{\omega+A}\psi - \overline{\psi}\gamma d_{\omega+A}[\iota_\xi A_0, \psi] \tag{5.319}$$

$$+ d_{\omega+A}[\iota_\xi A_0, \overline{\psi}]\gamma\psi\Big) \tag{5.320}$$

$$- \int_\Sigma \frac{i\lambda e_n e^2}{4}\Big(d_{\omega+A}\overline{\psi}\gamma[\iota_\xi A_0, \psi] + 2[A,\overline{\psi}]\gamma[\iota_\xi A_0, \psi] \tag{5.321}$$

$$+ 2[\iota_\xi A_0, \overline{\psi}]\gamma[A,\psi]\Big). \tag{5.322}$$

Hence, we get

$$\{P_\xi^{A,\psi}, H_\lambda^{A,\psi}\}_{YMS} = P^A_{Y^{(\nu)}} - L_{Y^{(\nu)}(\omega-\omega_0)_\nu} + H^A_{Y^{(n)}} \tag{5.323}$$

$$- M^A_{Y^{(\nu)}(A-A_0)_\nu} + p^\psi_{Y^{(\nu)}} - l^\psi_{Y^{(\nu)}(\omega-\omega_0)_\nu} \tag{5.324}$$

$$+ h^\psi_{Y^{(n)}} - \int_\Sigma \frac{i\mathrm{L}_\xi^{\omega_0}(\lambda e_n)e^2}{4}\Big(\overline{\psi}\gamma[A,\psi] - [A,\overline{\psi}])\gamma\psi\Big) \tag{5.325}$$

$$= P^A_{Y^{(\nu)}} - L_{Y^{(\nu)}(\omega-\omega_0)_\nu} + H^A_{Y^{(n)}} \tag{5.326}$$

$$- M^A_{Y^{(\nu)}(A-A_0)_\nu} + p^\psi_{Y^{(\nu)}} - l^\psi_{Y^{(\nu)}(\omega-\omega_0)_\nu} \tag{5.327}$$

$$+ h^\psi_{Y^{(n)}} + p^{A,\psi}_{Y^{(\nu)}} + h^{A,\psi}_{Y^{(n)}} - m^{A,\psi}_{Y^{(\nu)}(A-A_0)_\nu} \tag{5.328}$$

$$= P^{A,\psi}_{Y^{(\nu)}} - L^{A,\psi}_{Y^{(\nu)}(\omega-\omega_0)_\nu} + H^{A,\psi}_{Y^{(n)}} - M^{A,\psi}_{Y^{(\nu)}(A-A_0)_\nu} \tag{5.329}$$

Furthermore, we compute

$$\{H_\lambda^{A,\psi}, H_\lambda^{A,\psi}\}_{YMS} = \{H_\lambda + h_\lambda^A, H_\lambda + h_\lambda^A\}_A + \{H_\lambda + h_\lambda^\psi, H_\lambda + h_\lambda^\psi\}_\psi \tag{5.330}$$
$$- \{H_\lambda, H_\lambda\} + \iota_{\mathbb{h}^{A,\psi}} \delta(H_\lambda + h_\lambda^A + h_\lambda^\psi + h_\lambda^{A,\psi}) \tag{5.331}$$
$$+ \iota_{\mathbb{h}^{A,\psi}} \delta(H_\lambda + h_\lambda^A + h_\lambda^\psi + h_\lambda^{A,\psi}) - \iota_{\mathbb{h}^{A,\psi}} \iota_{\mathbb{h}^{A,\psi}} \varpi_{YMS} \tag{5.332}$$
$$= 0, \tag{5.333}$$

because all the components of the Hamiltonian vector field of $H_\lambda^{A,\psi}$ are proportional to λ, and $\lambda^2 = 0$.

To complete the proof, observe that all brackets involving R_τ have either been computed in the previous sections, or vanish due to the fact that R_τ has no Hamiltonian vector field along the spinor directions, or because $e_n\tau = 0$. □

Corollary 5.4.4. *If the boundary metric i^*g is non-degenerate, then the functionals in Definition 5.4.2 define a coisotropic submanifold.*

5.5 Yang–Mills-Higgs

The following case consists of a scalar field and a Yang–Mills field both coupled to gravity, with the addition of a Higgs-type potential.

Definition 5.5.1 (Higgs field)**.** Let $P_{\mathrm{SU}(n)}$ be a $\mathrm{SU}(n)$-principal bundle over M, with the fundamental representation given by $n\colon \mathrm{SU}(n) \to \mathrm{End}(\mathbb{C}^n)$ and its conjugate one $\bar{n}$ taken with respect to the canonical hermitian structure on $\mathbb{C}^n$. Then, we define the Higgs field ϕ (a complex scalar multiplet) as a section of the associated vector bundle $E_n := P_{\mathrm{SU}(n)} \times_n \mathbb{C}^n$, whereas, respectively, $\phi^\dagger$ is a section of $E_{\bar{n}} := P_{\mathrm{SU}(n)} \times_{\bar{n}} \mathbb{C}^n$.
We also introduce the field[23] $\Pi \in \Gamma(M, \mathcal{V} \otimes E_n) =: \Omega^{0,1}(E_n)$ and its conjugate $\Pi^\dagger \in \Omega^{(0,1)}(E_{\bar{n}})$.

Remark 5.5.2. In the remainder of this section, we will identify (sections of) the Lie algebra $\mathfrak{su}(n)$ with (sections of) the algebra of hermitian traceless matrices over $\mathbb{C}^n$, i.e.

$$\Gamma(M, \mathfrak{su}(n)) \simeq \Gamma(M, (E_n \otimes E_{\bar{n}})_{\mathrm{t,h}}) =: \Gamma(M, (E_n \otimes E_{\bar{n}})').$$

Furthermore, we will consider ϕ and Π to be such that the total degrees are $|\phi| = 0$ and $|\Pi| = 1$.

[23] Which turns out to be the associated momentum.

Remark 5.5.3. The canonical hermitian product on $\mathbb{C}^n$ induces a hermitian product on E_n (and hence on $\Gamma(E_n)$). We *symmetrize* it (i.e. add its complex conjugate) to account for the reality requirement

$$
\begin{aligned}
<\cdot,\cdot>\colon E_n \times E_n &\longrightarrow \mathbb{C} && (5.334)\\
(\,\phi\,,\,\varphi\,) &\longmapsto <\phi,\varphi> := \frac{1}{2}(\phi^\dagger\varphi + (-1)^{|\phi||\varphi|}\varphi^\dagger\phi). && (5.335)
\end{aligned}
$$

Furthermore, we denote the full interior product on $E_n \otimes \mathcal{V}$ by

$$
\begin{aligned}
(<\cdot,\cdot>)\colon (E_n \otimes \mathcal{V})^2 &\longrightarrow \mathbb{C} && (5.336)\\
(\Pi, \epsilon) &\longmapsto\; (<\Pi,\epsilon>) := \frac{1}{2}\eta_{ab}(\Pi_i^{\dagger a}\epsilon^{b,i} + \text{c.c}). && (5.337)
\end{aligned}
$$

Definition 5.5.4. Given $\alpha \in \Gamma(M, (E_n \otimes E_{\bar{n}})')$, we set $d_\alpha\phi := d\phi + [A,\phi]$ and $d_\alpha\phi^\dagger := d\phi^\dagger + [\alpha,\phi^\dagger]$, which in coordinates we read[24]

$$
\begin{aligned}
[\alpha,\phi]^i &:= ig_H(\alpha\phi)^i = ig\alpha^i_j\phi^j; && (5.338)\\
[\alpha,\phi^\dagger]_i &= -(-1)^{|\alpha|(|\phi|+1)} ig_H\phi^\dagger_j\alpha^j_i, && (5.339)
\end{aligned}
$$

where g_H is a constant related to the representation of $\mathrm{SU}(n)$.

The space of fields of the spinor Yang–Mills-Higgs Palatini–Cartan theory is given by

$$\mathcal{F}_{YMH} = \mathcal{F}_{YM} \times \Gamma(M, E_n) \times \Gamma(M, E_{\bar{n}}) \times \Omega^{0,1}(E_n) \times \Omega^{0,1}(E_{\bar{n}}), \tag{5.340}$$

whereas the action functional reads

$$S_{YMS} = S + S_H + S_A + S_{A,\psi}, \tag{5.341}$$

where

$$
\begin{aligned}
S_H &= \int_M \frac{e^3}{3!} <\Pi, d_A\phi> + \frac{e^4}{2\cdot 4!}(<\Pi,\Pi>) - \frac{q_H}{4\cdot 4!}e^4(<\phi,\phi> - v^2)^2 && (5.342)\\
S_{A,\psi} &= \int_M \frac{e^{N-1}}{2(N-1)!}\Big(\overline{\psi}\gamma[A,\psi] - [A,\overline{\psi}]\gamma\psi\Big), && (5.343)
\end{aligned}
$$

with $\overline{\psi}\gamma[A,\psi] = ig_i\overline{\psi}_I\gamma A^I_J\psi^J$ and q_H, v and g_i constants.

---

[24]the action of $\mathfrak{su}(n)$ on $\phi^\dagger$ is defined by requiring that $[\alpha, <\phi,\phi>] = 0$, since $<\phi,\phi>$ is assumed to be a $SU(n)$ scalar.

Remark 5.5.5. Notice that the first terms of S_H are formally equivalent to S_ϕ of Section 5.1, after substituting $d\phi \to d_A\phi$ and setting ϕ to a $\mathrm{SU}(n)$ multiplet. Then one can easily show that in this case the structural constraint reads

$$d_A\phi - (e, \Pi) = 0. \tag{5.344}$$

As a consequence the interaction between the YM field and the Higgs field is contained in S_H.

Alike the preceding section, the interaction term does not contain derivatives. Hence, the boundary structure is just the direct sum of the Yang–Mills and the (multiplet) scalar structures. In particular, it follows the following result.

Theorem 5.5.6. *The geometric phase space of the Yang–Mills-Higgs Palatini–Cartan theory is symplectomorphic to the space* $(\mathcal{F}^\partial_{YMH}, \varpi^\partial_{YMH})$*, where*

$$\mathcal{F}^\partial_{YMH} = \mathcal{F}^\partial_{YM} \times \Gamma(\Sigma, E_n|_\Sigma) \times \Gamma(\Sigma, E_{\bar{n}}|_\Sigma) \times \Omega^{0,1}_\Sigma(E_n|_\Sigma) \times \Omega^{0,1}_\Sigma(E_{\bar{n}}|_\Sigma) \tag{5.345}$$

with $(e, \omega, A, B, \phi, \phi^\dagger, \Pi, \Pi^\dagger) \in \mathcal{F}^\partial_{YMH}$ *satisfying*

$$\begin{cases} e_n(d_\omega e - p_\mathcal{T}(d_\omega e)) \in \operatorname{Im} W_1^{\Sigma,(1,1)} \\ p_\mathcal{K}\omega = 0 \\ d_A\phi - (e, \Pi) = 0 \\ p_W\Pi = 0 \\ F_A + \frac{1}{2}(e^2, B) = 0 \\ p_{\Omega_e^{0,1*}\wedge W} B = 0, \end{cases} \tag{5.346}$$

as defined in Lemma 4.2.19, Lemma 5.1.2, Lemma 5.2.3 and Remark 5.3.14, and where the corresponding symplectic form on $\mathcal{F}^\partial_{YMH}$ *is given by*

$$\varpi_{YMH} = \varpi + \varpi_A + \varpi_H, \tag{5.347}$$

where

$$\varpi_H = \int_\Sigma < \delta p, \delta\phi >, \tag{5.348}$$

with $p := \frac{e^3}{3!}\Pi$.

Before moving on to the constraint analysis, we provide some useful identities.

Lemma 5.5.7. *Let* $\phi, \varphi \in \Gamma(E_n)$*,* $\alpha \in \Gamma(\mathfrak{su}(n))$*. Then, we have*

$$< \alpha\varphi, \phi > = (-1)^{|\alpha||\varphi|} < \varphi, \alpha\phi > \tag{5.349}$$

$$< \varphi, [\alpha, \phi] > = \frac{ig_H}{2} Tr[(-1)^{|\alpha||\varphi|}\varphi\phi^\dagger - (-1)^{|\phi|(|\alpha|+|\varphi|)}\phi\varphi^\dagger)\alpha] \tag{5.350}$$

$$[L_\xi^{A_0}, d_A]\phi = L_\xi^{A_0} d_A\phi + d_A L_\xi^{A_0}\phi = [\iota_\xi F_{A_0} + L_\xi^{A_0}(A - A_0), \phi]. \tag{5.351}$$

Definition 5.5.8. Let $c \in \Omega_\Sigma^{0,2}[1]$, $\mu \in C^\infty(\Sigma, \mathfrak{g})[1]$, $\xi \in \mathfrak{X}(\Sigma)[1]$, $\lambda \in C^\infty(\Sigma)[1]$, and $\tau \in \mathcal{S}[1]$, and the other functionals be as in Definition 5.4.2. Then, we define the following functionals

$$L_c^{H,A} = L_c \tag{5.352}$$
$$P_\xi^{H,A} = P_\xi + p_\xi^A + p_\xi^H + p_\xi^{H,A} \tag{5.353}$$
$$H_\lambda^{H,A} = H_\lambda + h_\lambda^A + h_\lambda^H + h_\lambda^{V_H} + h_\lambda^{H,A} \tag{5.354}$$
$$M_\mu^{H,A} = M_\mu^A + m_\mu^{A,\psi} \tag{5.355}$$
$$R_\tau^{H,A} = R_\tau, \tag{5.356}$$

where

$$l_c^H = L_c^{\phi\to H} - L_c \tag{5.357}$$
$$p_\xi^H = P_\xi^{\phi\to H} - P_\xi \tag{5.358}$$
$$h_\lambda^H = H_\lambda^{\phi\to H} - H_\lambda, \tag{5.359}$$

with $\phi \to H$ denoting the formal substitution[25] of the scalar field of Section 5.1 with the Higgs field and

$$m_\mu^{H,A} = \int_\Sigma \frac{i}{2} g_H \mathrm{Tr}[\mu(\phi p^\dagger - p\phi^\dagger)] = \int_\Sigma < p, [\mu, \phi] > \tag{5.360}$$
$$p_\xi^{H,A} = \int_\Sigma - < p, \iota_\xi [A_0, \phi] > \tag{5.361}$$
$$h_\lambda^{V_H} = -\int_\Sigma \lambda e_n \frac{q_H}{24} e^3 (< \phi, \phi > - v^2)^2 = \int_\Sigma \lambda e_n \frac{e^3}{3!} V_H \tag{5.362}$$
$$h_\lambda^{H,A} = \int_\Sigma \lambda e_n \frac{e^2}{2} < \Pi, [A, \phi] > . \tag{5.363}$$

We refer to these as the constraints of the Yang–Mills-Higgs Palatini–Cartan theory.

Theorem 5.5.9. *The Poisson brackets of the constraints of Definition 5.5.8 read*

$$\{L_c^{H,A}, L_c^{H,A}\}_{YMH} = -\frac{1}{2} L_{[c,c]}^{H,A} \qquad \{P_\xi^{H,A}, P_\xi^{H,A}\}_{YMH} = \frac{1}{2} P_{[\xi,\xi]}^{H,A} - \frac{1}{2} L_{\iota_\xi \iota_\xi F_{\omega_0}}^{H,A}$$
$$\{L_c^{H,A}, P_\xi^{H,A}\}_{YMH} = L_{\mathcal{L}_\xi^{\omega_0} c}^{H,A} \qquad \{H_\lambda^{H,A}, H_\lambda^{H,A}\}_{YMH} \approx 0$$
$$\{L_c^{H,A}, R_\tau\}_{YMH} = -R_{p_\mathcal{S}[c,\tau]} \qquad \{R_\tau^{H,A}, P_\xi^{H,A}\}_{YMH} = R_{p_\mathcal{S} \mathcal{L}_\xi^{\omega_0} \tau}$$
$$\{R_\tau, H_\lambda^{H,A}\}_{YMH} \approx G_{\lambda\tau} + K_{\lambda\tau}^A \qquad \{R_\tau, R_\tau\}_{YMH} \approx F_{\tau\tau}$$
$$\{L_c^{H,A}, H_\lambda^{H,A}\}_{YMH} = -P_{X^{(a)}}^{H,A} + L_{X^{(a)}(\omega-\omega_0)_a}^{H,A} - H_{X^{(n)}}^{H,A}$$
$$\{P_\xi^{H,A}, H_\lambda^{H,A}\}_{YMH} = P_{Y^{(a)}}^{H,A} - L_{Y^{(a)}(\omega-\omega_0)_a}^{H,A} + H_{Y^{(n)}}^{H,A},$$

[25]This means that we are considering the natural pairing $<,>$ when needed.

with $X = [c, \lambda e_n]$, $Y = \mathcal{L}_\xi^{\omega_0}(\lambda e_n)$ and where the superscripts (a) and (n) describe their components with respect to e_a, e_n. Furthermore, $F_{\tau\tau}$, $G_{\lambda\tau}$ and $K^A_{\lambda\tau}$ are functionals of e, ω, A, B, τ and λ defined in Theorem 5.2.6 which are not proportional to any other constraint.

Proof. Once again, we use the results contained in the preceding sections for the components of the Hamiltonian vector fields. In particular, we have, upon the usual formal substitution, $\varkappa^H = \varkappa^\phi$. The residual components are computed using the results in Appendix B. We start with $M_\mu^{H,A}$. One can quite easily see that

$$\mathbb{m}_\phi^{H,A} = [\mu, \phi], \qquad \mathbb{m}_p^{H,A} = [\mu, p].$$

Now, since $l_c^{H,A} = 0$ and $\iota_{\mathbb{l}^\phi}\varpi_A = \iota_{\mathbb{l}^A}\varpi_H = 0$, one finds

$$\mathbb{l}^{H,A} = 0.$$

For $P_\xi^{H,A}$, we find

$$\delta p_\xi^{A,H} = \int_\Sigma - < \delta p, \iota_\xi[A_o, \phi] > + < [A_0, \iota_\xi p], \delta\phi >= \underbrace{\iota_{\mathbb{p}^\phi}\varpi_A}_{0} + \underbrace{\iota_{\mathbb{p}^A}\varpi_H}_{0} + \iota_{\mathbb{p}^{H,A}}\varpi_{YMH},$$

finding

$$\mathbb{p}_\phi^{H,A} = -\iota_\xi[A_0, \phi], \qquad \mathbb{p}_p^{H,A} = [A_0, \iota_\xi p].$$

All the other components of $\mathbb{p}^{H,A}$ vanish.

Regarding $H_\lambda^{H,A}$, we find that $\mathbb{h}^H = \mathbb{h}^\phi$ as expected, except for the components that inherit the Higgs potential term:

$$\mathbb{h}_p^H = d_\omega\Big(\frac{\lambda e_n}{2} e^2 \Pi\Big) + \frac{\lambda e_n}{2 \cdot 3!} q_h e^3 (< \phi, \phi > - v^2)\phi; \tag{5.364}$$

$$e\mathbb{h}_\omega^H = \lambda e_n \Big(e < \Pi, d\phi > + \frac{e^2}{4} < (\Pi, \Pi) > + \frac{e^2}{2} V_H\Big) - \frac{\lambda}{2} e^2 \Pi(\Pi, e_n) + \mathbb{V}_{h^H}. \tag{5.365}$$

For $h_\lambda^{H,A}$, we obtain

$$\delta h_\lambda^{H,A} = \int_\Sigma \lambda e_n \Big[e < \Pi, [A, \phi] > \delta e + \frac{e^2}{2}\Big(< \delta\Pi, [A, \phi] > \tag{5.366}$$

$$+ < [A, \Pi], \delta\phi > \Big)\Big] + \int_\Sigma \lambda e_n \frac{i g_H e^2}{4} \mathrm{Tr}[(\Pi\phi^\dagger - \phi\Pi^\dagger)\delta A] \tag{5.367}$$

$$= \underbrace{\iota_{\mathbb{h}^\phi}\varpi_A}_{0} + \underbrace{\iota_{\mathbb{h}^A}\varpi_H}_{0} + \iota_{\mathbb{h}^{H,A}}\varpi_{YMH}, \tag{5.368}$$

hence finding

$$\mathbb{h}_e^{A,H} = 0 \qquad e\mathbb{h}_\omega^{A,H} = -\frac{\lambda e_n}{2} e < \Pi, [A, \phi] > \tag{5.369}$$

$$\frac{e^3}{3!}\mathbb{h}_\phi^{A,H} = \frac{\lambda e_n}{2} e^2 [A.\phi] \qquad \frac{e^3}{3!}\mathbb{h}_\Pi^{A,H} = \frac{\lambda e_n}{2} e^2 [A, \Pi] \tag{5.370}$$

$$\mathbb{h}_A^{A,H} = 0 \qquad \mathbb{h}_\rho^{A,H} = \frac{ig_H}{4} \lambda e_n e^2 \mathrm{Tr}(\Pi\phi^\dagger - \phi\Pi^\dagger) \tag{5.371}$$

For the following computations, we use Appendix B. As before, we notice that, for all constraints, it holds

$$\iota_{\mathbb{X}+\mathbb{x}^H} \iota_{\mathbb{Y}+\mathbb{y}^H} \varpi_A + \iota_{\mathbb{X}+\mathbb{x}^A} \iota_{\mathbb{Y}+\mathbb{y}^A} \varpi_H = 0, \tag{5.372}$$

$$\iota_{\mathbb{X}} \iota_{\mathbb{Y}} \varpi_A = 0, \tag{5.373}$$

$$\iota_{\mathbb{X}} \iota_{\mathbb{Y}} \varpi_H = 0, \tag{5.374}$$

$$\iota_{\mathbb{x}^H} \iota_{\mathbb{y}^A} \varpi_{YMH} = 0, \tag{5.375}$$

$$\iota_{\mathbb{x}^A} \iota_{\mathbb{y}^H} \varpi_{YMH} = 0. \tag{5.376}$$

Hence, we can use

$$\{X^{H,A}, Y^{H,A}\}_{YMH} = \{X + x^A, Y + y^A\}_A + \{X + x^H, Y + y^H\}_H \tag{5.377}$$

$$- \{X, Y\} + \iota_{\mathbb{y}^{H,A}} \delta(X + x^A + x^H + x^{H,A}) \tag{5.378}$$

$$+ \iota_{\mathbb{x}^{H,A}} \delta(Y + y^A + y^H + y^{H,A}) - \iota_{\mathbb{x}^{H,A}} \iota_{\mathbb{y}^{H,A}} \varpi_{YMH}. \tag{5.379}$$

Applying it, we get

$$\{M_\mu^{A,H}, M_\mu^{A,H}\}_{YMH} = \{M_\mu^A, M_\mu^A\}_A + 2\iota_{\mathbb{m}^{H,A}} \delta(M_\mu^A + m_\mu^{H,A}) \tag{5.380}$$

$$- \iota_{\mathbb{m}^{H,A}} \iota_{\mathbb{m}^{H,A}} \varpi_{YMH} \tag{5.381}$$

$$= \frac{1}{2} M_{[\mu,\mu]}^A + \int_\Sigma < [\mu, p], [\mu, \phi] > + < p, [\mu, [\mu, \phi]] > \tag{5.382}$$

$$- \int_\Sigma < [\mu, p], [\mu, \phi] > \tag{5.383}$$

$$= \frac{1}{2} M_{[\mu,\mu]}^A + \frac{1}{2} \int_\Sigma < p, [[\mu, \mu], \phi] > \tag{5.384}$$

$$= \frac{1}{2} M_{[\mu,\mu]}^{H,A}, \tag{5.385}$$

$$\{L_c^{A,H}, M_\mu^{A,H}\}_{YMH} = \{M_\mu^A, L_c\}_A + \iota_{\mathbb{m}^{H,A}} \delta L_c = 0, \tag{5.386}$$

and

$$\{M_\mu^{H,A}, P_\xi^{H,A}\}_{YMH} = \{M_\mu^A, P_\xi + p_\xi^A\}_A + \iota_{\mathbb{p}^{H,A}}\delta(M_\mu^A + m_\mu^{H,A}) \tag{5.387}$$

$$- \iota_{\mathbb{m}^{H,A}}\iota_{\mathbb{p}^{H,A}}\varpi_{YMH} + \iota_{\mathbb{M}^A}\iota_{\mathbb{P}+\mathbb{p}^A}\varpi_H + \iota_{\mathbb{M}^A}\iota_{\mathbb{p}^H}\varpi_{YMH} \tag{5.388}$$

$$= M^A_{\mathrm{L}^{A_0}\mu} \tag{5.389}$$

$$+ \int_\Sigma \Big(< [A_0, \iota_\xi p], [\mu, \phi] > + < p, [\mu, [\iota_\xi A_0, \phi]] > \Big) \tag{5.390}$$

$$+ \int_\Sigma \Big(- < [\mu, p], \mathrm{L}_\xi^{\omega_0 + A_0}\phi > - < p, \mathrm{L}_\xi^{\omega_0+A_0}[\mu, \phi] > \Big) \tag{5.391}$$

$$- \int_\Sigma \Big(< [\mu, p], [\iota_\xi A_0, \phi] > + < [A_0, \iota_\xi p], [\mu, \phi] > \Big) \tag{5.392}$$

$$= M^A_{\mathrm{L}_\xi^{A_0}\mu} - \int_\Sigma < p, [\mathrm{L}_\xi^{A_0}\mu, \phi] >= M^{H,A}_{\mathrm{L}_\xi^{A_0}\mu}, \tag{5.393}$$

where we noticed that in the second step the first and third lines cancel out and where we also used

$$- < [\mu, p], \mathrm{L}_\xi^{\omega_0+A_0}\phi >=< p, [\mu, \mathrm{L}_\xi^{\omega_0+A_0}\phi] > . \tag{5.394}$$

Then, we compute

$$\{M_\mu^{H,A}, H_\lambda^{H,A}\}_{YMH} = \{M_\mu^A, H_\lambda^A\}_A + \iota_{\mathbb{h}^{H,A}}\delta(M_\mu^A + m_\mu^{H,A}) \tag{5.395}$$

$$+ \iota_{\mathbb{m}^{H,A}}\delta(H_\lambda + h_\lambda^A + h_\lambda^H + h_\lambda^{H,A}) - \iota_{\mathbb{m}^{H,A}}\iota_{\mathbb{h}^{H,A}}\varpi_{YMH} \tag{5.396}$$

$$= \int_\Sigma \frac{\lambda e_n e^2}{2}\Big(< \Pi, [d_A\mu, \phi] > + < [A, \Pi], [\mu, \phi] > \tag{5.397}$$

$$+ < \Pi, [\mu, [A, \phi]] > \Big) \tag{5.398}$$

$$- \int_\Sigma \frac{\lambda e_n e^2}{2}\Big(< [\mu, \Pi], d\phi > + < \Pi, d[\mu, \phi] > \Big) \tag{5.399}$$

$$+ \frac{\lambda e_n e^3}{3} < [\mu, \Pi], \Pi > \tag{5.400}$$

$$+ \int_\Sigma \frac{\lambda e_n e^3 q_H}{3}(< \phi, \phi > - v^2) < [\mu, \phi], \phi > \tag{5.401}$$

$$+ \int_\Sigma \frac{\lambda e_n e^2}{2}\Big(< [\mu, \Pi], [A, \phi] > + < \Pi, [A, [\mu, \phi]] > \Big) \tag{5.402}$$

$$- \int_\Sigma \frac{\lambda e_n e^2}{2}\Big(< [\mu, \Pi], [A, \phi] > + < [A, \Pi], [\mu, \phi] > \Big) \tag{5.403}$$

$$= 0, \tag{5.404}$$

having used the fact that $[\mu, < \phi, \phi >] = [\mu, < \Pi, \Pi >] = 0$ and the Jacobi identity for A, μ and ϕ. We also have

$$\{L_c^{A,H}, L_c^{A,H}\}_{YMH} = \{L_c^H, L_c^H\}_H + \{L_c^A, L_c^A\}_A - \{L_c, L_c\} \tag{5.405}$$
$$= -\frac{1}{2}L_{[c,c]} = -\frac{1}{2}L_{[c,c]}^{A,H} \tag{5.406}$$

and

$$\{L_c^{A,H}, P_\xi^{H,A}\}_{YMH} = \{L_c, P_\xi + p_\xi^H\}_H + \{L_c, P_\xi + p_\xi^A\}_A \tag{5.407}$$
$$- \{L_c, P_\xi\} + \iota_{\mathbb{p}^{H,A}}\delta L_c \tag{5.408}$$
$$= L^{H,A}_{\mathrm{L}^{\omega_0} c}. \tag{5.409}$$

Furthermore, we have

$$\iota_{\mathbb{h}^{H,A}}\delta L_c = \int_\Sigma c\Big[\frac{\lambda e_n}{4} < \Pi, [A,\phi] >, e^2\Big] = \int_\Sigma \Big[c, \lambda e_n\Big]\frac{e^2}{4} < \Pi, [A,\phi] >, \tag{5.410}$$

and hence, using $[c, \lambda e_n] = X = X^{(\nu)}e_\nu + X^{(n)}e_n$ and

$$-p^{H,A}_{X^{(\nu)}} + m^{H,A}_{X^{(\nu)}(A-A_0)_\nu} = \int_\Sigma \Big[c, \lambda e_n\Big]^{(\nu)} e_\nu \frac{e^2}{4} < \Pi, [A,\phi] >, \tag{5.411}$$

we get

$$\{L_c^{A,H}, H_\lambda^{H,A}\}_{YMH} = \{L_c, H_\lambda + h_\lambda^A\}_A + \{L_c, H_\lambda + h_\lambda^H\}_H \tag{5.412}$$
$$- \{L_c, H_\lambda\} + \iota_{\mathbb{h}^{H,A}}\delta L_c \tag{5.413}$$
$$= -P^A_{X^{(\nu)}} + L^A_{X^{(\nu)}(\omega-\omega_0)_\nu} - H^A_{X^{(n)}} \tag{5.414}$$
$$+ M^A_{X^\nu(A-A_0)_\nu} - P^H_{X^{(\nu)}} + L^H_{X^{(\nu)}(\omega-\omega_0)_\nu} \tag{5.415}$$
$$- H^H_{X^{(n)}} + P_{X^{(\nu)}} - L_{X^{(\nu)}(\omega-\omega_0)_\nu} \tag{5.416}$$
$$+ H_{X^{(n)}} - p^{H,A}_{X^{(\nu)}} - h^{H,A}_{X^{(n)}} + m^{H,A}_{X^{(\nu)}(A-A_0)_\nu} \tag{5.417}$$
$$= -P^{H,A}_{X^{(\nu)}} + L^{H,A}_{X^{(\nu)}(\omega-\omega_0)_\nu} - H^{H,A}_{X^{(n)}} + M^{H,A}_{X^\nu(A-A_0)_\nu}. \tag{5.418}$$

We compute

$$\begin{aligned}
\{P_\xi^{H,A}, P_\xi^{H,A}\}_{YMH} &= \{P_\xi + p_\xi^A, P_\xi + p_\xi^A\}_A + \{P_\xi + p_\xi^H, P_\xi + p_\xi^H\}_H && (5.419)\\
&- \{P_\xi, P_\xi\} + 2\iota_{\mathbb{P}^{H,A}}\delta(P_\xi + p_\xi^A + p_\xi^H + p_\xi^{H,A}) && (5.420)\\
&- \iota_{\mathbb{P}^{H,A}}\iota_{\mathbb{P}^{H,A}}\varpi_{YMH} && (5.421)\\
&= \frac{1}{2}P^A_{[\xi,\xi]} - \frac{1}{2}L^A_{\iota_\xi\iota_\xi F_{\omega_0}} && (5.422)\\
&- \frac{1}{2}M^A_{\iota_\xi\iota_\xi F_{A_0}} + \frac{1}{2}P^\psi_{[\xi,\xi]} - \frac{1}{2}L^\psi_{\iota_\xi\iota_\xi F_{\omega_0}} - \frac{1}{2}P_{[\xi,\xi]} && (5.423)\\
&+ \frac{1}{2}L_{\iota_\xi\iota_\xi F_{\omega_0}} + \int_\Sigma < [A_0, \iota_\xi p], \mathrm{L}_\xi^{\omega_0+A_0}\phi > && (5.424)\\
&+ < p, \mathrm{L}_\xi^{\omega_0+A_0}[\iota_\xi A_0, \phi] > && (5.425)\\
&- \int_\Sigma < [A_0, \iota_\xi p], [\iota_\xi A_0, \phi] > && (5.426)\\
&= \frac{1}{2}P^A_{[\xi,\xi]} - \frac{1}{2}L^A_{\iota_\xi\iota_\xi F_{\omega_0}} - \frac{1}{2}M^A_{\iota_\xi\iota_\xi F_{A_0}} && (5.427)\\
&+ \frac{1}{2}P^\psi_{[\xi,\xi]} - \frac{1}{2}L^\psi_{\iota_\xi\iota_\xi F_{\omega_0}} - \frac{1}{2}P_{[\xi,\xi]} && (5.428)\\
&+ \frac{1}{2}L_{\iota_\xi\iota_\xi F_{\omega_0}} + \int_\Sigma < p, [\iota_{[\xi,\xi]}A_0, \phi] > && (5.429)\\
&= \frac{1}{2}P^{H,A}_{[\xi,\xi]} - \frac{1}{2}L^{H,A}_{\iota_\xi\iota_\xi F_{\omega_0}} - \frac{1}{2}M^{H,A}_{\iota_\xi\iota_\xi F_{A_0}}, && (5.430)
\end{aligned}$$

where we made use of the following identity

$$\frac{1}{2}\iota_{[\xi,\xi]}A_0 = \iota_\xi d\iota_\xi A_0 - \frac{1}{2}\iota_\xi\iota_\xi dA_0. \tag{5.431}$$

We also have

$$\{P^{H,A}_\xi, H^{H,A}_\lambda\}_{YMH} = \{P_\xi + p^A_\xi, H_\lambda + h^A_\lambda\}_A + \{P_\xi + p^H_\xi, H_\lambda + h^H_\lambda\}_H \tag{5.432}$$

$$- \{P_\xi, H_\lambda\} + \iota_{\mathbb{h}^{H,A}}\delta(P_\xi + p^A_\xi + p^H_\xi + p^{H,A}_\xi) \tag{5.433}$$

$$+ \iota_{\mathbb{p}^{H,A}}\delta(H_\lambda + h^A_\lambda + h^H_\lambda + h^{H,A}_\lambda) \tag{5.434}$$

$$- \iota_{\mathbb{p}^{H,A}}\iota_{\mathbb{h}^{H,A}}\varpi_{YMH} \tag{5.435}$$

$$= P^A_{Y^{(\nu)}} - L^A_{Y^{(\nu)}(\omega-\omega_0)_\nu} + H^A_{Y^{(n)}} \tag{5.436}$$

$$- M^A_{Y^\nu(A-A_0)_\nu} + P^H_{Y^{(\nu)}} - L^H_{Y^{(\nu)}(\omega-\omega_0)_\nu} \tag{5.437}$$

$$+ H^H_{Y^{(n)}} - P_{Y^{(\nu)}} + L_{Y^{(\nu)}(\omega-\omega_0)_\nu} - H_{Y^{(n)}} \tag{5.438}$$

$$+ \int_\Sigma \mathrm{L}^{\omega_0+A_0}_\xi e\frac{\lambda e_n e}{2} < \Pi, [A,\phi] > \tag{5.439}$$

$$+ \int_\Sigma \frac{\lambda e_n e^2}{2} < [A,\Pi], \mathrm{L}^{\omega_0+A_0}_\xi \phi > \tag{5.440}$$

$$- \int_\Sigma \frac{\lambda e_n e^2}{2} < \Pi, \mathrm{L}^{\omega_0+A_0}_\xi [A,\phi] > \tag{5.441}$$

$$+ \int_\Sigma < \Pi, \mathrm{L}^{\omega_0+A_0}_\xi \Big(\frac{\lambda e_n e^2}{2}[A,\phi]\Big) > \tag{5.442}$$

$$+ \int_\Sigma \frac{\lambda e_n e^2}{2} < [\iota_\xi A_0, \Pi], d_A\phi > \tag{5.443}$$

$$+ \int_\Sigma \frac{\lambda e_n e^2}{2}\Big(< \Pi, d_A[\iota_\xi A_0, \phi] > \Big) \tag{5.444}$$

$$+ \int_\Sigma \frac{\lambda e_n e^3}{3} < [A,\Pi], \Pi > \tag{5.445}$$

$$+ \int_\Sigma \frac{\lambda e_n e^3 q_H}{3}(< \phi,\phi > - v^2) < [A,\phi],\phi > \tag{5.446}$$

$$= P^A_{Y^{(\nu)}} - L^A_{Y^{(\nu)}(\omega-\omega_0)_\nu} \tag{5.447}$$

$$+ H^A_{Y^{(n)}} - M^A_{Y^\nu(A-A_0)_\nu} + P^H_{Y^{(\nu)}} - L^H_{Y^{(\nu)}(\omega-\omega_0)_\nu} \tag{5.448}$$

$$+ H^H_{Y^{(n)}} - P_{Y^{(\nu)}} + L_{Y^{(\nu)}(\omega-\omega_0)_\nu} - H_{Y^{(n)}} \tag{5.449}$$

$$+ \int_\Sigma \mathrm{L}^{\omega_0+A_0}_\xi (\lambda e_n)\frac{e^2}{2} < \Pi, [A,\phi] > \tag{5.450}$$

$$= P^{H,A}_{Y^{(\nu)}} - L^{H,A}_{Y^{(\nu)}(\omega-\omega_0)_\nu} + H^{H,A}_{Y^{(n)}} - M^{H,A}_{Y^\nu(A-A_0)_\nu}, \tag{5.451}$$

having used (5.351).

Then, we calculate

$$\{H^{H,A}_\lambda, H^{H,A}_\lambda\}_{YMH} = \{H^H_\lambda, H^H_\lambda\}_H + \{H^A_\lambda, H^A_\lambda\}_A - \{H_\lambda, H_\lambda\} \tag{5.452}$$

$$+ 2\iota_{\mathbb{h}^{H,A}_\lambda}\delta H^{H,A}_\lambda - \iota_{\mathbb{h}^{H,A}_\lambda}\iota_{\mathbb{h}^{H,A}_\lambda}\varpi_{YMH} = 0, \tag{5.453}$$

where we implemented the fact all terms in $2\iota_{\mathbb{h}^{H,A}_{\lambda}}\delta H^{H,A}_{\lambda} - \iota_{\mathbb{h}^{H,A}_{\lambda}}\iota_{\mathbb{h}^{H,A}_{\lambda}}\Omega_{SM}$ contain either $(\lambda e_n)^2 = 0$ or $\lambda e_n d_\omega(\lambda e_n) = 0$.

What remains is to compute the brackets involving R_τ that are neither trivially vanishing nor directly reducible to cases already treated in the preceding sections. For instance, we have

$$\{R_\tau, h^{H,A}_\lambda\} = \mathbb{R}_\tau \int_\Sigma \lambda e_n \frac{e^2}{2} < \Pi, [A, \phi] > \tag{5.454}$$

$$= \int_\Sigma \lambda e_n [\tau, e] < \Pi, [A, \phi] > \tag{5.455}$$

$$= \int_\Sigma \lambda e_n [\tau, \hat{e}] < \Pi, [A, \phi] > \tag{5.456}$$

$$= \int_\Sigma \lambda \tau [e_n, \hat{e}] < \Pi, [A, \phi] >, \tag{5.457}$$

where $\hat{e}$ is the vielbein along the non-degenerate direction. Moreover, we used the fact that $e_n[\tau, e] = [e_n\tau, e] + \tau[e_n, e] = \tau[e_n, e]$, thanks to $e_n\tau = 0$ following the definitions. We notice that the integral contains no derivatives. Thus, we can diagonalize the vielbein as[26]

$$\hat{e}^a = \begin{cases} e^a_1 & = \delta^a_1 \\ e^a_2 & = \delta^a_2 \end{cases} \tag{5.458}$$

$$e^a_+ = \delta^a_3 - \delta^a_4 \tag{5.459}$$

$$e^a_n = \delta^a_3 + \delta^a_4, \tag{5.460}$$

with $a = 1, 2, 3, 4$. This leads to

$$\{R_\tau, h^{H,A}_\lambda\} = 0, \tag{5.461}$$

since $[e_n, \hat{e}] = 0$ because of our choice of diagonalization.

On top of that, we have

$$\{R_\tau, h^H_\lambda\} = \mathbb{R}_\tau \int_\Sigma \lambda e_n \left[\frac{e^2}{2} < \Pi, d\phi > \right. \tag{5.462}$$

$$\left. + \frac{e^3}{2 \cdot 3!}(< \Pi, \Pi >) - \frac{q}{24} e^3 (< \phi, \phi > -v^2)^2 \right] \tag{5.463}$$

$$= \int_\Sigma \lambda e_n [\tau, e] < \Pi, d\phi > \tag{5.464}$$

$$= 0, \tag{5.465}$$

[26]Note that, in this setting, the vielbein along the degenerate direction is e_+.

by noticing that all terms containing $e[\tau, e]$ are vanishing because of $e\tau = 0$ (by definition of $\mathcal{S}$) and $[e, e] = 0$. The integral in (5.464) has analogously been computed in Section 5.1.

In order to complete the proof, we notice that $\{R_\tau, m_\mu^H\} = 0$ because $\mathfrak{m}_e^H = \mathfrak{m}_\omega^H = 0$. □

Remark 5.5.10. Notice that, after choosing U(1) as the gauge group and setting $q_H = 0$, we obtain scalar electrodynamics coupled to gravity as a particular case of this theory.

Corollary 5.5.11. *If the boundary metric i^*g is non-degenerate, then the functionals in Definition 5.5.8 define a coisotropic submanifold.*

5.6 Yukawa

The space of fields of the Yukawa Palatini–Cartan theory is given by

$$\mathcal{F}_Y = \mathcal{F}_\phi \times \Gamma(\Pi S) \times \Gamma(\Pi \bar{S}), \tag{5.466}$$

or, equivalently,

$$\mathcal{F}_Y = \mathcal{F}_\psi \times C^\infty(M) \times \Omega^{0,1}, \tag{5.467}$$

where $\mathcal{F}_\phi$ and $\mathcal{F}_\psi$ are defined in Eq. (5.1) and Eq. (5.125) respectively. Whereas, the action functional reads

$$S_Y = S + S_\phi + S_\psi + S_{\phi,\psi}, \tag{5.468}$$

where

$$S_{\phi,\psi} = g_Y \int_M \frac{1}{2 \cdot 4!} e^4 \overline{\psi} \phi \psi, \tag{5.469}$$

with g_Y a constant.

As in the previous sections, we notice that the interaction term does not contain derivatives. It follows that the boundary structure is just the direct sum of the scalar and spinor structures. In particular, the geometric phase space is given by the following theorem.

Theorem 5.6.1. *The geometric phase space of the Yukawa Palatini–Cartan theory is symplectomorphic to the space $(\mathcal{F}_Y^\partial, \varpi_Y^\partial)$, where*

$$\mathcal{F}_Y^\partial \subset \mathcal{F}_\phi^\partial \times \Gamma(\Pi i^* S) \times \Gamma(\Pi i^* \bar{S}) \tag{5.470}$$

with $\mathcal{F}^{\partial}_{\phi}$ *defined in Theorem 5.1.3 and* $(e, \omega, \phi, \Pi, \psi, \overline{\psi}) \in \mathcal{F}^{\partial}_{Y}$ *satisfying*

$$\begin{cases} e_n(a_\psi - p_{\mathcal{T}} a_\psi) \in \operatorname{Im} W_1^{\Sigma,(1,1)} \\ p_{\mathcal{K}}\omega = 0 \\ d_A\phi - (e, \Pi) = 0 \\ p_W \Pi = 0, \end{cases} \tag{5.471}$$

as defined in Lemma 4.2.19, Lemma 5.1.2, and Remark 5.3.14, and where the corresponding symplectic form on $\mathcal{F}^{\partial}_{Y}$ *is given by*

$$\varpi_Y = \varpi + \varpi_\phi + \varpi_\psi. \tag{5.472}$$

Remark 5.6.2. From the previous computation, we already know that for a spinor field and a scalar field coupled to gravity, the constraint-generating expressions are those associated with variations δe and $\delta\omega$, as all remaining variations correspond to evolution equations. From the form of the Yukawa interaction term in the action, we deduce that no additional constraints arise beyond those of pure gravity, and that the only constraint modified by the interaction is H_λ.

This leads to the following definition.

Definition 5.6.3. Let $c \in \Omega_\Sigma^{0,2}[1]$, $\xi \in \mathfrak{X}(\Sigma)[1]$, $\lambda \in C^\infty(\Sigma)[1]$, and $\tau \in \mathcal{S}[1]$, and the other functionals be as in Definition 5.1.4 and Definition 5.3.17. Then, we define the following functionals

$$L_c^{\phi,\psi} = L_c + l_c^\psi \tag{5.473}$$

$$P_\xi^{\phi,\psi} = P_\xi + p_\xi^\phi + p_\xi^\psi \tag{5.474}$$

$$H_\lambda^{\phi,\psi} = H_\lambda + h_\lambda^\phi + h_\lambda^\psi + h_\lambda^{\phi,\psi} \tag{5.475}$$

$$R_\tau^{\phi,\psi} = R_\tau, \tag{5.476}$$

where

$$p_\xi^\phi = P_\xi^\phi - P_\xi \tag{5.477}$$

$$h_\lambda^\phi = H_\lambda^\phi - H_\lambda \tag{5.478}$$

and

$$h_\lambda^{\phi,\psi} = g_Y \int_\Sigma \lambda e_n \frac{1}{2 \cdot 3!} e^3 \overline{\psi} \phi \psi \tag{5.479}$$

We refer to these as the constraints of the Yukawa Palatini–Cartan theory.

Theorem 5.6.4. *The Poisson brackets of the constraints of Definition 5.6.3 read*

$$\begin{aligned}
&\{L_c^{\phi,\psi}, L_c^{\phi,\psi}\}_Y = -\frac{1}{2}L_{[c,c]}^{\phi,\psi} && \{P_\xi^{\phi,\psi}, P_\xi^{\phi,\psi}\}_Y = \frac{1}{2}P_{[\xi,\xi]}^{\phi,\psi} - \frac{1}{2}L_{\iota_\xi\iota_\xi F_{\omega_0}}^{\phi,\psi}\\
&\{L_c^{\phi,\psi}, P_\xi^{\phi,\psi}\}_Y = L_{\mathcal{L}_\xi^{\omega_0}c}^{\phi,\psi} && \{H_\lambda^{\phi,\psi}, H_\lambda^{\phi,\psi}\}_Y \approx 0\\
&\{L_c^{\phi,\psi}, R_\tau\}_Y = -R_{p_\mathcal{S}[c,\tau]} && \{R_\tau^{\phi,\psi}, P_\xi^{\phi,\psi}\}_Y = R_{p_\mathcal{S}\mathcal{L}_\xi^{\omega_0}\tau}\\
&\{R_\tau, H_\lambda^{\phi,\psi}\}_Y \approx G_{\lambda\tau} + K_{\lambda\tau}^\psi && \{R_\tau, R_\tau\}_Y \approx F_{\tau\tau}\\
&\{L_c^{\phi,\psi}, H_\lambda^{\phi,\psi}\}_Y = -P_{X^{(a)}}^{\phi,\psi} + L_{X^{(a)}(\omega-\omega_0)_a}^{\phi,\psi} - H_{X^{(n)}}^{\phi,\psi}\\
&\{P_\xi^{\phi,\psi}, H_\lambda^{\phi,\psi}\}_Y = P_{Y^{(a)}}^{\phi,\psi} - L_{Y^{(a)}(\omega-\omega_0)_a}^{\phi,\psi} + H_{Y^{(n)}}^{\phi,\psi},
\end{aligned}$$

with $X = [c, \lambda e_n]$, $Y = \mathcal{L}_\xi^{\omega_0}(\lambda e_n)$ *and where the superscripts* (a) *and* (n) *describe their components with respect to* e_a, e_n*. Furthermore,* $F_{\tau\tau}$*,* $G_{\lambda\tau}$ *and* $K_{\lambda\tau}^\psi$ *are functionals of* $e, \omega, \psi, \overline{\psi}, \tau$ *and* λ *defined in Theorem 5.3.18 which are not proportional to any other constraint.*

Proof. The proof will follow the usual path defined in the previous sections and in Appendix B, given that most of the results are taken from the computations of Theorem 5.1.5 and Theorem 5.3.18.

Let us start with the constraint $L_c^{\phi,\psi}$. We first notice that $\iota_{\mathbb{l}^\phi}\varpi_\psi = \iota_{\mathbb{l}^\psi}\varpi_\phi = 0$. The variation of the interaction term is $l_c^{\phi,\psi} = 0$, hence we also conclude

$$\mathbb{l}^{\phi,\psi} = 0. \tag{5.480}$$

For $P_\xi^{\phi,\psi}$, we work in the same way and find

$$\mathbb{p}^{\phi,\psi} = 0. \tag{5.481}$$

On the other hand, for $H_\lambda^{\phi,\psi}$ we get $\iota_{\mathbb{h}^\phi}\varpi_\psi = 0$ and

$$\iota_{\mathbb{h}^\psi}\varpi_\phi = \int_\Sigma \frac{1}{2}e^2\mathbb{h}_e^\psi \Pi\delta\phi. \tag{5.482}$$

Note that we will not need the explicit expression. The variation of the interaction term reads

$$\delta h_\lambda^{\phi,\psi} := g_Y \int_\Sigma \lambda e_n \Big(\frac{1}{4}e^2\delta e\overline{\psi}\phi\psi + \frac{1}{2\cdot 3!}e^3\delta\overline{\psi}\phi\psi \tag{5.483}$$

$$- \frac{1}{2\cdot 3!}e^3\overline{\psi}\delta\phi\psi - \frac{1}{2\cdot 3!}e^3\overline{\psi}\phi\delta\psi\Big). \tag{5.484}$$

Hence, we get:

$$\mathbb{h}_e^{\phi,\psi} = 0 \qquad e\mathbb{h}_\omega^{\phi,\psi} = 0 \tag{5.485}$$

$$\mathbb{h}_\phi^{\phi,\psi} = 0 \qquad \frac{1}{3!}e^3\mathbb{h}_\Pi^{\phi,\psi} = -\frac{1}{2}e^2\mathbb{h}_e^\psi\Pi + \frac{1}{2\cdot 3!}g_Y\lambda e_n e^3\overline{\psi}\psi \tag{5.486}$$

$$\gamma\mathbb{h}_\psi^{\phi,\psi} = -\frac{i}{2}g_Y\lambda e_n\phi\psi \qquad \mathbb{h}_{\overline{\psi}}^{\phi,\psi}\gamma = \frac{i}{2}g_Y\lambda e_n\overline{\psi}\phi \tag{5.487}$$

Before implementing the results of Appendix B, we note that for the whole set of constraints, we have

$$\iota_{\mathbb{X}+\mathbb{x}^\psi}\iota_{\mathbb{Y}+\mathbb{y}^\psi}\varpi_\phi + \iota_{\mathbb{X}+\mathbb{x}^\phi}\iota_{\mathbb{Y}+\mathbb{y}^\phi}\varpi_\psi = 0. \tag{5.488}$$

Then,

$$\{L_c^{\phi,\psi}, L_c^{\phi,\psi}\}_Y = \{L_c + l_c^\phi, L_c + l_c^\phi\}_\phi + \{L_c + l_c^\psi, L_c + l_c^\psi\}_\psi - \{L_c, L_c\} \tag{5.489}$$

$$+ \iota_{\mathbb{l}^\phi}\iota_{\mathbb{l}^\psi}\varpi_Y + \iota_{\mathbb{l}^\psi}\iota_{\mathbb{l}^\phi}\varpi_Y \tag{5.490}$$

$$= -\frac{1}{2}L_{[c,c]} - \frac{1}{2}l_{[c,c]}^\psi = -\frac{1}{2}L_{[c,c]}^{\phi,\psi} \tag{5.491}$$

where we used for the second line $\mathbb{l}_e^\phi = \mathbb{l}_e^\psi = 0$ and skipped the zero terms.

Similarly, we obtain

$$\{L_c^{\phi,\psi}, P_\xi^{\phi,\psi}\}_Y = \{L_c + l_c^\phi, P_\xi + p_\xi^\phi\}_\phi + \{L_c + l_c^\psi, P_\xi + p_\xi^\psi\}_\psi - \{L_c, P_\xi\} \tag{5.492}$$

$$+ \iota_{\mathbb{l}^\phi}\iota_{\mathbb{p}^\psi}\varpi_Y + \iota_{\mathbb{l}^\psi}\iota_{\mathbb{p}^\phi}\varpi_Y \tag{5.493}$$

$$= L_{\mathrm{L}_\xi^{\omega_0}c} + l_{\mathrm{L}_\xi^{\omega_0}c}^\psi = L_{\mathrm{L}_\xi^{\omega_0}c}^{\phi,\psi} \tag{5.494}$$

where we used for the second line $\mathbb{l}_e^\phi = \mathbb{p}_e^\phi = \mathbb{l}_e^\psi = \mathbb{p}_e^\psi = 0$, and

$$\{P_\xi^{\phi,\psi}, P_\xi^{\phi,\psi}\}_Y = \{P_\xi + p_\xi^\phi, P_\xi + p_\xi^\phi\}_\phi + \{P_\xi + p_\xi^\psi, P_\xi + p_\xi^\psi\}_\psi - \{P_\xi, P_\xi\} \tag{5.495}$$

$$+ \iota_{\mathbb{p}^\phi}\iota_{\mathbb{p}^\psi}\varpi_Y + \iota_{\mathbb{p}^\psi}\iota_{\mathbb{p}^\phi}\varpi_Y \tag{5.496}$$

$$= \frac{1}{2}P_{[\xi,\xi]} - \frac{1}{2}L_{\iota_\xi\iota_\xi F_{\omega_0}} + \frac{1}{2}p_{[\xi,\xi]}^\phi + \frac{1}{2}p_{[\xi,\xi]}^\psi - \frac{1}{2}l_{\iota_\xi\iota_\xi F_{\omega_0}}^\psi \tag{5.497}$$

$$= \frac{1}{2}P_{[\xi,\xi]}^{\phi,\psi} - \frac{1}{2}L_{\iota_\xi\iota_\xi F_{\omega_0}}^{\phi,\psi}. \tag{5.498}$$

For the brackets with $H_\lambda^{\phi,\psi}$, we have

$$\{L_c^{\phi,\psi}, H_\lambda^{\phi,\psi}\}_Y = \{L_c + l_c^\phi, H_\lambda + h_\lambda^\phi\}_\phi + \{L_c + l_c^\psi, H_\lambda + h_\lambda^\psi\}_\psi - \{L_c, H_\lambda\} \tag{5.499}$$

$$+ \iota_{\mathbb{h}^{\phi,\psi}}\delta(L_c + l_c^\psi), \tag{5.500}$$

and we compute the last term, yielding

$$\iota_{\mathbb{h}^{\phi,\psi}}\delta(L_c + l_c^\psi) = \int_\Sigma -ce[\mathbb{h}_\omega^{\phi,\psi}, e] - i\frac{e^3}{2\cdot 3!}\Big(- [c, \mathbb{h}_{\overline{\psi}}^{\phi,\psi}]\gamma\psi - \mathbb{h}_{\overline{\psi}}^{\phi,\psi}\gamma[c,\psi]\Big) \quad (5.501)$$

$$- i\frac{e^3}{2\cdot 3!}\Big(- [c, \overline{\psi}]\gamma\mathbb{h}_\psi^{\phi,\psi} + \overline{\psi}\gamma[c, \mathbb{h}_\psi^{\phi,\psi}]\Big) \quad (5.502)$$

$$= \int_\Sigma -\frac{1}{4}g_Y ce[\lambda e_n e\overline{\psi}\phi\psi, e] \quad (5.503)$$

$$- i\frac{e^3}{2\cdot 3!}\Big(- [c, \mathbb{h}_{\overline{\psi}}^{\phi,\psi}]\gamma\psi - \mathbb{h}_{\overline{\psi}}^{\phi,\psi}\gamma[c,\psi]\Big) \quad (5.504)$$

$$- i\frac{e^3}{2\cdot 3!}\Big(- [c, \overline{\psi}]\gamma\mathbb{h}_\psi^{\phi,\psi} + \overline{\psi}\gamma[c, \mathbb{h}_\psi^{\phi,\psi}]\Big) \quad (5.505)$$

$$= -\int_\Sigma \frac{e^3}{2\cdot 3!}g_Y[c, \lambda e_n]\overline{\psi}\phi\psi, \quad (5.506)$$

where we used the properties for the bracket $[\cdot,\cdot]$ on spinors. The result can be easily regarded as $h_{X^{(n)}}^{\phi,\psi}$. Hence, we get

$$\{L_c^{\phi,\psi}, H_\lambda^{\phi,\psi}\}_Y = -P_{X^{(\nu)}} + L_{X^{(\nu)}(\omega-\omega_0)_\nu} - H_{X^{(n)}} - p_{X^{(\nu)}}^\phi - h_{X^{(n)}}^\phi - p_{X^{(\nu)}}^\psi \quad (5.507)$$

$$+ l_{X^{(\nu)}(\omega-\omega_0)_\nu}^\psi - h_{X^{(n)}}^\psi - h_{X^{(n)}}^{\phi,\psi} \quad (5.508)$$

$$= -P_{X^{(\nu)}}^{\phi,\psi} + L_{X^{(\nu)}(\omega-\omega_0)_\nu}^{\phi,\psi} - H_{X^{(n)}}^{\phi,\psi} \quad (5.509)$$

Analogously, we write

$$\{P_\xi^{\phi,\psi}, H_\lambda^{\phi,\psi}\}_Y = \{P_\xi + p_\xi^\phi, H_\lambda + h_\lambda^\phi\}_\phi + \{P_\xi + p_\xi^\psi, H_\lambda + h_\lambda^\psi\}_\psi - \{P_\xi, H_\lambda\} \quad (5.510)$$

$$+ \iota_{\mathbb{h}^{\phi,\psi}}\delta(P_\xi + p_\xi^\psi + p_\xi^\phi) + \iota_{\mathbb{h}^\psi}\iota_{\mathbb{p}^\phi}\varpi_Y. \quad (5.511)$$

Let us consider the terms in the second row. We have

$$\iota_{\mathbb{h}^\psi}\iota_{\mathbb{p}^\phi}\varpi_Y = \int_\Sigma \frac{1}{2}e^2\mathbb{h}_e^\psi\Pi\mathbb{p}_\phi^\phi = -\int_\Sigma \frac{1}{2}e^2\mathbb{h}_e^\psi\Pi \mathrm{L}_\xi\phi.$$

On the other hand, we compute

$$\begin{aligned}
\iota_{\mathbb{h}^{\phi,\psi}}\delta(P_\xi + p_\xi^\psi + p_\xi^\phi) &= \int_\Sigma -\frac{1}{3!}e^3\mathbb{h}_\Pi^{\phi,\psi}\mathrm{L}_\xi\phi && (5.512)\\
&\quad - i\frac{e^3}{2\cdot 3!}\Big(\mathbb{h}_{\overline{\psi}}^{\phi,\psi}\gamma \mathrm{L}_\xi^{\omega_0}(\psi) + \mathrm{L}_\xi^{\omega_0}(\overline{\psi})\gamma\mathbb{h}_\psi^{\phi,\psi}\Big) && (5.513)\\
&\quad - \frac{i}{2\cdot 3!}\Big(\mathrm{L}_\xi^{\omega_0}(e^3\overline{\psi})\gamma\mathbb{h}_\psi^{\phi,\psi} + \mathbb{h}_{\overline{\psi}}^{\phi,\psi}\gamma\mathrm{L}_\xi^{\omega_0}(e^3\psi)\Big) && (5.514)\\
&= \int_\Sigma \frac{1}{2}e^2\mathbb{h}_e^\psi\Pi\mathrm{L}_\xi\phi - \frac{1}{2\cdot 3!}g_Y\lambda e_n e^3\overline{\psi}\psi\mathrm{L}_\xi\phi && (5.515)\\
&\quad + \frac{e^3}{2\cdot 3!}\Big(\frac{1}{2}g_Y\lambda e_n\overline{\psi}\phi\mathrm{L}_\xi^{\omega_0}(\psi) && (5.516)\\
&\quad - \frac{1}{2}g_Y\lambda e_n\mathrm{L}_\xi^{\omega_0}(\overline{\psi})\phi\psi\Big) && (5.517)\\
&\quad - \frac{1}{2\cdot 3!}\Big(\frac{1}{2}g_Y\lambda e_n\mathrm{L}_\xi^{\omega_0}(e^3\overline{\psi})\phi\psi && (5.518)\\
&\quad - \frac{1}{2}g_Y\lambda e_n\overline{\psi}\phi\mathrm{L}_\xi^{\omega_0}(e^3\psi)\Big) && (5.519)\\
&= \int_\Sigma -g_Y\frac{1}{2\cdot 3!}\lambda e_n\mathrm{L}_\xi^{\omega_0}(e^3\overline{\psi}\phi\psi) + \frac{1}{2}e^2\mathbb{h}_e^\psi\Pi\mathrm{L}_\xi\phi && (5.520)\\
&= \int_\Sigma g_Y\frac{1}{2\cdot 3!}\mathrm{L}_\xi^{\omega_0}(\lambda e_n)e^3\overline{\psi}\phi\psi + \frac{1}{2}e^2\mathbb{h}_e^\psi\Pi\mathrm{L}_\xi\phi, && (5.521)
\end{aligned}$$

where we used $\mathrm{L}_\xi^{\omega_0}\gamma = 0$. The second term in the last row cancels out with the one computed above, while the first is exactly $h_{Y(n)}^{\phi,\psi}$. Thus, collecting all the terms, we get

$$\begin{aligned}
\{P_\xi^{\phi,\psi}, H_\lambda^{\phi,\psi}\}_Y &= P_{Y^{(\nu)}} - L_{Y^{(\nu)}(\omega-\omega_0)_\nu} + H_{Y^{(n)}} + p_{Y^{(\nu)}}^\phi + h_{Y^{(n)}}^\phi + p_{Y^{(\nu)}}^\psi && (5.522)\\
&\quad - l_{Y^{(\nu)}(\omega-\omega_0)_\nu}^\psi + h_{Y^{(n)}}^\psi + h_{Y^{(n)}}^{\phi,\psi} && (5.523)\\
&= -P_{Y^{(\nu)}}^{\phi,\psi} + L_{Y^{(\nu)}(\omega-\omega_0)_\nu}^{\phi,\psi} - H_{Y^{(n)}}^{\phi,\psi}. && (5.524)
\end{aligned}$$

Finally, we consider

$$\begin{aligned}
\{H_\lambda^{\phi,\psi}, H_\lambda^{\phi,\psi}\}_Y &= \{H_\lambda + h_\lambda^\phi, H_\lambda + h_\lambda^\phi\}_\phi + \{H_\lambda + h_\lambda^\psi, H_\lambda + h_\lambda^\psi\}_\psi && (5.525)\\
&\quad - \{H_\lambda, H_\lambda\} + 2\iota_{\mathbb{h}^{\phi,\psi}}\delta(H_\lambda + h_\lambda^\phi + h_\lambda^\psi + h_\lambda^{\phi,\psi}) && (5.526)\\
&\quad - \iota_{\mathbb{h}^{\phi,\psi}}\iota_{\mathbb{h}^{\phi,\psi}}\varpi_Y + 2\iota_{\mathbb{h}^\phi}\iota_{\mathbb{h}^\psi}\varpi_Y. && (5.527)
\end{aligned}$$

All the terms of the first line are zero, as proved in the previous theorems. Furthermore, notice that $H_\lambda + h_\lambda^\phi + h_\lambda^\psi + h_\lambda^{\phi,\psi}$ is proportional to λ, as well as $\mathbb{h}^{\phi,\psi}$, $\mathbb{h}^\phi$ and $\mathbb{h}^\psi$. Since $\lambda^2 = 0$, we conclude that $\{H_\lambda^{\phi,\psi}, H_\lambda^{\phi,\psi}\} = 0$.

For what concerns the brackets with R_τ, we notice that they are either treated in Theorem 5.1.5 and Theorem 5.3.18 or trivially vanishing. □

Corollary 5.6.5. *If the boundary metric* i^*g *is non-degenerate, then the functionals in Definition 5.6.3 define a coisotropic submanifold.*

Chapter 6

Codimension-2 structure of gravity

6.1 Courant algebroid and Dirac structure

Definition 6.1.1 (Courant algebroid[1])**.** A *Courant algebroid* over a smooth manifold M is a vector bundle $E \to M$ equipped with a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$, a bilinear bracket $[\cdot, \cdot] \colon \Gamma(E) \times \Gamma(E) \to \Gamma(E)$, and an anchor map $\rho \colon \Gamma(E) \to \Gamma(TM)$, satisfying the following conditions for all $e_1, e_2, e_3 \in \Gamma(E)$ and $f \in C^\infty(M)$:

i. $[e_1, [e_2, e_3]] = [[e_1, e_2], e_3] + [e_2, [e_1, e_3]]$,

ii. $\rho([e_1, e_2]) = \{\rho(e_1), \rho(e_2)\}$,

iii. $[e_1, f e_2] = f[e_1, e_2] + (\rho(e_1) f) e_2$,

iv. $\langle e_1, [e_2, e_3] + [e_3, e_2] \rangle = \rho(e_1) \langle e_2, e_3 \rangle$,

v. $\rho(e_1) \langle e_2, e_3 \rangle = \langle [e_1, e_2], e_3 \rangle + \langle e_2, [e_1, e_3] \rangle$,

where $\{\cdot, \cdot\}$ is the Lie bracket of vector fields.

Remark 6.1.2. From the properties above, it follows that $[e_1, e_2] + [e_2, e_1] = \mathcal{D}\langle e_1, e_2 \rangle$, where $\mathcal{D} \colon C^\infty(M) \to \Gamma(E)$ is a differential operator defined as $\mathcal{D} = \rho^* \mathrm{d}$, with $\rho^* \colon \Gamma(T^*M) \to \Gamma(E)$ the co-anchor map of ρ defined by $\langle \rho^*(\alpha), e \rangle = \alpha(\rho(e))$, with $\alpha \in \Gamma(T^*M)$ and $e \in \Gamma(E)$, and d the de Rham differential operator. In other words, the bracket $[\cdot, \cdot]$ is not antisymmetric in general, and its lack of antisymmetry is measured by $\mathcal{D}$ and the pairing. Notice that it would be possible to modify

[1] See [Mei25], [Roy02] and references therein for more details.

the axioms of Definition 6.1.1 in order to account for an antisymmetric bracket. In this case, it would no longer satisfy the Jacobi-type identity of property (i), and the differential operator $\mathcal{D}$ would then measure its failure.

Remark 6.1.3. Notice the difference with a Lie algebroid. A Lie algebroid is a vector bundle $A \to M$ equipped with a Lie bracket $[\cdot,\cdot]$ on its space of sections and an anchor map $\rho\colon A \to TM$ satisfying the Leibniz rule

$$[a, fb] = f[a,b] + (\rho(a)f)\, b, \tag{6.1}$$

for all $a, b \in \Gamma(A)$ and $f \in C^\infty(M)$. In this setting, the bracket is skew-symmetric and satisfies both the Leibniz and the Jacobi identities.

Definition 6.1.4 (Dirac structure)**.** A *Dirac structure* on a manifold M is a subbundle $D \subseteq TM \oplus T^*M$ whose space of sections $\Gamma(D)$ is closed under the bracket $[\cdot,\cdot]$ and that is *maximally isotropic* with respect to the bilinear form $\langle\cdot,\cdot\rangle$, i.e. $\langle \mathbb{X} + \mathcal{X}, \mathbb{Y} + \mathcal{Y}\rangle = 0$ for all $\mathbb{X} + \mathcal{X}, \mathbb{Y} + \mathcal{Y} \in \Gamma(D)$ and D has maximal rank among all isotropic subbundles of $TM \oplus T^*M$.

6.2 Corner structure of gravity

Consider the pure gravitational setting, where we set the boundary metric i^*g to be non-degenerate.[2]

Definition 6.2.1. Let $c \in \Omega_\Sigma^{0,2}[1]$, $\xi \in \mathfrak{X}(\Sigma)[1]$ and $\lambda \in C^\infty(\Sigma)[1]$. Then, we define the following functionals

$$L_c = \int_\Sigma c e d_\omega e \tag{6.2}$$

$$P_\xi = \int_\Sigma \frac{1}{2}\iota_\xi(e^2)F_\omega + \iota_\xi(\omega - \omega_0) e d_\omega e \tag{6.3}$$

$$H_\lambda = \int_\Sigma \lambda e_n\Big(eF_\omega + \frac{\Lambda}{3!}e^3\Big). \tag{6.4}$$

We refer to these as the constraints of the Palatini–Cartan non-degenerate theory.

Remark 6.2.2. The geometric set-up we are working with is that of a stratified manifold—more precisely, a manifold with corners as defined in Definition 2.2.6 and specified in Remark 2.2.7. From now on, we will refer to the codimension-1 and codimension-2 strata as, respectively, the boundary and the corner of the manifold.

[2]Alternatively, one imposes the degeneracy constraint and work within that submanifold.

Notice that we can make the vielbein descend to the corner[3] by considering

$$\widetilde{\Omega}_{\Sigma}^{1,1} \ni e \mapsto e + e_m dx^m, \tag{6.5}$$

where, on the right-hand side[4], $e \in \widetilde{\Omega}_{\Gamma}^{1,1}$ and $e_m \in \Omega_{\Gamma}^{0,1}$ are such that, given a completion vielbein as in Definition 4.2.11, we have that $\{e(v_1), e(v_2), e_m, e_n\}$ is a basis of $j^*i^*\mathcal{V}$, where $\{v_1, v_2\}$ is a basis of $T\Gamma$. Notice, however, that e_m is not fixed, conversely to the case of Definition 4.2.11.
Analogously, we have

$$\omega \mapsto \omega + \omega_m dx^m \tag{6.6}$$
$$d \mapsto d + dx^m \partial_m. \tag{6.7}$$

By means of these decompositions, we have that the boundary constraints in Definition 6.2.1 yield the following additional conditions on the corner[5]

$$e_m d_\omega e = e d_\omega e_m - e d_{\omega_m} e \tag{6.8}$$
$$e_m F_\omega - e F_{\omega_m} = 0. \tag{6.9}$$

Whereas, the structural constraint in Eq. (4.29) (with $\alpha = d_\omega e$) descends to the corner as

$$e_n d_{\omega_m} e + e_n d_\omega e_m = e\sigma_m + e_m \sigma. \tag{6.10}$$

Definition 6.2.3. We define $\pi_\Gamma \colon \mathcal{F}^\partial \to \mathcal{F}^{\partial\partial}$ to be the restriction of the boundary fields to the corner.

Now, given a local functional X of Definition 6.2.1 and its Hamiltonian vector field $\mathbb{X} \in \mathfrak{X}(\Sigma)$, we can notice that the Hamiltonian equation in the presence of a codimension-2 corner becomes something of the form

$$\delta X = \iota_{\mathbb{X}} \varpi + \pi_\Gamma^* \mathcal{X}, \tag{6.11}$$

where $\mathcal{X}$ is a 1-form on $\mathcal{F}^{\partial\partial}$.

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[3]See [CC24] for other details.
[4]Given $i \colon \Sigma \to M$ and $j \colon \Gamma \to \Sigma$, we can define $\Omega_\Gamma^{i,j} := \Omega^i(\Gamma, \bigwedge^j (j^*i^*\mathcal{V}))$.
[5]We have to account for the transversal components of the zero locus of the constraints in Definition 6.2.1.

Definition 6.2.4. Let $c \in \Omega_\Gamma^{0,2}[1]$, $\zeta \in \mathfrak{X}(\Gamma)[1]$ and $\lambda, \eta \in C^\infty(\Gamma)[1]$. Then, the functionals in Definition 6.2.1 define the following 1-forms on $\mathcal{F}^{\partial\partial}$

$$\mathcal{J}_c = \int_\Gamma ce\delta e = \delta \int_\Gamma \frac{ce^2}{2} \tag{6.12}$$

$$\mathcal{E}_\zeta = \int_\Gamma \iota_\zeta \frac{e^2}{2}\delta\omega + \iota_\zeta(\omega - \omega_0)e\delta e = \delta \int_\Gamma \iota_\zeta \frac{e^2}{2}(\omega - \omega_0) \tag{6.13}$$

$$\mathcal{K}_\eta = \int_\Gamma \eta\Big[ee_m\delta\omega + (\omega - \omega_0)_m e\delta e\Big] \tag{6.14}$$

$$\mathcal{F}_\lambda = \int_\Gamma \lambda e_n e\delta\omega, \tag{6.15}$$

where we have considered $\iota_\xi \mapsto \iota_\zeta + \eta\partial_m$, with ∂_m the derivation transversal to Γ.

Remark 6.2.5. Notice that we can also project the Hamiltonian vector fields for the functionals of Definition 6.2.1 to the corner, eventually defining a subbundle $D \subseteq T\mathcal{F}^{\partial\partial} \oplus T^*\mathcal{F}^{\partial\partial}$ constructed out of such restricted vector fields together with the 1-forms of Definition 6.2.4. The construction of this projection is contained in the proof of Theorem 6.2.10.

Definition 6.2.6. Let $E = T\mathcal{F}^{\partial\partial} \oplus T^*\mathcal{F}^{\partial\partial}$. Then, the structure $(E, \langle\cdot,\cdot\rangle, [\cdot,\cdot], \rho)$, given by

- $\langle \mathbb{X} + \mathcal{X}, \mathbb{Y} + \mathcal{Y}\rangle = \iota_{\mathbb{X}}\mathcal{Y} - \iota_{\mathbb{Y}}\mathcal{X}$,
- $[\mathbb{X} + \mathcal{X}, \mathbb{Y} + \mathcal{Y}] = \{\mathbb{X}, \mathbb{Y}\} + \mathcal{L}_{\mathbb{X}}\mathcal{Y} + \iota_{\mathbb{Y}}\delta\mathcal{X}$ (Dorfman bracket), and
- $\rho = p_1$ (natural projection to $\Gamma(T\mathcal{F}^{\partial\partial})$),

is called the standard Courant algebroid on $\mathcal{F}^{\partial\partial}$.

Remark 6.2.7. Note that the introduction of odd Lagrange multipliers induces a parity shift, resulting in a sign change in the relations among odd quantities. Accordingly, we have structured the quantities in Definition 6.2.6 to reflect this grading.

Lemma 6.2.8. *If* $\delta(\iota_{\mathbb{X}}\mathcal{Y} - \iota_{\mathbb{Y}}\mathcal{X}) = 0$*, then* $[\mathbb{X} + \mathcal{X}, \mathbb{Y} + \mathcal{Y}] = [\mathbb{Y} + \mathcal{Y}, \mathbb{X} + \mathcal{X}]$.

Proof. The proof is given by direct computation. We calculate $[\mathbb{X} + \mathcal{X}, \mathbb{Y} + \mathcal{Y}] = \{\mathbb{X}, \mathbb{Y}\} + \iota_{\mathbb{X}}\delta\mathcal{Y} - \delta\iota_{\mathbb{X}}\mathcal{Y} + \iota_{\mathbb{Y}}\delta\mathcal{X} = \{\mathbb{Y}, \mathbb{X}\} + \iota_{\mathbb{X}}\delta\mathcal{Y} - \delta\iota_{\mathbb{Y}}\mathcal{X} + \iota_{\mathbb{Y}}\delta\mathcal{X} = \{\mathbb{Y}, \mathbb{X}\} + \mathcal{L}_{\mathbb{Y}}\mathcal{X} + \iota_{\mathbb{X}}\delta\mathcal{Y} = [\mathbb{Y} + \mathcal{Y}, \mathbb{X} + \mathcal{X}]$, where we used that $\delta\iota_{\mathbb{X}}\mathcal{Y} = \delta\iota_{\mathbb{Y}}\mathcal{X}$. □

Remark 6.2.9. This result coincides with the observation in Remark 6.1.2. Namely, we can construct an alternative bracket—the Courant bracket ([Cou90])—which is defined as the antisymmetrization[6] of the Dorfman bracket but does not satisfy the Jacobi identity. Once again, Lemma 6.2.8 essentially shows that, given the vanishing of the differential of the pairing, the Dorfman and Courant brackets are equivalent.

Theorem 6.2.10. *The subbundle $D \subseteq T\mathcal{F}^{\partial\partial} \oplus T^*\mathcal{F}^{\partial\partial}$ as defined in Remark 6.2.5 is closed under the Dorfman bracket and isotropic with respect to the natural pairing.*

Proof. By Definition 6.2.6, we know that $T\mathcal{F}^{\partial\partial} \oplus T^*\mathcal{F}^{\partial\partial}$ can be endowed with a Courant algebroid structure.

We begin by proving that the subbundle is isotropic. Once this is achieved, the proof of the closedness of the brackets can be simplified, as it will suffice to verify it for a smaller class of elements, thanks to the antisymmetry of the Courant bracket. In fact, due to Lemma 6.2.8, the Dorfman bracket and its antisymmetrized counterpart—the Courant bracket—are equivalent in an isotropic setting.

Thus, we obtain the following results.

$$\begin{aligned}
\langle \mathbb{J}_c + \mathcal{J}_c, \mathbb{E}_\zeta + \mathcal{E}_\zeta \rangle &= \iota_{\mathbb{J}_c} \int_\Gamma \iota_\zeta \frac{e^2}{2}\delta\omega + \iota_\zeta(\omega - \omega_0)e\delta e - \iota_{\mathbb{E}_\zeta}\int_\Gamma ce\delta e && (6.16)\\
&= \int_\Gamma \iota_\zeta \frac{e^2}{2} d_\omega c + \iota_\zeta(\omega-\omega_0)e[c,e] + ceL_\zeta^{\omega_0} e && (6.17)\\
&= \int_\Gamma -\frac{e^2}{2}\iota_\zeta d_\omega c + \frac{e^2}{2}[\iota_\zeta(\omega-\omega_0), c] + \frac{e^2}{2} L_\zeta^{\omega_0} c && (6.18)\\
&= 0, && (6.19)
\end{aligned}$$

$$\begin{aligned}
\langle \mathbb{J}_c + \mathcal{J}_c, \mathbb{K}_\eta + \mathcal{K}_\eta \rangle &= \iota_{\mathbb{J}_c}\int_\Gamma \eta e_m e\delta\omega + \eta(\omega-\omega_0)_m e\delta e - \iota_{\mathbb{K}_\eta}\int_\Gamma ce\delta e && (6.20)\\
&= \int_\Gamma \eta e_m e d_\omega c + \eta(\omega-\omega_0)_m[c, \frac{e^2}{2}] + ce(\eta d_{\omega_m^0} e - d\eta e_m) && (6.21)\\
&= \int_\Gamma d\eta e_m e c - \eta d_\omega(e_m e)c + c[\eta(\omega-\omega_0)_m, \frac{e^2}{2}] && (6.22)\\
&\quad + c\eta d_{\omega_m^0}\frac{e^2}{2} - ced\eta e_m && (6.23)\\
&= \int_\Gamma c\eta d_\omega(e_m e) + c\eta e d_{\omega_m} e && (6.24)\\
&= 0, && (6.25)
\end{aligned}$$

[6]We mean here a graded antisymmetrization.

where we imposed the condition Eq. (6.8),

$$\begin{aligned}
\langle \mathbb{J}_c + \mathcal{J}_c, \mathbb{F}_\lambda + \mathcal{F}_\lambda \rangle &= \iota_{\mathbb{J}_c} \int_\Gamma \lambda e_n e \delta\omega - \iota_{\mathbb{F}_\lambda} \int_\Gamma c e \delta e && (6.26)\\
&= \int_\Gamma \lambda e_n e d_\omega c - c d_\omega(\lambda e_n e) && (6.27)\\
&= 0, && (6.28)
\end{aligned}$$

$$\begin{aligned}
\langle \mathbb{E}_\zeta + \mathcal{E}_\zeta, \mathbb{K}_\eta + \mathcal{K}_\eta \rangle = \iota_{\mathbb{E}_\zeta} &\int_\Gamma \eta e_m e \delta\omega + \eta(\omega - \omega_0)_m e \delta e && (6.29)\\
- \iota_{\mathbb{K}_\eta} &\int_\Gamma \iota_\zeta \frac{e^2}{2} \delta\omega + \iota_\zeta(\omega - \omega_0) e \delta e && (6.30)\\
= &\int_\Gamma \eta e_m e(\iota_\zeta F_\omega + d_\omega \iota_\zeta(\omega - \omega_0)) - \eta(\omega - \omega_0)_m L_\zeta^{\omega_0} \frac{e^2}{2} && (6.31)\\
&- \iota_\zeta e e F_{\omega_m} - \iota_\zeta e e d_\omega(\eta(\omega - \omega_0)_m) && (6.32)\\
&+ \iota_\zeta(\omega - \omega_0) e \eta d_{\omega_m^0} e - \iota_\zeta(\omega - \omega_0) e d\eta e_m && (6.33)\\
= &\int_\Gamma \eta e_m \iota_\zeta e F_\omega - d\eta e_m e \iota_\zeta(\omega - \omega_0) && (6.34)\\
&+ \eta d_\omega(e_m e)\iota_\zeta(\omega - \omega_0) + \eta(\omega - \omega_0)_m d_{\omega_0} \iota_\zeta \frac{e^2}{2} && (6.35)\\
&- \iota_\zeta e e F_{\omega_m} - \eta d_\omega \iota_\zeta \frac{e^2}{2}(\omega - \omega_0)_m && (6.36)\\
&+ \iota_\zeta(\omega - \omega_0) e \eta d_{\omega_m^0} e - \iota_\zeta(\omega - \omega_0) e d\eta e_m && (6.37)\\
= &\int_\Gamma \eta e_m \iota_\zeta e F_\omega + \eta e d_{\omega_m} e \iota_\zeta(\omega - \omega_0) - \iota_\zeta e e F_{\omega_m} && (6.38)\\
&- \eta(\omega - \omega_0)_m[\omega - \omega_0, \iota_\zeta \frac{e^2}{2}] + \iota_\zeta(\omega - \omega_0) e \eta d_{\omega_m^0} e && (6.39)\\
= &\int_\Gamma \eta \iota_\zeta e(e_m F_\omega - e F_{\omega_m}) - \eta e d_{\omega_m} e \iota_\zeta(\omega - \omega_0) && (6.40)\\
&+ \eta \iota_\zeta(\omega - \omega_0)([(\omega - \omega_0)_m, \frac{e^2}{2}] - d_{\omega_m^0} \frac{e^2}{2}) && (6.41)\\
= &\, 0, && (6.42)
\end{aligned}$$

where we both used Eq. (6.8) and Eq. (6.9),

$$\langle \mathbb{E}_\zeta + \mathcal{E}_\zeta, \mathbb{F}_\lambda + \mathcal{F}_\lambda \rangle = \iota_{\mathbb{E}_\zeta} \int_\Gamma \lambda e_n e \delta\omega - \iota_{\mathbb{F}_\lambda} \int_\Gamma \iota_\zeta \frac{e^2}{2} \delta\omega + \iota_\zeta(\omega - \omega_0) e \delta e \tag{6.43}$$

$$= \int_\Gamma \lambda e_n e(-\iota_\zeta F_\omega + d\omega \iota_\zeta(\omega - \omega_0)) \tag{6.44}$$

$$- \iota_\zeta e \lambda e_n F_\omega - \iota_\zeta(\omega - \omega_0) d_\omega(\lambda e_n e) \tag{6.45}$$

$$= 0, \tag{6.46}$$

and

$$\langle \mathbb{F}_\lambda + \mathcal{F}_\lambda, \mathbb{K}_\eta + \mathcal{K}_\eta \rangle = \iota_{\mathbb{F}_\lambda} \int_\Gamma \eta e_m e \delta\omega + \eta(\omega - \omega_0)_m e \delta e - \iota_{\mathbb{K}_\eta} \int_\Gamma \lambda e_n e \delta\omega \tag{6.47}$$

$$= \int_\Gamma \eta e_m \lambda e_n F_\omega + \eta(\omega - \omega_0)_m d_\omega(\lambda e_n e) \tag{6.48}$$

$$- \lambda e_n e \eta F_{\omega_m} - \lambda e_n e d_\omega(\eta(\omega - \omega_0)_m) \tag{6.49}$$

$$= \int_\Gamma \eta \lambda e_n (e_m F_\omega - e F_{\omega_m}) \tag{6.50}$$

$$= 0, \tag{6.51}$$

where we implemented again Eq. (6.9), which concludes the proof of the isotropy.

In order to prove that the Dorfman bracket on $\Gamma(D)$ is involutive, we first have to complete the construction of D, i.e. we have to consider the Hamiltonian vector fields for the functionals in Definition 6.2.1[7]—i.e. vector fields over $\mathcal{F}^\partial$—and push them forward to $\Gamma(T\mathcal{F}^{\partial\partial})$. Therefore, we have to project the Hamiltonian vector fields on the boundary to the corner, taking into account its transversal direction. We achieve this result by means of Eq. (6.5) and Eq. (6.6), as well as $\iota_\xi \mapsto \iota_\zeta + \eta\partial_m$.

For J_c, we simply have

$$\mathbb{J}_{c,e} = [c, e] \qquad \mathbb{J}_{c,e_m} = [c, e_m] \tag{6.52}$$

$$\mathbb{J}_{c,\omega} = d_\omega c \qquad \mathbb{J}_{c,\omega_m} = d_{\omega_m} c. \tag{6.53}$$

[7]See, for example, [CCS21a] for a derivation of the Hamiltonian vector fields on the boundary for non-degenerate gravity.

Then, we observe the following results for the Lie derivatives

$$
\begin{aligned}
-L_\xi^{\omega_0} e \mapsto & -(\iota_\zeta + \eta\iota_{\partial_m})(d_{\omega_0} + dx^m d_{\omega_m^0})(e - e_m dx^m) && (6.54)\\
& + (d_{\omega_0} + dx^m d_{\omega_m^0})(\iota_\zeta + \eta\iota_{\partial_m})(e - e_m dx^m) && (6.55)\\
= & -(\iota_\zeta + \eta\iota_{\partial_m})(d_{\omega_0} e - d_{\omega_0} e_m dx^m + dx^m d_{\omega_m^0} e) && (6.56)\\
& + (d_{\omega_0} + dx^m d_{\omega_m^0})(\iota_\zeta e + \eta e_m) && (6.57)\\
= & -(\iota_\zeta d_{\omega_0} e - dx^m \iota_\zeta d_{\omega_0} e_m + dx^m \iota_\zeta d_{\omega_m^0} e && (6.58)\\
& - \eta d_{\omega_0} e_m + \eta d_{\omega_m^0} e) + d_{\omega_0}\iota_\zeta e + d_{\omega_0}\eta e_m && (6.59)\\
& - \eta d_{\omega_0} e_m + dx^m d_{\omega_m^0}\iota_\zeta e && (6.60)\\
& + dx^m(\partial_m \eta e_m - \eta d_{\omega_m^0} e_m) && (6.61)\\
= & -L_\zeta^{\omega_0} e - dx^m(-L_\zeta^{\omega_0} e_m + \iota_\zeta d_{\omega_m^0} e - d_{\omega_m^0}\iota_\zeta e) && (6.62)\\
& - \eta d_{\omega_m^0} e + d\eta e_m + dx^m d_{\omega_m^0}(\eta e_m) && (6.63)\\
= & -L_\zeta^{\omega_0} e + dx^m(L_\zeta^{\omega_0} e_m + \iota_{\partial_m \zeta} e) - \eta d_{\omega_m^0} e && (6.64)\\
& + d\eta e_m + dx^m d_{\omega_m^0}(\eta e_m), && (6.65)
\end{aligned}
$$

$$
\begin{aligned}
-L_\xi^{\omega_0}(\omega - \omega_0) \mapsto & -(\iota_\zeta + \eta\iota_{\partial_m})(d_{\omega_0} + dx^m d_{\omega_m^0})(\omega - \omega_0 + dx^m(\omega - \omega_m^0)) && (6.66)\\
& + (d_{\omega_0} + dx^m d_{\omega_m^0})(\iota_\zeta + \eta\iota_{\partial_m})(\omega - \omega_0 + dx^m(\omega - \omega_m^0)) && (6.67)\\
= & -(\iota_\zeta + \eta\iota_{\partial_m})(d_{\omega_0}(\omega - \omega_0) - dx^m d_{\omega_0}(\omega - \omega_0)_m && (6.68)\\
& + dx^m d_{\omega_m^0}(\omega - \omega_0)) + (d_{\omega_0} + dx^m d_{\omega_m^0})(\iota_\zeta(\omega - \omega_0) && (6.69)\\
& + \eta(\omega - \omega_0)_m) && (6.70)\\
= & -(\iota_\zeta d_{\omega_0}(\omega - \omega_0) - dx^m \iota_\zeta d_{\omega_0}(\omega - \omega_0)_m && (6.71)\\
& + dx^m \iota_\zeta d_{\omega_m^0}(\omega - \omega_0) - \eta d_{\omega_0}(\omega - \omega_0)_m && (6.72)\\
& + \eta d_{\omega_m^0}(\omega - \omega_0)) + d_{\omega_0}\iota_\zeta(\omega - \omega_0) && (6.73)\\
& + d\eta(\omega - \omega_0)_m - \eta d_{\omega_0}(\omega - \omega_0)_m && (6.74)\\
& + dx^m(\iota_{\partial_m \zeta}(\omega - \omega_0) + \iota_\zeta d_{\omega_m^0}(\omega - \omega_0) && (6.75)\\
& + d_{\omega_m^0}(\eta(\omega - \omega_0)_m)) && (6.76)\\
= & -L_\zeta^{\omega_0}(\omega - \omega_0) + dx^m(L_\zeta^{\omega_0}(\omega - \omega_0)_m && (6.77)\\
& + \iota_{\partial_m \zeta}(\omega - \omega_0)) - \eta d_{\omega_m^0}(\omega - \omega_0) && (6.78)\\
& + d\eta(\omega - \omega_0)_m + dx^m d_{\omega_m^0}(\eta(\omega - \omega_0)_m) && (6.79)
\end{aligned}
$$

and

$$-\iota_\xi F_{\omega_0} \mapsto -(\iota_\zeta + \eta\iota_{\partial_m})(d_{\omega_0} + dx^m\partial_m\omega_0 + dx^m d\omega_m^0 \tag{6.80}$$
$$+ \frac{1}{2}[\omega_0,\omega_0] + dx^m[\omega_m^0,\omega_0]) \tag{6.81}$$
$$= -(\iota_\zeta + \eta\iota_{\partial_m})(F_{\omega_0} + dx^m(\partial_m\omega_0 - d_{\omega_0}\omega_m^0)) \tag{6.82}$$
$$= -\iota_\zeta F_{\omega_0} - dx^m(\iota_\zeta\partial_m\omega_0 - L_\zeta^{\omega_0}\omega_m^0) - \eta\partial_m\omega_0 + \eta d_{\omega_0}\omega_m^0. \tag{6.83}$$

With these identities we can write the other Hamiltonian vector fields, which are given by

$$\mathbb{E}_{\zeta,e} = -L_\zeta^{\omega_0} e \qquad \mathbb{E}_{\zeta,e_m} = L_\zeta^{\omega_0} e_m + \iota_{\partial_m\zeta} e \tag{6.84}$$
$$\mathbb{E}_{\zeta,\omega} = -\iota_\zeta F_{\omega_0} - L_\zeta^{\omega_0}(\omega-\omega_0) \qquad \mathbb{E}_{\zeta,\omega_m} = L_\zeta^{\omega_0}\omega_m + \iota_{\partial_m\zeta}(\omega-\omega_0) - \iota_{\partial_m\zeta}\omega_0 \tag{6.85}$$

$$\mathbb{F}_{\lambda,e} = d_\omega(\lambda e_n) + \lambda\sigma \qquad \mathbb{F}_{\lambda,e_m} = d_{\omega_m}(\lambda e_n) + \lambda\sigma_m \tag{6.86}$$
$$e\mathbb{F}_{\lambda,\omega} = \lambda e_n F_\omega \qquad e\mathbb{F}_{\lambda,\omega_m} + e_m\mathbb{F}_{\lambda,\omega} = \lambda e_n(\partial_m\omega - d_\omega\omega_m) \tag{6.87}$$

$$\mathbb{K}_{\eta,e} = -\eta d_{\omega_m^0} e + d\eta e_m \qquad \mathbb{K}_{\eta,e_m} = d_{\omega_m^0}(\eta e_m) \tag{6.88}$$
$$\mathbb{K}_{\eta,\omega} = -\eta d_{\omega_m^0}\omega + d\eta(\omega-\omega^0)_m + \eta d\omega_m^0 \qquad \mathbb{K}_{\eta,\omega_m} = d_{\omega_m^0}(\eta(\omega-\omega^0)_m). \tag{6.89}$$

Therefore, the Dorfman brackets are given by the following computations.[8]

$$[\mathbb{J}_c + \mathcal{J}_c, \mathbb{J}_c + \mathcal{J}_c] = -\frac{1}{2}\mathbb{J}_{[c,c]} + \cancel{\frac{1}{2}\iota_{\mathbb{J}_c}\delta\mathcal{J}_c} - \frac{1}{2}\delta(\iota_{\mathbb{J}_c}\mathcal{J}_c) + \cancel{\frac{1}{2}\iota_{\mathbb{J}_c}\delta\mathcal{J}_c} \tag{6.90}$$
$$= -\frac{1}{2}\mathbb{J}_{[c,c]} - \frac{1}{2}\int_\Gamma c[c, e\delta e] = -\frac{1}{2}\mathbb{J}_{[c,c]} - \frac{1}{2}\int_\Gamma [c,c]e\delta e \tag{6.91}$$
$$= -\frac{1}{2}\mathbb{J}_{[c,c]} - \frac{1}{2}\mathcal{J}_{[c,c]}, \tag{6.92}$$

where we used the exactness property of $\mathcal{J}_c$,

$$[\mathbb{E}_\zeta + \mathcal{E}_\zeta, \mathbb{J}_c + \mathcal{J}_c] = \mathbb{J}_{L_\zeta^{\omega_0}c} - \delta(\iota_{\mathbb{E}_\zeta}\mathcal{J}_c) \tag{6.93}$$
$$= \mathbb{J}_{L_\zeta^{\omega_0}c} + \delta\int_\Gamma ceL_\zeta^{\omega_0}e \tag{6.94}$$
$$= \mathbb{J}_{L_\zeta^{\omega_0}c} + \delta\int_\Gamma \frac{e^2}{2}L_\zeta^{\omega_0}c \tag{6.95}$$
$$= \mathbb{J}_{L_\zeta^{\omega_0}c} + \mathcal{J}_{L_\zeta^{\omega_0}c}, \tag{6.96}$$

[8] The criteria for choosing if to compute $[\mathbb{X}+\mathcal{X},\mathbb{Y}+\mathcal{Y}]$ or $[\mathbb{Y}+\mathcal{Y},\mathbb{X}+\mathcal{X}]$ were purely selected on the base of computational convenience.

$$
\begin{aligned}
[\mathbb{J}_c + \mathcal{J}_c, \mathbb{K}_\eta + \mathcal{K}_\eta] &= \mathbb{J}_{\eta d_{\omega^0_m} c} + \iota_{\mathbb{J}_c} \delta \mathcal{K}_\eta - \delta(\iota_{\mathbb{J}_c} \mathcal{K}_\eta) && (6.97)\\
&= \mathbb{J}_{\eta d_{\omega^0_m} c} + \iota_{\mathbb{J}_c} \int_\Gamma - \eta \delta e_m e \delta\omega + \eta e_m \delta e \delta \omega - \eta e \delta \omega_m \delta e && (6.98)\\
&\quad - \delta(\int_\Gamma \eta e_m e d_\omega c - \eta e(\omega - \omega_0)_m [c, e]) && (6.99)\\
&= \mathbb{J}_{\eta d_{\omega^0_m} c} + \int_\Gamma \eta e [c, e_m] \delta\omega - \eta \delta e_m e d_\omega c + \eta e_m [c, e] \delta \omega && (6.100)\\
&\quad + \eta e_m \delta e d_\omega c + \eta e d_{\omega_m} c \delta e + \eta e \delta \omega_m [c, e] && (6.101)\\
&\quad + \eta \delta e_m e d_\omega c - \eta e_m \delta e d_\omega c - \eta e_m [\delta\omega, c] && (6.102)\\
&\quad + \eta(\omega - \omega_0)_m [c, e\delta e] - \eta \delta\omega_m [c, \frac{e^2}{2}] && (6.103)\\
&= \mathbb{J}_{\eta d_{\omega^0_m} c} + \int_\Gamma \eta d_{\omega^0_m} c e \delta e && (6.104)\\
&= \mathbb{J}_{\eta d_{\omega^0_m} c} + \mathcal{J}_{\eta d_{\omega^0_m} c}, && (6.105)
\end{aligned}
$$

$$\begin{aligned}
[\mathbb{F}_\lambda + \mathcal{F}_\lambda, \mathbb{J}_c + \mathcal{J}_c] &= -\mathbb{E}_{[c,\lambda e_n]^i} - \mathbb{R}_{[c,\lambda e_n]^m} - \mathbb{F}_{[c,\lambda e_n]^n} && (6.106)\\
&\quad + \mathbb{J}_{[c,\lambda e_n]^i(\omega-\omega_0)_i} + \mathbb{J}_{[c,\lambda e_n]^m(\omega-\omega_0)_m} && (6.107)\\
&\quad - \delta(\iota_{\mathbb{F}_\lambda}\mathcal{J}_c) + \iota_{\mathbb{J}_c}\delta\mathcal{F}_\lambda && (6.108)\\
&= -\mathbb{E}_{[c,\lambda e_n]^i} - \mathbb{R}_{[c,\lambda e_n]^m} - \mathbb{F}_{[c,\lambda e_n]^n} && (6.109)\\
&\quad + \mathbb{J}_{[c,\lambda e_n]^i(\omega-\omega_0)_i} + \mathbb{J}_{[c,\lambda e_n]^m(\omega-\omega_0)_m} && (6.110)\\
&\quad - \delta\int_\Gamma ce(d_\omega(\lambda e_n) + \lambda\sigma) + \iota_{\mathbb{J}_c}\int_\Gamma \lambda e_n \delta e \delta\omega && (6.111)\\
&= -\mathbb{E}_{[c,\lambda e_n]^i} - \mathbb{R}_{[c,\lambda e_n]^m} - \mathbb{F}_{[c,\lambda e_n]^n} && (6.112)\\
&\quad + \mathbb{J}_{[c,\lambda e_n]^i(\omega-\omega_0)_i} + \mathbb{J}_{[c,\lambda e_n]^m(\omega-\omega_0)_m} && (6.113)\\
&\quad - \delta\int_\Gamma d_\omega c(e\lambda e_n) - \int_\Gamma \lambda e_n[c,e]\delta\omega - \lambda e_n d_\omega c\delta e && (6.114)\\
&= -\mathbb{E}_{[c,\lambda e_n]^i} - \mathbb{R}_{[c,\lambda e_n]^m} - \mathbb{F}_{[c,\lambda e_n]^n} && (6.115)\\
&\quad + \mathbb{J}_{[c,\lambda e_n]^i(\omega-\omega_0)_i} + \mathbb{J}_{[c,\lambda e_n]^m(\omega-\omega_0)_m} && (6.116)\\
&\quad - \int_\Gamma [\delta\omega, c], \lambda e_n e + \lambda e_n d_\omega c\delta e && (6.117)\\
&\quad + \int_\Gamma \lambda e_n[c,e]\delta\omega + \lambda e_n d_\omega c\delta e && (6.118)\\
&= -\mathbb{E}_{[c,\lambda e_n]^i} - \mathbb{R}_{[c,\lambda e_n]^m} - \mathbb{F}_{[c,\lambda e_n]^n} && (6.119)\\
&\quad + \mathbb{J}_{[c,\lambda e_n]^i(\omega-\omega_0)_i} + \mathbb{J}_{[c,\lambda e_n]^m(\omega-\omega_0)_m} && (6.120)\\
&\quad - \mathcal{E}_{[c,\lambda e_n]^i} - \mathcal{R}_{[c,\lambda e_n]^m} - \mathcal{F}_{[c,\lambda e_n]^n} && (6.121)\\
&\quad + \mathcal{J}_{[c,\lambda e_n]^i(\omega-\omega_0)_i} + \mathcal{J}_{[c,\lambda e_n]^m(\omega-\omega_0)_m}, && (6.122)
\end{aligned}$$

$$\begin{aligned}
[\mathbb{E}_\zeta + \mathcal{E}_\zeta, \mathbb{E}_\zeta + \mathcal{E}_\zeta] &= \frac{1}{2}\mathbb{E}_{[\zeta,\zeta]} - \frac{1}{2}\mathbb{J}_{\iota_\zeta\iota_\zeta F_{\omega_0}} - \frac{1}{2}\delta(\iota_{\mathbb{E}_\zeta}\mathcal{E}_\zeta) && (6.123)\\
&= \frac{1}{2}\mathbb{E}_{[\zeta,\zeta]} - \frac{1}{2}\mathbb{J}_{\iota_\zeta\iota_\zeta F_{\omega_0}} && (6.124)\\
&\quad + \frac{1}{2}\delta\int_\Gamma \iota_\zeta\frac{e^2}{2}(\iota_\zeta F_{\omega_0} + L_\zeta^{\omega_0}(\omega-\omega_0)) + \iota_\zeta(\omega-\omega_0)L_\zeta^{\omega_0}\frac{e^2}{2} && (6.125)\\
&= -\frac{1}{2}\delta\int_\Gamma \frac{e^2}{2}\iota_\zeta\iota_\zeta F_{\omega_0} + \frac{e^2}{2}\iota_\zeta L_\zeta^{\omega_0}(\omega-\omega_0) && (6.126)\\
&\quad - \frac{e^2}{2}(\iota_{[\zeta,\zeta]}(\omega-\omega_0) + \iota_\zeta L_\zeta^{\omega_0}(\omega-\omega_0)) && (6.127)\\
&= \frac{1}{2}\mathbb{E}_{[\zeta,\zeta]} + \frac{1}{2}\mathcal{E}_{[\zeta,\zeta]} - \frac{1}{2}\mathbb{J}_{\iota_\zeta\iota_\zeta F_{\omega_0}} - \frac{1}{2}\mathcal{J}_{\iota_\zeta\iota_\zeta F_{\omega_0}}, && (6.128)
\end{aligned}$$

$$
\begin{aligned}
[\mathbb{E}_\zeta + \mathcal{E}_\zeta, \mathbb{K}_\eta + \mathcal{K}_\eta] &= \mathbb{K}_{\iota_\zeta d\eta} + \mathbb{J}_{\eta\iota_\zeta F_{\omega_m^0}} + \mathbb{E}_{\eta\partial_m\zeta} + \iota_{\mathbb{E}_\eta}\delta\mathcal{K}_\eta - \delta(\iota_{\mathbb{E}_\zeta}\mathcal{K}_\eta) && (6.129)\\
&= \mathbb{K}_{\iota_\zeta d\eta} + \mathbb{J}_{\eta\iota_\zeta F_{\omega_m^0}} + \mathbb{E}_{\eta\partial_m\zeta} && (6.130)\\
&\quad \iota_{\mathbb{E}_\zeta}\int_\Gamma - \eta\delta e_m e\delta\omega + \eta e_m \delta e\delta\omega - \eta e\delta\omega_m\delta e && (6.131)\\
&\quad - \delta\int_\Gamma \eta e_m e(-\iota_\zeta F_{\omega_0} - L_\zeta^{\omega_0}(\omega-\omega_0)) && (6.132)\\
&\quad + \eta e(\omega-\omega_0)_m L_\zeta^{\omega_0} e && (6.133)\\
&= \mathbb{K}_{\iota_\zeta d\eta} + \mathbb{J}_{\eta\iota_\zeta F_{\omega_m^0}} + \mathbb{E}_{\eta\partial_m\zeta} && (6.134)\\
&\quad + \int_\Gamma \eta L_\zeta^{\omega_0} e_m e\delta\omega - \eta\delta e_m e(-\iota_\zeta F_{\omega_0} - L_\zeta^{\omega_0}(\omega-\omega_0)) && (6.135)\\
&\quad - \eta e_m L_\zeta^{\omega_0} e\delta\omega + \eta e_m(-\iota_\zeta F_{\omega_0} - L_\zeta^{\omega_0}(\omega-\omega_0))\delta e && (6.136)\\
&\quad - \eta e(-L_\zeta^{\omega_0}\omega_m - \iota_{\partial_m\zeta}(\omega-\omega_0) + \iota_\zeta\partial_m\omega_0)\delta e && (6.137)\\
&\quad - \eta e\delta\omega_m L_\zeta^{\omega_0} e + \eta\delta e_m e(-\iota_\zeta F_{\omega_0} - L_\zeta^{\omega_0}(\omega-\omega_0)) && (6.138)\\
&\quad - \eta e_m \delta e(-\iota_\zeta F_{\omega_0} - L_\zeta^{\omega_0}(\omega-\omega_0)) && (6.139)\\
&\quad + \eta(\omega-\omega_0)_m L_\zeta^{\omega_0}(e\delta e) - \eta\delta\omega_m L_\zeta^{\omega_0}(\frac{e^2}{2}) && (6.140)\\
&\quad - \eta e_m e L_\zeta^{\omega_0}\delta\omega + \eta\iota_{\partial_m\zeta}\frac{e^2}{2}\delta\omega && (6.141)\\
&= \mathbb{K}_{\iota_\zeta d\eta} + \mathbb{J}_{\eta\iota_\zeta F_{\omega_m^0}} + \mathbb{E}_{\eta\partial_m\zeta} && (6.142)\\
&\quad + \int_\Gamma \iota_\zeta d\eta e_m e\delta\omega + \iota_\zeta d\eta e(\omega-\omega_0)_m\delta e && (6.143)\\
&\quad - \eta L_\zeta^{\omega_0}(\omega-\omega_0)_m e\delta e + \eta e(L_\zeta^{\omega_0}\omega_m + \iota_{\partial_m\zeta}(\omega-\omega_0) && (6.144)\\
&\quad + \iota_\zeta\partial_m\omega_0)\delta e && (6.145)\\
&= \mathbb{K}_{\iota_\zeta d\eta} + \mathbb{J}_{\eta\iota_\zeta F_{\omega_m^0}} + \mathbb{E}_{\eta\partial_m\zeta} && (6.146)\\
&\quad + \mathcal{K}_{\iota_\zeta d\eta} + \int_\Gamma \eta e(\iota_\zeta d\omega_m^0 + \iota_{\partial_m\zeta}(\omega-\omega_0) + \iota_\zeta\partial_m\omega_0)\delta e && (6.147)\\
&\quad + \eta\iota_{\partial_m\zeta}\frac{e^2}{2}\delta\omega && (6.148)\\
&= \mathbb{K}_{\iota_\zeta d\eta} + \mathbb{J}_{\eta\iota_\zeta F_{\omega_m^0}} + \mathbb{E}_{\eta\partial_m\zeta} && (6.149)\\
&\quad + \mathcal{K}_{\iota_\zeta d\eta} + \mathcal{J}_{\eta\iota_\zeta F_{\omega_m^0}} + \int_\Gamma \eta(\iota_{\partial_m\zeta}\frac{e^2}{2}\delta\omega && (6.150)\\
&\quad + \iota_{\partial_m\zeta}(\omega-\omega_0)\frac{\delta e^2}{2}) && (6.151)\\
&= \mathbb{K}_{\iota_\zeta d\eta} + \mathbb{J}_{\eta\iota_\zeta F_{\omega_m^0}} + \mathbb{E}_{\eta\partial_m\zeta} && (6.152)\\
&\quad + \mathcal{K}_{\iota_\zeta d\eta} + \mathcal{J}_{\eta\iota_\zeta F_{\omega_m^0}} + \mathcal{E}_{\eta\partial_m\zeta} && (6.153)
\end{aligned}
$$

$$[\mathbb{K}_\eta + \mathcal{K}_\eta, \mathbb{K}_\eta + \mathcal{K}_\eta] = \mathbb{K}_{\eta\partial_m\eta} + \iota_{\mathbb{K}_\eta}\delta\mathcal{K}_\eta - \frac{1}{2}\delta(\iota_{\mathbb{K}_\eta}\mathcal{K}_\eta) \tag{6.154}$$

$$= \mathbb{K}_{\eta\partial_m\eta} + \iota_{\mathbb{K}_\eta}\int_\Gamma - \eta\delta e_m e\delta\omega \tag{6.155}$$

$$+ \eta e_m\delta e\delta\omega - \eta e\delta\omega_m\delta e \tag{6.156}$$

$$+ \frac{1}{2}\delta\int_\Gamma \eta(ee_m\delta\omega + (\omega - \omega_0)_m e\delta e \tag{6.157}$$

$$= \mathbb{K}_{\eta\partial_m\eta} + \int_\Gamma \eta\partial_m\eta e_m e\delta\omega \tag{6.158}$$

$$+ \eta\delta e_m ed\eta(\omega - \omega_0)_m + \eta e_m\delta ed\eta(\omega - \omega_0)_m \tag{6.159}$$

$$+ \eta e\partial_m\eta(\omega - \omega_0)_m\delta e - \eta ed\eta e_m\delta\omega_m \tag{6.160}$$

$$+ \frac{1}{2}\delta\int_\Gamma \eta ee_m d\eta(\omega - \omega_0)_m + \eta(\omega - \omega_0)_m ed\eta e_m \tag{6.161}$$

$$= \mathbb{K}_{\eta\partial_m\eta} + \int_\Gamma \eta\partial_m\eta(e_m e\delta\omega + e(\omega - \omega_0)_m\delta e) \tag{6.162}$$

$$+ \eta d\eta\delta(e_m e(\omega - \omega_0)_m) \tag{6.163}$$

$$+ \delta\int_\Gamma \eta d\eta e_m e(\omega - \omega_0)_m \tag{6.164}$$

$$= \mathbb{K}_{\eta\partial_m\eta} + \int_\Gamma \eta\partial_m\eta(e_m e\delta\omega + e(\omega - \omega_0)_m\delta e) \tag{6.165}$$

$$+ \eta d\eta\delta(e_m e(\omega - \omega_0)_m) \tag{6.166}$$

$$- \int_\Gamma \eta d\eta\delta(e_m e(\omega - \omega_0)_m) \tag{6.167}$$

$$= \mathbb{K}_{\eta\partial_m\eta} + \int_\Gamma \eta\partial_m\eta(e_m e\delta\omega + e(\omega - \omega_0)_m\delta e) \tag{6.168}$$

$$= \mathbb{K}_{\eta\partial_m\eta} + \mathcal{K}_{\eta\partial_m\eta}, \tag{6.169}$$

$$[\mathbb{F}_\lambda + \mathcal{F}_\lambda, \mathbb{F}_\lambda + \mathcal{F}_\lambda] = \iota_{\mathbb{F}_\lambda}\delta\mathcal{F}_\lambda - \frac{1}{2}\delta(\iota_{\mathbb{F}_\lambda}\mathcal{F}_\lambda) \tag{6.170}$$

$$= \iota_{\mathbb{F}_\lambda}\int_\Gamma \lambda e_n\delta e\delta\omega - \frac{1}{2}\delta\int_\Gamma \lambda e_n\lambda e_n F_\omega \tag{6.171}$$

$$= \int_\Gamma \lambda e_n(d_\omega(\lambda e_n) + \lambda\sigma)\delta\omega + \lambda e_n\mathbb{F}_{\lambda,\omega}\delta e \tag{6.172}$$

$$= \int_\Gamma \lambda e_n(d_\omega\lambda e_n - \lambda d_\omega e_n)\delta\omega \tag{6.173}$$

$$= 0, \tag{6.174}$$

where we used $\lambda^2 = e_n^2 = 0$ and the fact that $\mathbb{F}_{\lambda,\omega}$ is proportional to λ,

$$
\begin{aligned}
[\mathbb{F}_\lambda + \mathcal{F}_\lambda, \mathbb{K}_\eta + \mathcal{K}_\eta] &= \mathbb{E}_{\eta\partial_m(\lambda e_n)^i} + \mathbb{F}_{\eta\partial_m(\lambda e_n)^n} + \mathbb{K}_{\eta\partial_m(\lambda e_n)^m} && (6.175)\\
&\quad - \mathbb{J}_{\eta\partial_m(\lambda e_n)^i(\omega-\omega_0)_i} - \mathbb{J}_{\eta\partial_m(\lambda e_n)^m(\omega-\omega_0)_m} && (6.176)\\
&\quad + \iota_{\mathbb{F}_\lambda}\delta\mathcal{K}_\eta - \delta(\iota_{\mathbb{F}_\lambda}\mathcal{K}_\eta) + \iota_{\mathbb{K}_\eta}\delta\mathcal{F}_\lambda && (6.177)\\
&= \mathbb{E}_{\eta\partial_m(\lambda e_n)^i} + \mathbb{F}_{\eta\partial_m(\lambda e_n)^n} + \mathbb{K}_{\eta\partial_m(\lambda e_n)^m} && (6.178)\\
&\quad - \mathbb{J}_{\eta\partial_m(\lambda e_n)^i(\omega-\omega_0)_i} - \mathbb{J}_{\eta\partial_m(\lambda e_n)^m(\omega-\omega_0)_m} && (6.179)\\
&\quad + \iota_{\mathbb{F}_\lambda}\int_\Gamma - \eta\delta e e_m\delta\omega - \eta\delta e_m e\delta\omega - \eta e\delta\omega_m\delta e && (6.180)\\
&\quad - \delta\int_\Gamma \eta e_m\lambda e_n F_\omega - \eta(\omega-\omega_0)_m d_\omega(\lambda e_n e) && (6.181)\\
&\quad + \iota_{\mathbb{K}_\eta}\int_\Gamma \lambda e_n\delta e\delta\omega && (6.182)\\
&= \mathbb{E}_{\eta\partial_m(\lambda e_n)^i} + \mathbb{F}_{\eta\partial_m(\lambda e_n)^n} + \mathbb{K}_{\eta\partial_m(\lambda e_n)^m} && (6.183)\\
&\quad - \mathbb{J}_{\eta\partial_m(\lambda e_n)^i(\omega-\omega_0)_i} - \mathbb{J}_{\eta\partial_m(\lambda e_n)^m(\omega-\omega_0)_m} && (6.184)\\
&\quad + \int_\Gamma \eta(d_\omega(\lambda e_n) + \lambda\sigma)e_m\delta\omega + \eta e_m\mathbb{F}_\omega\delta e && (6.185)\\
&\quad + \eta(d_{\omega_m}(\lambda e_n) + \lambda\sigma)e\delta\omega - \eta\delta e_m\lambda e_n F_\omega && (6.186)\\
&\quad + \eta e\mathbb{F}_\omega\delta e - \eta e(d_\omega(\lambda e_n) + \lambda\sigma)\delta\omega_m && (6.187)\\
&\quad + \int_\Gamma \eta\delta e_m\lambda e_n F_\omega + \eta e_m\lambda e_n d_\omega\delta\omega && (6.188)\\
&\quad - \eta\delta\omega_m d_\omega(\lambda e_n e) - \eta(\omega-\omega_0)_m[\delta\omega, \lambda e_n e] && (6.189)\\
&\quad + \eta(\omega-\omega_0)_m d_\omega(\lambda e_n\delta e) && (6.190)\\
&\quad + \int_\Gamma \lambda e_n(-\eta d_{\omega_m^0} e - d\eta e_m)\delta\omega && (6.191)\\
&\quad + \lambda e_n(-\eta d_{\omega_m^0}\omega + d\eta(\omega-\omega_0)_m)\delta e && (6.192)\\
&= \mathbb{E}_{\eta\partial_m(\lambda e_n)^i} + \mathbb{F}_{\eta\partial_m(\lambda e_n)^n} + \mathbb{K}_{\eta\partial_m(\lambda e_n)^m} && (6.193)\\
&\quad - \mathbb{J}_{\eta\partial_m(\lambda e_n)^i(\omega-\omega_0)_i} - \mathbb{J}_{\eta\partial_m(\lambda e_n)^m(\omega-\omega_0)_m} && (6.194)\\
&\quad + \int_\Gamma \eta(d_\omega(\lambda e_n) + \lambda\sigma)e_m\delta\omega + \eta\lambda e_n(d_\omega\omega_m - \partial_m\omega)\delta e && (6.195)\\
&\quad + \eta(d_{\omega_m}(\lambda e_n) + \lambda\sigma_m)e\delta\omega - \eta\delta e_m\lambda e_n F_\omega && (6.196)\\
&\quad - \eta d_\omega(\lambda e_n e)\delta\omega_m + \int_\Gamma \eta\lambda e_n F_\omega\delta e_m - d_\omega(\eta e_m\lambda e_n)\delta\omega && (6.197)\\
&\quad + \eta\delta\omega[(\omega-\omega_0)_m\lambda e_n e] + d_\omega(\eta(\omega-\omega_0)_m)\lambda e_n\delta e && (6.198)\\
&\quad + \int_\Gamma \lambda e_n(-\eta d_{\omega_m^0} e - d\eta e_m)\delta\omega && (6.199)\\
&\quad + \lambda e_n(-\eta d_{\omega_m^0}\omega + d\eta(\omega-\omega_0)_m)\delta e && (6.200)
\end{aligned}
$$

$$= \mathbb{E}_{\eta\partial_m(\lambda e_n)^i} + \mathbb{F}_{\eta\partial_m(\lambda e_n)^n} + \mathbb{K}_{\eta\partial_m(\lambda e_n)^m} \tag{6.201}$$
$$- \mathbb{J}_{\eta\partial_m(\lambda e_n)^i(\omega-\omega_0)_i} - \mathbb{J}_{\eta\partial_m(\lambda e_n)^m(\omega-\omega_0)_m} \tag{6.202}$$
$$+ \int_\Gamma \eta(d_\omega(\lambda e_n) + \lambda\sigma)e_m\delta\omega + \eta\lambda e_n(d_\omega\omega_m - \partial_m\omega)\delta e \tag{6.203}$$
$$+ \eta(d_{\omega_m}(\lambda e_n) + \lambda\sigma_m)e\delta\omega - \eta\delta e_m\lambda e_n F_\omega \tag{6.204}$$
$$- \eta d_\omega(\lambda e_n e)\delta\omega_m + \int_\Gamma d\eta e_m\lambda e_n\delta\omega \tag{6.205}$$
$$+ \eta\lambda(e_n d_{\omega_m} ee\sigma_m)\delta\omega - \eta e_m d_\omega(\lambda e_n)\delta\omega \tag{6.206}$$
$$+ \eta\delta\omega[(\omega-\omega_0)_m, \lambda e_n e] + d\eta(\omega-\omega_0)_m\lambda e_n\delta e \tag{6.207}$$
$$- \eta d_\omega(\omega-\omega_0)_m\lambda e_n\delta e + \int_\Gamma \lambda e_n(-\eta d_{\omega^0_m} e - d\eta e_m)\delta\omega \tag{6.208}$$
$$+ \lambda e_n(-\eta d_{\omega^0_m}\omega + d\eta(\omega-\omega_0)_m)\delta e \tag{6.209}$$
$$= \mathbb{E}_{\eta\partial_m(\lambda e_n)^i} + \mathbb{F}_{\eta\partial_m(\lambda e_n)^n} + \mathbb{K}_{\eta\partial_m(\lambda e_n)^m} \tag{6.210}$$
$$- \mathbb{J}_{\eta\partial_m(\lambda e_n)^i(\omega-\omega_0)_i} - \mathbb{J}_{\eta\partial_m(\lambda e_n)^m(\omega-\omega_0)_m} \tag{6.211}$$
$$+ \int_\Gamma \eta\lambda e_n(d_\omega\omega_m - \partial_m\omega)\delta e + \eta\partial_m(\lambda e_n)e\delta\omega \tag{6.212}$$
$$- \eta\lambda e_n d_{\omega^0_m} e\omega - \eta\lambda e_n d_{\omega^0_m}\omega\delta e + \eta\lambda e_n d_{\omega_m} e\delta\omega \tag{6.213}$$
$$+ \eta\delta\omega[(\omega-\omega_0)_m, \lambda e_n e] - \eta d_\omega\omega_m\lambda e_n\delta e \tag{6.214}$$
$$+ \eta d_\omega\omega^0_m\lambda e_n\delta e \tag{6.215}$$
$$= \mathbb{E}_{\eta\partial_m(\lambda e_n)^i} + \mathbb{F}_{\eta\partial_m(\lambda e_n)^n} + \mathbb{K}_{\eta\partial_m(\lambda e_n)^m} \tag{6.216}$$
$$- \mathbb{J}_{\eta\partial_m(\lambda e_n)^i(\omega-\omega_0)_i} - \mathbb{J}_{\eta\partial_m(\lambda e_n)^m(\omega-\omega_0)_m} \tag{6.217}$$
$$+ \mathcal{E}_{\eta\partial_m(\lambda e_n)^i} + \mathcal{F}_{\eta\partial_m(\lambda e_n)^n} + \mathcal{K}_{\eta\partial_m(\lambda e_n)^m} \tag{6.218}$$
$$- \mathcal{J}_{\eta\partial_m(\lambda e_n)^i(\omega-\omega_0)_i} - \mathcal{J}_{\eta\partial_m(\lambda e_n)^m(\omega-\omega_0)_m}, \tag{6.219}$$

and

$$
\begin{aligned}
[\mathbb{E}_\zeta + \mathcal{E}_\zeta, \mathbb{F}_\lambda + \mathcal{F}_\lambda] &= \mathbb{E}_{L_\zeta^{\omega_0}(\lambda e_n)^i} + \mathbb{F}_{L_\zeta^{\omega_0}(\lambda e_n)^n} + \mathbb{R}_{L_\zeta^{\omega_0}(\lambda e_n)^m} && (6.220)\\
&\quad - \mathbb{J}_{L_\zeta^{\omega_0}(\lambda e_n)^i(\omega-\omega_0)_i} - \mathbb{J}_{L_\zeta^{\omega_0}(\lambda e_n)^m(\omega-\omega_0)_m} && (6.221)\\
&\quad + \iota_{\mathbb{E}_\zeta}\delta\mathcal{F}_\lambda - \delta(\iota_{\mathbb{E}_\zeta}\mathcal{F}_\lambda) && (6.222)\\
&= \mathbb{E}_{L_\zeta^{\omega_0}(\lambda e_n)^i} + \mathbb{F}_{L_\zeta^{\omega_0}(\lambda e_n)^n} + \mathbb{R}_{L_\zeta^{\omega_0}(\lambda e_n)^m} && (6.223)\\
&\quad - \mathbb{J}_{L_\zeta^{\omega_0}(\lambda e_n)^i(\omega-\omega_0)_i} - \mathbb{J}_{L_\zeta^{\omega_0}(\lambda e_n)^m(\omega-\omega_0)_m} && (6.224)\\
&\quad + \iota_{\mathbb{E}_\zeta}\int_\Gamma \lambda e_n \delta e \delta\omega + \delta\int_\Gamma \lambda e_n e(L_\zeta^{\omega_0}(\omega-\omega_0) + \iota_\zeta F_{\omega_0}) && (6.225)\\
&= \mathbb{E}_{L_\zeta^{\omega_0}(\lambda e_n)^i} + \mathbb{F}_{L_\zeta^{\omega_0}(\lambda e_n)^n} + \mathbb{R}_{L_\zeta^{\omega_0}(\lambda e_n)^m} && (6.226)\\
&\quad - \mathbb{J}_{L_\zeta^{\omega_0}(\lambda e_n)^i(\omega-\omega_0)_i} - \mathbb{J}_{L_\zeta^{\omega_0}(\lambda e_n)^m(\omega-\omega_0)_m} && (6.227)\\
&\quad - \int_\Gamma \lambda e_n L_\zeta^{\omega_0} e\delta\omega + \lambda e_n(\iota_\zeta F_{\omega_0} + L_\zeta^{\omega_0}(\omega-\omega_0))\delta e && (6.228)\\
&\quad + \lambda e_n(-\iota_\zeta F_{\omega_0} - L_\zeta^{\omega_0}(\omega-\omega_0))\delta e + \lambda e_n e L_\zeta^{\omega_0}\delta\omega && (6.229)\\
&= \mathbb{E}_{L_\zeta^{\omega_0}(\lambda e_n)^i} + \mathbb{F}_{L_\zeta^{\omega_0}(\lambda e_n)^n} + \mathbb{R}_{L_\zeta^{\omega_0}(\lambda e_n)^m} && (6.230)\\
&\quad - \mathbb{J}_{L_\zeta^{\omega_0}(\lambda e_n)^i(\omega-\omega_0)_i} - \mathbb{J}_{L_\zeta^{\omega_0}(\lambda e_n)^m(\omega-\omega_0)_m} && (6.231)\\
&\quad + \int_\Gamma L_\zeta^{\omega_0}(\lambda e_n) e\delta\omega && (6.232)\\
&= \mathbb{E}_{L_\zeta^{\omega_0}(\lambda e_n)^i} + \mathbb{F}_{L_\zeta^{\omega_0}(\lambda e_n)^n} + \mathbb{R}_{L_\zeta^{\omega_0}(\lambda e_n)^m} && (6.233)\\
&\quad - \mathbb{J}_{L_\zeta^{\omega_0}(\lambda e_n)^i(\omega-\omega_0)_i} - \mathbb{J}_{L_\zeta^{\omega_0}(\lambda e_n)^m(\omega-\omega_0)_m} && (6.234)\\
&\quad + \int_\Gamma (L_\zeta^{\omega_0}(\lambda e_n)^i e_i + L_\zeta^{\omega_0}(\lambda e_n)^n e_n + L_\zeta^{\omega_0}(\lambda e_n)^m e_m) e\delta\omega && (6.235)\\
&= \mathbb{E}_{L_\zeta^{\omega_0}(\lambda e_n)^i} + \mathbb{F}_{L_\zeta^{\omega_0}(\lambda e_n)^n} + \mathbb{R}_{L_\zeta^{\omega_0}(\lambda e_n)^m} && (6.236)\\
&\quad - \mathbb{J}_{L_\zeta^{\omega_0}(\lambda e_n)^i(\omega-\omega_0)_i} - \mathbb{J}_{L_\zeta^{\omega_0}(\lambda e_n)^m(\omega-\omega_0)_m} && (6.237)\\
&\quad + \mathcal{E}_{L_\zeta^{\omega_0}(\lambda e_n)^i} + \mathcal{F}_{L_\zeta^{\omega_0}(\lambda e_n)^n} + \mathcal{R}_{L_\zeta^{\omega_0}(\lambda e_n)^m} && (6.238)\\
&\quad - \mathcal{J}_{L_\zeta^{\omega_0}(\lambda e_n)^i(\omega-\omega_0)_i} - \mathcal{J}_{L_\zeta^{\omega_0}(\lambda e_n)^m(\omega-\omega_0)_m}. && (6.239)
\end{aligned}
$$

□

Remark 6.2.11. In this setting, the isotropy is a sort of on-shell condition. In fact, Eq. (6.8) and Eq. (6.9) are, as we already mentioned, the transversal to the corner components of the constraints Definition 6.2.1 on the boundary and they are satisfied on their zero locus.

Remark 6.2.12. Notice that the only thing left to prove in order to show that the subbundle D forms a Dirac structure is that the isotropy is maximal, possibly on a suitable submanifold of $\mathcal{F}^{\partial\partial}$.

Appendix A

Linear maps, decompositions and contractions

Lemma A.0.1. *Let* $e_n \in \Omega_\Sigma^{0,1}$ *be as in Lemma 4.2.14 and* $\alpha \in \Omega_\Sigma^{2,1}$*. Then, we have*

$$\alpha = 0$$

if and only if

$$\begin{cases} \alpha \in \mathrm{Ker} W_1^{\Sigma,(2,1)} \\ e_n \alpha \in \mathrm{Im}\, W_1^{\Sigma,(1,1)}. \end{cases} \tag{A.1}$$

Proof. See [CCS21a]. □

Corollary A.0.2. *Let* $e_n \in \Omega_\Sigma^{0,1}$ *be as in Lemma 4.2.14 and* $\gamma \in \Omega_\Sigma^{2,2}$*. Then, we have the unique decomposition*

$$\gamma = e\sigma + e_n \alpha, \tag{A.2}$$

with $\sigma \in \Omega_\Sigma^{1,1}$ *and* $\alpha \in \mathrm{Ker} W_1^{\Sigma,(2,1)}$*.*

Proof. We define the map

$$W_1^{n,\Sigma,(i,j)} \colon \Omega_\Sigma^{i,j} \to \Omega_\Sigma^{i,j+1} \tag{A.3}$$

$$\kappa \mapsto e_n \kappa. \tag{A.4}$$

From Lemma A.0.1, we know that the map $W_1^{n,\Sigma,(2,1)}|_{\mathrm{Ker} W_1^{\Sigma,(2,1)}}$ is injective,[1] whereas, the proof of the injectivity of $W_1^{\Sigma,(1,1)}$ is given in [Can21]. Moreover,

[1] It is easy to see by setting $e_n \alpha = 0$.

Lemma A.0.1 basically states that the intersection $\mathrm{Im}W_1^{\Sigma,(1,1)} \cap \mathrm{Im}W_1^{n,\Sigma,(2,1)}|_{\mathrm{Ker}W_1^{\Sigma,(2,1)}}$ is trivial. We then have

$$\dim(\mathrm{Im}W_1^{\Sigma,(1,1)}) = \dim(\Omega_\Sigma^{1,1}) = 12 \tag{A.5}$$

and

$$\dim(\mathrm{Im}W_1^{n,\Sigma,(2,1)}|_{\mathrm{Ker}W_1^{\Sigma,(2,1)}}) = \dim(\mathrm{Ker}W_1^{\Sigma,(2,1)}) = 6, \tag{A.6}$$

since we know from [Can21] that $W_1^{\Sigma,(2,1)}$ is surjective. Given that

$$\dim(\Omega_\Sigma^{2,2}) = 18, \tag{A.7}$$

it follows the statement. □

Lemma A.0.3. *Let $e_n \in \Omega_\Sigma^{0,1}$ be as in Lemma 4.2.14 and $v \in \Omega_\Sigma^{1,2}$. Then, we have*

$$v = 0 \tag{A.8}$$

if and only if

$$\begin{cases} v & \in \mathrm{Ker}W_1^{\Sigma,(1,2)} \\ e_n v & \in \mathrm{Im}W_1^{\Sigma,(0,2)}. \end{cases} \tag{A.9}$$

Proof. This statement is the precise analogous of Lemma A.0.1 and the proof follows verbatim upon the substitution $W_1^{\Sigma,(1,1)} \to W_1^{\Sigma,(0,2)}$. □

Corollary A.0.4. *Let $e_n \in \Omega_\Sigma^{0,1}$ be as in Lemma 4.2.14 and $\theta \in \Omega_\Sigma^{1,3}$. Then, we have the unique decomposition*

$$\theta = ec + e_n\beta, \tag{A.10}$$

with $c \in \Omega_\Sigma^{0,2}$ and $\beta \in \mathrm{Ker}W_1^{\Sigma,(1,2)}$.

Proof. Given the map $W_1^{n,\Sigma,(1,2)}$ defined in Corollary A.0.2, from Lemma A.0.3, we know that the map $W_1^{n,\Sigma,(1,2)}|_{\mathrm{Ker}W_1^{\Sigma,(1,2)}}$ is injective, whereas, the proof of the injectivity of $W_1^{\Sigma,(0,2)}$ is given in [Can21]. Moreover, Lemma A.0.3 basically states that the intersection $\mathrm{Im}W_1^{\Sigma,(0,2)} \cap \mathrm{Im}W_1^{n,\Sigma,(1,2)}|_{\mathrm{Ker}W_1^{\Sigma,(1,2)}}$ is trivial. We then have

$$\dim(\mathrm{Im}W_1^{\Sigma,(0,2)}) = \dim(\Omega_\Sigma^{0,2}) = 6 \tag{A.11}$$

and

$$\dim(\mathrm{Im}W_1^{n,\Sigma,(1,2)}|_{\mathrm{Ker}W_1^{\Sigma,(1,2)}}) = \dim(\mathrm{Ker}W_1^{\Sigma,(1,2)}) = 6, \tag{A.12}$$

since we know from [Can21] that $W_1^{\Sigma,(1,2)}$ is surjective. Given that

$$\dim(\Omega_\Sigma^{1,3}) = 12, \tag{A.13}$$

it follows the statement. □

Proposition A.0.5. *Let* $\tau \in \mathcal{S}$. *Then,* $\tau = e_n\beta$ *with* $\beta \in \Omega_\Sigma^{1,2}[1]$ *such that* $e_n\beta \in \mathrm{Ker}\tilde{\varrho}^{1,3}$ *and* e_n *defined as above.*

Proof. From Lemma 4.2.15, in particular, we have that

$$p_{\mathcal{T}}\alpha = 0 \quad \Longrightarrow \quad \int_\Sigma \tau\alpha = 0 \quad \forall \tau \in \mathcal{S}, \tag{A.14}$$

for $\alpha \in \Omega_\Sigma^{2,1}$. Now, consider an $\alpha \in \Omega_\Sigma^{2,1}$ such that $p_{\mathcal{T}}\alpha = 0$ holds together with the structural constraint $e_n(\alpha - p_{\mathcal{T}}\alpha) = e\sigma$ (notice that this subset of $\Omega_\Sigma^{2,1}$ is in general non-trivial because we do not require the condition $\alpha \in \mathrm{Ker}W_1^{\Sigma,(2,1)}$ as in Lemma 4.2.14), then it follows that

$$\int_\Sigma \tau\alpha = \int_\Sigma ec\alpha + e_n\beta\alpha = \int_\Sigma ecp_{\mathcal{T}^C}\alpha + \beta e\sigma = \int_\Sigma ecp_{\mathcal{T}^C}\alpha, \tag{A.15}$$

where $p_{\mathcal{T}^C}$ is the projection onto a complement of $\mathcal{T}$. Since the right hand side of (A.14) must hold for all $\tau \in \mathcal{S}$, if the intersection $\mathcal{S} \cap \mathrm{Im}W_1^{\Sigma,(0,2)}$ were not trivial, we would have an absurdum. This implies $c \in \mathrm{Ker}W_1^{\Sigma,(0,2)}$ for all $\tau \in \mathcal{S}$, which, thanks to the injectivity of $W_1^{\Sigma,(0,2)}$, is equivalent to $c = 0$.

Lastly, the fact that $e_n\beta \in \mathrm{Ker}\tilde{\varrho}^{1,3}$ follows immediately from the definition of $\mathcal{S}$. □

Proposition A.0.6. *Let* $\tau \in \mathcal{S}$ *and* e *be a diagonal degenerate boundary vielbein, i.e.* $e^*\eta = i^*\tilde{g}$ *with* $\eta = \mathrm{diag}(1,1,1-1)$ *and* $i^*\tilde{g} = \mathrm{diag}(1,1,0)$. *Then, we have*

$$e_n[\tau, e] = 0. \tag{A.16}$$

Proof. Given $a = 1,2,3,4$ and let $\mu = 1,2,+$ be the coordinates on the boundary Σ such that we can write the diagonal degenerate boundary vielbein e as

$$\hat{e}^a = \begin{cases} e_1^a & = \delta_1^a \\ e_2^a & = \delta_2^a \end{cases} \tag{A.17}$$

$$e_+^a = \delta_3^a - \delta_4^a \tag{A.18}$$

$$e_n^a = \delta_3^a + \delta_4^a. \tag{A.19}$$

Then, the definition of $\tau \in \mathcal{S}$ implies the following relations

$$\tau_+^{abc} = 0 \quad \forall a,b,c \tag{A.20}$$

$$\tau_\mu^{123} = 0 \quad \mu = 1,2 \tag{A.21}$$

$$\tau_\mu^{124} = 0 \quad \mu = 1,2 \tag{A.22}$$

$$\tau_1^{234} = \tau_2^{134} \tag{A.23}$$

$$\tau_1^{134} = -\tau_2^{234}. \tag{A.24}$$

The proof follows simply by computing $e_n[\tau, e]$ in components implementing the explicit form of the diagonal vielbein above.[2] □

Lemma A.0.7. *Let*[3] $A \in \Omega_\Sigma^{k,i}$ *with* $2 \leq i \leq 4$. *Then, it holds*

$$\gamma\iota_\gamma\iota_\gamma A = (-1)^{|A|}(\iota_\gamma\iota_\gamma A\gamma + 4(i-1)[\gamma, A]). \tag{A.25}$$

Proof.

$$\begin{aligned}
\gamma\iota_\gamma\iota_\gamma A &= (i-2)!\gamma^a\gamma^b\gamma^c v_a\iota_{v_b}\iota_{v_c}A & \text{(A.26)}\\
&= -(i-2)!(\gamma^b\gamma^a\gamma^c + 2\eta^{ab}\gamma^c)v_a\iota_{v_b}\iota_{v_c}A & \text{(A.27)}\\
&= -(i-2)!(-\gamma^b\gamma^c\gamma^a + 4\eta^{ab}\gamma^c)v_a\iota_{v_b}\iota_{v_c}A & \text{(A.28)}\\
&= (-1)^{|A|}(\iota_\gamma\iota_\gamma A\gamma + 4(i-1)[\gamma, A]). & \text{(A.29)}
\end{aligned}$$

□

Remark A.0.8. This lemma introduces a relation between the action of the brackets over the Clifford algebra and $\mathcal{V}$-algebra. In particular, it is consistent a triviality condition on the bracket in the Clifford algebra, i.e.

$$[A, \overline{\psi}\gamma\psi] = (-1)^{|A|}\overline{\psi}[A, \gamma]_{\mathcal{V}}\psi = \overline{\psi}\gamma[A, \psi]_{Cl} + [\overline{\psi}, A]_{Cl}\gamma\psi, \tag{A.30}$$

where we occasionally added some redundancy with the labels of the specific algebras, even if we will not use them in general.

Lemma A.0.9. *Given* $A \in \Omega_\Sigma^{k,i}$ *and* $B \in \Omega_\Sigma^{l,j}$ *with* $i, j = 2, 3$ *such that* $i + j < 6$, *then we have*

$$B(\overline{\psi}\gamma[A, \psi] - [A, \overline{\psi}]\gamma\psi) = (-1)^{|A||B|}A(\overline{\psi}\gamma[B, \psi] - [B, \overline{\psi}]\gamma\psi). \tag{A.31}$$

Proof. The proof goes by direct computation of

$$\begin{aligned}
B\gamma\iota_\gamma\iota_\gamma A &= B\gamma\gamma^a\gamma^b[v_a, [v_b, A]] & \text{(A.32)}\\
&= (-1)^{|B|}([v_a, B]\gamma + (-1)^{|B|}B\gamma_a)\gamma^a\gamma^b[v_b, A] & \text{(A.33)}\\
&= (-1)^{|B|}([v_a, B]\gamma\gamma^a - 4(-1)^{|B|}B)\gamma^b[v_b, A] & \text{(A.34)}\\
&= -\Big([v_b, [v_a, A]]\gamma\gamma^a\gamma^b - (-1)^{|B|}([v_a, B]\gamma_b\gamma^a\gamma^b + 4[\gamma, B])\Big)A & \text{(A.35)}\\
&= -(-(-1)^{|B|}\gamma\iota_\gamma\iota_\gamma B - 6(-1)^{|B|}\iota_\gamma B)A & \text{(A.36)}\\
&= (-1)^{|B|}(-1)^{|A|(|B|+1)}A(\gamma\iota_\gamma\iota_\gamma B + 6\iota_\gamma B) & \text{(A.37)}
\end{aligned}$$

[2]We refer to [Tec19a] for further details about this kind of computations.
[3]Notice that this may be also a *shifted* variable, like τ for example.

and

$$
\begin{aligned}
B\iota_\gamma\iota_\gamma A\gamma &= (-1)^{|A||B|}\gamma^a\gamma^b[v_a,[v_b,A]]B\gamma && \text{(A.38)}\\
&= (-1)^{|A||B|}(-1)^{|A}\gamma^a\gamma^b[v_b,A]([v_a,B]\gamma+(-1)^{|B|}\gamma_a B) && \text{(A.39)}\\
&= -(-1)^{|A||B|}A\gamma^a\gamma^b\Big([v_b,[v_a,B]]\gamma-(-1)^{|B|}([v_a,B]\gamma_b-\gamma_a[v_b,B])\Big) && \text{(A.40)}\\
&= -(-1)^{|A||B|}A\Big(-\iota_\gamma\iota_\gamma B\gamma+(-1)^{|B|}(4[\gamma,B]+\gamma^a\gamma^b\gamma_a[v_b,B])\Big) && \text{(A.41)}\\
&= (-1)^{|A||B|}A(\iota_\gamma\iota_\gamma B\gamma-(-1)^{|B|}6\iota_\gamma B). && \text{(A.42)}
\end{aligned}
$$

Then, we can conclude the proof by considering the four possible parities of A and B. □

Appendix B

Properties of the Poisson brackets

Sum of two terms

Let $(\mathcal{W}, \varpi)$ be a symplectic manifold, $\mathcal{W}_f$ a manifold and ϖ_f a differential 2-form on $\mathcal{W}$ such that

$$(\mathcal{W} \oplus \mathcal{W}_f, = \Omega_f = \varpi + \varpi_f)$$

forms a symplectic manifold.[1] Moreover, let us suppose that we have two constraints of the form

$$X^f = X + x^f, \qquad Y^f = Y + y^f$$

where $X, Y \in C^\infty(\mathcal{W})$ and $x^f, y^f \in C^\infty(\mathcal{W} \oplus \mathcal{W}_f)$.

The first step towards the computation of the Poisson brackets is to find the Hamiltonian vector fields of the functions X^f and Y^f, i.e. vector fields satisfying

$$\iota_{\mathbb{X}^f} \Omega_f = \delta X^f, \qquad \iota_{\mathbb{Y}^f} \Omega_f = \delta Y^f.$$

Without loss of generality, we will consider only the case for X. Let us denote by $\mathbb{X}$ the Hamiltonian vector field of X with respect to ϖ:

$$\iota_{\mathbb{X}} \varpi = \delta X.$$

Then, if we look for an Hamiltonian vector field of the form

$$\mathbb{X}^f = \mathbb{X} + \mathbb{x}^f$$

we get that $\mathbb{x}^f$ must satisfy

$$\iota_{\mathbb{X}} \varpi_f + \iota_{\mathbb{x}^f} (\varpi + \varpi_f) = \delta x^f.$$

Then we have the following result.

[1] In our case $(\mathcal{W}, \varpi)$ is the geometric phase space of gravity while the index f denotes the spaces of some matter fields.

Lemma B.0.1. *The Poisson brackets of the functions X^f and Y^f with respect to the symplectic form Ω_f are given by the following expression*

$$\{X^f, Y^f\}_f = \{X, Y\} + \iota_{\mathbb{y}^f}\delta(X + x^f) + \iota_{\mathbb{x}^f}\delta(Y + y^f) - \iota_{\mathbb{x}^f}\iota_{\mathbb{y}^f}\Omega_f + \iota_{\mathbb{X}}\iota_{\mathbb{Y}}\varpi_f,$$

where $\{\bullet, \bullet\}_f$ means that we are considering the Poisson brackets with respect to $\varpi + \varpi_f$ and $\{\bullet, \bullet\}$ means that we are considering the Poisson brackets with respect to ϖ.

Proof. If we expand the Poisson brackets $\{X^f, Y^f\}_f$ we get eight terms as follows

$$\begin{aligned}\iota_{\mathbb{X}+\mathbb{x}^f}\iota_{\mathbb{Y}+\mathbb{y}^f}(\varpi + \varpi_f) = {} & \iota_{\mathbb{X}}\iota_{\mathbb{Y}}\varpi + \iota_{\mathbb{X}}\iota_{\mathbb{y}^f}\varpi + \iota_{\mathbb{x}^f}\iota_{\mathbb{Y}}\varpi + \iota_{\mathbb{x}^f}\iota_{\mathbb{y}^f}\varpi \\ & + \iota_{\mathbb{X}}\iota_{\mathbb{Y}}\varpi_f + \iota_{\mathbb{X}}\iota_{\mathbb{y}^f}\varpi_f + \iota_{\mathbb{x}^f}\iota_{\mathbb{Y}}\varpi_f + \iota_{\mathbb{x}^f}\iota_{\mathbb{y}^f}\varpi_f.\end{aligned}$$

It is then straightforward to observe that the first term in this expression corresponds to $\{X, Y\}$; the sum of the second, fourth, sixth, and eighth terms corresponds to $\iota_{\mathbb{y}^f}\delta(X + x^f)$, while the sum of the third, fourth, seventh, and eighth terms corresponds to $\iota_{\mathbb{x}^f}\delta(Y + y^f)$. Note, however, that the eighth term has been counted twice and must therefore be subtracted once, and the fifth term, which was omitted so far, must be added. □

Bibliography

[Can21] G. Canepa. “General Relativity on Stratified Manifolds in the BV-BFV Formalism”. *PhD Thesis* (Mar. 2021). DOI: http://user.math.uzh.ch/cattaneo/theses.html.

[CC24] G. Canepa and A. S. Cattaneo. “Corner Structure of Four-Dimensional General Relativity in the Coframe Formalism”. *Annales Henri Poincaré* 25.5 (2024), pp. 2585–2639. DOI: 10.1007/s00023-023-01360-8.

[CCF22] G. Canepa, A. S. Cattaneo, and F. Fila-Robattino. “Boundary structure of gauge and matter fields coupled to gravity” (June 2022). arXiv: 2206.14680 [math-ph].

[Can+23] G. Canepa, A. S. Cattaneo, F. Fila-Robattino, and M. Tecchiolli. “Boundary structure of the standard model coupled to gravity” (July 2023). arXiv: 2307.14955 [math-ph].

[CCS21a] G. Canepa, A. S. Cattaneo, and M. Schiavina. “Boundary structure of general relativity in tetrad variables”. *Adv. Theor. Math. Phys.* 25.2 (2021), pp. 327–377. DOI: 10.4310/ATMP.2021.v25.n2.a3.

[CCS21b] G. Canepa, A. S. Cattaneo, and M. Schiavina. “General Relativity and the AKSZ Construction”. *Commun. Math. Phys.* 385.3 (2021), pp. 1571–1614. DOI: 10.1007/s00220-021-04127-6.

[CCT21] G. Canepa, A. S. Cattaneo, and M. Tecchiolli. “Gravitational Constraints on a Lightlike Boundary”. *Annales Henri Poincare* 22.9 (2021), pp. 3149–3198. DOI: 10.1007/s00023-021-01038-z.

[CMR14] A. S. Cattaneo, P. Mnev, and N. Reshetikhin. “Classical BV Theories on Manifolds with Boundary”. *Communications in Mathematical Physics* 332.2 (2014), pp. 535–603. ISSN: 1432-0916. DOI: 10.1007/s00220-014-2145-3.

[CS19] A. S. Cattaneo and M. Schiavina. “BV-BFV approach to General Relativity: Palatini–Cartan–Holst action”. *Adv. Theor. Math. Phys.* 23.8 (2019), pp. 2025–2059. DOI: 10.4310/ATMP.2019.v23.n8.a3.

[Cou90] T. Courant. "Dirac manifolds". *Transactions of the American Mathematical Society* 319.2 (1990), pp. 631–661.

[Dir50] P. Dirac. "Generalized Hamiltonian Dynamics". *Canadian Journal of Mathematics* 2 (1950), pp. 4129–4148.

[Dir58] P. A. M. Dirac. "Generalized Hamiltonian Dynamics". *Proceedings of the Royal Society of London. Series A, Math. and Phys. Sciences* 246.1246 (1958), pp. 326–332.

[Ein16] A. Einstein. "Die Grundlage der allgemeinen Relativitätstheorie". *Annalen der Physik* 49 (1916), pp. 769–822. DOI: `10.1002/andp.19163540702`.

[Fat18] L. Fatibene. *Relativistic Theories, Gravitational Theories and General Relativity*. 2018. URL: `http://www.fatibene.org/book.html`.

[Fec11] M. Fecko. *Differential Geometry and Lie Groups for Physicists*. Cambridge University Press, 2011.

[Fig06] J. Figueroa–O'Farrill. *Connections on principal fibre bundles*. 2006.

[HT92] M. Henneaux and C. Teitelboim. *Quantization of gauge systems*. 1992. ISBN: 978-0-691-03769-1.

[K S16] K. Schwarzschild. "On the gravitational field of a mass point according to Einstein's theory" (Jan. 1916).

[KT79] J. Kijowski and W. M. Tulczyjew. "A Symplectic Framework for Field Theories". *Lect. Notes Phys.* 107 (1979).

[KN69] S. Kobayashi and K. Nomizu. *Foundations of Differential Geometry, Vol. 2*. New York: Wiley and Sons, 1969.

[MW74] J. Marsden and A. Weinstein. "Reduction of symplectic manifolds with symmetry". *Reports on Mathematical Physics* 5.1 (1974), pp. 121–130. ISSN: 0034-4877. DOI: `10.1016/0034-4877(74)90021-4`.

[Mei25] E. Meinrenken. "Introduction to Moduli Spaces and Dirac Geometry". *arXiv preprint arXiv:2506.04150* (2025). DOI: `10.48550/arXiv.2506.04150`.

[Mel] R. B. Melrose. *Differential Analysis on Manifolds with Corners*. Lecture notes. URL: `https://math.mit.edu/~rbm/18.158/daomwc.1/daomwc.1.pdf`.

[Pal19] A. Palatini. "Deduzione invariantiva delle equazioni gravitazionali dal principio di Hamilton". *Rendiconti del Circolo Matematico di Palermo (1884-1940)* 43.1 (1919), pp. 203–212. ISSN: 0009-725X. DOI: `10.1007/BF03014670`.

[Pfl01] M. J. Pflaum. *Analytic and Geometric Study of Stratified Spaces.* Vol. 1768. Lecture Notes in Mathematics. Springer, 2001.

[Roy02] D. Roytenberg. "On the structure of graded symplectic supermanifolds and Courant algebroids". *arXiv:math/0203110 [math.SG]* (2002).

[Sha97] R. W. Sharpe. *Differential Geometry: Cartan's Generalization of Klein's Erlangen Program.* New York: Springer, 1997.

[Tec19a] M. Tecchiolli. "Algebra of Constraints for the Linearized Palatini-Cartan Theory on a Light-Like Boundary". Master thesis. Nov. 2019. URL: http://user.math.uzh.ch/cattaneo/tecchiolli.pdf.

[Tec19b] M. Tecchiolli. "On the Mathematics of Coframe Formalism and Einstein–Cartan Theory—A Brief Review". *Universe* 5.10 (Sept. 2019), p. 206. ISSN: 2218-1997. DOI: 10.3390/universe5100206.

[Wer19] K. Wernli. *Lecture Notes on Spin Geometry.* 2019. arXiv: 1911.09766 [math.DG].