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On the Differential Geometry on Infinite Jet Bundles

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Contents

Introduction

Modern classical field theories are typically formulated in the Lagrangian framework, where the action principle determines the dynamics of fields. This principle, first applied to classical optical systems by Hamilton [Ham34], is grounded in the successive formulations of the principle of least action by Maupertuis, Lagrange, and Euler, and ultimately traces its origins to Fermat's principle (1662). The latter states that light travels along the path that can be traversed in the shortest time. Hamilton's idea is that the Euler–Lagrange equations characterize the critical points of the action functional, thereby describing the dynamics of a physical system entirely in terms of its Lagrangian.

In modern classical field theories, this formulation remains in use for two primary reasons. First, symmetries of physical systems, which play a pivotal role in modern physics, are naturally incorporated into the Lagrangian formalism as symmetries of the action. Secondly, in this framework the potential replaces the force. While force is a classical concept that does not generalize naturally to quantum mechanics, the potential does.

This allows one to quantize the classical field theory more straightforwardly via the quantum incarnation of the action principle that is Feynman's path integral(1) (cf. [Fey48]). The latter heuristically describes the propagator of the quantum particle via a sum over the classical paths, each contributing with a weight given by an exponential whose phase is the classical action of that path divided by $\hbar$.

The path integral is not rigorously defined (cf. [Maz09]), but it has been largely used in perturbative field theories so far. The perturbative approach has extreme predictive power, but cannot describe the core of the physical systems that would be accessible via non-perturbative formulations of the theories.

Developing such non-perturbative descriptions for strongly coupled systems, for example Yang-Mills theory coupled to fermionic quark matter (QCD), is one of the "Millennium Problems" posed by the Clay Mathematics Institute (cf. [GS25, Ch. 1]), and a major open problem in contemporary fundamental physics.

In [GS25], Sati and Giotopoulos aim to lay out missing modern mathematical foundations toward this goal. They propose supergeometric homotopy theory as a candidate for a mathematical notion of geometry that incorporates all the fundamental characteristics of the physical world, which is field-theoretic, smooth, local, gauged, non-perturbative, and contains fermions. To this end, they introduce the category **SmthSet** of smooth sets as the category of sheaves over the site of Cartesian spaces with respect to the differentiably-good open covers (Def. 2.2.1), in which they naturally formalize the action principle for local bosonic classical Lagrangian field theories. The article is actually the first of a series that, by enriching this category with infinitesimal structure and odd variables (cf. [Gri25]), aims to formalize fermionic field theories. They justify

(1) Feynman was inspired by the idea, presented by Dirac in [Dir33], that the Lagrangian formulation of quantum mechanics would have provided a way to interpret quantum dynamics as a superposition of classical paths.

this approach first by arguing that the core idea of topos-theoretic geometry is physical in an operational way: to know a geometric space, such as the spacetime of the observable universe, is not to define a set of points equipped with extra structures, but to probe it with particle trajectories, and more generally, with probebranes that traverse it. Intuitively, the smooth sets are the spaces that can be probed by smooth coordinate charts $\mathbb{R}^k$. Secondly, they claim that the classical approaches to the geometry of field theory, which exploit Fréchet manifolds and, more recently, diffeological spaces (cf. [Blo24]), are faithfully subsumed in SmthSet, where field-theoretic constructions find a more natural embodiment.

The aim of this project is to study the article [GS25] by Sati and Giotopoulos. Specifically, the focus is on Chapter 4, where they develop differential geometry on the infinite jet bundle in the category SmthSet. The goal is to find a way to define the full universal Cartan calculus categorically on it.

This would enable one to formalize the action principle for local Lagrangian field theories (LFTs) and implement the BV-BFV formalism, which is an algebraic formalism that allows for the rigorous quantization of gauge theories on the boundary and bulk of the base manifold (cf. [SCS23; CM20]), within SmthSet. Consequently, an interesting connection could be established between the categorical, non-perturbative formulation of field theory presented in [GS25] and the algebraic, perturbative approach developed in [SCS23; CM20].

Unfortunately, a full universal Cartan calculus on the infinite jet bundle $J^\infty_M F$ is not fully obtained. However, the Cartan calculus is defined on the subalgebra of classical globally finite order differential forms on $J^\infty_M F$, which is shown to be canonically embedded as a subalgebra of the smooth set-theoretic forms on $J^\infty_M F$ (Thm. 3.1). One could, in principle, transport the Cartan operations from the classical to the smooth set theoretic setting, but only when restricting to this subalgebra.

We start in **Chapter** 1 with a brief overview of category and sheaf theory. The basic notions of category, functor, and natural transformation are presented to state the fundamental Yoneda lemma (Prop. 1.1). This result is crucial to define smooth sets consistently in Section 2.2. Then, the general definition of limits and colimits is provided, along with some examples, and the relationships between limits and relevant functors are recalled. Finally, petit and gros sheaves and topoi, together with Grothendieck topology, are introduced in Section 1.3. Some results on categories of presheaves are stated.

Chapter 2 presents local Lagrangian bosonic theories and the action principle. Some notions of differential geometry on smooth manifolds, useful for this chapter and the following one, are collected in **Appendix** A. The lack of rigorous formalism outlines the structural requirements that the category of smooth sets should meet to formalize the geometry of the action principle rigorously. In Section 2.3, we demonstrate how different field-theoretical objects are naturally encoded within the category SmthSet, which is defined both heuristically and rigorously in Section 2.2. In particular, the field space, vector fields, and differential forms can be uniformly treated within SmthSet. However, the definitions of differential forms and their classifying space present some issues that we try to address in the following chapter and that, only partially solved for the case of the infinite jet bundle, prevents us from reaching the goal of finding a universal property of Cartan calculus.

In **Chapter** 3, we develop the differential geometry of the infinite jet bundle within the smooth-set framework. Starting from the definition of finite order jet bundles, we then define the infinite jet bundle as their projective limit within **LocProMfd**, a full subcategory of Fréchet manifolds (Def. 3.1.6), and consequently embed it into SmthSet. We then show that there is a canonical splitting of its tangent bundle, into vertical and horizontal subbundle, by means of the Cartan connection (Prop. 3.2.3). The splitting is essential to formulate the action principle. In particular, the horizontal component identifies total derivatives in the variation, ensuring the boundary term in the Euler–Lagrange formula. Subsequently, we prove (Thm. 3.1) that the subalgebra of classical globally finite order differential forms on $J^\infty_M F$ (Def. 3.2.4) is canonically embedded as a subalgebra of smooth set-theoretic forms on $J^\infty_M F$ (Def. 3.2.6). One could, in principle, transport the Cartan operations from the classical forms (Def. 3.2.5) to the smooth set theoretic setting, but only when restricting to this subalgebra. Finally, in Section 3.3, we demonstrate that the infinite jet bundle encodes an explicit global (i.e. chart-independent) description of local Lagrangians, fields, and forms.

Chapter 1

Preliminaries

This chapter summarizes the notions and results of category and sheaf theory used in the report. We refer to [Lei14; Rie17] and to [MM12] for a detailed discussion of category theory and sheaf theory, respectively.

1.1 Elements of Category Theory

In this section, the basic notions of category, functor, natural transformation, and the universal property of representable functors are introduced to state the Yoneda lemma.

Definition 1.1.1. A **category** $\mathscr{A}$ consists of:

i. a class $\mathrm{ob}(\mathscr{A})$ of **objects** (we equivalently write $A \in \mathscr{A}$ or $A \in \mathrm{ob}(\mathscr{A})$);

ii. for each $A, B \in \mathrm{ob}(\mathscr{A})$, a class $\mathrm{Hom}_{\mathscr{A}}(A, B)$ of **morphisms** from A to B;

iii. for each $A \in \mathrm{ob}(\mathscr{A})$, an **identity** $1_A \in \mathrm{Hom}_{\mathscr{A}}(A, A)$;

iv. for each $A, B, C \in \mathrm{ob}(\mathscr{A})$ a **composition** map

$$\mathrm{Hom}_{\mathscr{A}}(B, C) \times \mathrm{Hom}_{\mathscr{A}}(A, B) \to \mathrm{Hom}_{\mathscr{A}}(A, C), \ (f, g) \mapsto f \circ g$$

such that for each $f \in \mathrm{Hom}_{\mathscr{A}}(A, B)$, $g \in \mathrm{Hom}_{\mathscr{A}}(B, C)$, $h \in \mathrm{Hom}_{\mathscr{A}}(C, D)$, the following holds:

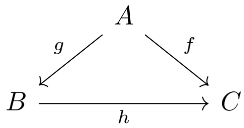
$$(h \circ g) \circ f = h \circ (g \circ f) \ \text{ and } \ f \circ 1_A = f = 1_B \circ f.$$

A **subcategory** $\mathscr{B}$ of $\mathscr{A}$ is a sub-class of objects and morphisms closed under composition.

This definition enables a uniform treatment of mathematical entities that may be fundamentally different. For instance, consider the class **Set** of sets together with maps, the class **Grp** of groups and group homomorphisms, and the class **SmthMfd** of smooth manifolds and smooth maps. They all satisfy Definition 1.1.1 and share universal properties that can be studied categorically.

More generally, category theory studies the universal properties of objects[(1)] that are related to the structure of the category, regardless of its specific realization. For this purpose, it is useful to introduce commutative diagrams in which we connect objects through arrows that represent morphisms. For example, the diagram

[(1)] The classical approaches to describe a universal property are representables, adjoint functors, and limits. For a formal definition and a detailed discussion, see [Rie17, Ch. 2.2] and [Lei14].



is said to commute if $f = h \circ g$. It is straightforward to generalize the concepts of surjective and injective morphisms between objects.

Definition 1.1.2. Let $\mathscr{A}$ be a category and $f \in \mathrm{Hom}_{\mathscr{A}}(A, B)$.
Then f is called **epic** if for all $C \in \mathscr{A}$ and all $g, g' : B \to C$, the following holds:

$$g \circ f = g' \circ f \;\Rightarrow\; g = g'.$$

In contrast, f is called **monic** if for all $C \in \mathscr{A}$ and all $h, h' : C \to A$, the following holds:

$$f \circ h = f \circ h' \;\Rightarrow\; h = h'.$$

Finally, f is an **isomorphism** if there exists $g \in \mathrm{Hom}_{\mathscr{A}}(B, A)$ such that $f \circ g = 1_B$ and $g \circ f = 1_A$. Then, $g = f^{-1}$ is called the **inverse** of f and we write $A \cong B$.

There are two particularly relevant constructions: the dual or opposite category and the product category.

Definition 1.1.3. Given a category $\mathscr{A}$, the **opposite** category $\mathscr{A}^{\mathrm{op}}$ is such that $\mathrm{ob}(\mathscr{A}) = \mathrm{ob}(\mathscr{A}^{\mathrm{op}})$ and $\mathrm{Hom}_{\mathscr{A}}(A, B) = \mathrm{Hom}_{\mathscr{A}^{\mathrm{op}}}(B, A)$ for all $A, B \in \mathrm{ob}(\mathscr{A}^{\mathrm{op}})$.

Definition 1.1.4. Given two categories $\mathscr{C}_1$ and $\mathscr{C}_2$, the **product category** $\mathscr{C}_1 \times \mathscr{C}_2$ is such that:

i. $\mathrm{ob}(\mathscr{C}_1 \times \mathscr{C}_2) = \mathrm{ob}(\mathscr{C}_1) \times \mathrm{ob}(\mathscr{C}_2)$;

ii. for all $A, B \in \mathrm{ob}(\mathscr{C}_1)$ and $A', B' \in \mathrm{ob}(\mathscr{C}_2)$,
$\mathrm{Hom}_{\mathscr{C}_1 \times \mathscr{C}_2}((A, A'), (B, B')) = \mathrm{Hom}_{\mathscr{C}_1}(A, B) \times \mathrm{Hom}_{\mathscr{C}_2}(A', B')$;

iii. for all $A \in \mathrm{ob}(\mathscr{C}_1)$ and $A' \in \mathrm{ob}(\mathscr{C}_2)$, $1_{(A,A')} = (1_A, 1_{A'})$;

iv. the composition is the componentwise composition from the contributing categories;

Remark 1.1.1. Informally, we can think of the opposite category as the original category with the arrows reversed. This is an application of the principle of duality for which every categorical definition or statement has a dual counterpart obtained by reversing the arrows.

We can further consider the category of categories[(2)] defining the morphisms between them as follows.

Definition 1.1.5. Let $\mathscr{A}$ and $\mathscr{B}$ be categories. A **covariant functor** $F : \mathscr{A} \to \mathscr{B}$ consists of:

i. a function $F : \mathrm{ob}(\mathscr{A}) \to \mathrm{ob}(\mathscr{B}),\; A \mapsto F(A)$;

ii. for each $A, A' \in \mathscr{A}$, a function $F : \mathrm{Hom}_{\mathscr{A}}(A, A') \to \mathrm{Hom}_{\mathscr{B}}(F(A), F(A')),\; f \mapsto F(f)$;

such that $F(1_A) = 1_{F(A)}$ for all $A \in \mathscr{A}$ and $F(f' \circ f) = F(f') \circ F(f)$ whenever $A \xrightarrow{f} A' \xrightarrow{f'} A''$ in $\mathscr{A}$.
F is said **faithful** (resp. **full**) if for all $A, A' \in \mathscr{A}$, the map

$$\mathrm{Hom}_{\mathscr{A}}(A, A') \to \mathrm{Hom}_{\mathscr{B}}(F(A), F(A')),\; f \mapsto F(f)$$

is injective (surjective).
A **contravariant functor** $F : \mathscr{A} \to \mathscr{B}$ is a covariant functor $F : \mathscr{A}^{\mathrm{op}} \to \mathscr{B}$.

(2) This idea conceals subtleties and set-theoretical issues related to the size of the considered categories (cf. [Lei14, Ch. 3.2]). For the sake of simplicity, we ignore them throughout these notes.

Note that functors can be composed and that the identity functor can be trivially defined for each category, so the class of categories and functors **CAT** is indeed a category.[3] Some concrete examples of functors are the forgetful functor Grp $\to$ Set that forgets the group structure on top of the underlying set, and the inclusion functor AbGrp $\to$ Grp where **AbGrp** is the subcategory of abelian groups.

The latter definition enables us to establish some relationships between categories.

Definition 1.1.6.

1. A subcategory $\mathscr{B} \subseteq \mathscr{A}$ is **embedded** in $\mathscr{A}$ if the inclusion functor $i : \mathscr{B} \to \mathscr{A}$ is fully faithful and injective on objects, i.e. for every $B_1, B_2 \in \mathscr{B}$ with $B_1 \neq B_2$, then $i(B_1) \neq i(B_2)$.

2. Two categories are **isomorphic** if they are isomorphic as objects in CAT.

3. Two categories $\mathscr{A}$,$\mathscr{B}$ are **equivalent** if there exists a fully faithful functor $F : \mathscr{A} \to \mathscr{B}$ such that for all $B \in \mathscr{B}$, there is $A \in \mathscr{A}$ with $F(A) \cong B$, i.e. F is **essentially surjective on objects.**[4]

Since the conditions for a categorical isomorphism are often too restrictive, we are typically more interested in categorical equivalence. The latter can also be stated in terms of natural transformations (cf. [Lei14, Def. 1.3.16 and Thm. 1.3.18]), i.e. morphisms of functors.

Definition 1.1.7. Let $\mathscr{A}$ and $\mathscr{B}$ be categories, and let $F, G : \mathscr{A} \to \mathscr{B}$ be functors. A **natural transformation** $\eta : F \Rightarrow G$ is a family of morphisms $\{\eta_A : F(A) \to G(A)\}_{A \in \mathscr{A}}$ in $\mathscr{B}$, called the **components** of η, such that for every $f : A \to A'$ in $\mathscr{A}$, the following diagram commutes:

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(A') \\ {\scriptstyle \eta_A}\downarrow & & \downarrow{\scriptstyle \eta_{A'}} \\ G(A) & \xrightarrow[G(f)]{} & G(A'). \end{array}$$

That is, $\eta_{A'} \circ F(f) = G(f) \circ \eta_A$. We write the following diagram to indicate that η is a natural transformation from F to G:

$$\mathscr{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathscr{B}, \quad \Downarrow \eta$$

Note that, considering the vertical composition of natural transformations, i.e. componentwise (cf. [Rie17, Lem. 1.7.1]), and taking the trivial natural transformation as the identity, we can define the category of functors $[\mathscr{A}, \mathscr{B}]$ between two categories $\mathscr{A}$ and $\mathscr{B}$. We say that two functors between $\mathscr{A}$ and $\mathscr{B}$ are naturally isomorphic if they are isomorphic in this category.

To state the Yoneda lemma, we first need to introduce the following notions.

Definition 1.1.8. Let $\mathscr{C}$ be a category. Its **hom-functor** is the functor $\mathrm{Hom}_{\mathscr{C}} : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathrm{Set}$, defined as follows:

i. it sends $(C, C') \in \mathscr{C}^{\mathrm{op}} \times \mathscr{C}$ to the set of morphisms $\mathrm{Hom}_{\mathscr{C}}(C, C')$;

[3] CAT is a large 2-category whose 1-morphisms are functors and whose 2-morphisms are natural transformations.

[4] Using the axiom of choice, one can show that this definition is equivalent to the existence of an usual equivalence of categories by functors.

ii. it sends $(f, g) : (C, C') \to (D, D')$ to $\mathrm{Hom}_{\mathscr{C}}(C, C') \to \mathrm{Hom}_{\mathscr{C}}(D, D')$, $q \mapsto g \circ q \circ f$, i.e.

$$\begin{array}{ccc} (C, C') & \longmapsto & \mathrm{Hom}_{\mathscr{C}}(C, C') \\ f\uparrow \ \downarrow g & \longmapsto & \downarrow g\circ - \circ f \\ (D, D') & \longmapsto & \mathrm{Hom}_{\mathscr{C}}(D, D'). \end{array}$$

It is said to be **internal** if it is of the form $[-, -] : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{C}$.
Given a hom-functor $\mathrm{Hom}_{\mathscr{C}} : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathrm{Set}$, for any object $C \in \mathscr{C}$, one defines:

1. the functor $H^C := \mathrm{Hom}_{\mathscr{C}}(C, -) : \mathscr{C} \to \mathrm{Set}$;
2. the functor $H_C := \mathrm{Hom}_{\mathscr{C}}(-, C) : \mathscr{C}^{\mathrm{op}} \to \mathrm{Set}$.

Functors $\mathscr{C} \to \mathrm{Set}$ are called **copresheaves**, and those naturally isomorphic to $\mathrm{Hom}_{\mathscr{C}}(C, -)$ for some $C \in \mathscr{C}$ are called **representable copresheaves**.
Functors of the form $\mathscr{C}^{\mathrm{op}} \to \mathrm{Set}$ are called **presheaves**, and those naturally isomorphic to $\mathrm{Hom}_{\mathscr{C}}(-, C)$ for some $C \in \mathscr{C}$ are called **representable presheaves**.
Finally, we define the **Yoneda embedding**, $H_\bullet : \mathscr{C} \to [\mathscr{C}^{\mathrm{op}}, \mathrm{Set}]$, as $H_\bullet(C) := H_C$ and $H_\bullet(f) := H_f$ for all $C, C' \in \mathscr{C}$ and $f \in \mathrm{Hom}_{\mathscr{C}}(C, C')$, where $H_f : H_C \Rightarrow H_{C'}$ is the natural transformation defined by post-composition via f.

The latter is injective on objects by construction and is fully faithful thanks to the following fundamental result (cf. [Lei14, Cor. 4.3.7]).

Theorem 1.1 (Yoneda Lemma)**.** *Let $\mathscr{C}$ be a category. For any presheaf $F : \mathscr{C}^{\mathrm{op}} \to \mathrm{Set}$ and any object $C \in \mathscr{C}$, there is a bijection*

$$\begin{aligned} \Phi : [H_C, F] &\longrightarrow F(C) \\ \alpha &\longmapsto \alpha_C(1_C) \end{aligned}$$

where $[H_C, F]$ is the set of natural transformation between H_C and F.
Equivalently, $[\mathscr{C}^{\mathrm{op}}, \mathrm{Set}](H_C, F) \cong F(C)$ naturally in both $C \in \mathscr{C}$ and $F \in [\mathscr{C}^{\mathrm{op}}, \mathrm{Set}]$.

A proof can be found in [Lei14, Thm. 4.2.1]. The dual version follows by inverting the arrows.

This result has a fundamental role since it tells us how any presheaf in $[\mathscr{C}^{\mathrm{op}}, \mathrm{Set}]$ relates to the representable presheaf H_C for any $C \in \mathscr{C}$. In particular, it shows how H_C is mapped to any other presheaf F via a natural transformation $H_C \Rightarrow F$. It turns out that such natural transformations can be seen as elements of $F(C)$. The naturality in the statement amounts to the following pair of assertions, where up and down $*$ means precomposition and postcomposition.

Naturality in F: for all $\beta : F \Rightarrow G$, the following diagram commutes:

$$\begin{array}{ccc} [\mathscr{C}(-, C), F] & \xrightarrow[\Phi_F]{\cong} & FC \\ {\scriptstyle \beta_*}\downarrow & & \downarrow{\scriptstyle \beta_C} \\ [\mathscr{C}(-, C), G] & \xrightarrow[\Phi_G]{\cong} & GC. \end{array}$$

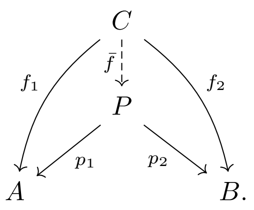
Naturality in C: for all $f : C' \to C$, the following diagram commutes:

$$\begin{array}{ccc} [\mathscr{C}(-, C), F] & \xrightarrow[\Phi_F]{\cong} & FC \\ {\scriptstyle H_f^*}\downarrow & & \downarrow{\scriptstyle Ff} \\ [\mathscr{C}(-, C'), F] & \xrightarrow[\Phi_F]{\cong} & FC'. \end{array}$$

1.2 Limits and Colimits

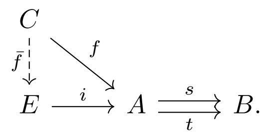
In this section, three useful examples of limits are presented to build up to the general definition. Consequently, we state the relations of the Yoneda embedding and the hom-functor with limits. Finally, some significant results for presheaves' categories are recalled.

Definition 1.2.1. Let $\mathscr{A}$ be a category and $A, B \in \mathscr{A}$. The **product** of A and B consists of a triple (P, p_1, p_2), where P is an object and p_1, p_2 are maps from P to A, B respectively, such that for any other such triple (C, f_1, f_2), there exists a unique map $\bar{f} : C \to P$ that makes the following diagram commutes:



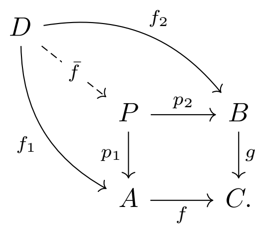
Examples of products are the Cartesian product in Set or the intersection in the poset of Set ordered by inclusions. The dual concept can be thought of as a sum (cf. [Lei14, Ch. 5.2]).

Definition 1.2.2. Let $\mathscr{A}$ be a category, and s, $t : A \to B$ in $\mathscr{A}$. An **equalizer** of s and t is a couple (E, i) of an object E and a map $i : E \to A$ with $s \circ i = t \circ i$ such that for any other such couple (C, f) there exists a unique map $\bar{f} : C \to E$ that makes the following diagram commutes:



One can check that, given an homomorphism $\theta : G \to H$, the kernel $\operatorname{Ker}\theta = \{g \in G \,|\, \theta(g) = 0_H\}$ is an equalizer in Grp. Similarly for $E = \{x \in A | s(x) = t(x)\} \subseteq A$ in Set.

Definition 1.2.3. Let $\mathscr{A}$ be a category and $B \xrightarrow{g} C \xleftarrow{f} A$ in $\mathscr{A}$. A **pullback** of this diagram is a triple (P, p_1, p_2) of an object $P \in \mathscr{A}$ and two maps $p_1 : P \to A$ and $p_2 : P \to B$ with $f \circ p_1 = g \circ p_2$ such that for any other such triple (D, f_1, f_2), there is a unique map $\bar{f} : D \to P$ that makes the following commutes:



An example of the latter is the preimage in Set.

Remark 1.2.1. All of the above definitions dualize by inverting the arrows. Additionally, products, equalizers, pullbacks, and their duals may not exist in some categories; however, if they do exist, they are unique up to isomorphisms. This observation holds in general for any limit (cf. [Rie17, Prop. 3.1.7]).

For each of the above definitions, the initial data can be represented as functors, respectively, from each of the following three categories to the category $\mathscr{A}$.

$$\bullet \qquad \bullet \qquad\qquad \bullet \rightrightarrows \bullet \qquad\qquad \begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \\ \bullet & & \end{array}$$

Note that the categories are made of arrows and dots. Moreover, they are small, i.e. the class of objects is a set, and for every two objects, the class of morphisms between them is also a set.

Definition 1.2.4. Let $\mathscr{A}$ be a category and I a small category of dots and arrows. A functor $I \to \mathscr{A}$ is called a **diagram in $\mathscr{A}$ of shape I**.

The next definition encompasses all three examples above, providing the general concept of a limit.

Definition 1.2.5. Let $\mathscr{A}$ be a category, I a small category of dots and arrows, and $D\colon I \to \mathscr{A}$ a diagram in $\mathscr{A}$.

1. A **cone** on D is an object $A \in \mathscr{A}$ (the **vertex** of the cone) together with a family $(A \xrightarrow{f_i} D(i))_{i\in I}$ of maps in $\mathscr{A}$ such that for all maps $i \xrightarrow{\alpha} j$ in I, the triangle

$$\begin{array}{ccccc} & & A & & \\ & \swarrow^{f_i} & & \searrow^{f_j} & \\ D(i) & & \xrightarrow[D(\alpha)]{} & & D(j) \end{array}$$

   commutes. The **cocone** is the dual concept. From now on $D(i) = D_i$.

2. A **limit**[5] of D is a cone $(L \xrightarrow{p_i} D(i))_{i\in I}$ such that for any other cone $(A \xrightarrow{f_i} D(i))_{i\in I}$ on D, there exists a unique map $\bar{f}\colon A \to L$ with $p_i \circ \bar{f} = f_i$ for all $i \in I$. The maps p_i are called the **projections** of the limit. The **colimit** is the dual concept.

Remark 1.2.2. We say that a category is (co)complete if it has all (co)limit. Interestingly, to assert that a category is (co)complete, it is sufficient to show that it admits (co)products and (co)equalizers, since any other limit can be constructed from these two (cf. [Lei14, Prop. 5.1.26]). Between the above examples, one can show[6] that Set and Grp are (co)complete, while SmthMfd is not since the intersection, defined through a pullback, of two smooth manifolds does not need to be one.

It is relevant to understand the interactions between functors and limits.

Definition 1.2.6. Let I be a small category of dots and arrows, and $F : \mathscr{A} \to \mathscr{B}$ a functor between the categories $\mathscr{A}$ and $\mathscr{B}$.

1. F **preserves limits of shape** I if the following holds: for all diagrams $D : I \to \mathscr{A}$ and all cones $(A \xrightarrow{\lambda_i} D(i))_{i\in I}$ on D, if $(A \xrightarrow{\lambda_i} D(i))_{i\in I}$ is a limit cone on D in $\mathscr{A}$, then $(F(A) \xrightarrow{F\lambda_i} F(D(i)))_{i\in I}$ is a limit cone on $F \circ D$ in $\mathscr{B}$. F **preserves limits** if it preserves limits of shape I for all small categories I.

[5] The terminology can be understood showing that a limit is a terminal object in an opportune category, i.e. in the category of cones over D (cf. [Rie17, Def. 3.1.6]).

[6] For a proof, see [Rie17, Thm. 3.2.6] and [Lei14, Lem. 5.3.6] with the following comment.

2. **Reflection of limits** is defined as in 1., but with the opposite implication in the condition.

3. F **creates limits** of shape I if, whenever $D : I \to \mathscr{A}$ is a diagram in $\mathscr{A}$, for any limit cone $(B \xrightarrow{q_i} F(D(i)))_{i\in I}$ on the diagram $F \circ D$ in $\mathscr{B}$, there exists a unique limit cone $(A \xrightarrow{p_i} D(i))_{i\in I}$ on D in $\mathscr{A}$ such that $F(A) = B$ and $F(p_i) = q_i$ for all $i \in I$. F **creates limits** if it creates limits of shape I for all small categories I.

The same terminology applies to colimits. The following result is particularly useful.

Proposition 1.2.1. *Let $\mathscr{C}$ be a category. Then:*

1. *The hom-functor* $\mathrm{Hom}_{\mathscr{C}} : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathrm{Set}$ (Def. 1.1.8) *preserves limits in both its arguments recalling that a limit in $\mathscr{C}^{\mathrm{op}}$ is a colimit in $\mathscr{C}$.*

2. *If it exists, the internal hom-functor* $[-,-] : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{C}$ (Def. 1.1.8) *preserves limits in the second variable, and sends colimits in the first variable to limits.*

3. *The Yoneda embedding* $H_\bullet : \mathscr{C} \to [\mathscr{C}^{\mathrm{op}}, \mathrm{Set}]$ (Def. 1.1.8) *preserves limits.*

The first and last statements are proven in [Lei14, Prop. 6.2.2 & Rmk. 6.2.3] and [Lei14, Cor. 6.2.12] respectively. For the second, one should use that for all $C \in \mathscr{C}$, $[C,-]$ is a right adjoint (cf. [Mac98, Ch. IV, Sect. 6]), use that right adjoints preserve limits (cf. [Rie17, Thm. 4.5.2]) and conclude dually.

Remark 1.2.3. The basic arithmetic operations on numbers and sets are $+$, $\times$, and exponentiation. Categorically, $+$ and $\times$ can be seen as coproduct and product, respectively. If it exists, the exponential Z^X in a category $\mathscr{C}$ can be described via the natural bijection

$$\mathrm{Hom}_{\mathscr{C}}(Y \times X, Z) \cong \mathrm{Hom}_{\mathscr{C}}(Y, Z^X). \tag{1.1}$$

This bijection completely determines Z^X up to isomorphism and defines the exponential functor $(-)^{(-)} : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{C}$ of the category $\mathscr{C}$.

Finally, since we mainly work with categories of presheaves, let us understand limits in this context. In particular, any category of presheaves is complete and cocomplete, with both limits and colimits being computed objectwise.

Proposition 1.2.2. *Let $\mathscr{C}$ be a category and write $[\mathscr{C}^{\mathrm{op}}, \mathrm{Set}]$ for its category of presheaves. Let I be a small category and consider any functor $F : I \longrightarrow [\mathscr{C}^{\mathrm{op}}, \mathrm{Set}]$, that is, a I-shaped diagram in the category of presheaves. Then:*

1. *the **limit** of F exists, and it is the presheaf that, for each object $C \in \mathscr{C}$, is given by the limit in* Set *of the values of F at C:*
$$\left(\lim_{i\in I}^{[\mathscr{C}^{\mathrm{op}},\mathrm{Set}]} F(i)\right)(C) \cong \lim_{i\in I}^{\mathrm{Set}} F(i)(C);$$

2. *the **colimit** of F exists, and it is the presheaf that, for each object $C \in \mathscr{C}$, is given by the colimit in* Set *of the values of F at C:*
$$\left(\mathrm{colim}_{i\in I}^{[\mathscr{C}^{\mathrm{op}},\mathrm{Set}]} F(i)\right)(C) \cong \mathrm{colim}_{i\in I}^{\mathrm{Set}} F(i)(C).$$

A proof of the proposition can be found in [MM12, pp. 22-23].

Fortunately, representable presheaves characterize the entire category of presheaves.

Proposition 1.2.3. *Let $\mathscr{C}$ be a category and write $[\mathscr{C}^{\mathrm{op}}, \mathrm{Set}]$ for its category of presheaves. Every presheaf $G \in [\mathscr{C}^{\mathrm{op}}, \mathrm{Set}]$ is a colimit of representable presheaves, i.e. $G \cong \mathrm{colim}_{i\in I}^{[\mathscr{C}^{\mathrm{op}},\mathrm{Set}]} H_{C_i}$.*

A proof of the proposition can be found in [MM12, Ch. 1, Sect. 5, Prop. 1].

1.3 Elements of Sheaf Theory

In this section, petit and gros sheaves are introduced together with the Grothendieck topology. Afterwards, some relevant properties are recalled.

Consider a fixed topological space X and the category $\mathbf{X_{cl}}$ of open subsets of X together with inclusions between them.[7] We want to encode the data of X_{cl} in the simpler category of sets Set via a presheaf (Def. 1.1.8) $F : X_{\text{cl}}^{\text{op}} \to \text{Set}$. However, the latter does not guarantee consistency on the intersections of open subsets. The additional locality and gluing conditions characterize sheaves and are necessary to obtain global information from local data uniquely.

Definition 1.3.1. Let X be a topological space. A presheaf of sets F on X consists of a set $F(U)$ for each open set $U \subseteq X$ and a restriction map $\text{res}_V^U : F(U) \to F(V)$ for each inclusion of open sets $V \subseteq U$, satisfying the functoriality axioms:

i. for every open set $U \subseteq X$, the restriction map res_U^U is the identity on $F(U)$;

ii. for open sets $W \subseteq V \subseteq U$, we have $\text{res}_W^V \circ \text{res}_V^U = \text{res}_W^U$.

A **petit sheaf** of sets[8] is a presheaf $F : X_{\text{cl}}^{\text{op}} \to \text{Set}$ that satisfies two additional axioms:[9]

1. let $U \subseteq X$ be open and $\{U_i\}_{i \in I}$ an open cover of U. If $s, t \in F(U)$ satisfy $s|_{U_i} = t|_{U_i}$ for all $i \in I$, then $s = t$; **(Locality condition)**
2. let $\{U_i\}_{i \in I}$ be an open cover of $U \subseteq X$ and $\{s_i \in F(U_i)\}_{i \in I}$ be a family such that for all $i, j \in I$, $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$. Then there exists $s \in F(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$. **(Gluing condition)**

For instance, the presheaf $C^0(-)$ assigning to each open $U \subseteq X$ the set $C^0(U)$ of real-valued continuous functions, with restriction maps being the usual restriction of functions, is also a sheaf. On the contrary, the constant presheaf $\underline{\mathbb{R}}^{\text{psh}}$, which assigns to every open set $U \subseteq X$ the set of constant functions from U to $\mathbb{R}$, does not satisfy the two additional sheaf axioms. For an n-dimensional C^∞-manifold M, we need to consider smooth functions, instead of continuous ones, to capture the smooth structure of the objects. The sheaf of C^∞-functions with the usual restriction is denoted as $\mathcal{O}_M$.

We now need the notion of sheaf morphisms to construct the category of sheaves on X.

Definition 1.3.2. Let F and G be two sheaves of sets on a topological space X. A **morphism** $\varphi : F \to G$, i.e. a natural transformation, consists of a morphism $\varphi_U : F(U) \to G(U)$ of sets for each open set $U \subseteq X$, such that it is compatible with restrictions, i.e. such that for every open subset $V \subseteq U$, the following diagram commutes:

$$\begin{array}{ccc} F(U) & \xrightarrow{\varphi_U} & G(U) \\ {\scriptstyle \text{res}_V^U}\downarrow & & \downarrow{\scriptstyle \text{res}_V'^U} \\ F(V) & \xrightarrow[\varphi_V]{} & G(V). \end{array}$$

[7] Here the suffix cl stands for "classical". It indicates that the site X_{cl} (Def. 1.3.4) is equipped with the usual classical topology, i.e. coverings are ordinary open coverings, so as to distinguish it from other possible choices of Grothendieck topologies on the same underlying space.

[8] One can alternatively store data defining a sheaf using a generic category instead of Set (cf. [Rot09, Ch. 5]).

[9] These axioms can be encoded in an equalizer: in the literature this universal definition of sheaf is usually used (cf. [Vis07, Ch. 2.3.3], [Rie17, App. E.4] and [MM12, Ch. 2]).

For example, the derivative gives a morphism of sheaves on $\mathbb{R}$ from $\mathcal{O}^n_{\mathbb{R}} \to \mathcal{O}^{n-1}_{\mathbb{R}}$. The category of petit sheaves of sets on a fixed topological space X together with such morphisms is a petit topos.

Considering the whole category **Top** of topological spaces and continuous functions between them, we can think of a sheaf on it as a presheaf $F : \mathrm{Top}^{\mathrm{op}} \to \mathrm{Set}$ such that for every $X \in \mathrm{Top}$, the restriction F_X to the subcategory X_{cl} is a sheaf in the sense of Definition 1.3.1. To formalize this, we need to introduce the Grothendieck topology on the category $\mathrm{Top}^{\mathrm{op}}$. The idea is to equip a category with covering families.

Definition 1.3.3. Let $\mathscr{C}$ be a category, $C \in \mathrm{Ob}(\mathscr{C})$. A **sieve** S on C is a subfunctor $S : \mathscr{C}^{\mathrm{op}} \to \mathrm{Set}$ of $\mathrm{Hom}_{\mathscr{C}}(-, C)$. That is, for all $C' \in \mathrm{Ob}(\mathscr{C})$, $S(C') \subseteq \mathrm{Hom}_{\mathscr{C}}(C', C)$ and for all arrows $f : C'' \to C'$, with $C', C'' \in \mathrm{Ob}(\mathscr{C})$, $S(f)$ is the restriction to $S(C')$ of $\mathrm{Hom}_{\mathscr{C}}(-, C)(f)$, where the latter is given by the precomposition with f.

Alternatively, a sieve S on C may be described as a family of morphisms in $\mathscr{C}$, all having codomain C, such that $f \circ g \in S$ whenever $f \in S$ and the composition $f \circ g$ is defined. Note that, if S is a sieve on C and $h \colon D \to C$ is any arrow in $\mathscr{C}$, then the pullback sieve $h^*(S)$ on D is defined by

$$h^*S = \{\, g \mid \mathrm{cod}(g) = D,\ h \circ g \in S \,\}.$$

Definition 1.3.4. A **Grothendieck topology** on a category $\mathscr{C}$ is a function J which assigns to each object $C \in \mathrm{Ob}(\mathscr{C})$ a collection $J(C)$ of sieves on C, in such a way that:

i the maximal sieve $t_C = \{f \mid \mathrm{cod}(f) = C\}$ is in $J(C)$;

ii if $S \in J(C)$, then the pullback sieve $h^*(S) \in J(D)$ for any arrow $h : D \to C$; **(Stability under pullback)**

iii if $S \in J(C)$ and R is any sieve on C such that $h^*(R) \in J(D)$ for all $h : D \to C$ in S, then $R \in J(C)$. **(Transitivity)**

A category $\mathscr{C}$ equipped with a Grothendieck topology J is called a **site**. If a sieve $S \in J(C)$, we say that S covers C. S on C is said to cover $f : D \to C$ if $f^*(S)$ covers D.

We now introduce the definition of a Grothendieck pretopology. As this notion is more intuitive, and every pretopology induces[(10)] a topology, we shall work with pretopologies to avoid additional and unnecessary convoluted structure (cf. [MM12, Ch. 3, Sect. 2]).

Definition 1.3.5. Let $\mathscr{C}$ be a category. A **Grothendieck pretopology** on $\mathscr{C}$ is the assignment to each object $U \in \mathscr{C}$ of a collection of families of morphisms $\{U_i \to U\}$, called **coverings** of U, satisfying the following conditions:

i. if $V \to U$ is an isomorphism in $\mathscr{C}$, then $\{V \to U\}$ is a covering; **(Isomorphism condition)**

ii. if $\{U_i \to U\}$ is a covering and $V \to U$ is any morphism in $\mathscr{C}$, then the fiber products $U_i \times_U V$, i.e. the pullback of $U_i \to U \leftarrow V$, exist, and the family of projections $\{U_i \times_U V \to V\}$ is a covering of V; **(Stability under pullback)**

iii. if $\{U_i \to U\}$ is a covering, and for each index i there is a covering $\{V_{ij} \to U_i\}$, then the collection of composites $\{V_{ij} \to U_i \to U\}$ is a covering of U. **(Transitivity)**

(10) Different pretopologies can give the same topology, but sheaf theory only depends on the topology. Therefore, we can work with pretopologies up to a certain equivalence (cf. [Vis07, Rmk. 2.25]).

One can show that for any topological space X, the category X_{cl} with open coverings is a site. Moreover, Top becomes a site if we take jointly surjective collections of local homeomorphisms $U_i \to U$ as coverings. For SmthMfd, a good choice of coverings is given by considering jointly surjective collections of local diffeomorphisms; however, later we use a more convenient choice to define the category of SmthSet (Def. 2.2.1).

Definition 1.3.6. Let $\mathscr{C}$ be a site, $F\colon \mathscr{C}^{\mathrm{op}} \to \mathrm{Set}$ be a presheaf and $\{U_i \xrightarrow{u_i} U\}_{i\in I}$ be a covering in $\mathscr{C}$. Consider the family $\{a_i \in F(U_i)\}_{i\in I}$, denote by $\mathrm{pr}_1\colon U_i \times_U U_j \to U_i$ and $\mathrm{pr}_2\colon U_i \times_U U_j \to U_j$ the canonical projections of the fibered product, and assume $\mathrm{pr}_1^* a_i = \mathrm{pr}_2^* a_j \in F(U_i \times_U U_j)$, for all $i, j \in I$. Then F is a **sheaf** if there exists a unique[(11)] $a \in F(U)$ such that $u_i^* a = a_i \in FU_i$ for all $i \in I$.
Let F, G be sheaves on a site $\mathscr{C}$. A **morphism of sheaves** $\varphi\colon F \to G$ is a natural transformation of functors. $\mathrm{Sh}(\mathscr{C})$ is the **category of sheaves** over the site $\mathscr{C}$ with such morphisms.

The above definition is a generalization of Definition 1.3.1 since the two coincide when specializing the second to the site X_{cl}.[(12)] When a sheaf is not petit, it is often called a gros sheaf. More explicitly, a gros sheaf is a sheaf defined on a site whose covering families do not arise from the ordinary classical topology of the underlying space, such as those of X_{cl}, but from a more general Grothendieck topology. The category of such sheaves is sometimes referred to as a gros topos. This distinct terminology highlights the qualitative differences between these two cases of the same mathematical definition of a sheaf.

The following summarizes the most important properties of a category of sheaves over a site. Note that, as for presheaves, (co)limits in the category of sheaves are computed objectwise.

Proposition 1.3.1. *Let $\mathscr{C}$ be a site with fixed Grothendieck topology and $\mathrm{Sh}(\mathscr{C})$ the category of sheaves over it. Then:*

1. $\mathrm{Sh}(\mathscr{C})$ *is (co)complete and closed under limits. Moreover, all sheaves are colimits of representable sheaves;*

2. $\mathrm{Sh}(\mathscr{C})$ *has an internal hom-functor and it has an exponential.*

Proofs of the statements can be found in [MM12, Ch. III, Sect. 4-6].

(11) Here, existence and uniqueness correspond respectively to the gluing and locality conditions (Def. 1.3.1).
(12) The connection between Definition 1.3.1 and Definition 1.3.6 is more evident when stating the sheaf condition in terms of an equalizer (cf. [Vis07, Ch. 2.3.3.]).

Chapter 2

Smooth Set Framework for a Bosonic Lagrangian Field Theory

In this chapter, we introduce classical Lagrangian bosonic theories and show how they can be naturally formulated in the setting of SmthSet (Def. 2.2.1). In particular, working within this category, the resulting geometry of field theory makes the differential geometry of the field space come out formally as we would expect if it were a finite-dimensional smooth manifold. This allows a rigorous reformulation of the action principle. However, the definitions of differential forms and their classifying space highlight some issues that are at the very core of these notes, and that are addressed in Chapter 3.

2.1 Lagrangian Field Theory

This section provides a brief overview of Lagrangian field theories and the classical action principle. The absence of a rigorous formalism motivates the structural requirements that the category of smooth sets must satisfy to rigorously formalize the geometry of the action principle.

Classically, the state of a system can be described by assigning the value of some physical observable to any point of a geometric space, typically spacetime. This assignment is supposed to be smooth.

Definition 2.1.1. A **field** is a smooth section (Def. A.1.2) of a smooth fiber bundle (Def. A.1.1) $\pi : F \to M$ where M is the spacetime and F is the configuration bundle. The set of all fields, called the **space of fields**, is the set of smooth maps $\varphi : M \to F$ with $\pi \circ \varphi = \mathrm{id}_M$ and is denoted by $\mathcal{F} := \Gamma_M(F) = \{\varphi : M \to F \,|\, \pi \circ \varphi = \mathrm{id}_M\}$.

The simplest example is the space of scalar fields $\mathcal{F} = C^\infty(M)$ where the fields are sections of the trivial fiber bundle $F := M \times \mathbb{R} \to M$. Taking instead $F = T^*M$ to be the cotangent bundle, we recover the field space of electromagnetism as $U(1)$-gauge potential 1-forms $\Omega^1_{\mathrm{dR}}(M)$.[(1)]

Physical laws are usually expressed as field equations of the form $f(\varphi) = 0$, where $f : \mathcal{F} \to V$ is a map to a vector space V that is typically a differential operator. The solutions of the field equation are called on-shell fields, and their set is denoted $\mathcal{F}_{\mathrm{shell}} \subset \mathcal{F}$. $\mathcal{F}_{\mathrm{shell}}$ is usually complicated to study: the action principle is particularly helpful in characterizing it.

[(1)] Note that connections do not fit exactly into this definition of a field; the notion must be slightly generalized to include differential forms taking values in a Lie algebra.

Principle 2.1.1. *There is a smooth function $S : \mathcal{F} \to \mathbb{R}$, called the* ***action****, such that $\varphi \in \mathcal{F}$ is a solution of the field equation if and only if it is a critical point of S.*

The action allows us to study field theories more easily, also exploiting the symmetries of the system that are manifest as symmetries of the action. Physical action functionals are usually given in an integral form:[(2)]

$$S(\varphi) := \int_M \mathcal{L}(\varphi)$$

where M is the d-dimensional spacetime manifold, assumed to be orientable for simplicity, and the Lagrangian $\mathcal{L}$ is defined as follows.

Definition 2.1.2. A **local Lagrangian** $\mathcal{L} : \mathcal{F} \to \Omega^d(M)$ is a smooth map of sections from the field space to the space of d-differential forms[(3)] on M (of dimension d), such that $\mathcal{L}(\varphi)$ at $m \in M$ depends smoothly on m and on the partial derivatives of φ only up to order k for some $k \geq 0$.[(4)]

Note that the degree of the Lagrangian form has to match the dimension of the manifold $\dim(M) = d$ for $S(\varphi)$ to be well defined. The dynamics of a physical theory is characterized by the Lagrangian and the type of fields we are considering in it.

Definition 2.1.3. A **Lagrangian field theory** (LFT) consists of a smooth bundle $F \to M$ and a local Lagrangian $\mathcal{L} : \mathcal{F} \to \Omega^d(M)$.

Examples of the Lagrangian formulation of a particle in a potential and of classical electromagnetism can be found in [Blo24, Ch. 1].

Using the locality of the Lagrangian, the action principle translates to:

$$\varphi \in \mathcal{F}_{\text{shell}} \ \text{ iff } \ \delta S|_\varphi = \int_M \langle \mathcal{E}\mathcal{L}(\varphi), \delta\varphi \rangle = 0.$$

Here $\mathcal{E}\mathcal{L}$ is the Euler-Lagrange differential operator symbolically given in local coordinates by

$$\mathcal{E}\mathcal{L}(\varphi) = \sum_{|I|=0}^{\infty} (-1)^{|I|} \partial_I \left(\frac{\delta \mathcal{L}(\varphi)}{\delta(\partial_I \varphi)} \right),$$

where I is the usual multindex.[(5)] Globally, this is a map of sections

$$\mathcal{E}\mathcal{L} := \mathcal{F} \longrightarrow \Gamma_M(V^*F \otimes \wedge^d T^*M),$$

where $V^*F \to F$ is the dual vertical bundle (Sect. A.2). The infinitesimal variation $\delta\varphi$ is identified as a section $\delta\varphi \in \Gamma_M(VF)$ to only account for the field variation along a fiber while the base manifold is kept fixed. The natural pairing $\langle -, - \rangle$ is the one induced by the fiber-wise non-degenerate duality pairing of VF with V^*F. Thus, φ is critical if and only if it solves the following equations, called Euler-Lagrange equations:

$$\mathcal{E}\mathcal{L}(\varphi) = 0_\varphi \in \Gamma_M(V^*F \otimes \wedge^d T^*M).$$

The latter is the field equation mentioned above, and the shell space can be thought of as an intersection, i.e. as the following pullback in some abstract category:

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(2) This integral is often divergent or not well defined (cf. [Blo24]), but in practice one does not need to compute it to characterize the on-shell field space.

(3) More generally, one lets $\mathcal{L}$ be a map taking values in the space of densities (cf. [Cat18]).

(4) The locality condition, here stated in a local chart, can be globally characterize using the infinite jet bundle (Def. 3.1.5 and Def. 3.3.1). It is necessary to ensure that the critical points are solutions of a PDE. Physically, this is equivalent to asking the Lagrangian not to include non-local interactions.

(5) For simplicity of notation, we are assuming a simple scalar theory, i.e. $\varphi \in C^\infty(M)$. Note that the infinite sum and the functional derivative are used here informally and would require a rigorous definition.

$$\begin{array}{ccc} \mathcal{F}_{\text{shell}} & \hookrightarrow & \mathcal{F} \\ \downarrow & & \downarrow \mathcal{EL} \\ \mathcal{F} & \xrightarrow{0_{\mathcal{F}}} & T^*_{\text{var}}\mathcal{F}. \end{array}$$

Here $T^*_{\text{var}}\mathcal{F} := \Gamma_M(V^*F \otimes \wedge^d T^*M) \to \mathcal{F}$ is the bundle of variational densities, not to be confused with the cotangent bundle of $\mathcal{F}$.

The above formulas and statements are not rigorously accurate. Indeed, these results hold in the case where all the spaces involved are nice smooth manifolds; however, $\mathcal{F}$ and $\mathcal{F}_{\text{shell}}$ typically do not belong to SmthMfd. Considering the simple case of scalar fields over a compact smooth manifold M, i.e. taking $F = M \times \mathbb{R}$, $\mathcal{F}$ is a Fréchet manifold (Def. 2.3.1) and $\mathcal{F}_{\text{shell}} = f^{-1}(0)$ is generally even more complicated. Moreover, physically we are usually interested in non-compact spacetime, and in this case $\mathcal{F}$ is not even a Fréchet manifold. Finally, the action principle, as stated in Principle 2.1.1, is not rigorously true (cf. [Blo24, Ch. 1.3]).

We aim to make sense of the above in terms of "smooth" spaces and maps within an appropriate category. The following are some requirements that the category of such generalized smooth spaces needs to satisfy.

1. Smooth manifolds and field-theoretical spaces such as the bundle F, the field space $\mathcal{F}$ or the d-dimensional differential forms $\Omega^d(M) = \Gamma_M(\wedge^d T^*M)$, should be objects of the category having therefore a comparable smooth structure.[(6)] To encode locality in global terms, the infinite jet bundle $J^\infty F$ and its space of sections should also be objects in it (Ch. 3).

2. The Lagrangian, the integral operation, and therefore the action, should be smooth maps in the category. For the same reason as above, infinite jet prolongation $j^\infty : \Gamma_M(F) \to \Gamma_M(J^\infty F)$ should be smooth.

3. The category must include a notion of smooth path of fields $\varphi_t : \mathbb{R} \to \mathcal{F}$, so that compositions like $S \circ \varphi_t$ are smooth in the classical sense.[(7)] The infinitesimal variation of the action functional at $\varphi = \varphi_0$ should then be rigorously defined via the usual derivative. Consequently, the action principle should be derived via the former so that

$$\partial_t(S \circ \varphi_t)|_{t=0} = \int \langle \mathcal{EL}(\varphi), \dot{\varphi}_0 \rangle \,, \tag{2.1}$$

where $\dot{\varphi}_0 = \partial_t \varphi_t|_{t=0} \in T_\varphi \mathcal{F} \to \Gamma_M(VF)$ defines the infinitesimal variation at $\varphi \in \mathcal{F}$ as the corresponding vertical tangent vector over it. Note that the variation formula (2.1) is only formal and becomes rigorous once $\Gamma_M(F)$ is interpreted as a smooth set and the integral as a natural transformation.

4. $\mathcal{F}_{\text{shell}}$ should inherit a natural smooth subspace structure, i.e. the category should have pullbacks.[(8)]

In the report, we introduce the category of smooth sets (Sect. 2.2) and demonstrate that it meets these demands (Sect. 2.3 and Ch. 3).

(6) As recalled later, this is one of the advantages in considering the category of all sheaves SmthSet and not only the concrete ones, i.e. the category **DflSp** (cf. [Blo24]).

(7) This assumes that the Cartesian space $\mathbb{R}^n$, for all $n \in \mathbb{N}$, should also be viewed as a generalized smooth space.

(8) This is guaranteed for SmthSet since limits exist and are computed objectwise (Prop. 1.3.1).

2.2 Smooth Sets

In this section, we build up to the formal definition of smooth sets as sheaves on the site CartSp[9] using physical intuition. Crucially, Yoneda lemma for smooth sets ensures the consistency of the definition.

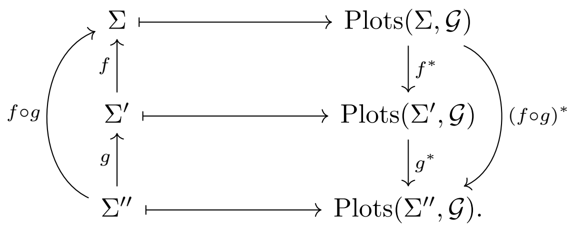
An intuitive approach to the definition of the so-called smooth sets can be given from a physical operational point of view, in analogy with the string-theoretical concept of probe branes. The idea is that we can explore a generalized space $\mathcal{G}$ through the trajectories, also called plots, of known probe branes.[10] In general, a brane of dimension p sweeps out a $(p+1)$-dimensional world volume. More formally, given any finite-dimensional manifold Σ, that is the probe brane's world volume, we call plot the smooth trajectory $\Sigma \to \mathcal{G}$. Instead of defining $\mathcal{G}$ as a set of points with extra structure, we follow the idea of characterizing $\mathcal{G}$ via the system of these plots.

The following heuristic requirements describe the minimum structure needed for this to be feasible and are formalized below by defining smooth sets as sheaves on the site CartSp.

1. For each probe manifold Σ, there should be a set of Σ-shaped smooth plots $\mathrm{Plots}(\Sigma, \mathcal{G})$.

2. Precomposition of a plot with a smooth map between probe manifolds should be a plot. Precomposition with the identity map should be the identity on the set of plots, and precomposition with two successive maps should satisfy the following:

$$\mathrm{SmthMfd}^{\mathrm{op}} \xrightarrow{\mathrm{Plots}(-,\mathcal{G})} \mathbf{Set}$$



One can recognize in the above the conditions characterizing a contravariant functor, therefore $\mathrm{Plots}(-, \mathcal{G})$ should be a presheaf on SmthMfd.

3. We should have a notion of locality and a way of consistently gluing information to go from local to global. Considering a differentiably-good open cover of the probe manifold $\{\Sigma_i \overset{\iota_i}{\hookrightarrow} \Sigma\}_{i\in I}$, i.e. a collection $\{\Sigma_i \subset \Sigma\}_{i\in I}$ of open subsets of Σ such that $\Sigma = \bigcup_{i\in I} \Sigma_i$ and such that all Σ_i, together with all their inhabited finite intersections, are diffeomorphic to an open ball, Σ-plots should be the same as I-tuples of the overlapping Σ_i-plots. That is, the following should be a bijective decomposition:

$$\begin{aligned} \mathrm{Plots}(\Sigma, \mathcal{G}) &\longrightarrow \left\{ (p_i \in \mathrm{Plots}(\Sigma_i, \mathcal{G}))_{i\in I} \,\middle|\, \forall i, j \in I \text{ with } p_i|_{\Sigma_i\cap\Sigma_j} = p_j|_{\Sigma_i\cap\Sigma_j} \right\} \\ (\Sigma \xrightarrow{p} \mathcal{G}) &\longmapsto (\Sigma_i \overset{\iota_i}{\hookrightarrow} \Sigma \xrightarrow{p} \mathcal{G})_{i\in I}. \end{aligned} \tag{2.2}$$

The latter is the sheaf condition (Def. 1.3.6) on the site SmthMfd with respect to the Grothendieck topology (Def. 1.3.4) induced by the pretopology of differentiably good open covers (Def. 1.3.5). The plots are maps from the defining site.

(9) **CartSp** is the category of Cartesian spaces with morphisms being smooth maps in the classical sense.
(10) For instance, we can sample the structure and geometry of spacetime via a one-dimensional trajectory given by the worldline of a point particle, which is a zero-dimensional brane.

Finally, note that by definition smooth n-dimensional manifolds are covered by Cartesian spaces $\mathbb{R}^n$. Therefore, using the sheaf condition of Equation (2.2), one can show the equivalence of categories Sh(SmthMfd)≅Sh(CartSp), i.e. it is sufficient to probe $\mathcal{G}$ using $\mathbb{R}^n$, $n \in \mathbb{N}$.

Definition 2.2.1. The category of **smooth sets** is the category of sheaves over the site of Cartesian spaces with respect to the differentiably-good open covers:

$$\mathbf{SmthSet} := \mathrm{Sh}(\mathrm{CartSp}), \;\; \mathcal{G}(-) := \mathrm{Plots}(-, \mathcal{G}).$$

Remark 2.2.1. We would like smooth manifolds to be smooth in the sense of SmthSet. This is true for any $M \in \mathrm{SmthMfd}$ by taking $\mathrm{Plots}(\mathbb{R}^k, M) := C^\infty(\mathbb{R}^k, M)$, $k \in \mathbb{N}$, since for all $f : \mathbb{R}^{k'} \to \mathbb{R}^k$, $k' \in \mathbb{N}$, the corresponding pullback of plots is the usual precomposition $(-) \circ f$ of smooth maps in SmthMfd. There is, however, a potential inconsistency in the definition: we bootstrapped Definition 2.2.1 assuming the existence of a map $\Sigma \to \mathcal{G}$, but now we know that these two objects live in the same category, so such map should now consistently send $\mathbb{R}^k$-plots of $y(M)$ to $\mathbb{R}^k$-plots of $\mathcal{G}$. In other words, the defining plots $\mathcal{G}(M)$ should coincide with the smooth maps $\mathrm{Hom}_{\mathrm{SmthSet}}(y(M), \mathcal{G})$, i.e. with the natural transformations between the two sheaves. This is the case thanks to Proposition 2.2.1, that is, the Yoneda Lemma for smooth sets.

Proposition 2.2.1. *For $\mathcal{G} \in \mathrm{SmthSet}$ and $M \in \mathrm{SmthMfd}$, there is a natural bijection between the M-plots of $\mathcal{G}$ and the smooth maps of smooth sets from $y(M)$ to $\mathcal{G}$:*

$$\mathcal{G}(M) \equiv \mathrm{Plots}(M, \mathcal{G}) \cong \mathrm{Hom}_{\mathrm{SmthSet}}(y(M), G).$$

Here, $y(M) := \mathrm{Hom}_{\mathrm{SmthMfd}}(-, M)$ is the image via Yoneda embedding of the smooth manifold M into the category of smooth sets.
Choosing $\mathcal{G} = y(N) \in \mathrm{SmthSet}$ for an arbitrary manifold N, the above immediately implies that the embedding functor

$$y : \mathrm{SmthMfd} \hookrightarrow \mathrm{SmthSet}, \;\; M \mapsto y(M) := \mathrm{Hom}_{\mathrm{SmthMfd}}(-, M)$$

is fully faithful. Hence, any results and constructions on finite-dimensional smooth manifolds may equivalently be phrased in terms of their smooth set incarnation and vice versa.

A proof of the statement can be found in [GS25, Prop. 2.5].

Consequently, Definition 2.2.1 is well posed and all the properties of categories of sheaves discussed in Chapter 1 apply. Note that this means that all limits and colimits exist in SmthSet and they are computed pointwise (Prop. 1.3.1). Consequently, requirement 4 in Section 2.1 is already fulfilled as far as we can express the critical locus as such pullback.[(11)]

2.3 Vector Fields and Differential Forms on the Field Space

In this section, we show that the category of SmthSet naturally encodes the notions of field space, its tangent bundle, vector fields, and differential forms on it. Finally, we present some issues related to the classifying space of de Rham forms.

The traditional approach to formalizing field spaces is to model them as infinite-dimensional Fréchet manifolds, i.e. modeled on Fréchet spaces instead of Cartesian spaces (cf. [KM97]).

(11) This goes beyond the scope of this report, and it is properly done in [GS25, Ch. 5]. An intuitive approach is given in the example after Definition 2.3.3

Definition 2.3.1. A **Fréchet manifold** is a Hausdorff topological space X with an atlas of coordinate charts over Fréchet spaces whose transitions are smooth mappings in the sense of Michal–Bastiani calculus (cf. [Ham82, Def. 3.6.1. and Ch. I.4]). A Fréchet space is a Hausdorff topological vector space whose topology may be induced by a countable family of semi-norms that are complete with respect to this family.

Consider the trivial bundle $F = M \times N$ where $N, M \in \mathrm{SmthMfd}$. If M is compact, the space of fields is inside the category of Fréchet manifolds $C^\infty(M, N) \in \mathbf{FrMfd}$,[12] whereas if M is not compact, this is not the case and the manifold description via infinite-dimensional charts becomes tricky (cf. [KM97]). In this setting, one can give sense to the smoothness of field-theoretical tools as presented above; however, this does not generalize straightforwardly to the non-compact case and brings heavy functional-analytical baggage.

A way to bypass the problem of defining the smooth structure of the field space is to utilize the technology of infinite jet bundles to encode local theories, which are more physically relevant (Ch. 3). More recently, attention has shifted to diffeological spaces (cf. [Blo24]), in which field theories and the technology of the infinite jet bundle are better understood. The category of diffeological spaces is the category of concrete sheaves, i.e. with a notion of underlying set of points,[13] over CartSp with respect to the same Grothendieck topology we used to define SmthSet. However, this brings two drawbacks: the categorical background needed to work in DflSp is heavy and the latter does not allow for a uniform treatment of all field-theoretical objects (cf. [GS25, Rmk. 3.8]).

SmthSet is a further abstraction in which SmthMfd, FrMfd and DflSp are fully faithfully included. It should allow to solve these drawbacks by subsuming and combining these approaches, and should naturally generalize to include fermionic fields and infinitesimal structure (cf. [Gri25]).

To build up to the general definitions in SmthSet, we start from FrMfd and then use the following result.

Proposition 2.3.1. *Consider the category* FrMfd *of infinite-dimensional Fréchet manifolds. The embedding along* CartSp $\hookrightarrow$ SmthMfd $\hookrightarrow$ FrMfd *defines a fully faithful embedding*

$$y\colon \mathrm{FrMfd} \longrightarrow \mathrm{SmthSet},\ G \longmapsto \mathrm{Hom}_{\mathrm{FrMfd}}(-, G)|_{\mathrm{CartSp}} \tag{2.3}$$

where on the right-hand side we consider smooth Fréchet maps.
Moreover, let M, N be finite-dimensional manifolds, with M compact. Then there exists a canonical bijection

$$\mathrm{Hom}_{\mathrm{FrMfd}}(S, C^\infty(M, N)_{\mathrm{FrMfd}}) \cong_{\mathrm{Set}} \mathrm{Hom}_{\mathrm{SmthMfd}}(S \times M, N),$$

for any smooth manifold $S \in$ SmthMfd*, that is moreover natural in S.*

For a proof of this proposition, see the references of [GS25, Prop. 2.8 and Prop. 2.9].

The latter exponential law has to be consistent, via Yoneda embedding, with the one given by the honest internal hom-function (Def. 1.1.8) of SmthSet. This is true if the smooth structure in the sense of Fréchet manifolds is equivalent to the one given by the hom-functor in SmthSet.

Definition 2.3.2. Let $\mathcal{G}, \mathcal{H} \in \mathrm{SmthSet}$, the **smooth mapping set** $[\mathcal{G}, \mathcal{H}] \in \mathrm{SmthSet}$ is defined by

$$[\mathcal{G}, \mathcal{H}](y(\mathbb{R}^k)) := \mathrm{Hom}_{\mathrm{SmthSet}}(y(\mathbb{R}^k) \times \mathcal{G}, \mathcal{H}).$$

(12) One can see this considering the countable family of seminorms each given by the sum of the supremum norms of all the derivatives up to some order $k \in \mathbb{N}$. M being compact ensures that these seminorms remain finite.

(13) One can intuitively think of this as follows: the smooth set $\mathcal{G}$ is concrete if there exists a set of point $G_s \in \mathrm{Set}$ such that for each probe $\mathbb{R}^k \in \mathrm{CartSp}$, $\mathcal{G}(\mathbb{R}^k) \subset \mathrm{Hom}_{\mathrm{Set}}(\mathbb{R}^k, G_s)$.

The exponential property on SmthSet

$$\mathrm{Hom}_{\mathrm{SmthSet}}(\mathcal{X}, [\mathcal{G}, \mathcal{H}]) \cong_{\mathrm{SmthSet}} \mathrm{Hom}_{\mathrm{SmthMfd}}(\mathcal{X} \times \mathcal{G}, \mathcal{H}),$$

descend from Yoneda lemma (Prop. 2.2.1) and generalize to all smooth sets using the relations between hom-functor and limits and the fact that any sheaf is a colimit of representable presheaves (Prop. 1.2.1 and Prop. 1.3.1). Indeed, Yoneda preserves Fréchet mapping space.

Proposition 2.3.2. *Let $y(M), y(N) \in \mathrm{SmthSet}$ for $M, N \in \mathrm{SmthMfd}$ with M compact. The embedding of the Fréchet manifold $C^\infty(M, N)_{\mathrm{FrMfd}}$ in smooth sets is isomorphic to the mapping smooth set $[y(M), y(N)]$:*

$$y(C^\infty(M, N)_{\mathrm{FrMfd}}) \cong_{\mathrm{SmthSet}} [y(M), y(N)].$$

For a proof, see [GS25, Prop. 2.11].

Thus, the internal hom-functor provides a means of defining the correct smooth structure within SmthSet, thereby bypassing functional analytical technology and extending the Fréchet mapping space even to the case when M is not compact.

We can now define smooth sets of smooth sections.

Definition 2.3.3. Let $\pi : F \to M$ be a fiber bundle of smooth manifolds, with set of smooth sections $\Gamma_M(F)$. The **smooth set of sections** $\mathcal{F} = \mathbf{\Gamma}_M(F) \in \mathrm{SmthSet}$ is defined by

$$\mathcal{F}(\mathbb{R}^k) \equiv \mathbf{\Gamma}_M(F)(\mathbb{R}^k) := \{\varphi^k : \mathbb{R}^k \times M \to F \,|\, \pi \circ \varphi^k = \mathrm{pr}_2\}$$

where $\mathbb{R}^k \in \mathrm{CartSp}$ and $\mathrm{pr}_2 : \mathbb{R}^k \times M \to M$ is projection onto M. That is, $\varphi^k : \mathbb{R}^k \times M \to F$ is such that

$$\begin{array}{ccc} & & F \\ & \overset{\varphi^k}{\nearrow} & \downarrow{\scriptstyle \pi} \\ \mathbb{R}^k \times M & \xrightarrow{\mathrm{pr}_2} & M \end{array}$$

commutes, and so equivalently $\mathcal{F}(\mathbb{R}^k) \cong_{\mathrm{Set}} \Gamma_{M \times \mathbb{R}^k}(\mathrm{pr}_2^* F)$ where $\mathrm{pr}_2^* F$ denotes the pullback bundle (Def. A.1).

Remark 2.3.1. Note that, up to this point, $\mathcal{F} \equiv \Gamma_M(F)$ denotes the set-theoretic field space, i.e. the set of smooth sections of the fiber bundle $\pi : F \to M$, where F is total space. From now on, unless stated otherwise, we set $\mathcal{F} \equiv \mathbf{\Gamma}_M(F)$ to denote the smooth set–theoretic field space, which we distinguish from the mere set of sections by using the bold symbol.

In other words,(14) $\mathcal{F}(\mathbb{R}^k)$ is the set of smoothly $\mathbb{R}^k$-parametrized sections of $F \to M$. As a motivating example, let's consider the trivial bundle $F = M \times N$, where M and N are smooth manifolds, with M being compact. The field of off-shell fields is the Fréchet manifold $C^\infty(M, N)$, i.e. the space of smooth functions from M to N. To see that this is indeed a smooth set, we need to build $\mathbb{R}^k$-shaped plots into it. We define a plot as a smoothly $\mathbb{R}^k$-parametrized family of fields φ^k, that is, via the map $\varphi^k : \mathbb{R}^k \times M \to N, (x, m) \mapsto \varphi^k_x(m) \equiv \varphi^k(x, m)$, smooth in both variables. Therefore $\varphi^k \in \mathrm{Plots}(\mathbb{R}^k, C^\infty(M, N)) := C^\infty(\mathbb{R}^k \times M, N)$.

This simple case allows us to get a glimpse of the next steps to take. Considering a smooth 1-parameter family of fields $\varphi_t \equiv \varphi^1 : \mathbb{R} \times M \to N$, $(t, m) \mapsto \varphi^1_t(m)$, one can see that for an

(14)This interpretation holds only for concrete smooth sets.

action functional to be a smooth map, it must take 1-parameter families of fields to 1-parameter families of real numbers:

$$\begin{array}{rcl} \Gamma_M(M \times N) & \xrightarrow{S} & \mathbb{R} \\ S_*^{\mathbb{R}} : C^\infty(\mathbb{R} \times M, N) & \longrightarrow & C^\infty(\mathbb{R}, \mathbb{R}) \\ \varphi_t & \longmapsto & (t_0 \mapsto S(\varphi_{t=t_0}))\,. \end{array}$$

Through the usual derivative on $\mathbb{R}$, we can define the variation δS at φ_0, along the family φ_t, as $\delta_{\varphi_t} S := \frac{\mathrm{d}}{\mathrm{d}t} S(\varphi_t)\big|_{t=0}$. Consequently, the critical locus is the following:

$$\mathrm{Crit}(S) = \left\{ \varphi \in C^\infty(M, N) \;\middle|\; \frac{\mathrm{d}}{\mathrm{d}t} S(\varphi_t)\Big|_{t=0} = 0,\ \forall \varphi_t \in C^\infty(\mathbb{R} \times M, N) \text{ such that } \varphi_{t=0} = \varphi \right\}.$$

More generally, the notion of smooth sets allows us to perform standard operations of finite-dimensional differential geometry pointwise on ordinary manifolds, and thereby extend these notions to smooth sets like $\mathbf{\Gamma}_M(F)$. This idea is used in [GS25] to construct the variation of the action and demonstrate how the action can indeed be viewed as a smooth map of smooth sets. In [GS25, Ch. 5] the critical locus is also shown to have a natural smooth subset structure, in more detail, satisfying requirements 3 in Section 2.1.

Remark 2.3.2. An arbitrary fiber bundle $F \to M$ might have no global sections, so Definition 2.3.3 might be null. However, noticing that the assignment of local smooth sections $U \mapsto \Gamma_U(F)$ defines a petit sheaf on M with values in SmthSet, one might instead consider the functor

$$\begin{array}{rl} \mathbf{\Gamma}_{(-)}(F) : M_{\mathrm{cl}}^{\mathrm{op}} \times \mathrm{CartSp}^{\mathrm{op}} & \longrightarrow \mathrm{Set} \\ (U \times \mathbb{R}^k) & \longmapsto \Gamma_U(F)(\mathbb{R}^k) \cong \Gamma_{U \times \mathbb{R}^k}(\mathrm{pr}_2^* F). \end{array}$$

Since all statements and definitions regarding the smooth set of fields $\mathcal{F} = \mathbf{\Gamma}_M(F)$ functorially apply for each such open set U, we can continue to work with Definition 2.3.3.

We would now like to define the tangent vectors intuitively as first-order infinitesimal smooth curves[(15)] in the field space. First, notice that smooth real-valued functions on $\mathcal{F}$ are defined as maps of smooth sets $C^\infty(\mathcal{F}) = \mathrm{Hom}_{\mathrm{SmthSet}}(\mathcal{F}, y(\mathbb{R}))$ and therefore the algebra structure of $C^\infty(\mathcal{F})$ follows pointwise from that of $y(\mathbb{R})$. Moreover, we can define the induced derivation

$$C^\infty(\mathcal{F}) \longrightarrow \mathbb{R},\ f \mapsto \partial_t (f \circ \varphi_t)|_{t=0},$$

since for any $\varphi_t \in \mathbf{\Gamma}_M(F)(\mathbb{R}) = \mathrm{Hom}_{\mathrm{SmthSet}}(y(\mathbb{R}), \mathbf{\Gamma}_M(F))$, the composition $f \circ \varphi_t$ defines a smooth map $\mathbb{R} \to \mathbb{R}$. Note that, by the section condition, for each $m \in M$ we have a smooth curve $\varphi_t(m) : \mathbb{R}_t \to F$ whose image is contained in the fiber over m. Thus,

$$\partial_t \varphi_t(m)|_{t=0} \in V_{\varphi_0(m)} F$$

defines a vertical tangent vector at $\varphi_0(m) \in F$. Varying over $m \in M$, we get a smooth section of the vertical bundle $\pi_F : TF \to F$ (Sect. A.2)

$$\begin{array}{ccc} & & VF \\ & \overset{\partial_t \varphi_t|_{t=0}}{\nearrow} & \downarrow{\scriptstyle \pi_F} \\ M & \xrightarrow{\varphi_0} & F. \end{array}$$

(15) This intuition should be incorporated in a rigorous definition enriching the category with infinitesimal structure (cf. [Gri25]).

Equivalently, the above diagram defines a section of the pullback bundle $\varphi_0^* VF$. We therefore interpret $\Gamma_M(\varphi_0^* VF)$ as the tangent space at φ_0 getting the following:

$$T(\Gamma_M(F)) := \bigcup_{\varphi_0 \in \Gamma_M(F)} \Gamma_M(\varphi_0^* VF) \cong_{\mathrm{Set}} \Gamma_M(VF).$$

The next result ensures that any tangent vector field is represented by a line-plot.

Theorem 2.1. *For any $\mathcal{Z}_\varphi \in \Gamma_M(VF)$ covering $\varphi = \pi_F \circ \mathcal{Z}_\varphi$, $\pi_F : TF \to F$, i.e. such that*

$$\begin{array}{ccc} & & VF \\ & \overset{\mathcal{Z}_\varphi}{\nearrow} & \downarrow{\scriptstyle \pi_F} \\ & \overset{\varphi}{\nearrow} & F \\ & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{\mathrm{id}_M} & M \end{array}$$

commutes, there exists a $\varphi_t : \mathbb{R} \times M \to F$ such that $\varphi_0 = \varphi$ and $\partial_t \varphi_t|_{t=0} = \mathcal{Z}_\varphi$. That is the following map is surjective:

$$\mathbf{\Gamma}_M(F)(\mathbb{R}) \to T(\Gamma_M(F)), \; \varphi_t \mapsto \partial_t \varphi_t|_{t=0}.$$

A proof can be found in [GS25, Lem. 2.18]. The existence of φ_t, hence the surjectivity of the above map, is guaranteed under suitable regularity assumptions on $F \to M$. In particular, since $F \to M$ is a fiber bundle, the result holds due to the geometrical and topological properties of its sections and its vector fields.

Remark 2.3.3. Unlike the finite-dimensional case, there is no obvious reason why the derivation should depend only on the corresponding tangent vector. More explicitly, the derivation might depend on the representative $\mathbb{R}$-plot. Since this potential issue is solved by working with local vectors (Sect. 3.3) or within the infinitesimally thickened category (cf. [GS25, Rmk. 2.19]), it precisely motivates the use of local or infinitesimal smooth sets.

This directly leads to the definition of the smooth tangent bundle to the field space.

Definition 2.3.4. The **smooth tangent bundle** to a field space $\mathcal{F} = \mathbf{\Gamma}_M(F)$ is defined by

$$T\mathcal{F} := \mathbf{\Gamma}_M(VF), \tag{2.4}$$

as the smooth set of sections of VF via Definition 2.3.3.

More concretely, $\mathbb{R}^k$-plots of the tangent bundle $T\mathcal{F}$ correspond to pairs $(\mathcal{Z}_{\varphi_0^k}, \varphi_0^k)$ of $\mathbb{R}^k$-parametrized sections over M such that the following commutes:

$$\begin{array}{ccc} & & VF \\ & \overset{\mathcal{Z}_{\varphi_0^k}}{\nearrow} & \downarrow{\scriptstyle \pi_F} \\ \mathbb{R}^k \times M & \xrightarrow{\varphi_0^k} & F. \end{array}$$

From this, one can see that the fiber-wise $\mathbb{R}$-linear structure of the plain set bundle $\Gamma_M(VF) \to \Gamma_M(F)$ extends plot-wise to a smooth $\mathbb{R}$-linear map

$$+_{T\mathcal{F}} : T\mathcal{F} \times_{\mathcal{F}} T\mathcal{F} \longrightarrow T\mathcal{F}, \; (\mathcal{Z}^1_{\varphi_0^k}, \mathcal{Z}^2_{\varphi_0^k}) \mapsto \mathcal{Z}^1_{\varphi_0^k} + \mathcal{Z}^2_{\varphi_0^k}. \tag{2.5}$$

Similarly for the scalar multiplication. Furthermore, the projection $\Gamma_M(VF) \to \Gamma_M(F)$ extends to the smooth projection map

$$\pi_{\mathcal{F}} : T\mathcal{F} \to \mathcal{F}, \ (\mathcal{Z}_{\varphi_0^k}, \varphi_0^k) \mapsto \varphi_0^k.$$

This allows us to define vector fields on the field space as geometrical smooth sections of the tangent bundle.

Definition 2.3.5. The set of **smooth vector fields** on the field space $\mathcal{F} = \mathbf{\Gamma}_M(F)$ is defined as smooth sections of its tangent bundle

$$\mathfrak{X}(\mathcal{F}) := \{\mathcal{Z} : \mathcal{F} \to T\mathcal{F} \mid \pi_{\mathcal{F}} \circ \mathcal{Z} = \mathrm{id}_{\mathcal{F}}\}.$$

That is, smooth maps $\mathcal{Z} : \mathcal{F} \to T\mathcal{F}$ such that the following diagram of smooth sets commutes:

$$\begin{array}{ccc} & & T\mathcal{F} \\ & {\scriptstyle \mathcal{Z}} \nearrow & \big\downarrow {\scriptstyle \pi_{\mathcal{F}}} \\ \mathcal{F} & \xrightarrow{\ \mathrm{id}_{\mathcal{F}}\ } & \mathcal{F}. \end{array}$$

In concrete smooth sets, one can understand this as follows. On $*$-plots, such a section defines a vector field in the usual sense, i.e. a map of sets $\mathcal{Z} : \Gamma_M(F) \longrightarrow \Gamma_M(VF)$ which assigns to every field configuration $\varphi \in \Gamma_M(F)$ a tangent vector $\mathcal{Z}_\varphi \in T_\varphi\big(\Gamma_M(F)\big) = \Gamma_M(\varphi^* VF)$. Being a smooth map, the section also sends smooth $\mathbb{R}^k$-plots of field configurations, i.e. smoothly $\mathbb{R}^k$-parametrized sections $\varphi^k : \mathbb{R}^k \times M \to F$, to smooth $\mathbb{R}^k$-plots of tangent vectors, i.e. smoothly $\mathbb{R}^k$-parametrized sections $\mathcal{Z}_{\varphi^k} : \mathbb{R}^k \times M \to VF$.

Vector fields on the field space $\mathcal{F}$ should be interpreted as infinitesimal smooth diffeomorphisms, in direct analogy with the finite-dimensional case. Note that this would allow us to incorporate the necessary tools to discuss the symmetries of field theory in this setting. This can be done (cf. [GS25, Ch. 2]), but it goes beyond the scope of these notes. Moreover, physically relevant smooth vector fields on field spaces are local vector fields: we introduce them in Sect. 3.3, but we refer to [GS25, Ch. 6] for an in-depth analysis of their role in infinitesimal symmetries and Noether theorems.

The fiber-wise linear structure (Eq. (2.5)) of the smooth tangent bundle $T\mathcal{F}$ on a field space $\mathcal{F}$ allows us to define differential forms as fiber-wise linear and antisymmetric maps out of $T\mathcal{F}$.

Definition 2.3.6. The **set of differential m-forms** on $\mathcal{F} = \mathbf{\Gamma}_M(F)$ is defined as

$$\Omega^m(\mathcal{F}) := \mathrm{Hom}^{\mathrm{fib.lin.an.}}_{\mathrm{SmthSet}}\left(T^{\times m}\mathcal{F}, y(\mathbb{R})\right).$$

That is, m-forms are smooth real-valued, fiber-wise linear antisymmetric maps with respect to the fiber-wise linear structure (Eq. (2.5)) on the m-fold fiber product[(16)]

$$T^{\times m}\mathcal{F} := T\mathcal{F} \times_{\mathcal{F}} \cdots \times_{\mathcal{F}} T\mathcal{F}$$

of the tangent bundle over $\mathcal{F}$.

The collection of differential forms of all degrees forms a graded $\mathbb{R}$-vector space (Def. A.3.1)

$$\Omega^\bullet(\mathcal{F}) := \bigoplus_{m \in \mathbb{N}} \Omega^m(\mathcal{F}).$$

(16) Here the product is the pullback (Def. 1.2.3) of $T\mathcal{F} \xrightarrow{\pi_{\mathcal{F}}} \mathcal{F} \xleftarrow{\pi_{\mathcal{F}}} T\mathcal{F}$.

The latter has a well-defined notion of a wedge product $\wedge$ and contraction $\iota_{\mathcal{Z}}$ along any vector field $\mathcal{Z} \in \mathfrak{X}(\mathcal{F})$.[17] However, there is no obvious definition for a de Rham differential $\mathrm{d}_{\mathcal{F}} : \Omega^m \to \Omega^{m+1}$. Indeed, if there were one, it would define a derivative on $C^\infty(\mathcal{F})$ through the Lie derivative, in contrast with Remark 2.3.3.

We want to solve this by trying to define de Rham forms abstractly in SmthSet, i.e. via a notion that naturally allows one to consider an m-form on an arbitrary smooth set.

Definition 2.3.7. For each $m \in \mathbb{N}$, we define the **moduli space of de Rham m-forms** $\mathbf{\Omega}^m_{\mathrm{dR}} \in \mathrm{SmthSet}$ by

$$\mathbf{\Omega}^m_{\mathrm{dR}}(\mathbb{R}^k) := \Omega^m(\mathbb{R}^k). \tag{2.6}$$

That is, the assignment of m-forms on each $\mathbb{R}^k \in \mathrm{CartSp}$.

One can see that $\mathbf{\Omega}^m_{\mathrm{dR}}$ is a smooth set. Firstly, it is a presheaf since forms pullback along maps of manifolds. Furthermore, it is a sheaf since locally defined forms can be consistently glued. Note that $\mathbf{\Omega}^m_{\mathrm{dR}}$ for $m > 0$ is the first non-concrete smooth set we encounter: for $m = 0$ it is concrete since $\mathbf{\Omega}^0_{\mathrm{dR}} = y(\mathbb{R})(\mathbb{R}^k)$, however for $m > 0$ this is not the case anymore. Indeed, once fixed $m \in \mathbb{N}$, it is possible to show that $\mathbf{\Omega}^n_{\mathrm{dR}}(\mathbb{R}^k) = \{*\}$ for all $k < m$ since the only form in this case is the null one. On the contrary, for $k \geq m$, $\mathbf{\Omega}^n_{\mathrm{dR}}(\mathbb{R}^k) = \Omega^m(\mathbb{R}^k)$ is an infinite set.

We can now introduce the operations $\mathrm{d}_{\mathrm{dR}} : \mathbf{\Omega}^m_{\mathrm{dR}} \longrightarrow \mathbf{\Omega}^{m+1}_{\mathrm{dR}}$ and $\wedge : \mathbf{\Omega}^n_{\mathrm{dR}} \times \mathbf{\Omega}^m_{\mathrm{dR}} \longrightarrow \mathbf{\Omega}^{n+m}_{\mathrm{dR}}$ on the moduli space for any $n, m \geq 0$. They are defined plot-wise for all $\mathbb{R}^k$ respectively by

$$\begin{aligned} \Omega^m(\mathbb{R}^k) &\to \Omega^{m+1}(\mathbb{R}^k) \qquad \text{and} \qquad & \Omega^m(\mathbb{R}^k) \times \Omega^n(\mathbb{R}^k) &\to \Omega^{m+n}(\mathbb{R}^k) \\ \omega_{\mathbb{R}^k} &\mapsto \mathrm{d}_{\mathbb{R}^k}\omega_{\mathbb{R}^k} & (\omega_{\mathbb{R}^k}, \omega'_{\mathbb{R}^k}) &\mapsto \omega_{\mathbb{R}^k} \wedge_{\mathbb{R}^k} \omega'_{\mathbb{R}^k}. \end{aligned} \tag{2.7}$$

Here $\omega_{\mathbb{R}^k} \in \Omega^m(\mathbb{R}^k)$, $\omega'_{\mathbb{R}^k} \in \Omega^n(\mathbb{R}^k)$, and $\mathrm{d}_{\mathbb{R}^k}$, $\wedge_{\mathbb{R}^k}$ are the usual de Rham derivative and wedge product on $\mathbb{R}^k$ respectively. Both are smooth maps since $\mathrm{d}_{\mathbb{R}^k}$, $\wedge_{\mathbb{R}^k}$ commute with pullbacks of manifolds. Due to the plot-wise properties of these two operations, we obtain the following proposition.

Proposition 2.3.3. *The differential graded commutative $\mathbb{R}$-algebra (DGCA) (Def. A.3.5) structure of forms $\Omega^\bullet(\mathbb{R}^k)$ on each Cartesian space $\mathbb{R}^k \in \mathrm{CartSp}$ induces a DGCA structure on the moduli space of forms $\mathbf{\Omega}^\bullet_{\mathrm{dR}} \in \mathrm{SmthSet}$. That is, the triple $(\mathbf{\Omega}^\bullet_{\mathrm{dR}}, \mathrm{d}_{\mathrm{dR}}, \wedge)$ forms a DGCA in* SmthSet *as a module* (Def. A.2.2) *over $y(\mathbb{R}) \in \mathrm{SmthSet}$:*

$$(\mathbf{\Omega}^\bullet_{\mathrm{dR}}, \mathrm{d}_{\mathrm{dR}}, \wedge) \in \mathrm{DGCA}(\mathrm{SmthSet}).$$

We now define n-forms on SmthSet as functor represented by the moduli space. This can be justified in analogy with the following: for any $M \in \mathrm{SmthMfd}$, using Proposition 2.2.1 and that $\mathrm{Sh}(CartSp) \cong \mathrm{Sh}(\mathrm{SmthMfd})$, one can see that the following holds:

$$\mathrm{Hom}_{\mathrm{SmthSet}}(y(M), \mathbf{\Omega}^n_{\mathrm{dR}}) \cong \Omega^n(M) \cong \mathrm{Hom}^{\text{fib.lin.}}_{\mathrm{SmthMfd}}(T^{\times n}M, \mathbb{R}). \tag{2.8}$$

Definition 2.3.8. The set of **smooth de Rham n-forms** on a generalized smooth space $\mathcal{F} \in \mathrm{SmthSet}$ is defined by

$$\Omega^n_{\mathrm{dR}}(\mathcal{F}) := \mathrm{Hom}_{\mathrm{SmthSet}}(\mathcal{F}, \mathbf{\Omega}^n_{\mathrm{dR}}).$$

More concretely, a form is a natural transformation sending $\mathbb{R}^k$-plots of $\mathcal{F}$ to $\mathbb{R}^k$-plots of $\mathbf{\Omega}^n_{\mathrm{dR}}$, i.e. to forms on $\mathbb{R}^k$. This allows us to define the differential and the wedge product as simple compositions with the universal ones defined in Equation (2.7).

[17] Thanks to the fiber-wise linear structure of $T\mathcal{F}$ and graded structure of $\Omega^\bullet(\mathcal{F})$, they can be defined as usual (App. A).

Definition 2.3.9.

1. The **de Rham 1-form differential** $\mathrm{d}_{\mathrm{dR}}S \in \Omega^1_{\mathrm{dR}}(\mathcal{F})$ of a smooth map $S : \mathcal{F} \to y(\mathbb{R})$ is defined as
$$\mathrm{d}_{\mathrm{dR}}S : \mathcal{F} \xrightarrow{S} y(\mathbb{R}) \cong \mathbf{\Omega}^0_{\mathrm{dR}} \xrightarrow{\mathrm{d}_{\mathrm{dR}}} \mathbf{\Omega}^1_{\mathrm{dR}}.$$

2. Similarly, the **de Rham differential** $\mathrm{d}_{\mathrm{dR}}\omega \in \Omega^{n+1}_{\mathrm{dR}}(\mathcal{F})$ of an n-form $\omega \in \Omega^n_{\mathrm{dR}}(\mathcal{F})$ is defined as
$$\mathrm{d}_{\mathrm{dR}}\omega : \mathcal{F} \xrightarrow{\omega} \mathbf{\Omega}^n_{\mathrm{dR}} \xrightarrow{\mathrm{d}_{\mathrm{dR}}} \mathbf{\Omega}^{n+1}_{\mathrm{dR}}.$$

3. Finally, the **wedge product** of two forms $\omega \in \Omega^n_{\mathrm{dR}}(\mathcal{F})$, $\omega' \in \Omega^m_{\mathrm{dR}}(\mathcal{F})$ is defined as
$$\omega \wedge \omega' : \mathcal{F} \xrightarrow{(\omega,\omega')} \mathbf{\Omega}^n_{\mathrm{dR}} \times \mathbf{\Omega}^m_{\mathrm{dR}} \xrightarrow{\wedge} \mathbf{\Omega}^{n+m}_{\mathrm{dR}}.$$

Remarkably, the collection of differential forms on any smooth space $\mathcal{F}$ inherits the structure of a DGCA over the real numbers $(\Omega^\bullet_{\mathrm{dR}}(\mathcal{F}), \mathrm{d}_{\mathrm{dR}}, \wedge) \in \mathrm{DGCA}_{\mathbb{R}}$ by Proposition 2.3.3. However, if we take $\mathcal{F} = \mathbf{\Gamma}_M(F)$, we don't have a notion of contraction operation with corresponding vector fields, in evident contrast with the case of Definition 2.3.6.

Finally, the set of smooth de Rham forms on a smooth space $\Omega^n_{\mathrm{dR}}(\mathcal{F})$ may be promoted to a smooth set using the internal hom (Prop. 2.3.2) as follows:

$$\mathbf{\Omega}^n_{\mathrm{dR}}(\mathcal{F})(\mathbb{R}^k) = \mathrm{Hom}_{\mathrm{SmthSet}}(y(\mathbb{R})^k \times \mathcal{F}, \mathbf{\Omega}^n_{\mathrm{dR}}). \tag{2.9}$$

This is the reason why we say that a notion of Cartan calculus on the smooth de Rham forms would be universal. Note that this smooth set is really large and its plots cannot be interpreted merely as smoothly $\mathbb{R}^k$-parametrized n-forms on $\mathcal{F}$ (cf. [GS25, pp. 25-26]). Moreover, even if there is no natural notion of a cotangent bundle for a general smooth set $\mathcal{F}$, we still wish to think of the de Rham forms as of sections of a would-be bundle.(18) This is in continuity with the case of a manifold $M \in \mathrm{SmthMfd}$, where 1-forms are naturally identified with sections of the cotangent bundle $\Omega^1(M) \cong \Gamma_M(T^*M)$. Analogously for n-forms, where the vertical smooth structure on n-forms (Def. 2.3.8) coincides with that on sections of a vertical exterior bundle, as in Definition 2.3.3.

Remark 2.3.4. To recap, we aim to derive the complete Cartan calculus and its associated bi-complex for forms and vector fields over the field space to formalize the action principle for Lagrangian field theories. We define the usual forms in Definition 2.3.6 and notice that, in this setting, while we have a good definition of contraction and wedge product, we lack the notion of differential. On the other hand, defining forms from a universal point of view through the smooth moduli space, as in Definition 2.3.8, allows us to determine the wedge product and differential naturally. However, we do not have a notion of contraction. Moreover, the relation between these two definitions is unclear.

Nevertheless, we have not yet utilized the fact that, practically speaking, one often uses local forms and local fields(19) in a local Lagrangian field theory. That is, the variation of the action functional in a local theory only requires the existence of a bi-complex structure and Cartan calculus on the subset of local forms and local vector fields on $\mathcal{F} \times M$. Through the technology associated with the infinite jet bundle $J^\infty_M F$, which encodes locality in a globalized fashion, we build the complete Cartan calculus, with respect to local vector fields, on local forms in the sense of Definition 2.3.6 (Ch. 3). Then, the intuition is that the space of local forms on $J^\infty_M F$ in the

(18) This is possible introducing a vertical smooth structure (cf. [GS25, Rmk. 2.35]).
(19) The term "local" gains a rigorous meaning in Section 3.3.

sense of Definition 2.3.6 is canonically identified with the smooth set-theoretical forms defined via the moduli space on it (Def. 2.3.8). Unfortunately, this is not yet proven in the most general case[20] (Thm. 3.1), and therefore this setting cannot already be applied straightforwardly.

[20] The authors of [GS25] suggest that the solution to this issue and a complete proof of this identification might be found once the category is enriched with infinitesimal structure via synthetic geometry technology.

Chapter 3

Differential Geometry on the Infinite Jet Bundle and Locality

In Chapter 2, we introduce smooth de Rham m-forms (Def. 2.3.8) to find a universal notion of forms with associated Cartan calculus since the latter is not defined for the classical Definition 2.3.6. The attempt is unsuccessful; indeed, while there is no notion of differential for the latter, there is no intuitive way of defining a contraction for the former. Furthermore, the connection between the two definitions remains unclear. We now introduce the infinite jet bundle technology to address this issue.

After a quick overview of finite jet bundles, we introduce the infinite jet bundle as a smooth set. This allows us to rigorously define the tangent bundle, vector fields, and classical forms on it. We then show that the tangent bundle has a canonical smooth horizontal splitting (Sect. A.2 and Prop. 3.2.3) that is essential for building the variational bi-complex (cf. [GS25, Ch. 5]). Subsequently, we define the Cartan calculus on the classical forms of the infinite jet bundle and prove that differential forms of globally finite order on it can be seen as a subalgebra of de Rham forms on $J^\infty_M F$. This result bridges Definitions 2.3.6 and 2.3.8, giving a way to construct the universal Cartan calculus, but unfortunately only when restricted to such a subalgebra. Finally, local vector fields and forms are introduced to demonstrate how the infinite jet bundle technology can be utilized to discuss locality globally, i.e. without referring to a specific local coordinate chart, and to define the local Cartan calculus on $\mathcal{F} \times M$.

3.1 Infinite Jet Bundle as a Smooth Set

In this section, we provide a brief overview of finite jet bundles. We refer to [Blo24] for a more detailed discussion. Consequently, we introduce the infinite jet bundle as a smooth set exploiting the category LocProMfd of local pro-manifolds (cf. [GP+17; DGV16; KS17]).

Definition 3.1.1. Let $\pi : F \to M$ be a smooth fiber bundle. Two local sections of it (Def. A.1.2), defined on a neighborhood of m, have the same **k-jet** at m, denoted by $j^k_m\varphi = j^k_m\varphi'$, if they have the same value and partial derivatives up to k-th order at m.

This is a good definition, i.e. independent of the chosen chart, since if the derivatives agree in one chart, they agree in any due to the chain rule. Moreover, it is an equivalence relation on local sections in a neighborhood of m with k-jets $j^k_m\varphi = [\varphi]$ being the equivalence classes:

$$\varphi \sim \varphi' \in \Gamma(U, F) \Leftrightarrow \partial_I\varphi^a(m) = \partial_I\varphi'^a(m) \quad \forall 0 \leq |I| \leq k.$$

Here, I denotes a symmetric multi-index and a is the index of the coordinate chart $\{m^\mu, u^a\}$ in a trivialization of the fiber bundle $\pi : F \to M$.

We can now define the set of k-jets at m and the jet bundle as generalizations of the tangent space and tangent bundle. Indeed, first-order jets describe tangent planes, while higher-order jets correspond to higher-degree polynomial approximations of submanifolds through m.

Definition 3.1.2. The **set of k-jets at m** is defined as

$$J^k_m F := \{j^k_m \varphi = [\varphi] \,|\, \forall \text{ open } U \ni m, \text{ and all } \varphi \in \Gamma(U, F)\}.$$

The **k-jet bundle** is given by

$$J^k_M F := \bigcup_{m \in M} J^k_m F \in \mathrm{SmthMfd}$$

with induced[(1)] charts $\{m^\mu, \{u^a_I\}_{|I| \le k}\} := \{m^\mu, u^a, u^a_\mu, u^a_{\mu_1\mu_2}, \cdots, u^a_{\mu_1 \dots \mu_k}\}$. The extra coordinates are the partial derivatives of φ at m. In other words, the k-jet bundle represents k-jets of local sections of $F \to M$.

One can also see $J^k_m F$ as the fiber over m of $\pi_k : J^k F \to M$ (cf. [Blo24, Prop. 3.1.10]). There are two important maps that one can define on jet bundles.

Definition 3.1.3. Let $\pi : F \to M$ be a fiber bundle, $\Gamma_M(F)$ the set of its global sections and fix $k \in \mathbb{N}$. Then, we define the following maps:

1. The **k, l-forgetful map** for all $k > l \ge 0$ is the projection $\pi^k_l : J^k_M F \to J^l_M F$, $j^k_m \varphi \mapsto j^l_m \varphi$;
2. The **k-th jet prolongation map** is defined as $j^k : \Gamma_M(F) \to \Gamma_M(J^k F)$, $\varphi \mapsto j^k \varphi$ such that $j^k \varphi(m) := j^k_m \varphi$.

Note that all the forgetful maps are surjective submersions, i.e. smooth surjective maps whose pushforward is an everywhere surjective linear map, whereas in general prolongation maps are not even surjective (cf. [Blo24, Ch. 3]). Moreover, we get the diagram of smooth manifolds

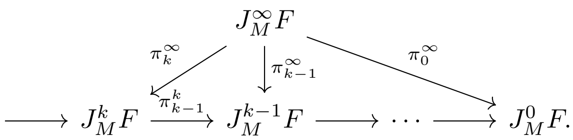
$$\to J^k_M F \to J^{k-1}_M F \to \cdots \to J^1_M F \to J^0_M F \cong F. \tag{3.1}$$

The arrows are the appropriate forgetful maps and therefore they are all surjective submersions.

To better work with jet bundles of finite but arbitrarily large order, we introduce the following definition.

Definition 3.1.4. Let $\pi : F \to M$ be a smooth fiber bundle. Two local sections of it φ and φ', defined on a neighborhood of m, have the same ∞-jet at m, denoted by $j^\infty_m \varphi = j^\infty_m \varphi'$, if they have the same k-jet at m for all $k \ge 0$.

Then, everything follows as before except for the smooth structure of the ∞-jet bundle $J^\infty_M F$. Indeed, the above definition can be given as a projective limit of the diagram (3.1) within Set, but the limit does not exist in SmthMfd. This can be seen by noticing that for every $k \ge 0$, the forgetful maps of sets $\pi^\infty_k : J^\infty F \to J^k F$, $j^\infty_m \varphi \mapsto j^k_m \varphi$, satisfying $\pi^k_{k-1} \circ \pi^\infty_k = \pi^\infty_{k-1}$, define the commutative diagram



(1) The charts are induced by the above chart on the trivialization of the fiber bundle.

As can be checked in jet coordinates, any other cone over the diagram (3.1) induces a unique map to $J^\infty F$, which shows that $J^\infty F$ is the categorical limit of the sequence (3.1). However, the limit does not exist in SmthSet since it is necessarily infinite-dimensional. Therefore, projections and infinite jet prolongation at this level exist only set-theoretically.

To give a smooth structure to this limit, we therefore have to embed SmthMfd as a subcategory into an ambient category in which such limits exist compatibly with the set-theoretical limits. Moreover, morphisms out of $J^\infty_M F$ should descend to a finite jet bundle, i.e. they should factor through a classically smooth map out of a finite jet bundle in the sequence (3.1). While in [Blo24; Leó18] the ∞-jet bundle is studied within the category of pro-manifolds (Rmk. 3.1.1), in our setting it is more natural to work in the category of locally pro-manifolds. The latter is a full subcategory of FrMfd where the limit exists as an infinite-dimensional paracompact manifold, as proven in [Sau89, Prop. 7.2.6] and [Tak79, Prop. 2.1]. The fact that the maps in diagram (3.1) are surjective submersions is essential for the existence of the limit within FrMfd.

Definition 3.1.5. The **infinite jet bundle** $J^\infty_M F$ is the paracompact and Hausdorff Fréchet manifold defined by the limit

$$J^\infty_M(F) := \lim_k^{\mathrm{FrMfd}} J^k_M(F) \in \mathrm{FrMfd},$$

whose local model is $\mathbb{R}^\infty = \lim_k^{\mathrm{FrMfd}} \mathbb{R}^k \in \mathrm{FrMfd}$, with local coordinates $\{x^\mu, \{u^a_I\}_{0\leq |I|}\} := \{x^\mu, u^a, u^a_{\mu_1}, u^a_{\mu_1\mu_2}, \dots\}$.

Consequently, the forgetful maps defining the cone of the projective limit are smooth Fréchet maps. In particular, the limit lives in LocProMfd, a full subcategory of FrMfd (c.f. [GS25, Sect. 3.1]).

Definition 3.1.6. We define the category of **locally pro-manifolds**, **LocProMfd** $\hookrightarrow$ FrMfd, to be the full subcategory of Fréchet manifolds consisting of projective limits of finite-dimensional manifolds.

Note that LocProMfd, being a full subcategory of FrMfd, is also fully faithfully embedded into SmthSet via the Yoneda embedding of FrMfd into SmthSet (Prop. 2.3.1). Therefore, $J^\infty_M F \in \mathrm{LocProMfd} \hookrightarrow \mathrm{FrMfd} \xrightarrow{y} \mathrm{SmthSet}$ where the first arrow is an inclusion and the second is the Yoneda embedding (Eq. (2.3)). Also, finite order jet bundles are objects of SmthMfd, and hence can be viewed in SmthSet through the Yoneda embedding (Prop. 2.2.1). Then, smooth maps between jet bundles, whether of finite or infinite order, correspond to morphisms of sheaves. This, together with the smooth set characterizations of field space, vector bundles, and forms (Ch. 2), shows that SmthSet satisfies requirement 1 of Section 2.1.

We have restricted ourselves to this subcategory because it is enough to describe the infinite jet bundle and to embed it into SmthSet, allowing at the same time to easily characterize smooth Fréchet maps into the infinite jet bundle and out of it.

Proposition 3.1.1. *Let* $\Sigma \in \mathrm{SmthMfd}$ *be a finite-dimensional manifold. This proposition characterize* $\mathrm{Hom}_{\mathrm{FrMfd}}\big(J^\infty_M(F), \Sigma\big)$ *and* $\mathrm{Hom}_{\mathrm{FrMfd}}\big(\Sigma, J^\infty_M(F)\big)$*, respectively in 1. and 2..*

1. *A map of sets* $f : \Sigma \longrightarrow J^\infty_M F$ *is smooth if and only if, for every* $k \in \mathbb{N}$*, the composite* $\pi^\infty_k \circ f : \Sigma \to J^k_M F$ *is smooth. Furthermore, smooth maps* $f : \Sigma \to J^\infty_M F$ *are in one-to-one correspondence with families of smooth maps* $\{f_k : \Sigma \to J^k_M F\}_{k\in\mathbb{N}}$ *such that for all* $k_2 \geq k_1$ *the following commutes:*

$$\begin{array}{ccc} \Sigma & & \\ {\scriptstyle f_{k_2}}\downarrow & \searrow^{f_{k_1}} & \\ J^{k_2}_M F & \xrightarrow{\pi^{k_2}_{k_1}} & J^{k_1}_M F. \end{array}$$

2. *Let $\pi^\infty_k : J^\infty_M F \to J^k_M F$ be the canonical projection for each $k \in \mathbb{N}$. A function of sets $f : J^\infty_M F \to \mathbb{R}$ is smooth if and only if, locally around every point $x \in J^\infty_M F$, it factors through some projection π^∞_k. Equivalently, for each $x \in J^\infty_M F$ there exists $k \in \mathbb{N}$, a neighborhood $U \subset J^k_M F$ of $\pi^\infty_k(x) \in J^k_M F$, and a smooth function of* SmthMfd $\tilde{f}^k_U : J^k_M F|_U \to \mathbb{R}$ *such that the following diagram commutes:*

$$\begin{array}{ccc} J^\infty_M F|_{(\pi^\infty_k)^{-1}(U)} & & \\ {\scriptstyle \pi^\infty_k}\downarrow & \searrow^{f} & \\ J^k_M F|_U & \xrightarrow{\ \tilde{f}^k_U\ } & \mathbb{R}. \end{array}$$

The result is readily extended if the target manifold $\mathbb{R}$ is replaced by $\mathbb{R}^n$, and consequently by any finite-dimensional manifold $\Sigma \in$ SmthMfd.

The first part follows directly from the universal cone property of the limit of the sequence (3.1), whereas the proof of the second is a bit trickier and can be found in [KS17, Prop. 2.29].

In particular, the result *1.* holds for all $\Sigma \in \mathrm{CartSp} \hookrightarrow \mathrm{SmthMfd}$, allowing us to describe the smooth set incarnation of J^∞_M via the embedding of Proposition 2.3.1:

$$y(J^\infty_M F) := \mathrm{Hom}_{\mathrm{FrMfd}}(-, J^\infty_M F) \cong \lim^{\mathrm{SmthSet}}_k (J^k_M F).$$

Here, the morphism in FrMfd is the one in Proposition 3.1.1 and the second equivalence comes from the fact that the limit is computed objectwise. Therefore, $\mathbb{R}^n$-plots of $y(J^\infty_M F)$ are

$$y(J^\infty_M F)(\mathbb{R}^n) \cong \{\{s^n_k : \mathbb{R}^n \to J^k_M F \,|\, \pi^k_{k-1} \circ s^n_k = s^n_{k-1}\}_{k\in\mathbb{N}}\}.$$

One can show (cf. [GS25]) that the infinite jet prolongation is smooth and uniquely extends to a smooth map

$$\begin{aligned} j^\infty : \mathbf{\Gamma}_M(F) &\longrightarrow \mathbf{\Gamma}_M(J^\infty F) \\ \varphi^k &\longmapsto j^\infty \varphi^k. \end{aligned}$$

Finally, we can characterize smooth maps of locally pro-manifolds $\mathrm{Hom}_{\mathrm{LocProMfd}}(J^\infty_M F, G^\infty)$, with $G^\infty = \lim_{\mathrm{FrMfd}} G_j$, as follows.

Proposition 3.1.2. *Let $G^\infty = \lim_{\mathrm{FrMfd}} G^j$ be any locally pro-manifold.*

1. *A map of sets $f : J^\infty_M F \to G^\infty$ is smooth if and only if*

$$\mathrm{p}^\infty_j \circ f : J^\infty_M F \to G^\infty \xrightarrow{\mathrm{p}^\infty_j} G^j$$

is smooth for each $j \in \mathbb{N}$, and hence if each $\mathrm{p}^\infty_j \circ f$ factors locally around every $x \in J^\infty_M F$ through $\pi^\infty_k : J^\infty_M F \to J^k_M F$ for some $k \in \mathbb{N}$, where $\mathrm{p}^\infty_j : G^\infty \to G^j$ denotes the universal cone projections of G^∞.

2. *Furthermore, smooth maps $f : J^\infty_M F \to G^\infty$ are in one-to-one correspondence with compatible families of smooth maps $\{f_j : J^\infty_M F \to G^j \mid j \in \mathbb{N}\}$ such that for each pair $j_2 \geq j_1$ the diagram*

$$\begin{array}{ccc} J^\infty_M F & & \\ {\scriptstyle f_{j_2}}\downarrow & \searrow^{f_{j_1}} & \\ G^{j_2} & \xrightarrow[\mathrm{p}^{j_2}_{j_1}]{} & G^{j_1} \end{array}$$

commutes, and hence, such that furthermore each f_j locally factors through $\pi^\infty_k : J^\infty_M F \to J^k_M F$ for some $k \in \mathbb{N}$.

The proof can be found in [GS25, Prop. 3.7].

Remark 3.1.1. In [Blo24; Leó18], the infinite jet bundle is studied within the category of pro-manifolds **ProMfd**. ProMfd can be informally thought of as an extension of SmthMfd, adding only well-behaved limits. Then, J^∞_M is not thought of as a Fréchet manifold, but simply as a formal limit of finite-dimensional manifolds. Smooth functions on it are those that globally factor through some $J^k_M F$ (cf. [Leó18, Sect. 1.2]), i.e. the smooth functions $C^\infty_{\text{glb}}(J^\infty_M F)$ on the formal limit are the union over $k \in \mathbb{N}$ of those on $J^k_M F$. Note that the latter is a subalgebra of the algebra of locally finite order functions that we characterized via Proposition 3.1.1:

$$C^\infty_{\text{glb}}(J^\infty_M F) \hookrightarrow C^\infty(J^\infty_M F).$$

This somehow justifies (cf. [KS17, Sect. 2.2]) the name of the category LocProMfd, which is, however, really different from ProMfd, being a full subcategory of FrMfd.

Moreover, in [Blo24; Leó18] the smooth structure of the space of sections of smooth fiber bundles is described within the category of concrete smooth sets, i.e. DflSp. Therefore, to obtain a uniform description of the relevant field-theoretical objects, the two approaches must be combined in the category **ProDflSp** of pro-diffeological spaces.
This approach has the following three main drawbacks ([GS25, Rmk. 3.8]).

1. $C^\infty_{\text{glb}}(J^\infty_M F)$ do not form a petit sheaf on the underlying topological space, in contrast to $C^\infty(J^\infty_M F)$.
2. Concrete sheaves do not naturally generalize to include fermionic fields or infinitesimal structure.
3. Treating the relevant field-theoretical objects in different categorical footing results in the introduction of heavy categorical machinery that can be avoided within the more natural setting of SmthSet.

3.2 Towards Cartan Calculus on the Infinite Jet Bundle

In this section, the tangent bundle, vector fields, and classical forms on the infinite jet bundle are defined within the setting of smooth sets. We show that there is a canonical smooth horizontal splitting of the tangent bundle and define the Cartan calculus on the classical forms. Finally, we prove that differential forms of globally finite order on the infinite jet bundle are naturally identified as a subalgebra of de Rham forms defined via the classifying space (Def. 2.3.7).

3.2.1 Tangent Bundle, Vector Fields and Horizontal Splitting

We recall the definition of the tangent bundle and vector field on $J^\infty_M F$ before introducing the Cartan connection.

The definition of the infinite jet tangent bundle is analogous to the definition of the infinite jet bundle in SmthSet. First, notice that the diagram (3.1) induces the diagram of finite-dimensional tangent bundles

$$\to T(J^k_M F) \xrightarrow{\mathrm{d}\pi^k_{k-1}} T(J^{k-1}_M F) \to \cdots \to T(J^1_M F) \xrightarrow{\mathrm{d}\pi^1_0} T(J^0_M F) \cong TF$$

with the maps being the pushforwards of the projections $\{\pi^k_{k-1} : J^k_M F \to J^{k-1}_M F\}_{k\in\mathbb{N}}$. The diagram is in SmthMfd and can be embedded in SmthSet by the Yoneda embedding (Prop. 2.2.1).

Definition 3.2.1. The **smooth infinite jet tangent bundle** $T\big(y(J^\infty_M F)\big) \in \mathrm{SmthSet}$ is defined by

$$T\big(y(J^\infty_M F)\big) := \lim^{\mathrm{SmthSet}}_{k\in\mathbb{N}} y\big(T(J^k_M F)\big). \tag{3.2}$$

The latter is a concrete smooth set. Indeed, each point, or tangent vector, $X_s \in T_s(J^\infty_M F)$ at $s = j^\infty_p \varphi \in J^\infty_M F$ is represented by a family of tangent vectors $\{X^k_s \in T_{\pi^\infty_k(s)}(J^k_M F)\}_{k\in\mathbb{N}}$ on each finite order tangent bundle at $\pi^\infty_k(s) = \pi^\infty_k(j^\infty_p\varphi) = j^k_p\varphi \in J^k_M F$. Such a family is compatible along the pushforward projections, i.e. $\mathrm{d}\pi^k_{k-1} X^k_s = X^{k-1}_s$. In a local coordinate chart $\{x^\mu, \{u^a_I\}_{0\leq |I|}\}$ of $J^\infty_M F \in \mathrm{LocProMfd}$, we can think of this family as the formal sum

$$X_s = X^\mu \frac{\partial}{\partial x^\mu}\Big|_s + \sum_{|I|=0}^{\infty} Y^a_I \frac{\partial}{\partial u^a_I}\Big|_s .$$

Here $\{X^\mu, Y^a_I\} \subset \mathbb{R}$, with each X^k corresponding to the case where the sum is terminated at order $|I| = k$.

From this definition one can recover the notion of tangent vectors as infinitesimal curves.

Proposition 3.2.1. *The set of tangent vectors $T\big(y(J^\infty_M F)\big)(*)$ is in bijection with equivalence classes of curves in $J^\infty_M F \in \mathrm{LocProMfd}$:*

$$T\big(y(J^\infty_M F)\big)(*) \cong \mathrm{Hom}_{\mathrm{FrMfd}}(\mathbb{R}, J^\infty_M F)/\sim_{O(t^1)} \cong\ y(J^\infty F)(\mathbb{R})/\sim_{O(t^1)}, \tag{3.3}$$

where the equivalent relation is agreement up to first order derivatives at $0 \in \mathbb{R}$.

A proof can be found in [GS25, Lem. 4.2].

Even if the latter approach is more intuitive, we would rather use Definition 3.2.1 since it allows us to consider not only poits, but any $\mathbb{R}^n$-plot of the tangent bundle of $J^\infty_M F$

$$T\big(y(J^\infty_M F)\big)(\mathbb{R}^n) = \lim^{\mathrm{SmthSet}}_{k}\ y\big(T(J^k_M F)\big)(\mathbb{R}^n) := \lim^{\mathrm{Set}}_{k} \mathrm{Hom}_{\mathrm{SmthMfd}}\big(\mathbb{R}^n, T(J^k_M F)\big). \tag{3.4}$$

This is represented by

$$\bigcup_{s^n\in J^\infty_M F(\mathbb{R}^n)} T_{s^n}(J^\infty_M F) := \bigcup_{s^n\in J^\infty_M F(\mathbb{R}^n)} \Big\{\{X^k_{s^n} : \mathbb{R}^n \to T(J^k_M F) \mid \mathrm{d}\pi^k_{k-1}\circ X^k_{s^n} = X^{k-1}_{s^n}\}_{k\in\mathbb{N}}\Big\},$$

where $X^k_{s^n}(x) \in T_{\pi^\infty_k\circ s^n(x)}(J^k_M F)\ \forall x \in \mathbb{R}^n$. In local charts, we can write the formal sum

$$X_{s^n} = X^\mu_{s^n} \frac{\partial}{\partial x^\mu}\Big|_{s^n} + \sum_{|I|=0}^{\infty} Y^a_{I,s^n} \frac{\partial}{\partial u^a_I}\Big|_{s^n},$$

where now $\{X^\mu_{s^n}, Y^a_{I,s^n}\}_{0\leq|I|}$ are smooth functions on $\mathbb{R}^n$, with $X^k_{s^n}$ corresponding to the case where the sum is terminated at order $|I| = k$.

We now define fiberwise addition and $y(\mathbb{R})$-multiplication as the natural transformations

$$\begin{aligned} + : T\big(y(J^\infty_M F)\big)\times_{y(J^\infty_M F)} T\big(y(J^\infty_M F)\big) \to T\big(y(J^\infty_M F)\big),\ \big(\{X^k_{s^n}\}_{k\in\mathbb{N}}, \{\tilde{X}^k_{s^n}\}_{k\in\mathbb{N}}\big) \mapsto \{X^k_{s^n} + \tilde{X}^k_{s^n}\}_{k\in\mathbb{N}} \\ \cdot : y(\mathbb{R}) \times T\big(y(J^\infty_M F)\big) \to T\big(y(J^\infty_M F)\big),\ \big(f_n, \{X^k_{s^n}\}_{k\in\mathbb{N}}\big) \mapsto \{f_n \cdot X^k_{s^n}\}_{k\in\mathbb{N}} \end{aligned} \tag{3.5}$$

where $f_n : \mathbb{R}^n \to \mathbb{R}$ is a plot of $\mathbb{R}$. This is possible thanks to the $C^\infty(\mathbb{R}^n)$-linear structure on the fiber $\mathbb{R}^n$-plots over each $\mathbb{R}^n$-plot of $J^\infty_M F$ that is induced by the linear structures on each $T(J^k_M F)$, natural in $\mathbb{R}^n$. To understand this, note that there is a canonical smooth bundle projection

$$\mathrm{p} : T\big(y(J^\infty_M F)\big) \to y(J^\infty_M F).$$

Then, the above means that the $\mathbb{R}^n$-plots of $T(J^\infty_M F)$ over a single $\mathbb{R}^n$-plot of $J^\infty_M F$ have a $C^\infty(\mathbb{R}^n)$-linear structure. Naturality ensures that the operations of the module commute with pulling back along a smooth change of plots $f : \mathbb{R}^n \to \mathbb{R}^m$, allowing us to define the two above morphisms of smooth sets.

We now define vector fields on the infinite jet bundle, internally to SmthSet, as the geometrical smooth sections of its tangent bundle. Consequently, we show that this recovers the usual algebraic definition as derivatives on $C^\infty(J^\infty_M)$.(2)

Definition 3.2.2. The **set of smooth vector fields** on the infinite jet bundle is defined as smooth sections of its tangent bundle:

$$\mathfrak{X}\big(y(J^\infty_M F)\big) := \Gamma_{J^\infty_M F}\big(T(J^\infty_M F)\big) = \big\{ X : y(J^\infty_M F) \to T(y(J^\infty_M F)) \;\big|\; \mathrm{p} \circ X = \mathrm{id}_{y(J^\infty_M F)} \big\}.$$

By the limit property for $T\big(y(J^\infty_M F)\big)$, a vector field X corresponds to a compatible family $\{X^k : y\big(J^\infty_M F\big) \to y\big(T(J^k_M F)\big)\}_{k\in\mathbb{N}}$ such that the following commutes for all $k \in \mathbb{N}$:

$$\begin{array}{ccc} & & y\big(T(J^k_M F)\big) \\ & \overset{X^k}{\nearrow} & \big\downarrow {\scriptstyle \mathrm{d}\pi^k_{k-1}} \\ T(J^\infty_M F) & \xrightarrow{X^{k-1}} & y\big(T(J^{k-1}_M F)\big). \end{array}$$

Since the embedding of LocProMfd into SmthSet is fully faithful, each map of such a family can be thought of as a smooth map in LocProMfd. The latter is, via Proposition 3.1.1, locally of finite order. In other words, it is, locally around each $s \in J^\infty_M F$, the pullback of some smooth function of a finite-order jet bundle via the appropriate projection. In particular, such a family $\{X^k : J^\infty_M F \to T(J^k_M F)\}_{k\in\mathbb{N}}$ may be represented in a local chart of $J^\infty_M F$ by the formal sum

$$X = X^\mu \frac{\partial}{\partial x^\mu} + \sum_{|I|=0}^{\infty} Y^a_I \frac{\partial}{\partial u^a_I}.$$

Here $\big\{ X^\mu, \{Y^a_I\}_{0\leq|I|} \big\} \subset C^\infty(J^\infty_M F)$ is a family of locally defined smooth functions, with each X^k corresponding to the truncation of the sum at order $|I| = k$.

We denote the vector subspace of globally finite order vector fields,(3) i.e. those that correspond to a compatible family in which every map $X^k : J^\infty_M F \to T(J^k_M F)$ is globally of finite order, by

$$\mathfrak{X}_{\mathrm{glb}}(J^\infty M F) \;\subset\; \mathfrak{X}(J^\infty M F).$$

Note that, $\mathfrak{X}(J^\infty_M F)$ has a $C^\infty(J^\infty_M F)$-module structure (Def. A.2.2) induced by the fiber-wise $y(\mathbb{R})$-linear structure on $T\big(y(J^\infty_M F)\big)$ through the composition with the operations in Equation (3.5). In local coordinates, this is represented as a formal sum of indices and multiplication by $C^\infty(J^\infty_M F)$. Similarly, $\mathfrak{X}_{\mathrm{glb}}(J^\infty_M F)$ inherit the structure of a $C^\infty_{\mathrm{glb}}(J^\infty_M F)$-module.

The following result establishes a connection between our definition of a vector space and the more intuitive interpretation of vector fields as derivations on the jet bundle.

Proposition 3.2.2. *Vector fields on $J^\infty_M F$ are in* 1-1 *correspondence with derivations of the algebra of smooth functions:*

$$\mathfrak{X}(J^\infty_M F) \;\cong\; \mathrm{Der}\big(C^\infty(J^\infty_M F)\big).$$

(2) Once the spaces are enriched with infinitesimal structure, this definition is naturally recovered as the synthetic tangent bundle of the infinite jet bundle (cf. [Gri25]).

(3) These are the vector fields that usually appear in field-theoretic examples, but restricting to them is not enough for some applications.

For the proof, see [GS25, Lem. 4.5].

The latter result restricts to the global case $\mathfrak{X}_{\text{glb}}(J^\infty_M F) = \text{Der}\big(C^\infty_{\text{glb}}(J^\infty_M F)\big)$. Notably, this point of view allows to define a Lie algebra structure on $\mathfrak{X}(J^\infty_M F)$ (cf. [GS25]) with the usual bracket defined by

$$[X, \tilde{X}](f) := X(\tilde{X}(f)) - \tilde{X}(X(f)) \tag{3.6}$$

for any $f \in C^\infty(J^\infty_M)$ and $X, \tilde{X} \in \mathfrak{X}(J^\infty_M F)$.

Remarkably,[(4)], the infinite jet tangent bundle $T(J^\infty_M F) \to J^\infty_M F$ has a smooth canonical horizontal splitting (App. A). Indeed, once the vertical subbundle is introduced, we prove that it fits into a short exact sequence (Eq. (3.8)) that has a canonical splitting induced by the Cartan connection (Prop. 3.2.3).

To this end, first notice that the total projection $\pi^\infty : y(J^\infty_M F) \xrightarrow{\pi^\infty_0} y(F) \xrightarrow{\pi} y(M)$ to the base M induces the following smooth pushforward

$$\text{d}\pi^\infty : T\big(y(J^\infty_M F)\big) \longrightarrow y(TM). \tag{3.7}$$

The action of this map on $\mathbb{R}^n$-plots is given by

$$\big\{\, X^k_{s^n} : \mathbb{R}^n \to T(J^k_M F) \;\big|\; \text{d}\pi^k_{k-1} \circ X^k_{s^n} = X^{k-1}_{s^n} \,\big\}_{k\in\mathbb{N}} \longmapsto \text{d}\pi^k \circ X^k_{s^n} \in\; y(TM)(\mathbb{R}^n),$$

where $X^k_{s^n}$ is a representative in the representing set of the plots and $\text{d}\pi^k : T(J^k_M F) \to TM$ is the pushforward of $\pi \circ \pi^k_0 =: \pi^k : J^k_M F \to M$. In local coordinates, we can concretely understand the action of the map on $X_s \in T\big(y(J^\infty_M F)\big)(*)$ at $s \in J^\infty_M F$ writing

$$X_s = X^\mu \frac{\partial}{\partial x^\mu}\Big|_s + \sum_{|I|=0}^{\infty} Y^a_I \frac{\partial}{\partial u^a_I}\Big|_s \longmapsto X^\mu \frac{\partial}{\partial x^\mu}\Big|_{\pi^\infty(s)}.$$

This, in turn, allows us to define the smooth vertical subbundle.

Definition 3.2.3. The **smooth vertical subbundle** of $T\big(y(J^\infty_M F)\big)$ is defined as the equalizer of $\text{d}\pi^\infty$ and the canonical zero map $0_M : T\big(y(J^\infty_M F)\big) \to y(TM)$:

$$VJ^\infty_M F := \text{eq}\left(T\big(y(J^\infty_M F)\big) \overset{\text{d}\pi^\infty}{\underset{0_M}{\rightrightarrows}} y(TM)\right).$$

$\mathbb{R}^n$-plots of the above are represented by compatible families

$$\left\{ X^k_{s_n} : \mathbb{R}^n \to T(J^k_M F) \;\middle|\; \text{d}\pi^k_{k-1} \circ X^k_{s_n} = X^{k-1}_{s_n},\ \text{d}\pi^k \circ X^k_{s_n} = (\pi^\infty \circ s_n, 0) \right\}_{k\in\mathbb{N}}.$$

For instance, a vertical tangent vector at $s \in J^\infty_M F$, namely $X_s \in VJ^\infty_M F(*)$, is represented in a local coordinate chart by

$$X_s = 0 + \sum_{|I|=0}^{\infty} Y^a_I \frac{\partial}{\partial u^a_I}\Big|_s.$$

Furthermore, note that smooth sections of the vertical sub-bundle $VJ^\infty F \to J^\infty F$ (Def. 3.2.2), define the subbandle of vertical vector fields on $J^\infty MF$:

$$\mathfrak{X}_V(J^\infty_M F) := \Gamma_{J^\infty_M F}\big(VJ^\infty_M F\big).$$

[(4)] This is not true in general for infinite-dimensional manifolds, and even if the split always exist in the finite-dimensional case (cf. [GSM09, Thm. 1.1.12]) this is not canonical because it corresponds to an arbitrary choice of connection.

In a local chart, these are represented by

$$X = 0 + \sum_{|I|=0}^{\infty} Y_I^a \frac{\partial}{\partial u_I^a} \quad \text{where} \quad Y_I^a \in C^\infty(J_M^\infty F).$$

It follows that $\mathfrak{X}_V(J_M^\infty F)$ is closed under the Lie bracket of $\mathfrak{X}(J_M^\infty F)$ (Eq. (3.6)).

Finally, we write the following natural short exact sequence (Sect. A.2) of smooth sets[(5)] over $y(J_M^\infty F)$

$$0_{J_M^\infty F} \longrightarrow VJ_M^\infty F \longrightarrow T\big(y(J_M^\infty F)\big) \to y\big(J_M^\infty F\big) \times_{y(M)} y(TM) \longrightarrow 0_{J_M^\infty F}. \tag{3.8}$$

This is a short exact sequence of $C^\infty(\mathbb{R}^n)$-modules of fiber $\mathbb{R}^n$-plots over each $\mathbb{R}^n$-plot of $J_M^\infty F$.[(6)] The third map is naturally given on $\mathbb{R}^n$-plots by

$$\{X_{s^n}^k : \mathbb{R}^n \to T\big(J_M^k F\big) \,|\, \mathrm{d}\pi_{k-1}^k \circ X_{s^n}^k = X_{s^n}^{k-1}\} \longmapsto (s^n, \mathrm{d}\pi^k \circ X_{s^n}^k).$$

The crucial property of the infinite jet bundle is that the above sequence has a canonical splitting

$$H : y\big(J_M^\infty F\big) \times_{y(M)} y\big(TM\big) \longrightarrow T\big(y(J_M^\infty F)\big).$$

That is, there exists a canonical connection (Sect. A.2) on $J_M^\infty F$, usually referred to as the Cartan connection.[(7)]

Before stating and proving Proposition 3.2.3, let us work out the description of the splitting at the point set level.
Let $j_p^k\varphi \in J_M^k F$ and choose a representative local section $\tilde{\varphi} : U \subset M \to F$ such that $j^k\tilde{\varphi}(p) = j_p^k\varphi$ (Def. 3.1.3). The induced pushforward map

$$\mathrm{d}\big(j^k\tilde{\varphi}\big)_p : \; T_pM \longrightarrow T_{j_p^k\varphi}(J_M^k F)$$

is given in local coordinates by the usual Cartan horizontal lift

$$\begin{aligned} X^\mu \frac{\partial}{\partial x^\mu}\Big|_p \longmapsto & X^\mu \left(\frac{\partial}{\partial x^\mu}\bigg|_{j_p^k\varphi} + \sum_{|I|=0}^{k} \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^I}\tilde{\varphi}^a(p) \cdot \frac{\partial}{\partial u_I^a}\bigg|_{j_p^k\varphi} \right) \\ &= X^\mu \left(\frac{\partial}{\partial x^\mu}\bigg|_{j_p^k\varphi} + \sum_{|I|=0}^{k} u_{I+\mu}^a(J_p^{k+1}\tilde{\varphi}) \cdot \frac{\partial}{\partial u_I^a}\Big|_{j_p^k\varphi} \right). \end{aligned}$$

Here $\{x^\mu\}$ and $\{x^\mu, \{u_I^a\}_{|I|\leq k}\}$ are the local coordinates of M and $J_M^k F$ respectively. The expression depicts how a vector over the base manifold M is horizontally lifted to a vector over the infinite jet bundle, compatibly with respect to the structure of the latter. Note that, such a map depends smoothly only on the $(k+1)$-jet at $p \in M$ of the chosen representative section $\tilde{\varphi}$. Therefore the assignment defines, , for each $k \in \mathbb{N}$, a smooth map of bundles over $J_M^k F$

$$H^k : J_M^{k+1} F \times_M TM \longrightarrow T(J_M^k F), \; \big(j_p^{k+1}\varphi, X_p\big) \longmapsto \mathrm{d}\big(j^k\varphi\big)_p(X_p). \tag{3.9}$$

(5)The fibered product is in SmthSet, with $\mathbb{R}^n$-plots being pairs of plots that project to the same plot in M. The coordinate charts take the form $\{x^\mu, \dot{x}^\mu, u^a, u_\mu^a, u_{\mu_1\mu_2}^a, \cdots\}$, with $\{\dot{x}^\mu\}$ denoting the fiber coordinates on TM.

(6)More concretely, exactness is checked plotwise: for each plot s^n of $J_M^\infty F$, the $\mathbb{R}^n$-plots of the smooth sets in the sequence above it, i.e. in the fiber of s^n, form a short exact sequence of $C^\infty(\mathbb{R}^n)$-modules.

(7)The Cartan connection is the Ehresmann connection that chooses the complementary horizontal subbundle as $HJ_M^\infty F$. This is canonical for the infinite jet bundle, but it is not when considering a finite order jet bundle. In any case, it is possible to equivalently describe $HJ_M^k F$, with $k \in \mathbb{N} \cup \{\infty\}$, in terms of Cartan distribution (Fn. (9), p. 36) or associated foliation. However, the Cartan distribution has desirable properties only on the infinite jet bundle, and not on the finite order ones (cf. [Leó18; Blo24]).

Here $\varphi : U \subset M \to F$ on the right-hand side is any representative local section with $j^{k+1}\varphi(p)$ equal to the given jet. This fits inside the following commutative diagram:

$$\begin{array}{ccc} J^{k+1}_M F \times_M TM & \xrightarrow{H^k} & T(J^k_M F) \\ {\scriptstyle \pi^{k+1}_k \times \mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle d\pi^k_{k-1}} \\ J^k_M F \times_M TM & \xrightarrow{H^{k-1}} & T(J^{k-1}_F). \end{array} \tag{3.10}$$

Notably, there is an injective point-set map

$$J^\infty_M F \times_M TM \to T\big(y(J^\infty_M F)\big)(*), \quad \big(j^\infty_p\varphi, X_p\big) \mapsto \mathrm{d}\big(j^\infty\varphi\big)_p(X_p) := \big\{\mathrm{d}\big(j^k\varphi\big)_p(X_p)\big\}_{k\in\mathbb{N}} \tag{3.11}$$

which splits the $*$-plot sequence of Equation (3.8). Again, $\varphi : U \subset M \to F$ is a representative local section of $j^\infty_p\varphi$. In the coordinate chart of $T_{j^\infty_p\varphi}J^\infty_M F$, the map takes the form

$$\left(j^\infty_p\varphi, X^\mu \frac{\partial}{\partial x^\mu}\bigg|_p\right) \longmapsto X^\mu\left(\frac{\partial}{\partial x^\mu}\bigg|_{j^\infty_p\varphi} + \sum_{|I|=0}^{\infty} u^a_{I+\mu}(j^\infty_p\varphi)\cdot\frac{\partial}{\partial u^a_I}\bigg|_{j^\infty_p\varphi}\right),$$

where and $\{x^\mu, \{u^a_I\}_{0\le|I|}\}$ are the local coordinates of $J^\infty_M F$. More generally, the map of Equation (3.11) extends to a smooth splitting of the corresponding smooth sets. Note that such splitting is not global on arbitrary jet bundles, but is canonical on $J^\infty_M F$ due to its properties.[(8)]

Proposition 3.2.3. *The family of smooth bundle maps* $\{H^k : J^{k+1}_M F \times_M TM \to T(J^k_M F)\}_{k\in\mathbb{N}}$ *determines a map, called* ***Cartan connection****, of smooth sets*

$$H : y(J^\infty_M F) \times_{y(M)} y(TM) \longrightarrow T\big(y(J^\infty_M F)\big), \tag{3.12}$$

which splits the corresponding exact sequence in Equation (3.8).
Moreover, the split is canonical, i.e. there is a canonical isomorphism of smooth sets over $y(J^\infty_M F)$

$$VJ^\infty_M F \times_{y(J^\infty_M F)} \big(y(J^\infty_M F) \times_{y(M)} y(TM)\big) \xrightarrow{\sim} T\big(y(J^\infty_M F)\big).$$

The induced splitting is $T\big(y(J^\infty_M F)\big) \cong VJ^\infty_M F \oplus HJ^\infty_M F$*, where the plots of the smooth subbandle* $HJ^\infty_M F$*, called* ***Cartan distribution****,*[(9)] *are given by the image of the Cartan connection*[(10)] (Eq. (3.12)).

Proof. We prove only the first part of the proposition, since the second comes as a corollary. The proof is a variation of Proposition 3.1.2. By the limit property of the infinite jet tangent bundle (Eq. (3.2)) and the fully faithful embedding of LocProMfd into SmthSet, we compute

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{SmthSet}}\big(y(J^\infty_M F) \times_{y(M)} y(TM), T\big(y(J^\infty_M F)\big)\big) \cong \\ &\cong \mathrm{Hom}_{\mathrm{SmthSet}}\big(y(J^\infty_M F \times_M TM), \lim_k^{\mathrm{SmthSet}} y(T(J^k_M F))\big) \cong \\ &\cong \lim_k^{\mathrm{Set}} \mathrm{Hom}_{\mathrm{SmthSet}}\big(y(J^\infty_M F \times_M TM), y(TJ^k_M F)\big) \cong \\ &\cong \lim_k^{\mathrm{Set}} \mathrm{Hom}_{\mathrm{FrMfd}}\big(J^\infty_M F \times_M TM, TJ^k_M F\big). \end{aligned}$$

(8) In particular, this is due to the contact structure of $J^\infty_M F$, i.e. the one forms annihilating the Cartan distribution.

(9) A distribution is a smooth assignment of a vector subspace of the tangent space to each point of the manifold $m \mapsto \Delta_m \subseteq T_m M$ (Fn. (7), p. 35).

(10) The direct sum here is the fibered product $VJ^\infty_M F \times_{y(J^\infty_M F)} \big(y(J^\infty_M F) \times_{y(M)} y(TM)\big)$ computed in SmthSet, with the linear structure being fiberwise for each $\mathbb{R}^n$-plot over the induced plot of the base $y(J^\infty_M F)$, i.e. the direct sum of $C^\infty(\mathbb{R}^n)$-modules of fiber $\mathbb{R}^n$-plots over each $\mathbb{R}^n$-plot of $y(J^\infty_M F)$.

In particular, in the second line we used the fact that the Yoneda embedding preserves limits. In the third line we applied the property of the hom-functor (Prop. 1.2.1). Hence a smooth map $f : y(J^\infty_M F \times_M TM) \to T\big(y(J^\infty_M F)\big)$ corresponds to a family of smooth Fréchet maps $\{f^k : J^\infty_M F \times_M TM \to T(J^k_M F)\}_{k\in\mathbb{N}}$ such that $\mathrm{d}\pi^k_{k-1} \circ f^k = f^{k-1}$, and vice versa.
By Proposition 3.1.1, the set-theoretic maps

$$(\pi^\infty_{k+1} \times \mathrm{id})^* H^k : J^\infty_M F \times_M TM \longrightarrow J^{k+1}_M F \times_M TM \longrightarrow T(J^k F)$$

are smooth Fréchet maps for each $k \in \mathbb{N}$, being the pullback of globally[(11)] finite order maps. Furthermore, by the commutativity of diagram (3.10), they satisfy

$$\begin{aligned}\mathrm{d}\pi^k_{k-1} \circ (\pi^\infty_{k+1} \times \mathrm{id})^* H^k &= \mathrm{d}\pi^k_{k-1} \circ H^k \circ (\pi^\infty_{k+1} \times \mathrm{id}) = H^{k-1} \circ (\pi^{k+1}_k \times \mathrm{id}) \circ (\pi^\infty_{k+1} \times \mathrm{id}) \\ &= H^{k-1} \circ (\pi^\infty_k \times \mathrm{id}) = (\pi^\infty_k \times \mathrm{id})^* H^{k-1}.\end{aligned}$$

Thus the family $\{(\pi^\infty_{k+1}\times\mathrm{id})^* H^k : J^\infty_M F \times_M TM \longrightarrow T(J^k_M F)\}_{k\in\mathbb{N}}$ uniquely corresponds to a map $H : y(J^\infty_M F \times_M TM) \to T\big(y(J^\infty_M F)\big)$. The underlying point set map is that of Equation (3.11), and since this splits the $*$-plot sequence, it follows that the $\mathbb{R}^n$-plot sequences split too. ■

In a local coordinate chart for $J^\infty_M F$, the explicit action on $\mathbb{R}^n$-plots may be seen as

$$\Big(s^n, X^\mu \frac{\partial}{\partial x^\mu}\Big|_{\pi^\infty \circ s^n}\Big) \longmapsto X^\mu \Big(\frac{\partial}{\partial x^\mu}\Big|_{s^n} + \sum_{|I|=0}^{\infty} u^a_{I+\mu} \circ s^n \cdot \frac{\partial}{\partial u^a_I}\Big|_{s^n}\Big).$$

Here $s^n : \mathbb{R}^n \to J^\infty_M F$ is an $\mathbb{R}^k$-plot of the infinite jet bundle, and $\{X^\mu : \mathbb{R}^n \to M\}$ are the components of the corresponding plot in TM.

The second part of the above proposition implies the splitting, in the sense of $C^\infty(J^\infty_M F)$-modules, of smooth vector fields $\mathfrak{X}(J^\infty_M F) \cong \mathfrak{X}_V(J^\infty_M F) \oplus \mathfrak{X}_H(J^\infty_M F)$, where the two components denote smooth sections of the corresponding bundles (Def. 3.2.2). Hence, any vector field X on $J^\infty_M F$ decomposes uniquely as $X = X_V + X_H$, where X_V and X_H are sections of the vertical and horizontal smooth sub-bundles, respectively. The latter can be represented in local coordinates as

$$X_V = \sum_{|I|=0}^{\infty} \Big(Y^a_I - X^\mu \cdot u^a_{I+\mu}\Big) \cdot \frac{\partial}{\partial u^a_I} \quad \text{and} \quad X_H = X^\mu \Big(\frac{\partial}{\partial x^\mu} + \sum_{|I|=0}^{\infty} u^a_{I+\mu} \frac{\partial}{\partial u^a_I}\Big).$$

In particular, if $\tilde{X}^\mu \frac{\partial}{\partial x^\mu} \in \Gamma_M(TM)$, its horizontal lift is $(\pi^\infty)^* \tilde{X}(\frac{\partial}{\partial x^\mu} + \sum_{|I|=0}^\infty u^a_{I+\mu} \frac{\partial}{\partial u^a_I}) \in \mathfrak{X}_H(J^\infty_M F)$. It is helpful to denote the local basis for horizontal vector fields on $J^\infty_M F$, i.e. the horizontal lifts of the local coordinate vector fields $\{\frac{\partial}{\partial x^\mu}\}$ on M, by

$$D_\mu := \frac{\partial}{\partial x^\mu} + \sum_{|I|\geq 0} u^a_{I+\mu} \frac{\partial}{\partial u^a_I}. \tag{3.13}$$

If $f : J^\infty_M F \to \mathbb{R}$ is a smooth function and $\varphi \in \Gamma_M(F)$ is a smooth section, then $f \circ j^\infty \varphi \in C^\infty(M)$. The vector fields $\{D_\mu\}$ encode the action of $\{\frac{\partial}{\partial x^\mu}\}$ on $f \circ j^\infty \phi$ via the chain rule, that is:

$$\begin{aligned}D_\mu(f) \circ j^\infty \varphi &= \left(\frac{\partial f}{\partial x^\mu} + \sum_{|I|=0}^{\infty} u^a_{I+\mu} \frac{\partial f}{\partial u^a_I}\right) \circ j^\infty \varphi \\ &= \frac{\partial f}{\partial x^\mu} \circ j^\infty \varphi + \sum_{|I|=0}^{\infty} \frac{\partial}{\partial x^\mu} \frac{\partial \varphi^a}{\partial x^I} \cdot \left(\frac{\partial f}{\partial u^a_I} \circ j^\infty \varphi\right) = \frac{\partial}{\partial x^\mu}\big(f \circ j^\infty \varphi\big).\end{aligned}$$

(11) In this particular case, the maps are globally of finite order, but this more generally works for locally finite order maps.

By construction, the vertical vector fields $\mathfrak{X}_V(J^\infty F)$ are closed under the Lie bracket defined in Equation (3.6). Crucially, the Cartan connection is flat. That is, the horizontal vector fields are closed under the Lie bracket

$$[X_H^1, X_H^2] \in \mathfrak{X}_H(J^\infty_M F), \quad \forall\, X_H^1, X_H^2 \in \mathfrak{X}_H(J^\infty_M F),$$

as can be checked in local coordinates. This means that the Cartan distribution has zero curvature and that fields over the infinite jet bundle behave well when parallelly transported along a local path of the base manifold M, i.e. the transport is locally independent of the path. This greatly simplifies the horizontal lifting.

Note that the splitting descends on the subspace of global finite order vector fields $\mathfrak{X}_{\mathrm{glb}}(J^\infty_M F) \cong \mathfrak{X}_{\mathrm{glb},V}(J^\infty_M F) \oplus \mathfrak{X}_{\mathrm{glb},H}(J^\infty_M F)$, with the same local representation formulas and properties.

3.2.2 Differential Forms and Cartan Calculus

We now classically define differential forms on the infinite jet bundle as smooth antisymmetric $y(\mathbb{R})$-linear maps $T\big(y(J^\infty_M F)\big) \to y(\mathbb{R})$, exploiting the fiber-wise linear structure of $J^\infty_M F$. This definition should be equivalent to the definition of de Rham forms on the infinite jet bundle in the sense of Definition 2.3.8. Although this identification is not yet fully established in the present framework,[(12)] we prove (Thm. 3.1) that the subalgebra of globally finite-order classical differential forms on $J^\infty_M F$ (Def. 3.2.4) admits a canonical identification with a subalgebra of de Rham forms on it (Def. 3.2.6).

Definition 3.2.4. The set of **differential m-forms** on the infinite jet bundle is defined by

$$\Omega^m(J^\infty_M F) := \mathrm{Hom}^{\text{fib.lin.an.}}_{\mathrm{SmthSet}}\Big(T^{\times m}(J^\infty_M F),\, y(\mathbb{R})\Big).$$

That is, m-forms are smooth real-valued, fiber-wise linear antisymmetric maps, with respect to the fiber-wise linear structure induced by the operations in Equation (3.5), on the m-fold fiber product

$$T^{\times m}(J^\infty_M F) \;:=\; T(J^\infty_M F) \times_{y(J^\infty_M F)} \cdots \times_{y(J^\infty_M F)} T(J^\infty_M F)$$

of the infinite jet tangent bundle over the infinite jet bundle.

Concretely, the above m-fold fiber product is the smooth set with $\mathbb{R}^n$-plots given by m-tuples of plots covering the same plot in $y(J^\infty_M F)$. Explicitly, the plots are

$$T^{\times m}(J^\infty_M F)(\mathbb{R}^n) \;=\; \big\{(X^1_{s^n}, \cdots, X^m_{s^n}) \in T(J^\infty_M F)(\mathbb{R}^n) \times \cdots \times T(J^\infty_M F)(\mathbb{R}^n)\big\},$$

where $p \circ X^1_{s^n} = \cdots = p \circ X^m_{s^n} = s^n \in y(J^\infty M F)(\mathbb{R}^n)$, with $X^i_{s^n} \in T(J^\infty_M F)(\mathbb{R}^n)$ (Eq. (3.4)). Note that sections of such m-fold fibered product are m-tuples of vector fields (Def. 3.2.2). Accordingly, any $\omega \in \Omega^m(J^\infty_M F)$ defines, via pre-composition, a map of $C^\infty(J^\infty_M F)$-modules

$$\begin{aligned} \omega :\; \mathfrak{X}(J^\infty_M F) \times \cdots \times \mathfrak{X}(J^\infty_M F) &\longrightarrow C^\infty(J^\infty_M F) \\ \big((X^1, \dots, X^m) : y(J^\infty_M F) \to T^{\times m}(J^\infty_M F)\big) &\longmapsto \big(\omega \circ (X^1, \dots, X^m) : y(J^\infty_M F) \to y(\mathbb{R})\big). \end{aligned}$$

On the right-hand side, we identify $C^\infty(J^\infty_M F) = \mathrm{Hom}_{\mathrm{FrMfd}}(J^\infty_M F, \mathbb{R})$ since the Yoneda embedding is fully faithful.

Consider now $\omega \in \Omega^1(J^\infty_M F)$. In local coordinates around $s \in J^\infty_M F$, we can write

$$X^\mu \frac{\partial}{\partial x^\mu} + \sum_{|I|=0}^\infty Y^a_I \frac{\partial}{\partial u^a_I} \mapsto X^\mu \cdot \omega\bigg(\frac{\partial}{\partial x^\mu}\bigg) + \sum_{|I|=0}^\infty Y^a_I \cdot \omega\bigg(\frac{\partial}{\partial u^a_I}\bigg).$$

(12) In [GS25], it is claimed that this identification will be clear in the extended topos of thickened smooth set.

The right-hand side sum, to be a well-defined smooth function on $J^\infty_M F$ for all vector fields X, must terminate for some finite $|I| = k_s \in \mathbb{N}$, i.e. $\omega(\frac{\partial}{\partial u^a_I}) = 0$ for $|I| \geq k_s$. By further restricting to a smaller neighborhood $U_s \subset J^\infty_M F$ around s, without loss of generality, we write

$$\omega = \omega_\mu \mathrm{d}x^\mu + \sum_{|I|=0}^{k_s} \omega^I_a \mathrm{d}u^a_i$$

where $\{\omega_\mu, \{\omega^I_a\}_{|I|\leq k_s}\} \subset C^\infty(J^\infty_M F)$ are locally defined and of finite order k_s. Extending this reasoning to the general case, we represent $\omega \in \Omega^m(J^\infty_M F)$, in local coordinates in a neighborhood U_s around a point $s \in J^\infty_M F$, by

$$\omega = \sum_{p+q=m} \sum_{I_1,\dots,I_q=0}^{k_s} \omega^{I_1\cdots I_q}_{\mu_1\cdots\mu_p\ a_1\cdots a_q} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p} \wedge \mathrm{d}u^{a_1}_{I_1} \wedge \cdots \wedge \mathrm{d}u^{a_q}_{I_q}.$$

Here the sum terminates at some finite order k_s, the coefficients are locally defined functions on $J^{k_s}_M F$, and the wedge is the usual exterior product.(13) That is, the map $\omega : \mathfrak{X}(J^\infty_M F) \times \cdots \times \mathfrak{X}(J^\infty_M F) \longrightarrow C^\infty(J^\infty_M F)$ is locally, for any $s' \in U_s \subset J^\infty_M F$, necessarily of the form

$$\omega\big(X^1, \cdots, X^m\big)(s') = \omega_{k_s}\big(\pi^\infty_{k_s}(s')\big)\Big(\mathrm{d}\pi^\infty_{k_s}\big(X_1(s')\big), \cdots, \mathrm{d}\pi^\infty_{k_s}\big(X_m(s')\big)\Big) \tag{3.14}$$

for some local form $\omega_{k_s} \in \Omega^m(J^{k_s}_M F)$ and $k_s \in \mathbb{N}$. Therefore, we can interpret m-forms on $J^\infty_M F$ as being locally the pullback of finite order forms, and represent them by compatible families of locally defined m-forms on finite order jet bundles

$$\big\{\omega_{k_s} \in \Omega^m\big(U_{\pi^\infty_{k_s}(s)}\big) \big| U_{\pi^\infty_{k_s}(s)} \subset J^{k_s}_M F\big\}_{s \in J^\infty_M F}.$$

Compatible here means that for any two $s, s' \in J^\infty_M F$ with $k_{s'} \geq k_s$ such that(14) $(\pi^{k_{s'}}_{k_s})^{-1}\big(U_{k_s}\big) \cap U_{k_{s'}} \neq \varnothing \subset J^{k_{s'}}_M F$, we have $(\pi^{k_{s'}}_{k_s})^* \omega_{k_s} = \omega_{k_{s'}}$. Equation (3.14) determines the form of the corresponding map out of the tangent bundle (Def. 3.2.4) on $*$-plots

$$\omega : T^{\times m}(J^\infty_M F)(*) \to \mathbb{R},\ (X^1_s, \cdots, X^m_s) \mapsto \omega_{k_s}\big(\pi^\infty_{k_s}(s)\big)\big(\mathrm{d}\pi^\infty_{k_s}(X^1_s), \cdots, \mathrm{d}\pi^\infty_{k_s}(X^m_s)\big),$$

for some locally defined form $\omega_{k_s} \in \Omega^m\big(J^{k_s}_M F\big)$ around $\pi^\infty_{k_s}(s) \in J^{k_s}_M F$. The action on any $\mathbb{R}^n$-plot follows immediately by substituting the $*$-plots denoted by s with $\mathbb{R}^n$-plots s^n.

Note that the globally finite order m-forms, i.e. those determined as a pullback of a single globally defined m-form $\omega_k \in \Omega^m(J^k_M F)$, fit in the above description. They form a vector subspace that we denote by $\Omega^m_{\mathrm{glb}}(J^\infty_M F) \subset \Omega^m(J^\infty_M F)$ in analogy to the case of smooth functions (or 0-forms).(15)

Remark 3.2.1. Equivalently, one could define the above within LocProMfd and consequently embed it in SmthSet. Recall that the local pro-manifold $J^\infty_M F$ is paracompact and has partitions of unity (cf. [Tak79]). By extending tangent vectors to vector fields, via a partition of unity,

(13)In this case it is the usual wedge product since in the local coordinates we are considering forms locally of finite order, i.e. over the smooth manifold $J^{k_s}_M F$.

(14)$(\pi^{k_{s'}}_{k_s})^{-1}$ is the preimage of $\big(U_{k_s}\big)$ via the projection.

(15)Both $\Omega^m_{\mathrm{glb}}(J^\infty_M F)$ and $\Omega^m(J^\infty_M F)$ have a natural $C^\infty(J^\infty MF)$-module structure, but only the latter also defines a petit sheaf on the topological space $|J^\infty_M F|$.

it follows that any $C^\infty(J^\infty MF)$-linear map $\mathfrak{X}(J^\infty_M F)\times\cdots\times\mathfrak{X}(J^\infty_M F)\to C^\infty(J^\infty_M F)$ defines a fiberwise linear smooth map $T^{\times m}(J^\infty_M F)\to\mathbb{R}$. Thus, there is in fact a bijection

$$\Omega^m(J^\infty_M F) = \mathrm{Hom}^{\text{fib.lin.an.}}_{\text{SmthSet}}\big(T^{\times m}(J^\infty_M F),\mathbb{R}\big) \tag{3.15}$$
$$\cong \mathrm{Hom}^{\text{antis.}}_{C^\infty(J^\infty_M F)\text{-Mod}}\big(\mathfrak{X}(J^\infty_M F)\times\cdots\times\mathfrak{X}(J^\infty_M F),\, C^\infty(J^\infty_M F)\big), \tag{3.16}$$

where "antis." denotes the fact that the morphisms are antisymmetric.

Note that the wedge product can be defined as for Definition 2.3.6. Furthermore, it is possible to algebraically define the Cartan calculus[(16)] on $\Omega^\bullet(J^\infty_M F) := \bigoplus_{m\in\mathbb{N}}\Omega^m(J^\infty_M F)$.

Definition 3.2.5. Let $\omega\in\Omega^m(J^\infty_M F)$, $\omega'\in\Omega^{m'}(J^\infty_M F)$ (Def. 3.2.4) for some $m,\ m'\in\mathbb{N}$, $f\in C^\infty(J^\infty_M F)$ and $X\in\mathfrak{X}(J^\infty_M F)$.

1. The **contraction** is defined as $\iota_X\omega := \omega(X,-,\cdots,-)\in\Omega^{m-1}(J^\infty_M F)$.
2. The **de Rham differential** $\mathrm{d}:\Omega^m(J^\infty_M F)\to\Omega^{m+1}(J^\infty_M F)$ is given by (Thm. A.3.2)

$$\begin{aligned}\mathrm{d}\omega(X^0,\cdots,X^m) = &\sum_{i=0}^m(-1)^i X^i\Big(\omega(X^0,\cdots,\widehat{X^i},\cdots,X^m)\Big)\\ &+\sum_{i<j}(-1)^{i+j}\omega\big([X^i,X^j],X^0,\cdots,\widehat{X^i},\cdots,\widehat{X^j},\cdots,X^m\big).\end{aligned} \tag{3.17}$$

   It follows that $\mathrm{d}^2=0$, and so $\big(\Omega^\bullet(J^\infty_M F),\mathrm{d}\big)$ defines a **cochain complex.**[(17)]
3. The **Lie derivative** of a function $f\in C^\infty(J^\infty_M F)$ along X is defined by $\mathbb{L}_X(f) := X(f)\in C^\infty(J^\infty_M F)$. It extends to m-forms by $\mathbb{L}_X := [\mathrm{d},\iota_X]$ (App. A), that is:

$$\mathbb{L}_X:\ \Omega^m(J^\infty_M F)\to\Omega^m(J^\infty_M F),\ \omega\mapsto \mathrm{d}(\iota_X\omega)+\iota_X(\mathrm{d}\omega).$$

The maps d, $\mathbb{L}_X$ and ι_X take the usual coordinate form around any point $s\in J^\infty_M F$, whereby ω and X are locally of finite order. One can show that the de Rham differential, the contraction by a vector field, and the Lie derivative with respect to a vector field are graded derivations of degree $+1$, -1, and 0, respectively. The usual Cartan calculus identities (Thm. A.3.1) between them also follow, since they hold locally around every point $s\in J^\infty_M F$ for the corresponding finite order representatives.

At the end of Chapter 2, we introduced the notion of de Rham m-forms as maps into the classifying space $\mathbf{\Omega}^m_{\mathrm{dR}}$ (Def. 2.3.8). This was an attempt to find a universal notion of forms with associated Cartan calculus, which was not defined for the classical Definition 2.3.6. We say that for the former, the Cartan calculus would be universal because Definition 2.3.6 can be upgraded to an object in the category of SmthSet (Eq. (2.9)). The attempt was unsuccessful; indeed, while there was no notion of a differential for the latter, there was no intuitive way to define a contraction for the former. Furthermore, the connection between the two definitions was not clear.

We then introduced infinite jet bundle technology to address this issue. We straightforwardly defined the Cartan calculus for forms on $J^\infty_M F$ in the sense of Definition 3.2.4 (Def. 3.2.5). Let us now introduce the de Rham forms on the infinite jet bundle in the sense of 2.3.8.

(16) The Cartan calculus also naturally descend to $\Omega^\bullet_{\mathrm{glb}}(J^\infty_M F)$.

(17) It turns out that the global and local cohomologies are isomorphic, since the relative cochains are quasi-isomorphic (cf. [Tak79] and [GS25, Prop. 4.14]).

Definition 3.2.6. The set of **smooth de Rham n-forms on the infinite jet bundle** $J^\infty_M F \in \mathrm{SmthSet}$ is defined by

$$\Omega^n_{\mathrm{dR}}(J^\infty_M F) := \mathrm{Hom}_{\mathrm{SmthSet}}(J^\infty_M F, \mathbf{\Omega}^n_{\mathrm{dR}}),$$

where $\mathbf{\Omega}^n_{\mathrm{dR}}$ is the moduli space of de Rham n-forms (Def. 2.3.7).

Remark 3.2.2. The goal of the project is to find a notion of universal Cartan calculus on $\Omega^\bullet_{\mathrm{dR}}(J^\infty_M F) := \bigoplus_{m\in\mathbb{N}} \Omega^m_{\mathrm{dR}}(J^\infty_M F)$ (Def. 3.2.6). We talk about universal Cartan calculus since $(\Omega^\bullet_{\mathrm{dR}}(J^\infty_M F), \mathrm{d}_{\mathrm{dR}}, \wedge) \in \mathrm{DGCA}_\mathbb{R}$ can be generalized to an object in SmthSet (Eq. (2.9)). The de Rham differential and wedge of Definition 2.3.9 extend accordingly to smooth maps (cf. [GS25, pp. 25-26]). As discussed in Chapter 2, in this setting there is no notion of contraction. If we were to find a relation between classical forms on $J^\infty_M F$ (Def. 3.2.4) and the smooth set-theoretical notion of forms on $J^\infty_M F$ (Def. 3.2.6), we could consistently define the universal Cartan calculus in the latter via Definition 3.2.5. Unfortunately, in this setting, the relation is evident only restricting to the case of globally finite order forms.

Theorem 3.1. *The subalgebra $\Omega^\bullet_{\mathrm{glb}}(J^\infty_M F) \hookrightarrow \Omega^\bullet(J^\infty_M F)$ of globally finite order differential forms* (Def. 3.2.4) *is canonically identified with a subalgebra of de Rham forms on the infinite jet bundle* (Def. 3.2.6). *That is, there is a canonical DGCA injection*

$$\Omega^\bullet_{\mathrm{glb}}(J^\infty_M F) \hookrightarrow \Omega^\bullet_{\mathrm{dR}}(J^\infty_M F).$$

Proof. This follows by the finite-dimensional manifold identification of Equation (2.8). Let $\omega \in \Omega^m_{\mathrm{glb}}(J^\infty_M F) \hookrightarrow \Omega^m(J^\infty_M F)$ be a differential form of globally finite order. More explicitly

$$\omega = (\pi^\infty_k)^* \omega_k = \omega_k \circ \mathrm{d}\pi^\infty_k : T^{\times m}(J^\infty_M F) \xrightarrow{d\pi^\infty_k} T^{\times m}(J^k_M F) \xrightarrow{\omega_k} \mathbb{R}$$

for a unique $\omega_k \in \Omega^m(J^k_M F)$, where k is the minimal such order. In particular, ω_k is a differential form on a finite-dimensional manifold. Therefore, by the Yoneda Lemma (Prop. 2.2.1) it corresponds uniquely to a map

$$\widetilde{\omega}_k : y(J^k_M F) \longrightarrow \mathbf{\Omega}^m_{\mathrm{dR}}$$

into the classifying space, i.e. a de Rham differential form on $y(J^k_M F)$. Precomposing with the projection $y(\pi^\infty_k) : y(J^\infty_M F) \to y(J^k_M F)$, we get

$$\widetilde{\omega} := \widetilde{\omega}_k \circ y(\pi^\infty_k) : y(J^\infty_M F) \longrightarrow \mathbf{\Omega}^m_{\mathrm{dR}}.$$

This is the unique de Rham m-form on $y(J^\infty_M F)$ corresponding to the traditional pullback form $\omega = \omega_k \circ \mathrm{d}\pi^\infty_k$. Such an assignment defines the desired injective algebra map

$$\Omega^\bullet_{\mathrm{glb}}(J^\infty_M F) \hookrightarrow \Omega^\bullet_{\mathrm{dR}}(J^\infty_M F).$$

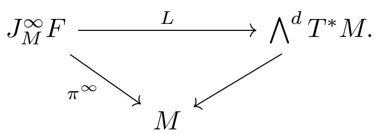
It follows that, under this identification, the classifying-space de Rham differential d_{dR} corresponds to the traditional differential d of globally finite order forms on $J^\infty_M F$, and similarly for the corresponding wedge products. Thus, the DGCA of globally finite order forms on $J^\infty_M F$ embeds into the de Rham forms (Prop. 2.3.3) on $J^\infty_M F$ defined via the classifying space $\mathbf{\Omega}^\bullet_{\mathrm{dR}}$. ■

This result gives a way to construct the universal Cartan calculus via Definition 3.2.5, but only when restricted to the subalgebra of globally finite order differential forms. The authors of [GS25] expect, but do not prove, that de Rham forms actually exhaust all differential forms on $J^\infty_M F$. In other words, the above inclusion should extend to a canonical bijection between all differential forms on $J^\infty_M F$ and those defined via the classifying space.

3.3 Local Lagrangians, Vector Fields and Differential Forms

In this section, we demonstrate how the infinite jet bundle encodes explicit global description of local Lagrangian densities, local vectors, and forms. Such a description turns out to be crucial for the formal implementation of the action principle in classical local field theories. This can also be implemented in other settings, for instance within ProDflSp (cf. [Blo24]). However, SmthSet affords a more natural treatment of all field-theoretic objects within a single category (Rmk. 3.1.1). In particular, the local Lagrangian density and local vector fields are morphisms in SmthSet, i.e. natural transformations between sheaves, that factor through the infinite jet bundle. Since the integration over M can also be viewed as a morphism of sheaves, the action functional itself defines a smooth map in SmthSet. Finally, local forms are morphisms in SmthSet obtained as pullbacks, along an appropriate smooth map, of differential forms on the infinite jet bundle; these, too, are natural transformations. For a more detailed treatment, we refer to [GS25, Ch. 3, 5, 6 and Ch. 7].

Definition 3.3.1. A **local Lagrangian density** is a map of smooth sections $\mathcal{L} : \Gamma_M(F) \to \Omega^d(M)$, $\varphi \mapsto L \circ j^\infty \varphi$ where $j^\infty : \Gamma_M(F) \to \Gamma_M(J^\infty F)$ is the jet prolongation, d is the dimension of M, and L is a smooth bundle map



The Lagrangian form is local if it locally factors through finite order jet bundles. This definition does indeed reflect the formulas written in the physics literature (Def. 2.1.2). Locally, the value of the local Lagrangian density $\mathcal{L}(\varphi)$ on a field $\varphi \in \Gamma_M(F)$ may be represented by

$$\mathcal{L}(\varphi) = L \circ j^\infty \varphi = L\big(x^\mu, \varphi^a, \{\partial_I \varphi^a\}_{|I|\leq k}\big) = \bar{L}\big(x^\mu, \varphi^a, \{\partial_I \varphi^a\}_{|I|\leq k}\big) \cdot \mathrm{d}x^1 \cdots \mathrm{d}x^d,$$

for some smooth function $\bar{L} \in C^\infty(J^\infty MF)$. One can show that it canonically extends to a smooth map of smooth sets $\mathcal{L} : \mathbf{\Gamma}_M(F) \longrightarrow \mathbf{\Gamma}_M(\wedge^d T^*M)$, where the latter is the smooth set whose plots can be interpreted as $\mathbb{R}^k$-parametrized d-forms on M (cf. [GS25, Lem. 3.11]).

Then, we define a bosonic smooth, local Lagrangian field theory to be a pair $(\mathcal{F}, \mathcal{L})$. $\mathcal{F} := y(\Gamma_M(F)) \in$ SmthSet is the smooth field space of sections of a finite–dimensional fiber bundle F over the spacetime M, and $\mathcal{L} = L \circ j^\infty$ is the smooth Lagrangian density defined by a smooth bundle map $L : J^\infty_M F \to \wedge^d T^*M$. Note that the majority of fundamental physical theories are described by Lagrangians that factor globally through a finite degree jet bundle. However, generic algebraic operations on fixed order Lagrangians, such as integration by parts, result in objects that necessarily factor through higher jet bundles, and so it is natural not to fix an order in the jet bundle and consider the infinite limit instead.

Integration defines another natural map of smooth sets. More precisely, if the base spacetime M is compact and oriented, then integration along M defines a smooth map

$$\int_M : \mathbf{\Gamma}_M(\wedge^d T^*M) \longrightarrow y(\mathbb{R}).$$

Here, the integration is plot-wise along M while keeping the $\mathbb{R}^k$-dependence fixed.

The composition of a smooth Lagrangian $\mathcal{L}$ with the smooth integration map defines the action functional as a map of smooth sets

$$S = \int_M \circ \mathcal{L} : \mathbf{\Gamma}_M(F) \longrightarrow y(\mathbb{R}). \tag{3.18}$$

Therefore, SmthSet satisfies also requirement 2 in Section 2.1.[18]

Since the classical field theories are assumed to be local, the relevant objects are local.[19] We might therefore restrict to local vector fields[20] and forms.[21]

Definition 3.3.2. A smooth vector field $\mathcal{Z} \in \mathfrak{X}(\mathcal{F}) = \Gamma_{\mathcal{F}}(T\mathcal{F})$ on the field space is **local** if it is given by $\mathcal{Z} = Z \circ j^\infty$ for some smooth bundle map

$$\begin{array}{ccc} J^\infty_M F & \xrightarrow{\quad Z \quad} & VF \\ & {\scriptstyle \pi^\infty_0} \searrow \quad \swarrow {\scriptstyle \pi_F} & \\ & F & \end{array}$$

over the total space F of the field bundle $F \to M$. The subset of local vector fields is denoted by $\mathfrak{X}_{\text{loc}}(F) \subset \mathfrak{X}(F)$.

The name is justified since, for each $\varphi \in \Gamma_M(F)$, the value of the tangent vector $\mathcal{Z}_\varphi$ depends only locally on φ via its jet $j^\infty_x \varphi$ at each $x \in M$. By Proposition 3.1.1, such bundle maps are locally represented by finite sums

$$Z = Z^a \frac{\partial}{\partial u^a}\,,$$

where $\{Z^a\} \subset C^\infty(J^\infty_M F)$ are locally defined smooth functions on the infinite jet bundle, and $\{\frac{\partial}{\partial u^a}\}$ is the local coordinate basis for vertical tangent vectors on F.

To define local forms, we first consider $\mathcal{F} \times M \in$ SmthSet and define the tangent bundle on it to be $T(\mathcal{F} \times M) := T(\mathcal{F}) \times T(M) \in$ SmthSet. Then, the by definition splitting can be thought of as horizontal splitting along $\text{pr}_2 : \mathcal{F} \times M \to M$, i.e.

$$V(\mathcal{F} \times M) \times_{\mathcal{F}\times M} H(\mathcal{F} \times M) := \big(T\mathcal{F} \times M\big) \times_{\mathcal{F}\times M} \big(\mathcal{F} \times TM\big) \;\cong\; T\mathcal{F} \times TM. \tag{3.19}$$

By definition, differential m-forms on the spacetime manifold M, the field space $\mathcal{F}$ and the infinite jet bundle $J^\infty_M F$ are given by fiber-wise linear bundle maps out of their tangent bundle in SmthSet. We therefore use this definition for differential forms on $\mathcal{F} \times M$ as well. As usual, the collection of differential forms of all degrees forms a graded $\mathbb{R}$-vector space

$$\Omega^\bullet(\mathcal{F} \times M) \;:=\; \bigoplus_{m\in\mathbb{N}} \Omega^m(\mathcal{F} \times M).$$

At this point, the development of Cartan calculus on $\mathcal{F} \times M$ reaches the same problems that we encounter for $\mathcal{F}$ in Sect. 2.3. Namely, sticking to this definition of forms, we lack of a notion of differential, whereas defining them through the classifying space (Def. 2.3.8) we cannot find an intuitive notion of contraction.[22]

The resolution, suggested by Sati and Giotopoulos, is that local classical field theory only requires the existence of the Cartan calculus and the corresponding bicomplex structure on the

[18] Smoothness of the jet prolongation is shown in Section 3.2

[19] The concept of locality is intuitively linked to finite order jet bundles. The following definitions make rigorous sense of this.

[20] [GS25, Prop. 6.17] shows that local vector fields do capture the infinitesimal version of spacetime covariant symmetries, thus justifying the focus solely on local vector fields.

[21] The usual manipulations of classical local field theories may be concisely expressed via the bicomplex of local differential forms on the product smooth set $F \times M$ of off-shell fields and the spacetime, which arises as the pullback of the variational bicomplex on $J^\infty_M F$ (cf. [GS25, Ch. 5]).

[22] Moreover, even though there is a notion of horizontal/vertical 1-forms on $\mathcal{F} \times M$ as the ones that vanishes respectively on the vertical/horizontal subbandle, it is not obvious that every 1-form ω on $F \times M$ splits accordingly. Similarly, for any m-form, creating a non-trivial obstacle towards building the variational bicomplex.

classical forms over $J^\infty_M F$ (Def. [GS25, Def. 5.4]). The latter is a splitting of the graded algebra of forms into horizontal and vertical forms[(23)] and a compatible grading structure given by vertical and horizontal differentials such that $\mathrm{d} = \mathrm{d}_H + \mathrm{d}_V$ (Def. [GS25, Def. 5.3]). This structure can then be pulled back along ev^∞ (Eq. 3.20) to the subset of local forms on $\mathcal{F} \times M$ (Def. [GS25, Def. 7.5]). This local bicomplex does exist within smooth sets. More explicitly, consider the smooth evaluation map

$$\mathrm{ev} : \mathbf{\Gamma}_M(J^\infty F) \times M \longrightarrow J^\infty_M F, \ (\tilde{\varphi}^k, p^k) \longmapsto \tilde{\varphi}^k \circ (\mathrm{id}_{\mathbb{R}^k}, p^k) \in J^\infty_M F(\mathbb{R}^k)$$

where φ^k and p^k are $\mathbb{R}^k$-plots of $\mathbf{\Gamma}_M(J^\infty F)$ and $y(M)$ respectively. Precomposing along the smooth jet prolongation $j^\infty : \mathcal{F} \to \mathbf{\Gamma}_M(J^\infty F)$, we may define the smooth prolongated evaluation map with values in $J^\infty_M F$.

Definition 3.3.3. The **prolongated evaluation map** $\mathrm{ev}^\infty : \mathcal{F} \times M \to J^\infty_M F$ is defined as the composition of maps of smooth sets

$$\mathrm{ev}^\infty : \mathcal{F} \times M \xrightarrow{(j^\infty, \mathrm{id}_M)} \mathbf{\Gamma}_M(J^\infty F) \times M \xrightarrow{\mathrm{ev}} J^\infty_M F. \tag{3.20}$$

At the level of $*$–plots this is given by $\mathrm{ev}^\infty(\varphi, p) = j^\infty \varphi(p) \in J^\infty_M F$, and similarly for higher plots.

Differentiating the latter, we can define the pushforward map along ev^∞ on $*$–plots as

$$\begin{aligned} \mathrm{dev}^\infty : T\mathcal{F} \times TM &\to VJ^\infty_M F \oplus HJ^\infty_M F \cong TJ^\infty_M F \\ (\mathcal{Z}_\varphi, X_p) &\mapsto j^\infty \mathcal{Z}_\varphi(p) + \big(\mathrm{d}j^\infty\varphi\big)_p X_p, \end{aligned}$$

and similarly for higher plots. Next, notice that by construction the pushforward dev^∞ respects the natural splitting of the tangent bundles $T\mathcal{F} \times TM$ from Equation (3.19), and that of $TJ^\infty_M F$ from Proposition 3.2.3, as bundles over M:

$$\begin{array}{ccc} V(\mathcal{F} \times M) \times_{\mathcal{F}\times M} H(\mathcal{F} \times M) & \xrightarrow{\quad \mathrm{def}^\infty \quad} & VJ^\infty_M F \times_{J^\infty_M F} HJ^\infty_M F. \\ & \searrow \quad M \quad \swarrow & \end{array}$$

The idea now is to define the subset of **local forms** on $\mathcal{F} \times M$ by pulling back forms on $J^\infty_M F$ via $\mathrm{ev}^\infty : \mathcal{F} \times M \to J^\infty_M F$. This means that the pullback local form $(\mathrm{ev}^\infty)^*\omega \in \Omega^\bullet_{loc}(\mathcal{F} \times M)$ of a differential form $\omega \in \Omega^\bullet(J^\infty_F)$ is defined by the composition

$$(\mathrm{ev}^\infty)^*\omega : \ T\mathcal{F} \times TM \xrightarrow{\mathrm{dev}^\infty} TJ^\infty_M F \xrightarrow{\omega} \mathbb{R}.$$

Note that, forms on $J^\infty_M F$ have a well-defined bigrading (cf. [GS25, Ch. 5]), induced by the splitting of the tangent bundle (Prop. 3.2.3), and the pushforward map dev^∞ preserve the respective splittings (cf. [GS25, Ch. 7]). Therefore, it follows that the bigrading structure of forms on $J^\infty_M F$ induces one on local forms over $\mathcal{F} \times M$. The latter has a well-defined corresponding Cartan calculus (cf. [GS25, Prop. 7.7]). Moreover, the subalgebra $\Omega^\bullet_{\mathrm{loc,glb}}(\mathcal{F} \times M) \hookrightarrow \Omega^\bullet_{\mathrm{dR}}(\mathcal{F} \times M)$, arising by pulling back the globally finite order differential forms $\Omega^\bullet_{\mathrm{glb}}(J^\infty_M F)$, is canonically identified with a subalgebra of de Rham forms (Def. 2.3.8) on $\mathcal{F} \times M$. That is, there is a canonical injective DGCA map

$$\Omega^\bullet_{\mathrm{loc,glb}}(\mathcal{F} \times M) \ \hookrightarrow \ \Omega^\bullet_{\mathrm{dR}}(\mathcal{F} \times M),$$

(23) This is induced by the tangent bundle splitting. Vertical/horizontal forms are intuitively the ones that vanish on the horizontal/vertical subbundle respectively.

that furthermore respects the corresponding bi-complex structures (cf. [GS25, Lem. 7.9]). If it is actually the case that $\Omega^\bullet(J^\infty_M F) \cong \Omega^\bullet_{\mathrm{dR}}(J^\infty_M F)$ (Thm. 3.1) as maps of smooth sets, then the above embedding would canonically extend to local forms on $\mathcal{F} \times M$ induced by any locally finite order forms on $J^\infty_M F$.

Conclusions

This report aims to study the differential geometry on the infinite jet bundle within the framework of smooth sets as done in [GS25]. Throughout the work, we demonstrate how SmthSet is a category that naturally incorporates the action principle and fundamental field-theoretical objects, with a focus on bosonic LFTs. The strength of this framework lies in its ability to carry out the standard operations of finite-dimensional differential geometry pointwise on ordinary manifolds, thereby extending them to smooth sets such as the field space $\mathcal{F}$.

Two approaches to differential forms are compared. On one hand, differential m-forms can be defined as real-valued smooth fiberwise linear antisymmetric maps on the m-fold fiber product of the tangent bundle over $\mathcal{F}$ (Def. 2.3.6); this definition admits a wedge product and contraction, but lacks a de Rham differential. On the other hand, the universal definition via the de Rham moduli space (Def. 2.3.8) straightforwardly incorporates both the wedge product and a differential, but not the contraction. Moreover, the relation between the two definitions is unclear.

To address this, the infinite jet bundle, whose structure globally encodes locality, is introduced. We prove that its tangent bundle admits a canonical splitting (Prop. 3.2.3). Subsequently, we algebraically define the full Cartan calculus on forms, regarded again as real-valued smooth fiberwise linear antisymmetric maps. The authors of [GS25] conjecture that there exists a canonical bijection between forms on the infinite jet bundle and those defined via the classifying space. This would allow the universal Cartan calculus to be categorically formulated on the infinite jet bundle. At present, however, only a canonical injection is available, and only for the subalgebra of globally finite-order forms (Thm. 3.1).

While the framework of SmthSet is promising, it does not yet provide a straightforward way to define the Cartan calculus universally on forms of the infinite jet bundle. Nevertheless, the authors argue that this is the correct setting to capture the essential features of the physical world, which is fundamentally field-theoretic, smooth, local, gauged, non-perturbative, and contains fermions, once enriched with infinitesimal structure and odd variables, giving rise to the category of thickened super smooth sets **ThSupSmthSet** (cf. [Gri25]).

Even in its restricted form, the SmthSet framework subsumes previous approaches via Fréchet and diffeological spaces. In particular, the space of on-shell fields can be realized as a smooth set using the variational bicomplex on the infinite jet bundle (cf. [GS25, Ch. 5]). This in turn induces the local variational bicomplex (cf. [GS25, Ch. 7]) on the space of local forms on $\mathcal{F} \times M$. Together with local vector fields, this suffices to formulate the action principle for bosonic classical LFTs and to study the presymplectic structure on $\mathcal{F} \times M$. Finally, Chapter 6 of [GS25] shows how local infinitesimal symmetries, described by local vector fields on $\mathcal{F}$, combine with vertical vector fields on the infinite jet bundle to yield Noether's first and second theorems.

Appendix A

Appendix

In this appendix, we review some key concepts of differential geometry on smooth manifolds. First, vector and fiber bundles are introduced with some specific examples. Then we define connections in relation to the splitting of the tangent bundles. To conclude, we recall the Cartan calculus in the setting of graded linear algebra. We assume that the reader is familiar with the definitions of smooth manifolds, pullback and pushforward, tangent and cotangent spaces and bundles, vector fields and forms, and integration on manifolds. For a detailed discussion, we refer to [Lee03; Nak18; GSM09; Cat18].

A.1 Fiber Bundles

We define fiber and vector bundles, sections, bundle maps, and introduce the pullback bundle.

A fiber bundle is intuitively a topological space that looks locally like a direct product of two topological spaces. The tangent bundle $TM := \bigcup_{p\in M} T_pM$ of a manifold M, modeled on $\mathbb{R}^n$, is an example of such bundles that locally looks like $\mathbb{R}^n \times \mathbb{R}^n$ (cf. [Nak18, Ch. 9]). In particular, it is a vector bundle of rank n.

Definition A.1.1. A smooth **fiber bundle** (F, π, M, E) consists of the following elements:

i. a smooth manifold F called the **total space**;

ii. a smooth manifold M called the **base space**;

iii. a smooth manifold E called the **typical fiber**;

iv. a smooth surjection $\pi : F \to M$ called the **projection**. The inverse image $\pi^{-1}(p) = E_p \cong E$, is called the **fiber at** $p \in M$;

v. an open covering $\{U_i\}$ of M with diffeomorphisms, called **local trivialisations**, $\phi_i : U_i \times E \longrightarrow \pi^{-1}(U_i)$ such that $\pi \circ \phi_i(p, t) = p$,

vi. at each point $p \in M$, $\phi_{i,p}(t) \equiv \phi_i(p, t)$ is a diffeomorphism $\phi_{i,p} : E \longrightarrow E_p$. On each overlap $U_i \cap U_j \neq \varnothing$, we require the **transition map** $g_{ij}(p) := \phi_{i,p}^{-1} \circ \phi_{j,p} : E \longrightarrow E$ to be a diffeomorphism and $g_{ij} : p \mapsto g_{ij}(p)$ to be smooth such that $\phi_j(p, t) = \phi_i\big(p,\, g_{ij}(p)\, t\big)$.

If the typical fiber is a real k-dimensional vector space $E \cong \mathbb{R}^k$, the restrictions of the local trivializations $\phi_{i,p}$ are linear isomorphisms, and the transition maps g_{ij} are smooth functions assigning to each point $p \in U_i \cap U_j$ an invertible matrix in $\mathrm{GL}(k, \mathbb{R})$, then the fiber bundle is called a **vector bundle**.

From the definition, it can be checked that the functions g_{ij} satisfy the cocycle condition $g_{ij}(p) = g_{ik}(p) \circ g_{ik}(p)$ for all $p \in U_i \cap U_k \cap U_j$. We often use a shorthand notation $F \to M$ to denote a fiber bundle (F, π, M, E). Moreover, strictly speaking, the definition of a fiber bundle should be independent of the special covering $\{U_i\}$ of M. In the mathematical literature, this definition is employed to define a coordinate bundle $(F, \pi, M, E, \{U_i\}, \{\phi_i\})$. Then, two coordinate bundles $(F, \pi, M, E, \{U_i\}, \{\phi_i\})$ and $(F, \pi, M, E, \{V_j\}, \{\psi_j\})$ are said to be equivalent if $(F, \pi, M, E, \{U_i\} \cup \{V_j\}, \{\phi_i\} \cup \{\psi_j\})$ is again a coordinate bundle. A fiber bundle is defined as an equivalence class of coordinate bundles. We shall assume a definite covering and make no distinction between a coordinate bundle and a fiber bundle.

If the bundle is globally trivial, i.e. $F \cong M \times E$, the fiber bundle is called a **trivial bundle**.

Definition A.1.2. Let $\pi : F \to M$ be a fiber bundle. A **section** $s : M \to F$ is a smooth map which satisfies $\pi \circ s = \mathrm{id}_M$. Clearly, $s(p) = s|_p$ is an element of $E_p = \pi^{-1}(p)$. The set of smooth sections on M is denoted by $\Gamma_M(F) = \Gamma(M, F)$. If $U \subset M$, **local sections** are sections $s : U \to F$ defined only on U. The condition reads $\pi \circ s = \mathrm{id}_U$. $\Gamma(U, F)$ denotes the set of local sections on U.

Note that $\pi \circ s = \mathrm{id}_M$ and compositions of sections preserve smoothness. Remarkably, $\Gamma_M(F)$ is a $C^\infty(M)$-module (Def. A.2.2) when F is a vector bundle. For example, $\Gamma(M, TM)$ is identified with the set of vector fields $\mathfrak{X}(M)$ and it is a $C^\infty(M)$-module. It should be noted that not all fiber bundles admit global sections.

Definition A.1.3. Let $F \xrightarrow{\pi} M$ and $F' \xrightarrow{\pi'} M'$ be fiber bundles. A smooth map $\bar{f} : F' \to F$ is called a **bundle map** if it maps each fiber E'_p of F' onto E_q of F. Then $\bar{f}$ naturally induces a smooth map $f : M' \to M$ such that $f(p) = q$. That is, the following diagram commutes:

$$
\begin{array}{ccc}
F' & \xrightarrow{\bar{f}} & F \\
{\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\
M' & \xrightarrow[f]{} & M
\end{array}
\qquad
\left(
\begin{array}{ccc}
u & \xmapsto{\bar{f}} & \bar{f}(u) \\
{\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\
p & \xmapsto[f]{} & q
\end{array}
\right)
$$

Note that a smooth map $\bar{f} : F' \to F$ is not necessarily a bundle map. It may map $u, v \in E'_p$ of F' to $\bar{f}(u)$ and $\bar{f}(v)$ on different fibers of F, so that $\pi(\bar{f}(u)) \neq \pi(\bar{f}(v))$.

A particular bundle we use is the **pullback bundle**. Let $F \xrightarrow{\pi} M$ be a fiber bundle with typical fiber E. If a map $f : N \to M$ is given, the pair (F, f) defines a new fiber bundle over N with the same fiber E as follows. Let $f^*F \subset N \times F$ be the subspace consisting of points (p, u) such that $f(p) = \pi(u)$. That is, the pullback[(1)] of F by f can be written as

$$f^*F \equiv \{(p, u) \in N \times F \mid f(p) = \pi(u)\}.$$

The fiber E_p of f^*F is just a copy of the fiber $E_{f(p)}$ of F. If we define $f^*F \xrightarrow{\pi_1} N$, $\pi_1(p, u) = p$, and $f^*F \xrightarrow{\pi_2} E$, $\pi_2(p, u) = u$, the pullback f^*F may be endowed with the structure of a fiber

(1) This is the categorical pullback (Def. 1.2.3) within SmthMfd.

bundle, and we obtain the following bundle map:

$$\begin{array}{ccc} f^*F & \xrightarrow{\pi_2} & F \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ N & \xrightarrow[f]{} & M \end{array} \qquad \left(\begin{array}{ccc} (p,u) & \xmapsto{\pi_2} & u \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ p & \xmapsto[f]{} & f(p) \end{array} \right). \tag{A.1}$$

The commutativity of the diagram follows since $\pi(\pi_2(p,u)) = \pi(u) = f(p) = f(\pi_1(p,u))$ for $(p,u) \in f^*F$. Regarding the trivializations, let $\{U_i\}$ be a covering of M and $\{\phi_i\}$ be local trivializations. Then $\{f^{-1}(U_i)\}$ defines a covering of N such that f^*F is locally trivial. Indeed, take $u \in F$ such that $\pi(u) = f(p) \in U_i$ for some $p \in N$. If $\phi_i^{-1}(u) = (f(p), f_i)$, we find $\psi_i^{-1}(p,u) = (p, f_i)$, where ψ_i is the local trivialization of f^*F.

A.2 Splitting of the Tangent Bundle and Connections

We define the vertical tangent bundle and the splitting of an exact sequence of vector bundles, and we heuristically show how the latter is related to a connection, namely the Cartan connection.

Let $\pi_F : TF \to F$ be the tangent bundle of a fiber bundle $\pi : F \to M$. The bundle coordinates (p^μ, f^a) on F induce the coordinates $(p^\mu, f^a, \dot{p}^\mu, \dot{f}^a)$ on TF (cf. [GSM09, Sect. 1.1.3]). The subbundle(2) $VF = \ker(\mathrm{d}\pi)$ of tangent bundle $\pi_F : TF \to F$, where $\mathrm{d}\pi : TF \to TM$, consists of the vectors tangent to fibers of F. It is called the **vertical tangent bundle** of F and is provided with the coordinates $(p^\mu, f^a, \dot{f}^a)$. One can show that it fits in the following **short exact sequence** of vector bundles

$$0 \to VF \xrightarrow{\iota} TF \xrightarrow{\mathrm{d}\pi} \pi^*M \to 0$$

where $\pi^*M \cong F \times_M TM$ is the pullback bundle (Eq. (A.1))

$$\begin{array}{ccc} \pi^*M & \longrightarrow & TM \\ \downarrow & & \downarrow{\scriptstyle \pi_M} \\ F & \xrightarrow[\pi]{} & M. \end{array}$$

A short exact sequence here means that the arrows are bundle maps such that the image of each morphism is the kernel of the next morphism. The sequence is said to **split** if there exist a bundle map $H : \pi^*M \to TF$ such that $\mathrm{d}\pi \circ H = \mathrm{id}_{\pi^*M}$ or equivalently $TF \cong \iota(VF) \oplus h(\pi^*M)$, where the sum has to be interpreted as direct sum compatible with the vector bundle structure (cf. Whitney sum [GSM09, Sect. 1.1.3]). The latter direct sum suggests identifying π^*M with the horizontal tangent subbundle HF, that is, the image of the lifting map H. In other words, the splitting is used to lift tangent vectors from M to horizontal vectors in TF, consistently with the bundle structure.(3) Such a lift can be described explicitly as

$$T_uF \ni X_u^H := \sigma(u)\big(X_{\pi(u)}\big) = H\big(u, X_{\pi(u)}\big) \quad \text{with} \quad X_{\pi(u)} \in T_{\pi(u)}M,$$

(2) A subbundle of a given vector bundle is a submanifold of the total space with vector bundle structure over the same base.

(3) This is used to depict how fields change under a spacetime diffeomorphism described by fields in TM.

where the connection split $\sigma : F \to \mathrm{Hom}(TM, TF)$ assign to each point $u \in F$ a bundle map in $\mathrm{Hom}(TM, TF)$.

This splitting is essential for comparing vertical field variations, which are useful for formalizing the action principle in field theories, separately from horizontal variations. It ensures that when one computes the variation of the action, the change of fields can be decomposed into genuine field transformation contributions (vertical) and variation solely due to a diffeomorphism of the base (horizontal).

The theorem [GSM09, Thm 1.1.12] states that every exact sequence of vector bundles is split in the case of finite-dimensional manifolds, even if the splitting is not canonical,[(4)] but this is not always true for the infinite-dimensional case.[(5)]

The choice of a complementary horizontal bundle defines an **Ehresmann connection** on the bundle. It can be shown that an Ehresmann connection uniquely determines a horizontal lift and therefore the splitting. That is, for a fiber bundle $\pi : F \to M$, a connection defines a splitting of the tangent bundle $TF \cong VF \oplus HF$, that is a vector bundle over F. $VF = \ker(\mathrm{d}\pi)$ denotes the vertical bundle, and HF is the horizontal distribution determined by the connection, called Cartan connection. Such connection is more general than a linear one, but it coincides with a linear connection, when considering a vector bundle $\pi : F \to M$, if the horizontal distribution is linear in the fibers, i.e. compatible with the vector space structure of each fiber. Nevertheless, the two concept are analogous in the scope: determining the parallel transport along M. Presenting the relation between the two type of connections goes beyond the ambit of the project.[(6)] Still, it can be intuitively understood by defining the horizontal lift as follows, showing an analogy with parallel transport (cf. [Tec19]).

Definition A.2.1. Let $\pi : F \to M$ be a fiber bundle, $p \in M$ and $f \in F$ such that $\pi(f) = p$. Given a smooth curve $\gamma : \mathbb{R} \to M$ such that $\gamma(0) = p$, we define a lift of γ through f as the curve $\tilde{\gamma}$, satisfying $\tilde{\gamma}(0) = f$ and $\pi(\tilde{\gamma}(t)) = \gamma(t)\ \forall t$. If F is smooth, then a lift is horizontal if every tangent to $\tilde{\gamma}$ lies in a fiber of HF, namely $\dot{\tilde{\gamma}}(t) \in HF_{\tilde{\gamma}(t)}$ for all $t \in \mathbb{R}$.

The splitting of the tangent bundle induces a splitting of the smooth vector fields into vertical or horizontal, depending on whether they belong to $\Gamma_F(VF)$ or $\Gamma_F(HF)$, respectively. That is a splitting of the $C^\infty(F)$-module $\mathfrak{X}(F) \cong \mathfrak{X}_V(F) \oplus \mathfrak{X}_H(F)$. The splitting of the tangent bundle induces also a splitting of differential forms into vertical and horizontal forms. The former are those forms on F that vanish when contracted with vectors tangent to the base M, i.e. when restricted to the horizontal subbundle. They are usually denoted by $\Omega^\bullet(F/M)$. The latter are those that vanish when restricted to the vertical subbundle. When the de Rham differential itself decomposes into horizontal and vertical parts, this structure endows the space of differential forms on F with the structure of a bicomplex.

For completeness, we define the module of a commutative ring.

Definition A.2.2. Let R be a commutative ring, and let 1 denote its multiplicative identity. An R**-module** is an abelian group $(M, +)$ together with a scalar multiplication $R \times M \to M$, $(r, m) \mapsto r \cdot m$, such that for all $r, s \in R$ and $m, n \in M$:

i. $r \cdot (m + n) = r \cdot m + r \cdot n$,

ii. $(r + s) \cdot m = r \cdot m + s \cdot m$,

iii. $(rs) \cdot m = r \cdot (s \cdot m)$,

(4) The connection becomes canonical only after fixing additional geometric data, such as a metric.
(5) For the infinite jet bundle there is a canonical splitting due to the properties of the latter (Sect. 3.2).
(6) For an extensive treatment of connections on fiber bundles see [Tec19; GSM09].

iv. $1 \cdot m = m$.

One can show that $C^\infty(F)$ forms a ring and that the action

$$(f \cdot X)(p) = f(p)\, X_p, \quad \text{for all } f \in C^\infty(M),\ X \in \mathfrak{X}(M),\ p \in M.$$

makes $\mathfrak{X}(M)$ into a $C^\infty(M)$-module. Equivalently for $\mathfrak{X}(F)$.

A.3 Cartan Calculus and Graded Linear Algebra

We present how the Cartan calculus can be understood within the context of graded linear algebra. We assume the reader to be at ease with the definition of forms and vector fields, tensor algebra, exterior algebra and wedge product, de Rham differential, contraction, Lie bracket, and Lie derivative. We use the conventions for which the degree of the interior product ι_X along $X \in \mathfrak{X}(M)$ and of the de Rham differential d are -1 and 1 respectively, the graded commutator is defined as $[A, B] = A \circ B - (-1)^{|A||B|}\, B \circ A$, and the Lie derivative is given by $\mathbb{L}_X = [d, \iota_X]$. We refer to [Cat18; Lee03] for a detailed discussion of these notions.

The idea of graded linear algebra is to generalize the usual concepts of linear algebra to collections of vector spaces.

Definition A.3.1.

1. A **graded vector space** $V^\bullet$ is a collection $\{V^k\}_{k\in\mathbb{Z}}$ of vector spaces.
2. A **morphism** $\varphi : V^\bullet \to W^\bullet$ is a collection of linear maps $\varphi^k : V^k \to W^k$ for all k.
3. A **graded morphism** $\varphi : V^\bullet \to W^\bullet$ of degree r is a collection of linear maps $\varphi^k : V^k \to W^{k+r}$ for all k. If $W^\bullet = V^\bullet$, φ is a **graded endomorphism**.

For instance, let $M \in$ SmthMfd and V a vector space. Then, the vector spaces $V^{\otimes k}, \Lambda^k V$ and $\Omega^k(M)$ define graded vector spaces $T^\bullet(V)$,[7] $\Lambda^\bullet V$ and $\Omega^\bullet(M)$. We can represent them as a direct sum of the vector spaces over $k \in \mathbb{Z}$.[8] For example $T^\bullet(V) := \bigoplus_{k\in\mathbb{Z}} V^{\otimes k}$ and $\Omega^\bullet(M) := \bigoplus_{k\in\mathbb{Z}} \Omega^k(M)$. In this setting, the de Rham differential, the contraction by a vector field, and the Lie derivative by a vector field are examples of graded morphisms of degree $+1$, -1 and 0 respectively. Indeed, by definition $\mathbb{L}_X = [d, \iota_X]$, and the differential and contraction acts on homogeneous elements elemets of $\Omega^\bullet(M)$ as

$$\begin{aligned} \Omega^m(M) &\xrightarrow{\mathrm{d}} \Omega^{m+1}(M) \\ \omega &\mapsto \mathrm{d}\omega \end{aligned} \qquad \text{and} \qquad \begin{aligned} \Omega^m(M) &\xrightarrow{\iota_X} \Omega^{m-1}(M) \\ \omega &\longmapsto \iota_X\omega := \omega(X, -, \cdots, -), \end{aligned}$$

for all $\omega \in \Omega^m(M)$, $X \in \mathfrak{X}(M)$.

Definition A.3.2.

1. A graded endomorphism of degree -1 that squares to zero is called a **boundary operator**.

[7] This T stands for tensor and should not be confused with some tangent bundle of a smooth manifold.

[8] The choice of $\mathbb{Z}$ for the grading and the sign conventions we have used are those needed for differential forms. More generally, one may define a G-graded vector space as a collection $\{V^k\}_{k\in G}$ of vector spaces, where G is a set. Then, all the definitions of this section have sense if G is assumed to have some nice properties (cf. [Cat18, Sect. 9.3]).

2. A graded endomorphism of degree $+1$ that squares to zero is called a **coboundary operator**.

3. A graded vector space endowed with a boundary or a coboundary operator is called a **complex**.

The graded vector space of forms over a smooth manifold, together with the de Rham differential, forms the de Rham complex $(\Omega^\bullet(M), \mathrm{d})$.

Definition A.3.3. A **graded algebra** is a graded vector space $A^\bullet$ together with a collection of bilinear maps

$$A^k \times A^l \to A^{k+l},\ (a, b) \mapsto ab,$$

for all $k, l \in \mathbb{N}$. The graded algebra is called

1. **associative** if $(ab)c = a(bc)$ for all a, b, c;

2. **graded commutative** if $ab = (-1)^{kl} ba, \quad$ for $a \in A^k, b \in A^l$;

3. **graded skew-commutative** if $ab = -(-1)^{kl} ba, \quad$ for $a \in A^k, b \in A^l$.

The graded tensor algebra $T^\bullet(V)$ is associative, with the bilinear operation being the tensor product, but neither graded commutative nor graded skew-commutative. The graded algebras $\Lambda^\bullet V$ and $\Omega^\bullet(M)$ are associative and graded commutative thanks to the exterior product $\wedge$. Indeed, the exterior product of the exterior algebra induces, by pointwise multiplication, the exterior product of differential forms:

$$\begin{aligned} \Omega^m(M) \times \Omega^n(M) &\xrightarrow{\wedge} \Omega^{m+n}(M) \\ (\omega, \omega') &\mapsto \omega \wedge \omega' \end{aligned}$$

for all $\omega \in \Omega^m(M)$, $\omega' \in \Omega^n(M)$. Note that $\deg(\omega \wedge \omega') = \deg(\omega) + \deg(\omega')$. Moreover, the following holds:

$$\omega \wedge \omega' = (-1)^{mn} \omega' \wedge \omega. \tag{A.2}$$

This can be shown starting from elementary 1-forms $\mathrm{d}f, \mathrm{d}g \in \Omega^1(M)$, where $f, g \in C^\infty(M)$. In local coordinates $\{x^i\}$ around $p \in M$, using Einstein summation convention and the exterior algebra pointwise product, it holds:

$$\begin{aligned} (\mathrm{d}f \wedge \mathrm{d}g)|_p &= \partial_i f(p)\, \mathrm{d}x^i|_p \wedge \partial_j g(p)\, \mathrm{d}x^j|_p = \partial_i f(p)\, \partial_j g(p)\, \mathrm{d}x^i|_p \wedge \mathrm{d}x^j|_p \\ &= \partial_i f(p)\, \partial_j g(p)\, (-1)\, \mathrm{d}x^j|_p \wedge \mathrm{d}x^i|_p = (-1)(\mathrm{d}g \wedge \mathrm{d}f)|_p. \end{aligned}$$

It is possible to extend this to any form using linearity and induction.

Definition A.3.4. A graded derivation of degree r of a graded algebra $A^\bullet$ is a graded morphism $D : A^\bullet \to A^\bullet$ of degree r that satisfies the graded Leibniz rule

$$D(ab) = D(a)b + (-1)^{rk} a D(b),$$

for all $a \in A^k$, $b \in A^l$.

The de Rham differential, the contraction by a vector field, and the Lie derivative with respect to a vector field are graded derivations of degree $+1$, -1, and 0, respectively. The graded Leibniz rule can be checked in coordinates as in the sketch of proof of Equation (A.2).

Definition A.3.5. A coboundary operator that is also a derivation is called **differential**. The triple $(A^\bullet, \cdot, \delta)$ of a graded algebra, with bilinear product $\cdot$, together with a differential δ, is called **differential graded algebra** (DGA).

The de Rham complex $(\Omega^\bullet(M), \mathrm{d}, \wedge)$ is a DGCA, where C stands for commutative. Note that the collection of all graded derivations of a graded algebra $A^\bullet$ forms a new graded vector space, denoted by $\mathrm{Der}^\bullet(A^\bullet)$. Indeed, the set $\mathrm{Der}^r(A^\bullet)$ of graded derivations of degree r on a graded algebra $A^\bullet$ is a vector space, actually a subspace of $\mathrm{End}^r(A^\bullet)$, for all $r \in \mathbb{N}$. We further denote the graded vector space of endomorphisms of a graded algebra as $\mathrm{End}^\bullet(A^\bullet)$.

Definition A.3.6. A **graded Lie algebra** (GLA) is a graded algebra $\mathfrak{g}^\bullet$ whose product, usually called the graded Lie bracket and denoted by $[\ ,\ ]$, is graded skew-commutative and satisfies the graded Jacobi identity $[a, [b, c]] = [[a, b], c] + (-1)^{kl}[b, [a, c]]$ for all $a \in \mathfrak{g}^k, b \in \mathfrak{g}^l, c \in \mathfrak{g}^m$.

Note that $(A^\bullet, [\ ,\ ])$ is a GLA when defining the backet as $[a, b] := ab - (-1)^{kl}ba$ for any $a \in A^k$, $b \in A^l$. Similarly, $(\mathrm{End}^\bullet(V^\bullet), [\ ,\ ])$ is a GLA when $V^\bullet$ is a graded vector space.

Definition A.3.7.

1. A **graded subspace** $W^\bullet$ of a graded vector space $V^\bullet$ is a collection $W^k \subset V^k$ of subspaces for all k.

2. A **graded subalgebra** $B^\bullet$ of a graded algebra $A^\bullet$ is a graded subspace that is closed under the product. Usually, a graded subalgebra of a graded Lie algebra is called a **graded Lie subalgebra**.

One can show that $(\mathrm{Der}^\bullet(A^\bullet), [\ ,\ ])$ is a graded Lie subalgebra of $(\mathrm{End}^\bullet(A^\bullet), [\ ,\ ])$, where $A^\bullet$ is a graded algebra, verifying that the former is closed under the Lie bracket.

Theorem A.3.1. *The span over $\mathbb{R}$ of the set $\{\mathrm{d}, \iota_X, \mathbb{L}_X : X \in \mathfrak{X}(M)\}$ is a graded Lie subalgebra of $\mathrm{Der}^\bullet(\Omega^\bullet(M))$. More precisely, the following relations hold for all $X, Y \in \mathfrak{X}(M)$:*

$$[\mathrm{d}, \mathrm{d}] = 0, \quad [\mathrm{d}, \iota_X] = \mathbb{L}_X, \quad [\mathrm{d}, \mathbb{L}_X] = 0,$$

$$[\iota_X, \iota_Y] = 0, \quad [\iota_X, \mathbb{L}_Y] = \iota_{[X,Y]}, \quad [\mathbb{L}_X, \mathbb{L}_Y] = \mathbb{L}_{[X,Y]}.$$

The proof can be found in [Cat18, Thm. 9.29].

The second relation is known as Cartan's formula and is very useful to compute the Lie derivative of a differential form. An application of the theorem is a very explicit formula for computing the de Rham differential.

Theorem A.3.2. *Given $k+1$ vector fields $X_0, X_1, \ldots, X_k$, $k \geq 0$, and a k-form ω, we have*

$$\begin{aligned}\mathrm{d}\omega(X_0, \ldots, X_k) = &\sum_{i=0}^{k} (-1)^i X_i(\omega(X_0, \ldots, \hat{X}_i, \ldots, X_k)) + \\ &+ \sum_{0 \leq i < j \leq k} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k)\end{aligned}$$

where the cap stands for omission.

A proof of the theorem can be found for example in [Cat18, Prop. 9.32].

These constructions provide the geometric background for the categorical formulations developed in Chapter 3.

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