Representations of the Quantized Corner Algebra in 4-Dimensional BF Theory

Master's Thesis by
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December 6, 2025

Post-Submission Version

This version of the Master's thesis contains minor corrections and additional comments that were implemented after the official submission date. For the original version, which was handed in to the ETH Zürich on September 11, 2025, please contact the author.[1]

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December 6, 2025

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Acknowledgments

I want to thank my supervisors Alberto Cattaneo, Giovanni Canepa and Filippo Fila-Robattino for their guidance and many fruitful discussions on this journey. Their incredible insight and dedication have been nothing short of inspiring. Furthermore, I want to thank Denis Nesterov for helpful comments regarding the literature of infinite-dimensional Lie algebras.

I am also very appreciative of the infrastructure and highly stimulating environment provided by the Institute of Theoretical Physics at ETH Zürich.

Finally, I want to thank my friends at ETH Zürich, specifically Eric Goldhahn, Matan Kaufmann, Danilo Lucic, Jan Obermeier, Somesh Vijayananda and my partner Ann-Sophie Frind. Their company made my final semester unforgettable. I am also extremely grateful for the continued encouragement and support from my family.

Contents

Abstract

The quantum BV-BFV framework [CMR17] has become an indispensable tool to perturbatively quantize gauge theories on manifolds with boundaries. It is also possible to extend the formalism to manifolds with corners, to which the QFT assigns an A_∞ algebra. The corner algebra then acts on state spaces that now satisfy a cutting and gluing formula. This fact allows one to classify state spaces by the representation theory of the algebra and to reduce the complexity of calculations tremendously. In this Master's thesis, we aim to develop explicit representations for the quantized corner structure of 4-dimensional BF theory on the sphere and the torus. In the abelian case, the modules of the corner algebra are described by the Heisenberg algebra with an infinite-dimensional center and are easily classified. In the non-abelian case, we construct modules of a larger algebra first and attempt to obtain the physical ones by an appropriate restriction. The larger algebra is characterized by an extension of a double-loop algebra over a non-semisimple Lie algebra. These algebras are not well studied, so we are prompted to develop new representation-theoretic methods. We introduce a modified version of the Verma module construction to develop explicit representations in terms of second-order differential operators on a space of polynomials. From a mathematical point of view, these representations are quite interesting and it would be worth studying the precise connection to free field realizations. Unfortunately, the representations do not descend to the physical algebra in a straightforward way, leaving some remaining questions open. In principle, one can now check the cutting and gluing formula explicitly.

Introduction

Quantum field theory (QFT) is the theoretical framework that describes the matter content of the universe and its interactions on a fundamental level. Notoriously, QFTs are extremely difficult to treat mathematically rigorously and to perform actual computations with. However, a major paradigm shift occurred when Atiyah [Ati88], Segal [Seg04] and others discovered the functorial aspects of QFTs. This new perspective allowed for a rigorous definition of topological quantum field theories (TQFTs)[2] and conformal field theories (CFTs) and unveiled an intimate connection between mathematical invariants of manifolds and physical theories [Wit89]. This framework and its extensions hint toward a universal structure present in all QFTs.

The rough idea in the functorial framework is the following: a d-dimensional (abbreviated by d-dim.) QFT should be a functor from a geometric category to an algebraic category that is compatible with the respective structure. Loosely speaking, to an appropriate $(d-1)$-dim. manifold[3] Σ, it assigns a vector space called the state space $\mathcal{H}(\Sigma)$ and to an appropriate d-dim. manifold M, it assigns the amplitude $Z(M) \in \mathcal{H}(\partial M)$ (or state, partition function). If $\partial M = \overline{\Sigma}_{\text{in}} \sqcup \Sigma_{\text{out}}$, one can show that it induces a linear map $Z(M)\colon \mathcal{H}(\Sigma_{\text{in}}) \longrightarrow \mathcal{H}(\Sigma_{\text{out}})$, which describes the evolution of the in-states to the out-states.[4] Functoriality then incorporates the principle of locality and properties of evolution. For a detailed explanation, consider [CR18; CMR17; Res10]. As an example, in the case where the QFT is topological, the source is the symmetric monoidal category of d-dim. bordisms Bord_d, the target is the symmetric monoidal category $\text{Vect}_{\mathbb{C}}$ and the functor is braided monoidal. In general, one can modify the category of bordisms to allow for more structure, such as a Riemannian metric, spin or conformal structure, etc. The target category is usually the category of vector spaces over $\mathbb{C}$, but there are other possibilities such as the category of super vector spaces or more abstract categories. Currently, however, there is no generally accepted mathematical definition of a QFT.

A step in this direction is achieved by the perturbative quantization scheme in the BV-BFV formalism (see Section 1) extended to manifolds with boundary [CMR17]. A big advantage of using the BV-BFV framework is its ability to perturbatively quantize a large class of theories, including gauge theories, by using standard constructions. Furthermore, it is compatible with cutting and gluing.[5] Therefore, it is capable of producing much of the data of these functors in terms of perturbative path integrals. In particular,

[2] TQFTs are types of QFTs and are called topological, because they are independent of the underlying metric-structure of space time. Furthermore, they exhibit no local degrees of freedom. This feature and the insensitivity to the local structure of spacetime makes them particularly tameable.

[3] We will generally assume that manifolds are compact, smooth and oriented.

[4] $\overline{\Sigma}$ denotes Σ but with the opposite orientation.

[5] See Footnote 8 and Formula (1.1).

the proposed extension of the formalism to corners promises to be very useful for two reasons. State spaces on $(d-1)$-dim. manifolds that bound a corner must constitute a representation of an associated corner algebra. Furthermore, cutting and gluing of $(d-1)$-dim. manifolds allows one to compute the state space on manifolds with complicated topology in terms of more elementary pieces. In this Master's thesis, we aim to construct representations of the quantized corner algebra in 4-dim. BF Theory.

In Section 1, we quickly review the BV-BFV formalism and sketch the perturbative quantization scheme. We then provide two examples, Chern-Simons theory and 4-dim. BF theory, of which we investigate the corner structure. In both cases, the quantized corner structure is essentially encoded in an infinite-dimensional Lie algebra. The first example illustrates the general idea of how we aim to obtain representations in an easy case. A main recurring theme will be to choose a basis of a Lie subalgebra made of finite Fourier modes. The second example is much harder and will be the main object of study of this thesis. In essence, we attempt to describe representations of a larger algebra and then take the quotient with respect to the quantized constraints. We end the section with a heuristic discussion of how a set of constraints can be quantized in our situation.

In Section 2, we investigate the corner Lie algebra of abelian BF on the torus. We state isomorphism theorems before and after reduction by the constraints. They are described by the infinite-dimensional oscillator (infinite-dimensional Heisenberg) algebra with an additional abelian summand. At the end, we suggest possible representations that are potential state spaces of the theory on a solid torus, for example.

In Section 3, we investigate the corner Lie algebra of non-abelian BF on the torus. The Lie algebra is classified as a central extension of a double-loop algebra over the isochronous Galilean Lie algebra. Unfortunately, there are few constructive results from the literature in this case, so we have to introduce a new construction. We modify the construction of the Verma modules to obtain representations in terms of second-order differential operators on a space of polynomials. These modules have nice properties such as a grading and a vacuum-like vector. Finally, we investigate whether these modules descend to the physical quotient algebra. After some technical details, we prove that they do not descend, as the constraints have a non-zero action. One can still hope to restrict to subrepresentations, but most likely, these modules are irreducible. In the last part of the section, we investigate the connection of BF to gravity on the corner.

In Section 4, we repeat the analysis of Section 2 but for the sphere. Instead of a basis made of finite Fourier modes, we use the Hodge decomposition and spherical harmonics. We prove two isomorphism results that are very similar to the abelian case on the torus.

In Section 5, we state the current standings of investigating non-abelian BF on the sphere. Due to a difficult integral, we are unable to completely characterize the Lie algebra.

1 Corner Structure in the BV-BFV Formalism

In this section, we briefly sketch the main idea of the perturbative quantization scheme in the BV-BFV formalism and its extension to corners. Afterwards, we look at two examples: Chern-Simons theory and 4-dim. BF theory, reviewing[6] [CC23]. The former example will illustrate how to obtain modules of the quantized corner algebra in a simple case. The latter example requires much more work and will be the main object of study of the thesis.

The BV formalism introduced in [BV83; BV81] is a framework to quantize complicated gauge theories, generalizing the Faddeev-Popov and BRST methods. Its main feature is to lift the gauge symmetry to an extended space of fields that carries a cohomological operator and an odd symplectic structure. This allows one to define the perturbative path integral using a robust gauge-fixing procedure and to determine observables in terms of the cohomology of the quantized operator. For details, consider e.g. the book [Mne19]. The BFV formalism introduced in [BF83; BV77] is the Hamiltonian counterpart to the BV formalism. It allows for a cohomological resolution of the reduced phase space which is much more robust and defined even in singular cases. Under mild assumptions, the BV formalism can be adapted to the case where the spacetime M has a boundary. The BV structure on the bulk then induces a compatible BFV structure on the boundary [CMR14; CMR11]. The formalism can be further extended to manifolds with strata of higher codimension (corners, etc.).

Let us make this more concrete.

Definition 1.1. A **BF^{k}V manifold**[7] for $k \in \mathbb{N}$ is a quadruple $(\mathcal{F}, \omega, S, Q)$, where

- $\mathcal{F}$ is a $\mathbb{Z}$-graded manifold
- $\omega \in \Omega^2(\mathcal{F})_{k-1}$ is a symplectic form of degree $k-1$
- $S \in C^\infty(\mathcal{F})_k$ is a function of degree k called the BF^{k}V action satisfying the Classical Master Equation (CME) $\{S, S\} = 0$, where $\{-,-\}$ is the Poisson bracket induced by ω
- $Q = \{S, -\}$ is a vector field of degree 1 which is cohomological, i.e. $Q^2 = 0$, by virtue of the CME

In the special case $k = 0$, we will write BV $=$ BF^{0}V. Furthermore, we will also sometimes call the BF^{k}V manifold a BF^{k}V structure instead. The fully extended clas-

[6] Except for Section 1.3.4, the results are not my work and only intended as a setup to the research question of the thesis.

[7] This definition is essentially the same as [Mne19, Def. 4.8.1.] with the convention that instead of the symplectic form ω having degree k, we take the action S to have degree k.

sical BV-BFV formalism applied to a d-dim. field theory assigns a BF^{k}V structure $(\mathcal{F}_N, \omega_N, S_N, Q_N)$ to every closed $(d-k)$-dim. manifold N.

Subsequently, in [CMR17], the authors introduced a general perturbative quantization scheme in the BV-BFV formalism that works on manifolds with boundaries and is compatible with cutting and gluing.[8] The formalism can be further extended to work on manifolds with corners such that it is compatible with cutting and gluing. The general theory of quantization with corners is still a work in progress, but a worked-out example discussing 2-dim. Yang-Mills theory can be found in [IM19].

1.1 The Quantum BV-BFV Formalism

In the following, we outline the rough idea of the perturbative quantization scheme on manifolds with boundaries and corners (see [CMR16, Section 2.3.]). A quantum BV-BFV package constructed from a given d-dim. field theory assigns :

- To a $(d-1)$-dim. closed manifold Σ, the state space $\mathcal{H}_\Sigma$ defined as the quantization of the the symplectic manifold $\mathcal{F}^\partial_\Sigma$, where one fixes a polarization $\mathcal{F}^\partial_\Sigma \xrightarrow{p} \mathcal{B}_\Sigma$. Since $\mathcal{F}^\partial_\Sigma$ has a symplectic structure, $\mathcal{H}_\Sigma$ can in principle be defined using geometric quantization techniques. Furthermore, $\mathcal{H}_\Sigma$ is assumed to carry a differential defined by the quantization $\Omega_\Sigma = \widehat{S}_\Sigma$ of the classical BFV action S_Σ.

- To a d-dim. manifold M with a boundary, the partition function
$$Z_M(b) = \int_{\mathcal{L}\subset\mathcal{F}_b} e^{\frac{i}{\hbar}S_M}\mu_M^{\frac{1}{2}} \in \mathcal{H}_{\partial M}$$
$\mathcal{F}_M$ denotes the classical space of BV fields[9] and S_M the BV action, $\mathcal{F}_b$ the preimage of $b \in \mathcal{B}_{\partial M}$ under the composed projection $\mathcal{F}_M \xrightarrow{\pi} \Phi_{\partial M} \xrightarrow{p} \mathcal{B}_{\partial M}$ and $\mathcal{L} \in \mathcal{F}_b$ a gauge-fixing Lagrangian submanifold. Lastly, $\mu_M^{\frac{1}{2}}$ is a reference half-density on $\mathcal{F}_M$. However, this construction is ill-defined in the presence of zero-modes and needs to be adapted accordingly, c.f. [CMR16, Section 2.3.].

The perturbative quantization scheme has been successfully applied in a wide variety of field theories like: BF theory and the Poisson sigma model [CMR17], Chern-Simons theory [CMW23], the relational symplectic groupoid [CMW17] and 2-dim. Yang-Mills theory [IM19].

One expects the framework to extend to the corner as follows. The quantum BV-BFV package from a given d-dim. field theory assigns, in addition to the standard package, the following data:

[8] This is meant in the following way: Given a decomposition $M = M_1 \sqcup M_2$ of spacetime, one can compute the partition function $Z(M)$ by knowing $Z(M_1)$ and $Z(M_2)$.

[9] In many cases, the field space is defined as the space of sections of an appropriate sheaf or bundle like a vector or spin bundle.

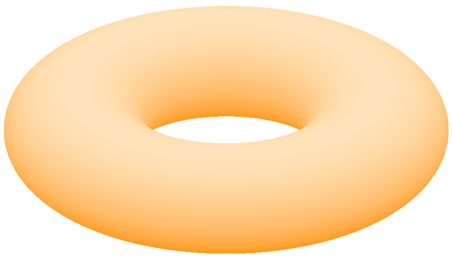
- To a closed $(d-2)$-dim. manifold Γ, an A_∞ algebra A_Γ defined by the deformation quantization of the classical P_∞ algebra P_Γ.[10] P_Γ is determined by the $\mathrm{BF}^2\mathrm{V}$ structure induced on the corner (c.f. 1.1) and a choice of polarization.
- To a $(d-1)$-dim. manifold Σ with a boundary, an action by A_Γ on the space of states $\mathcal{H}_\Sigma$ turning it into a module.

This is expected to be compatible with cutting and gluing in the following way: Given a decomposition $\Sigma = \Sigma_1 \sqcup \Sigma_2$ then the state spaces satisfy

$$\mathcal{H}_\Sigma \cong \mathcal{H}_{\Sigma_1} \otimes_{A_\Gamma} \mathcal{H}_{\Sigma_2} \,. \tag{1.1}$$

An example in which this is worked out for a theory is the paper [IM19], mentioned previously. The extension to corners is extremely useful for two reasons. Firstly, state spaces arising from $(d-1)$-dim. manifolds with a boundary are immediately restricted by the representation theory of the corner algebra. Secondly, the cutting and gluing formula (1.1) allows one to determine the state spaces of complicated $(d-1)$-dim. manifolds from more elementary pieces.

Example 1.2. *In a 4-dim. theory, an example of a corner of spacetime is the 2-dim. torus* $T^2 := S^1 \times S^1$.



In this example, the corner algebra A_{T^2} *acts on the state space* $\mathcal{H}_{\boldsymbol{T^2}}$ *associated to the 3-dim. solid torus* $\boldsymbol{T^2}$*, a 3-dim. manifold with boundary, for example. One can then glue two solid tori together along their common boundary and obtain the state space on the resulting lens space (c.f. [Rol00, Chapter 9.B]) using the gluing formula* (1.1).

This concludes the exposition on the quantization scheme using BV-BFV. Now, we turn to two examples, where we investigate the corner structure.

1.2 Example: Chern-Simons Theory

In this example, we want to outline some of the main ideas that are used to construct the quantized corner algebra and subsequently find representations thereof. To this end,

[10] In many cases, one can choose a suitable polarization such that the structure trivializes after the first two brackets, see [CC23, Section 3.3. & 3.4.] which are reviewed in 1.2.1 and 1.3.2.

we examine Chern-Simons theory with Lie algebra $\mathfrak{su}(2)$. Chern-Simons theory is the prototypical topological gauge theory. For an extensive discussion, see [Fre95]. We are, however, only interested in using it to highlight some constructions, so we refrain from a general discussion. First, we introduce the usual description on the bulk, assuming the spacetime has no boundary. We then specialize to the corner description, which was worked out in detail in [CC23] and quantize the associated affine Poisson algebra. Subsequently, we develop a convenient description of the corner algebra in terms of Fourier modes. It turns out that modules of the quantized corner algebra are given by representations of the affine Lie algebra $\widehat{\mathfrak{sl}}_2$, therefore providing possible state spaces on surfaces that have a circle as their boundary.

1.2.1 BF^{2}V Structure

Let M be a closed, oriented 3-manifold and fix a $SU(2)$-bundle P over M. From low-dim. homotopy theory, one knows that the bundle is automatically trivial. Thus, the space of connection 1-forms $\mathcal{A}_P$ is simply isomorphic to $\Omega^1(M) \otimes \mathfrak{su}(2)$. Let us also fix an invariant, non-degenerate inner product on $\mathfrak{su}(2)$ denoted by $(-,-)$. The action functional of Chern-Simons theory is then defined as

$$S_M = \frac{1}{2}\int_M (A \overset{\wedge}{,} \mathrm{d}A) + \frac{1}{3}(A \overset{\wedge}{,} [A \overset{\wedge}{,} A]) \,,$$

where $A \in \Omega^1(M) \otimes \mathfrak{su}(2)$ is a $\mathfrak{su}(2)$ connection 1-form called the gauge field. Furthermore, the notations $(- \overset{\wedge}{,} -)$ and $[- \overset{\wedge}{,} -]$ denote the wedge product of the underlying forms combined with the respective operations on the Lie algebra factor.[11] One can check that the integrand is indeed a differential 3-form and can be integrated over the orientable manifold M. From now on, we will consider the wedge product and Lie-algebra pairing to be implicit unless it is ambiguous.

We are interested in the corner algebra of this theory. To this end, we employ the BV-BFV formalism. We will closely follow [CC23] to examine the BF^{2}V structure. Let $\Gamma \cong S^1$, a potential corner of M, the corner data is then given by:

- The symplectic space of corner fields: $\mathcal{F}^{\partial\partial}_\Gamma = C^\infty(\Gamma)[1] \otimes \mathfrak{su}(2) \oplus \Omega^1(\Gamma) \otimes \mathfrak{su}(2)$
- The corner action: $S^{\partial\partial}_\Gamma = \int_\Gamma \frac{1}{2} c \mathrm{d}_A c$, where $(c, A) \in \mathcal{F}^{\partial\partial}_\Gamma$

The symbol $C^\infty(\Gamma)[1]$ denotes the 1-shifted[12] sheaf of functions and is called the space of ghosts for the gauge field. d_A denotes the exterior derivative twisted by the connection

[11] For more information on vector bundles, principal bundles and connections, consider [Tau11], for example.

[12] This essentially just means the functions have odd parity, meaning they anticommute. For more information on graded geometry in field theory see [Mne19, Section 4.2.2.].

1-form A, also called the exterior covariant derivative. As mentioned previously, the wedge product and pairing are left implicit. Note that the corner action has degree $+2$ in line with the BF^{2}V structure (c.f. Definition 1.1). To quantize, one possible polarization is $\mathcal{F}_\Gamma^{\partial\partial} \cong T^*[1]\mathcal{B}$, the shifted cotangent bundle of $\mathcal{B} := \Omega^1(\Gamma) \otimes \mathfrak{su}(2)$. The shifted cotangent bundle carries a canonical symplectic structure which, in particular, defines a Poisson bracket on $C^\infty(T^*[1]\mathcal{B})$. This Poisson algebra of functions can itself be canonically identified with the algebra of multivector fields on $\mathcal{B}$ [CC23, Section 2.2.]. The multivector field π for the corner action then defines a P_∞ structure on $\mathcal{B}$ by virtue of the Master equation. In the present case, one obtains a Poisson bivector field $\pi_2 = \int_\Gamma \left(\frac{1}{2} \frac{\delta}{\delta A} \mathrm{d} \frac{\delta}{\delta A} + \frac{1}{2} A \left[\frac{\delta}{\delta A}, \frac{\delta}{\delta A} \right] \right)$ on $\mathcal{B}$.[13] The action of the Poisson bivector field on linear functionals defines the bracket

$$\left\{ \int_\Gamma fA, \int_\Gamma gA \right\}_2 = \int_\Gamma (f \mathrm{d}g + [f,g]A) \,.$$

Using the Leibniz rule, one can extend the bracket to polynomial functionals. Our goal is now to quantize this affine Poisson structure (APS) and find modules for the resulting algebra. Recall that a linear Poisson structure (LPS) is equivalently described by the standard Lie-Poisson structure on the vector space $\mathcal{G}^*$, the dual of a Lie algebra $\mathcal{G}$.[14] On the other hand, an APS on $\mathcal{G}^*$ defines a 1-dim. central extension of $\mathcal{G}$ denoted by $\widehat{\mathcal{G}}$. Finally, the original APS induces a LPS on $\widehat{\mathcal{G}}^*$ [Bha90]. In our example, $\mathcal{G} = C^\infty(\Gamma) \otimes \mathfrak{su}(2)$ and the 2-cocyle defining the central extension is given by $c(f,g) = \int_\Gamma f \mathrm{d}g \, \mathrm{Z}$, for a basis Z of $\mathbb{R}$. From the theory of deformation quantization, one expects the universal enveloping algebra (UEA) $\mathcal{U}(\widehat{\mathcal{G}})$ to provide a quantization[15] of the classical Poisson algebra $C^\infty(\widehat{\mathcal{G}}^*) = C^\infty(\mathcal{B})$ [Gut83]. Finally, since the representation theory of a Lie algebra and its UEA are equivalent, we are left with the task of finding representations of $\widehat{\mathcal{G}}$. We also speak of the quantized corner algebra to refer to $\mathcal{U}(\widehat{\mathcal{G}})$ and $\widehat{\mathcal{G}}$.

[13] In this context, A is regarded as a coordinate on the latter. To make this statement precise, one can employ the variational bicomplex formalism, where d is the horizontal and δ the vertical differential. However, this is out of the scope of this work. For a rigorous treatment see [Chr24] and [Del+99], for example.

[14] This always holds in finite dimensions, but also works for the case at hand.

[15] Technically, we choose the subalgebra of polynomial functions to quantize. However, in infinite dimensions, the space of polynomials on $\widehat{\mathcal{G}}_\Lambda^*$ is strictly bigger than the space of symmetric tensors of $\widehat{\mathcal{G}}_\Lambda$. Therefore, we will actually quantize the latter. This is okay, because under some assumptions, one can recover those functions by finding a suitable topology on the quantization and taking the completion with respect to that topology.[ESW15, Section 2.2]

1.2.2 The Lie Algebra $\widehat{\mathcal{G}}(\mathfrak{su}(2))$

The Lie algebra describing the quantized corner structure $\mathfrak{su}(2)$ Chern-Simons theory on a circle is given by the vector space:

$$\widehat{\mathcal{G}}(\mathfrak{su}(2)) = C^\infty(S^1) \otimes \mathfrak{su}(2) \oplus \mathbb{R}\,,$$

with bracket

$$[f \oplus r, g \oplus s]_{\widehat{\mathcal{G}}_{(\mathfrak{su}(2))}} = [f, g] \oplus \frac{1}{2\pi i} \int_{S^1} f \mathrm{d}g\, \mathrm{Z}\,.$$

At this point, one can already guess that this Lie algebra will turn out to be an affine Lie algebra. However, in BF theory this is quite different, so we shall continue on with our analysis. Note that a rescaling of the 2-cocycle associated to the central extension produces isomorphic Lie algebras, because the extension only depends on the second Lie algebra cohomology group $H^2(\mathcal{G}(\mathfrak{su}(2)), \mathbb{R})$. Fix a basis $\{t_\mu\}_{1\leq\mu\leq 3}$ of $\mathfrak{su}(2)$, such that $(t_\mu, t_\nu) = \delta_{\mu,\nu}$ and $[t_\mu, t_\nu] = \varepsilon^\lambda_{\mu\nu} t_\lambda$, where $\varepsilon^\lambda_{\mu\nu}$ denotes the Levi-Civita symbol. Instead of the entire algebra, we will consider the Lie subalgebra of the complexification[16] $(\widehat{\mathcal{G}}(\mathfrak{su}(2)))_{\mathbb{C}}$ that consists of finite Fourier expansions

$$f = \sum_{\substack{1\leq\mu\leq 3 \\ m\in\mathbb{Z}}} f_{\mu m} \left(t_\mu \otimes e^{im\theta} \right), \quad \text{for } f_{\mu m} \in \mathbb{C}\,,$$

where θ is the coordinate[17] on the circle with orientation given by the volume form $\mathrm{d}\theta$ and with only finitely many non-zero coefficients in the sum. By abuse of notation, we denote the vector subspace by the same symbol and drop the complexification symbol. Now that the vector space $\widehat{\mathcal{G}}(\mathfrak{su}(2))$ has countable dimension, we can choose a suitable Hamel basis: $\mathrm{J}_{\mu m} \coloneqq t_\mu \otimes e^{im\theta}$. A quick computation yields the bracket relations:

$$\begin{aligned} [\mathrm{J}_{\mu m}, \mathrm{J}_{\nu n}] &= [t_\mu, t_\nu] \otimes e^{i(m+n)\theta} + \frac{1}{2\pi i} \int_{S^1} ((t_\mu, t_\nu) i n e^{i(m+n)\theta} \mathrm{d}\theta)\mathrm{Z} \\ &= \varepsilon^\lambda_{\mu\nu} \mathrm{J}_{\lambda m+n} + n \delta_{\mu,\nu} \delta_{m,-n} \mathrm{Z}\,, \end{aligned}$$

and all other brackets vanish. This is precisely the structure of the affine Lie algebra $\widehat{\mathfrak{sl}}_2$ without the derivation element. Its representation theory has been widely studied (see [KR87] for a starting point). Consequently, we could now produce possible state spaces for surfaces that have a circle as their boundary.

1.3 Example: BF Theory

BF theory is a topological field theory originally introduced in [BT91; Hor89] and will be the main focus of the present work. Its name stems from the shape of its action

[16] In the following work, we will only discuss complex representations.

[17] The coordinate function θ is not defined globally, but the smooth function $e^{im\theta}$ is.

functional (see (1.2)) that includes an adjoint-valued form B and the curvature F_A of a principal connection A. After briefly motivating why BF theory is interesting to study, we explore the bulk and subsequently the corner description of the 4-dimensional version, following the work of [CC23]. We end the example with a heuristic discussion of quantizing a Poisson submanifold defined by constraint functionals.

1.3.1 Motivation

Since its introduction, BF theory has been the subject of intense research. Similarly to other topological field theories, it can be used to define invariants of knots and manifolds, studied in [Bae96; Cat+95; CR01; CM94], for example. However, unlike other topological field theories, it can be defined consistently in any dimension. In dimension $d = 2$, it is related to $(1 + 1)$-dim. Yang-Mills theory. In dimension $d = 3$, it is a special case of Chern-Simons theory. If the Lie algebra defining BF (see (1.2)) is $\mathfrak{g} = \mathfrak{so}(1, 2)$ (or $\mathfrak{so}(3)$), it is related to $1 + 2$ (Euclidean) gravity with cosmological constant Λ[18] in the coframe formulation. In dimension $d = 4$, BF theory with $\mathfrak{g} = \mathfrak{so}(1, 3)$ (or $\mathfrak{so}(4)$) is related to $1 + 3$ (Euclidean) gravity with cosmological constant Λ in the coframe formalism in a very non-trivial way [Ple77]. The precise connection of the respective P_∞ structure describing the space of corner fields was investigated in [CC23]. One could, therefore, hope to infer state spaces for gravity from those constructed for 4-dim. BF theory.

1.3.2 BF^{2}V Structure

Let M be a closed, oriented 4-manifold, G a Lie group with Lie algebra $\mathfrak{g}$ that has a non-degenerate invariant inner product [19]. Fix a principal G-bundle P and denote $\mathcal{A}_P$ the space of connections. The action functional of BF theory with a cosmological term is then defined by

$$S_M = \int_M (B \overset{\wedge}{,} F_A) + \frac{\Lambda}{2}(B \overset{\wedge}{,} B)\,, \tag{1.2}$$

where $B \in \Omega^2(M, \mathrm{ad}P)$ is an adjoint-valued 2-form, F_A the curvature of the connection 1-form $A \in \mathcal{A}_P$ and $\Lambda \in \mathbb{R}$ the cosmological constant. Pure BF theory is the special case when $\Lambda = 0$. The integrand is indeed a differential 4-form and can be integrated over the orientable manifold M. As in the previous example, we will consider the wedge product and Lie-algebra pairing implicitly.

[18] In dimension 3 and 4, one can add a so-called cosmological term to the BF action (1.2). In this context, one speaks of the original theory as pure BF.

[19] In pure BF theory, this is not necessary, as one can define B to take values in the dual bundle instead.

To describe the corner algebra of this theory, we employ the BV-BFV formalism. We will closely follow [CC23] to define the BF^{2}V structure and subsequently the P_∞ structure. Their results for BF theory can be summarized as follows. Let Γ be a compact, oriented 2-manifold, and we assume that the pullback bundle $\mathcal{A}_{P|_\Gamma}$ is trivial for simplicity, meaning isomorphic to $\Omega^1(\Gamma)\otimes\mathfrak{g}$. The corner data is then given by

- The space of corner fields:

$$\mathcal{F}^{\partial\partial}_\Gamma := \left(\Omega^1(\Gamma)\oplus\Omega^2(\Gamma)\oplus\Omega^2[-1](\Gamma)\oplus\Omega^1[1](\Gamma)\oplus\Omega^0[1](\Gamma)\oplus\Omega^0[2](\Gamma)\right)\otimes\mathfrak{g}\,,$$

  where the square brackets denote the respective shifts and the coordinate functions are labeled $(A, B, B^+, c, \tau, \phi)$ respectively. In the BV-BFV formalism for BF theory, one has to include ghosts and ghosts-for-ghosts to take care of the gauge symmetry, explaining the additional fields (see [CC23] for details).

- The corner action:

$$S^{\partial\partial}_\Gamma := \int_\Gamma \left(\frac{1}{2}B[c,c] + \tau(\mathrm{d}_{A_0}c + [a,c]) + \phi\left(F_{A_0} + \mathrm{d}_{A_0}a + \frac{1}{2}[a,a] + [c,B^+]\right)\right.$$
$$\left. + \Lambda\left(\frac{1}{2}\tau\tau + B\phi\right)\right)\,,$$

  where we split $a := A - A_0$ and d_{A_0} is the exterior derivative twisted by the reference connection A_0. This splitting is mostly relevant for a non-trivial bundle, but to match notation with [CC23], we keep it that way.

To find a suitable quantization, one should choose a polarization. One possible choice is to realize $\mathcal{F}^{\partial\partial}_\Gamma = T^*[1]\mathcal{B}$, where $\mathcal{B} := \mathcal{F}^{\partial\partial}_\Gamma|_{c=\phi=\tau=0}$. The corresponding multivector field of the corner action is $\pi = \pi_1 + \pi_2$, where

$$\pi_1 = \int_\Gamma (F_A + \Lambda B)\frac{\delta}{\delta B^+}\,,$$
$$\pi_2 = \int_\Gamma \left(\frac{1}{2}B\left[\frac{\delta}{\delta B},\frac{\delta}{\delta B}\right] + \frac{\delta}{\delta a}\mathrm{d}_{A_0}\frac{\delta}{\delta B} + a\left[\frac{\delta}{\delta a},\frac{\delta}{\delta B}\right] + B^+\left[\frac{\delta}{\delta B^+},\frac{\delta}{\delta B}\right] + \frac{1}{2}\Lambda\frac{\delta}{\delta a}\frac{\delta}{\delta a}\right)\,.$$

In other words, it defines a differential graded, affine Poisson algebra. Ultimately, we are interested in the degree-0 cohomology induced by the differential π_1, because it describes the physical part of the corner fields. The degree-0 cohomology of π_1 is given by the degree zero part of the corner fields, namely $C^\infty(\mathcal{B}_0)$, where $\mathcal{B}_0 := \Omega^1(\Gamma)\otimes\mathfrak{g}\oplus\Omega^2(\Gamma)\otimes\mathfrak{g}$ modulo the Poisson ideal generated by $\int_\Gamma f(F_A + \Lambda B)$ for $f\in\Omega^0(\Gamma)\otimes\mathfrak{g}$. Therefore, we will focus on describing the degree zero part $\mathcal{B}_0$ and quotient by the constraints. The space $\mathcal{B}_0$ is a Poisson submanifold with brackets defined on linear functionals by

$$\left\{\int_\Gamma \alpha a, \int_\Gamma \beta a\right\}_2 = \Lambda\int_\Gamma \alpha\beta\,,$$

$$\left\{\int_\Gamma \alpha a, \int_\Gamma fB\right\}_2 = \int_\Gamma (\alpha \mathrm{d}_{A_0} f + [\alpha, f]a)\,,$$
$$\left\{\int_\Gamma fB, \int_\Gamma gB\right\}_2 = \int_\Gamma [f,g]B\,.$$

Since $\mathcal{B}_0$ is a vector space and the Poisson structure is affine, we can equivalently describe it by the dual $\widehat{\mathcal{G}}_\Lambda^*$ of a Lie algebra $\mathcal{G}_\Lambda$ with central extension $\widehat{\mathcal{G}}_\Lambda$ defined by the 2-cocycle $c_\Lambda(\alpha \oplus f, \beta \oplus g) = \int_\Gamma (\alpha \mathrm{d}_{A_0} g - \beta \mathrm{d}_{A_0} f + \Lambda \alpha\beta)\mathrm{Z}$. From the theory of deformation quantization, one expects the UEA $\mathcal{U}(\widehat{\mathcal{G}}_\Lambda)$ to provide a quantization (see Footnote 15).

1.3.3 The Lie Algebra $\widehat{\mathcal{G}}_\Lambda$

Thus, the Lie algebra describing the quantized corner structure of 4-dim. BF theory on Γ is given by the vector space

$$\widehat{\mathcal{G}}_\Lambda = \Omega^0(\Gamma) \otimes \mathfrak{g} \oplus \Omega^1(\Gamma) \otimes \mathfrak{g} \oplus \mathbb{R} \tag{1.3}$$

with brackets

$$[f \oplus \alpha \oplus r, g \oplus \beta \oplus s]_{\widehat{\mathcal{G}}_\Lambda} = [f,g] \oplus (\mathrm{ad}_f \beta - \mathrm{ad}_g \alpha) \oplus \frac{-1}{(2\pi)^2} \int_\Gamma (\alpha \mathrm{d}_{A_0} g - \beta \mathrm{d}_{A_0} f + \Lambda \alpha\beta)\mathrm{Z}\,, \tag{1.4}$$

where $\mathrm{ad}_f \alpha \coloneqq [f, \alpha]$ stands for the 1-form produced by the pointwise commutator. However, one still somehow needs to take the quotient with respect to the constraints. In the following section, we argue that the Poisson ideal $I_{F_A + \Lambda B}$ in $C^\infty(\mathcal{B}_0)$ induces a two-sided ideal $\mathcal{I}_{F_A + \Lambda B}$ in the UEA $\mathcal{U}(\widehat{\mathcal{G}}_\Lambda)$ (technically in an appropriate completion of the UEA). The physically relevant algebra describing the quantization of the physical space of corner fields is therefore given heuristically by the quotient $\mathcal{U}(\widehat{\mathcal{G}}_\Lambda)|_{F_A + \Lambda B = 0} \coloneqq \mathcal{U}(\widehat{\mathcal{G}}_\Lambda) / \mathcal{I}_{F_A + \Lambda B}$. We will refer to this as the physical quotient algebra. One expects that the quantized corner algebra acts on the state spaces, so we are interested in finding its representations. Physically admissible representations should be those that descend to the quotient and provide examples of possible state spaces in 4-dim. BF theory and, hopefully, also gravity.

1.3.4 Constraint $F_A + \Lambda B = 0$

To describe the physical space of fields on the corner, one has to consider the cohomology induced by π_1 in degree 0, or equivalently, reduce to the Poisson submanifold $P_\Gamma \coloneqq \{(A, B) \in \mathcal{B}_0 | F_A + \Lambda B = 0\}$. This submanifold can be obtained by taking the quotient with respect to the Poisson ideal generated by the constraint functionals defined by $\widehat{f} \coloneqq \int_\Gamma f(F_A + \Lambda B)$. Their Poisson bracket with the linear functionals is the following:

$$\left\{\int_\Gamma f(F_A + \Lambda B), \int_\Gamma \alpha a\right\} = 0\,,$$

$$\left\{\int_\Gamma f(F_A+\Lambda B), \int_\Gamma gB\right\}_2 = \int_\Gamma [f,g](F_A+\Lambda B)\,.$$

These relations show that the constraints indeed generate a Poisson ideal, which we denote by $I_{F_A+\Lambda B}$. Therefore, one has the isomorphism of Poisson algebras.

$$C^\infty(P_\Gamma) \cong {C^\infty(\mathcal{B}_0)}\big/{I_{F_A+\Lambda B}}\,.$$

The question we are aiming to answer is essentially the following: How is a Poisson submanifold or a quotient quantized with respect to the ambient Poisson manifold? This is a difficult question, but has been answered in [Cat08, Section 7.5.] and [CF07] if there are no anomalies. However, this approach involves heavy technical machinery and is difficult to describe explicitly. Instead, we will take a less sophisticated approach. To understand what requirements we need to impose on the corresponding quantization, i.e. the UEA $\mathcal{U}(\widehat{\mathcal{G}}_\Lambda)$, we make the following observation: $\widehat{f}$ is a polynomial in the fields, i.e. on $\widehat{\mathcal{G}}^*_\Lambda$. In other words:

$$\widehat{f} \in (\widehat{\mathcal{G}}^*_\Lambda)^* \oplus \left(\widehat{\mathcal{G}}^*_\Lambda \otimes \widehat{\mathcal{G}}^*_\Lambda\right)^*\,. \tag{1.5}$$

Heuristically, $\widehat{f} \in \widehat{\mathcal{G}}_\Lambda \oplus \mathrm{Sym}^2(\widehat{\mathcal{G}}_\Lambda)$, which could in principle be made rigorous for some appropriate completion of the tensor product. By the linear symmetrization isomorphism

$$\mathrm{Sym}^\bullet(\widehat{\mathcal{G}}_\Lambda) \cong \mathcal{U}(\widehat{\mathcal{G}}_\Lambda),$$

we can expect this functional to describe an element in (the completion of) the UEA of $\widehat{\mathcal{G}}_\Lambda$. These elements will also generate an ideal, denoted $\mathcal{I}_{F_A+\Lambda B}$, in the corresponding UEA, as we will see. Finally, we expect the physical quantized corner algebra to be the quotient of the UEA with respect to the generated ideal.

To summarize, we expect the vanishing ideal $I_{F_A+\Lambda B}$ defining the Poisson submanifold P_Γ, to get quantized by an ideal $\mathcal{I}_{F_A+\Lambda B}$ in the UEA. Said ideal can then be used to (heuristically) define the physical corner algebra by the quotient

$$A_\Gamma \equiv \mathcal{U}(\widehat{\mathcal{G}}_\Lambda)|_{F_A+\Lambda B=0} := {\mathcal{U}(\widehat{\mathcal{G}}_\Lambda)}\big/{\mathcal{I}_{F_A+\Lambda B}}\,.$$

In the abelian case, i.e. $\mathfrak{g}=\mathbb{R}$, the ideals of constraints $\mathcal{I}_{\mathrm{d}A}$ & $\mathcal{I}_{\mathrm{d}A+\Lambda B}$ are a subset of the center of the Lie algebra $\widehat{\mathcal{G}}$ & $\widehat{\mathcal{G}}_\Lambda$ respectively. This special case makes the selection of physically admissible representations much easier.

1.3.5 Goal of the Thesis

Thus, our course is set. We are aiming to determine or construct representations of the infinite-dim. corner Lie algebra $\widehat{\mathcal{G}}_\Lambda$ and select those that descend to the physical quotient.

To obtain some concrete description, we investigate the cases where $\Gamma \cong T^2$ and $\Gamma \cong S^2$ for the Lie algebras $\mathfrak{g} = \mathbb{R}$ and $\mathfrak{g} = \mathfrak{su}(2)$.[20] We denote the corner Lie algebra by $\widehat{\mathcal{G}}_\Lambda$ and $\widehat{\mathcal{G}}_\Lambda(\mathfrak{su}(2))$ respectively to emphasize the choice. For $\Lambda = 0$, the Lie algebra is denoted by $\widehat{\mathcal{G}}$. At the end, we briefly consider $\mathfrak{so}(1,3)$ on the torus and discuss the connections to gravity.

[20]A full discussion of non-abelian BF on the sphere is still work in progress.

2 Abelian BF on $\Gamma \cong T^2$

In this section, we explore the quantized corner algebra of 4-dim. abelian BF on a torus. First, to make the Lie algebra more manageable, we reduce to the Lie subalgebra generated by finite Fourier expansions and work out the bracket relations. The Lie algebras for a zero and non-zero cosmological constant turn out to be isomorphic to a known infinite-dim. Lie algebra. After classifying the Lie algebras, we discuss the consequence of imposing the on-shell constraints. Finally, we provide an example of a representation that realizes a possible space of states for a compact, oriented 3-manifold Σ (for example, a solid torus) such that $\partial\Sigma \cong \Gamma$.

2.1 The Lie Algebra $\widehat{\mathcal{G}}_\Lambda$

The Lie algebra describing 4-dim. abelian BF on a torus is given by the vector space:

$$\widehat{\mathcal{G}}_\Lambda = \Omega^0(T^2) \oplus \Omega^1(T^2) \oplus \mathbb{R}\,,$$

with brackets

$$[f \oplus \alpha \oplus r, g \oplus \beta \oplus s]_{\widehat{\mathcal{G}}_\Lambda} = -\frac{1}{(2\pi)^2}\int_{T^2}(\alpha \mathrm{d}g - \beta \mathrm{d}f + \Lambda\alpha\beta)\mathrm{Z}\,, \tag{2.1}$$

where $\Lambda \in \mathbb{R}$ is the cosmological constant and Z the central charge. This is obtained from (1.3) and (1.4) by setting $\mathfrak{g} = \mathbb{R}$ respectively.

Following what we have done for Chern-Simons theory in Section 1.2.1, we will again consider the Lie subalgebra of the complexification $(\widehat{\mathcal{G}}_\Lambda)_\mathbb{C}$ defined by finite Fourier modes. These elements are of the form

$$\alpha = \sum_{m,n\in\mathbb{Z}} \alpha^{(\theta)}_{mn} e^{im\theta} e^{in\varphi}\mathrm{d}\theta + \alpha^{(\varphi)}_{mn} e^{im\theta} e^{in\varphi}\mathrm{d}\varphi\,, \quad \alpha^{(\theta)}_{mn}, \alpha^{(\varphi)}_{mn} \in \mathbb{C}\,,$$
$$f = \sum_{m,n\in\mathbb{Z}} f_{mn} e^{im\theta} e^{in\varphi}\,, \qquad f_{mn} \in \mathbb{C}\,,$$

where (θ, φ) are coordinates[21] on the torus with orientation given by the volume form $\mathrm{d}\theta \wedge \mathrm{d}\varphi$ and with only finitely many non-zero coefficients. The space of differential 1-forms $\Omega^1(T^2)$ is a $C^\infty(T^2)$-module. It has a basis given by $\{\mathrm{d}\theta, \mathrm{d}\varphi\}$.

With a slight abuse of notation, we denote the Lie subalgebra by the same symbol and drop the complexification symbol. Now that the vector space $\widehat{\mathcal{G}}_\Lambda$ has countable dimension, we can choose a suitable basis: $\mathrm{E}_{mn} \coloneqq e^{im\theta}e^{in\varphi}$, $\Phi_{mn} \coloneqq e^{im\theta}e^{in\varphi}\mathrm{d}\varphi$ and $\Theta_{mn} \coloneqq e^{im\theta}e^{in\varphi}\mathrm{d}\theta$ for $m, n \in \mathbb{Z}$.

From now on, the summation convention will be adopted unless there is a chance of confusion. Furthermore, we drop the subscript of the Lie bracket in (2.1). A quick computation yields the following lemma.

[21]The coordinate functions θ and φ are not defined globally, but the induced 1-forms $\mathrm{d}\theta$ and $\mathrm{d}\varphi$ are.

Lemma 2.1. *The brackets in the finite Fourier mode algebra are given by:*

$$[\mathrm{E}_{kl}, \Phi_{mn}] = im\delta_{k,-m}\delta_{l,-n}\mathrm{Z}\,, \tag{2.2}$$

$$[\mathrm{E}_{kl}, \Theta_{mn}] = -in\delta_{k,-m}\delta_{l,-n}\mathrm{Z}\,, \tag{2.3}$$

$$[\Phi_{kl}, \Theta_{mn}] = \Lambda\delta_{k,-m}\delta_{l,-n}\mathrm{Z}\,, \tag{2.4}$$

and all other brackets vanish.

Proof. The proof of the bracket relations is straightforward. Evaluating the bracket on the basis elements and remembering the chosen orientation on T^2 yields:

$$\begin{aligned}
[\mathrm{E}_{kl}, \Phi_{mn}] &= \frac{1}{(2\pi)^2}\int_{T^2} \Phi_{mn}\mathrm{d}\mathrm{E}_{kl}\mathrm{Z} \\
&= -\frac{1}{(2\pi)^2}\left(\int_{T^2} ike^{i(m+k)\theta}e^{i(n+l)\varphi}\mathrm{d}\theta\wedge\mathrm{d}\varphi\right)\mathrm{Z} \\
&= im\delta_{k,-m}\delta_{l,-n}\mathrm{Z} \\
[\mathrm{E}_{kl}, \Theta_{mn}] &= \frac{1}{(2\pi)^2}\int_{T^2} \Theta_{mn}\mathrm{d}\mathrm{E}_{kl}\mathrm{Z} \\
&= \frac{1}{(2\pi)^2}\left(\int_{T^2} ile^{i(m+k)\theta}e^{i(n+l)\varphi}\mathrm{d}\theta\wedge\mathrm{d}\varphi\right)\mathrm{Z} \\
&= -in\delta_{k,-m}\delta_{l,-n}\mathrm{Z} \\
[\Phi_{kl}, \Theta_{mn}] &= -\frac{1}{(2\pi)^2}\int_{T^2} \Lambda\Phi_{kl}\Theta_{mn}\mathrm{Z} \\
&= \frac{1}{(2\pi)^2}\left(\int_{T^2} \Lambda e^{i(k+m)\theta}e^{i(l+n)\varphi}\mathrm{d}\theta\wedge\mathrm{d}\varphi\right)\mathrm{Z} \\
&= \Lambda\delta_{k,-m}\delta_{l,-n}\mathrm{Z}
\end{aligned}$$

□

Next, we would like to classify this Lie algebra for zero and non-zero cosmological constant.

2.2 Classification of $\widehat{\mathcal{G}}$ & $\widehat{\mathcal{G}}_\Lambda$

The following theorem establishes a connection of $\widehat{\mathcal{G}}$ and $\widehat{\mathcal{G}}_\Lambda$ with a known infinite-dim. Lie algebra.

Theorem 2.2. *There is an isomorphism of Lie algebras:*

$$\widehat{\mathcal{G}} \cong \widehat{\mathcal{G}}_\Lambda \cong \mathcal{A}\oplus\mathfrak{a}\,,$$

where $\mathcal{A}$ is the infinite-dim. oscillator algebra (or infinite-dim. Heisenberg algebra) and $\mathfrak{a}$ is the countably infinite-dim. abelian Lie algebra.[22]

[22]Essentially, just the $\mathbb{C}$-vector space of countably-infinite dimension, which is unique up to isomorphism.

Proof. The proof consists of redefining the generators such that they satisfy the generic oscillator algebra relations. Since (2.4) depends on Λ, the exact transformations differ for zero and non-zero cosmological constant, so we differentiate between two cases.

Case $\Lambda = 0$ Observe that both the Θ-generators and Φ-generators couple in the same way (up to scaling) to the E-generators. We can therefore define two new families of generators by normalizing appropriately and taking the sum and the difference of the original generators. The new family arising from the sum behaves the same as the original up to a scaling factor and is adorned with a check. The second family arising from the difference now has trivial bracket with all other generators and is therefore part of the center. These generators will be adorned with a hat. This observation is made precise with the following definitions:

$$
\begin{aligned}
\mathrm{F}^{\pm}_{kl} &:= \frac{1}{2}\left(-\frac{1}{l}\Theta_{kl} \pm \frac{1}{k}\Phi_{kl}\right) && k \neq 0, l \neq 0\,,\\
\check{\Phi}_k &:= \frac{1}{k}\Phi_{k0} && k \neq 0\,,\\
\widehat{\Phi}_l &:= \Phi_{0l} && l \in \mathbb{Z}\,,\\
\check{\Theta}_l &:= -\frac{1}{l}\Theta_{0l} && l \neq 0\,,\\
\widehat{\Theta}_k &:= \Theta_{k0} && k \in \mathbb{Z}\,,\\
\widehat{\mathrm{E}} &:= \mathrm{E}_{00}\,.
\end{aligned}
$$

Computing the bracket relations of the new generators using Equations (2.2)-(2.4), one obtains:

$$
\begin{aligned}
[\mathrm{E}_{kl}, \mathrm{F}^{+}_{mn}] &= i\delta_{k,-m}\delta_{l,-n}\mathrm{Z},\\
[\mathrm{E}_{0l}, \check{\Theta}_n] &= i\delta_{l,-n}\mathrm{Z},\\
[\mathrm{E}_{k0}, \check{\Phi}_m] &= i\delta_{k,-m}\mathrm{Z},
\end{aligned}
$$

and zero otherwise. To make the connection to the oscillator algebra manifest, we can denote the non-central generators:

$$
\begin{aligned}
\mathrm{a}^{\dagger}_l &:= \check{\Theta}_l && l \neq 0\,,\\
\mathrm{b}^{\dagger}_k &:= \check{\Phi}_k && k \neq 0\,,\\
\mathrm{c}^{\dagger}_{kl} &:= \mathrm{F}^{+}_{kl} && k \neq 0, l \neq 0\,,\\
\mathrm{a}_l &:= \mathrm{E}_{0-l} && l \neq 0\,,\\
\mathrm{b}_k &:= \mathrm{E}_{-k0} && k \neq 0\,,\\
\mathrm{c}_{kl} &:= \mathrm{E}_{-k-l} && k \neq 0, l \neq 0\,.
\end{aligned}
$$

The final relations are the following:

$$[\mathrm{a}_l, \mathrm{a}_l^\dagger] = [\mathrm{b}_k, \mathrm{b}_k^\dagger] = [\mathrm{c}_{kl}, \mathrm{c}_{kl}^\dagger] = i\mathrm{Z}\,,$$

and zero otherwise.[23] The abelian summand is spanned by the central elements excluding Z, i.e. $\mathfrak{a} := \mathrm{span}_{\mathbb{C}}\Big(\Big\{\mathrm{F}^-_{kl}, \widehat{\Phi}_n, \widehat{\Theta}_m, \widehat{\mathrm{E}}\,\Big|\, k,l,m,n \in \mathbb{Z} \text{ and } k,l \neq 0\Big\}\Big)$. The resulting Lie algebra is isomorphic to the direct sum $\mathcal{A} \oplus \mathfrak{a}$.

Case $\Lambda \neq 0$ Now the Θ-generators and Φ-generators couple non-trivially in (2.4). However, we can do a similar transformation to separate the generators into central- and oscillator-type elements. In terms of the generators for the $\Lambda = 0$ case, one can come up with:

$$
\begin{aligned}
\mathrm{u}_l^\dagger &:= \mathrm{a}_l^\dagger && l \neq 0\\
\mathrm{v}_k^\dagger &:= \mathrm{b}_k^\dagger && k \neq 0\\
\mathrm{w}_{kl}^\dagger &:= \mathrm{c}_{kl}^\dagger && k,l \neq 0\\
\mathrm{u}_l &:= \frac{1}{2}\left(\frac{l}{i\Lambda}\widehat{\Phi}_{-l} + \mathrm{a}_l\right) && l \neq 0\\
\mathrm{v}_k &:= \frac{1}{2}\left(\frac{k}{i\Lambda}\widehat{\Theta}_{-k} + \mathrm{b}_k\right) && k \neq 0\\
\mathrm{w}_{kl} &:= -\frac{1}{2}\left(\frac{2kl}{i\Lambda}\mathrm{F}^-_{-k-l} + \mathrm{c}_{kl}\right) && k,l \neq 0\\
\widehat{\mathrm{u}}_l &:= \frac{1}{2}\left(\frac{l}{i\Lambda}\widehat{\Phi}_{-l} - \mathrm{a}_l\right) && l \neq 0\\
\widehat{\mathrm{v}}_k &:= \frac{1}{2}\left(\frac{k}{i\Lambda}\widehat{\Theta}_{-k} - \mathrm{b}_k\right) && k \neq 0\\
\widehat{\mathrm{w}}_{kl} &:= \frac{1}{2}\left(\frac{2kl}{i\Lambda}\mathrm{F}^-_{-k-l} - \mathrm{c}_{kl}\right) && k,l \neq 0\\
\bar{\Phi} &:= \frac{1}{i\Lambda}\widehat{\Phi}_0\\
\bar{\Theta} &:= \widehat{\Theta}_0
\end{aligned}
$$

The final relations are the following:

$$[\mathrm{u}_l, \mathrm{u}_l^\dagger] = [\mathrm{v}_k, \mathrm{v}_k^\dagger] = [\mathrm{w}_{kl}, \mathrm{w}_{kl}^\dagger] = [\bar{\Theta}, \bar{\Phi}] = i\mathrm{Z}\,,$$

and zero otherwise. The abelian summand is spanned by the central elements $\widehat{\mathrm{w}}_{kl}, \widehat{\mathrm{u}}_l, \widehat{\mathrm{v}}_k$ and $\widehat{\mathrm{E}}$ the same as before. Again, the resulting Lie algebra is isomorphic to the direct sum $\mathcal{A} \oplus \mathfrak{a}$ and as such, also isomorphic to the Lie algebra for $\Lambda = 0$. □

[23] Note that we could absorb the i-factor by a redefinition of generators, however, for later convenience, we leave it as is.

By Dixmier's Lemma, any central element, say $\widehat{\mathrm{E}}$ in a countably infinite-dim. Lie algebra acts by a multiple of the identity on an irreducible representation. In this case, we denote the proportionality factor by $\chi_{\widehat{\mathrm{E}}} \in \mathbb{C}$ and call it a charge.

Next, we want to understand the appropriate reduction necessary to impose the constraints.

2.3 Constraints $\mathrm{d}A = 0$ & $\mathrm{d}A + \Lambda B = 0$

The constraints simplify a lot in the abelian theory. In particular, the constraints (2.5) and (2.6) are linear and central in the Poisson algebra and thus define central elements in the Lie algebra. These elements, $\mathcal{I}_{\mathrm{d}A} \subset Z(\widehat{\mathcal{G}})$ and $\mathcal{I}_{\mathrm{d}A+\Lambda B} \subset Z(\widehat{\mathcal{G}}_\Lambda)$, quantize the constraints (recall the heuristics in Section 1.3.4 and see Proof 2.3). One can form the quotient Lie algebras $\widehat{\mathcal{G}}|_{\mathrm{d}A=0} := \widehat{\mathcal{G}}/\mathcal{I}_{\mathrm{d}A}$ and $\widehat{\mathcal{G}}_\Lambda|_{\mathrm{d}A+\Lambda B=0} := \widehat{\mathcal{G}}_\Lambda/\mathcal{I}_{\mathrm{d}A+\Lambda B}$ in a straightforward way and they are characterized by the following proposition:

Proposition 2.3. *There is an isomorphism of Lie algebras:*

$$\widehat{\mathcal{G}}|_{\mathrm{d}A=0} \cong \mathcal{A} \oplus \mathbb{C}^3, \quad \widehat{\mathcal{G}}_\Lambda|_{\mathrm{d}A+\Lambda B=0} \cong \mathcal{A}\,.$$

Proof.

Case $\Lambda = 0$ The constraint ideal in the classical Poisson algebra is generated by the functionals:

$$\hat{f} := \int_\Gamma f \mathrm{d}A = -\int_\Gamma (\mathrm{d}f)A\,, \tag{2.5}$$

for any smooth function $f \in C^\infty(M)$. But these functionals correspond precisely to the elements $\mathrm{d}f \in \widehat{\mathcal{G}}$. Furthermore, any closed 1-form α, and in particular exact ones, must have trivial bracket with the other functionals. This follows directly from the definition of the bracket in (2.1) which only depends on $\mathrm{d}\alpha$ via Stokes' Theorem. Therefore, the set of constraints lies in the center: $\mathcal{I}_{\mathrm{d}A} := \{\mathrm{d}f | f \in C^\infty(M)\} \subset Z(\widehat{\mathcal{G}})$, where $Z(\widehat{\mathcal{G}})$ denotes the center. We can examine the constraints explicitly using the Lie subalgebra of Fourier modes that we have examined in the previous section.

In the usual basis, the constraints imply the following for all $k, l \in \mathbb{Z}$:

$$\begin{aligned} 0 &\overset{!}{=} \mathrm{dE}_{kl} \\ &= il\Phi_{kl} + ik\Theta_{kl} \\ &= \begin{cases} -2ikl\mathrm{F}^-_{kl}\,, & \text{for } k \neq 0, l, \neq 0 \\ il\widehat{\Phi}_l\,, & \text{for } k = 0, l \neq 0, \\ ik\widehat{\Theta}_k\,, & \text{for } k \neq 0, l = 0, \\ 0\,, & \text{for } k = l = 0\,. \end{cases} \end{aligned}$$

Taking the quotient with respect to the span of these elements, one is left with:

$$\widehat{\mathcal{G}}|_{\mathrm{d}A=0} \cong \mathcal{A} \oplus \mathbb{C}^3,$$

which describes the quantization of the degree zero cohomology of corner fields in abelian BF for $\Lambda = 0$. In other words, we are left with the Lie algebra spanned by the generators $\mathrm{a}_l^\dagger, \mathrm{a}_l, \mathrm{b}_k^\dagger, \mathrm{b}_k, \mathrm{c}_{kl}^\dagger, \mathrm{c}_{kl}, \mathrm{Z}, \widehat{\Theta}_0, \widehat{\Phi}_0, \widehat{\mathrm{E}}$, where the latter three span the abelian summand.

Remark 2.4. It makes sense that the center (ignoring the extension) is 3-dim. Before the quotient, the center consisted of constant functions and closed 1-forms. By taking the quotient, we essentially obtain the corresponding cohomology groups of the torus, i.e. $\mathbb{R}^3 \cong H^0(T^2, \mathbb{R}) \oplus H^1(T^2, \mathbb{R})$ ignoring the complexification.

To summarize, restricting to the submanifold $\mathrm{d}A = 0$ manifests itself in the quantization as selecting the irreducible representations of $\widehat{\mathcal{G}}$, where the charges $\chi_{\mathrm{F}_{kl}^-} = \chi_{\widehat{\Phi}_l} = \chi_{\widehat{\Theta}_k} = 0$ for $k \neq 0, l \neq 0$ are set to zero.

Case $\Lambda \neq 0$ Similarly to the $\Lambda = 0$ case, the constraints amount to setting most charges in $\mathfrak{a}$ to zero. The constraint ideal is generated by the functionals:

$$\widehat{f} := \int_\Gamma f(\mathrm{d}A + \Lambda B) = -\int_\Gamma (\mathrm{d}f)A + \int_\Gamma \Lambda f B \tag{2.6}$$

And therefore, we just need to set the combination of $\mathrm{d}f - \Lambda f \in \widehat{\mathcal{G}}_\Lambda$ to zero. The set of constraints $\mathcal{I}_{\mathrm{d}A+\Lambda B}$ again lies in the center for a similar reason. In the usual basis, the constraints imply the following for all $k, l \in \mathbb{Z}$:

$$\begin{aligned}
0 &\stackrel{!}{=} \mathrm{dE}_{kl} - \Lambda \mathrm{E}_{kl} \\
&= ik\Theta_{kl} + il\Phi_{kl} - \Lambda \mathrm{E}_{kl} \\
&= \begin{cases} -2ikl\mathrm{F}_{kl}^- - \Lambda \mathrm{c}_{-k-l}\,, & \text{for } k \neq 0, l, \neq 0 \\ il\widehat{\Phi}_l - \Lambda \mathrm{a}_{-l}\,, & \text{for } k = 0, l \neq 0, \\ ik\widehat{\Theta}_k - \Lambda \mathrm{b}_{-k}\,, & \text{for } k \neq 0, l = 0, \\ -\Lambda\widehat{\mathrm{E}}\,, & \text{for } k = l, = 0, \end{cases} \\
&= \begin{cases} 2\Lambda\widehat{\mathrm{w}}_{-k-l}\,, & \text{for } k \neq 0, l \neq 0, \\ 2\Lambda\widehat{\mathrm{u}}_{-l}\,, & \text{for } k = 0, l \neq 0, \\ 2\Lambda\widehat{\mathrm{v}}_{-k}\,, & \text{for } k \neq 0, l = 0, \\ -\Lambda\widehat{\mathrm{E}}\,, & \text{for } k = l = 0\,. \end{cases}
\end{aligned}$$

Taking the quotient with respect to the span of these elements, one is left with:

$$\widehat{\mathcal{G}}_\Lambda|_{\mathrm{d}A+\Lambda B=0} \cong \mathcal{A}\,,$$

which describes the quantization of the degree zero cohomology of corner fields in abelian BF for $\Lambda \neq 0$. In other words, we are left with the oscillator Lie algebra spanned by the generators $\mathrm{u}_l^\dagger, \mathrm{u}_l, \mathrm{v}_k^\dagger, \mathrm{v}_k, \mathrm{w}_{kl}^\dagger, \mathrm{w}_{kl}, \bar{\Theta}, \bar{\Phi}, \mathrm{Z}$.

To summarize, restricting to the submanifold $\mathrm{d}A+\Lambda B = 0$ manifests itself as selecting the irreducible representation, where the charges $\chi_{\hat{\mathrm{w}}_{kl}} = \chi_{\hat{\mathrm{u}}_l} = \chi_{\hat{\mathrm{v}}_k} = 0$ for $k \neq 0, l \neq 0$ are set to zero. □

2.4 Representations of $\widehat{\mathcal{G}}|_{\mathrm{d}A}$ & $\widehat{\mathcal{G}}_\Lambda|_{\mathrm{d}A+\Lambda B=0}$

By choosing a representation of $\mathcal{A}$, e.g. the bosonic Fock space representation (c.f. [KR87, Section 2.2]), one immediately obtains a set of representations of $\widehat{\mathcal{G}}|_{\mathrm{d}A}$ & $\widehat{\mathcal{G}}_\Lambda|_{\mathrm{d}A+\Lambda B=0}$ parametrized by the action of the abelian summand. In physics, bosonic and fermionic Fock spaces are Hilbert spaces which describe the quantum mechanical state spaces of arbitrarily many, indistinguishable particles. We will consider the term bosonic Fock space as the one defined in [KR87, Section 2.2]. In particular, the bosonic Fock space does not include any inner product or completion. One still has to explore whether these representations can be usefully extended to the full corner algebra, because we have restricted ourselves to finite Fourier modes. If the modules extend, they would describe possible state spaces of 4-dim. abelian BF theory on surfaces that bound a torus (recall Example 1.2). It would then be interesting to investigate the details of 1.2 and how the state space on the solid torus decomposes in terms of the corner algebra modules.

3 Non-Abelian BF on $\Gamma \cong T^2$

In this section, we explore the quantized corner algebra of 4-dim. non-abelian BF on a torus with Lie algebra $\mathfrak{g} = \mathfrak{su}(2)$. We proceed similarly to the previous section and start by considering the Lie subalgebra generated by finite Fourier expansions and working out the bracket relations. The Lie algebras for zero and non-zero cosmological constant turn out to be isomorphic to certain central extensions of the double-loop algebra[24] over the 9-dim. isochronous Galilean Lie algebra $\mathfrak{igal}(3)$. After classifying the respective algebras, we discuss the heuristic two-sided constraint ideals defining the physical quotient algebra. Obtaining representations of the quantized corner algebra is very difficult because $\mathfrak{igal}(3)$ is neither semisimple nor solvable. To the best of our knowledge, there are no useful results concerning representations of the types of Lie algebras at hand. Fortunately, the standard highest-weight module construction only fails in a controllable way. Exploiting this fact, we use the induced module construction applied to a so-called modified triangular decomposition (MTD) of the Lie algebra to construct representations. The so constructed representations turn out to have nice properties such as: having a vacuum-like vector, being graded, etc. One can then explicitly realize these representations as second-order differential operators on the space of polynomials in countably-infinite variables, essentially constituting a free field realization (c.f. the discussion in Section 3.3.1). Finally, we address how to modify the modules so that the ideal of constraints has a well-defined action. However, with this modification, the constraints cannot be imposed and the representation does not descend to the physical quotient algebra. It is not yet clear if this can be remedied or if there is a deeper reason why the construction fails, e.g. the fact that the representations only represent the finite Fourier mode subalgebra. Another possibility, although seemingly unlikely, is that it might be an artifact of all of the heuristics that led to the construction of these constraints.

3.1 The Lie Algebra $\widehat{\mathcal{G}}_\Lambda(\mathfrak{su}(2))$

The Lie algebra describing 4-dim. non-abelian BF on a torus with $\mathfrak{g} = \mathfrak{su}(2)$ is given by the vector space:

$$\widehat{\mathcal{G}}_\Lambda(\mathfrak{su}(2)) = \Omega^0(T^2) \otimes \mathfrak{su}(2) \oplus \Omega^1(T^2) \otimes \mathfrak{su}(2) \oplus \mathbb{R}\,,$$

with brackets

$$[f \oplus \alpha \oplus r, g \oplus \beta \oplus s]_{\widehat{\mathcal{G}}_\Lambda(\mathfrak{su}(2))} = [f,g] \oplus (\mathrm{ad}_f \beta - \mathrm{ad}_g \alpha) \oplus \frac{-1}{(2\pi)^2} \int_{T^2} (\alpha \mathrm{d}g - \beta \mathrm{d}f + \Lambda \alpha\beta)\mathrm{Z}\,, \quad (3.1)$$

[24]Given any Lie algebra $\mathfrak{g}$, we can construct its double loop algebra $\mathfrak{g}[z, z^{-1}, w, w^{-1}] := \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}, w, w^{-1}]$, where the latter factor is the ring of Laurent polynomials in two variables valued in $\mathbb{C}$. Note that only a sum of finitely many powers of the variables are allowed in the ring.

where we choose the trivial reference connection $A_0 = 0$.[25] Furthermore, we drop the subscript of the Lie bracket in (2.1) We want to have an explicit description in terms of generators. Therefore, let us fix a basis $\{t_\mu\}_{1\leq\mu\leq 3}$ of $\mathfrak{su}(2)$, such that $(t_\mu, t_\nu) = \delta_{\mu,\nu}$ and $[t_\mu, t_\nu] = \varepsilon^{\lambda}_{\mu\nu} t_\lambda$, where $(-,-)$ is an invariant, non-degenerate inner product on $\mathfrak{su}(2)$ and ε is the Levi-Civita symbol. Such a basis exists because $\mathfrak{su}(2)$ is compact and simple.

Instead of the entire algebra, we will consider the Lie subalgebra of $(\widehat{\mathcal{G}}_\Lambda)_\mathbb{C}$ that consists of finite Fourier modes. These elements are of the form

$$
\begin{aligned}
f &= \sum_{\substack{1\leq\mu\leq 3\\ m,n\in\mathbb{Z}}} f_{\mu mn}\left(t_\mu \otimes e^{im\theta}e^{in\varphi}\right),\\
\alpha &= \sum_{\substack{1\leq\mu\leq 3\\ m,n\in\mathbb{Z}}} \alpha^{(\theta)}_{\mu mn}\left(t_\mu \otimes e^{im\theta}e^{in\varphi}\right)\mathrm{d}\theta + \alpha^{(\varphi)}_{\mu mn}\left(t_\mu \otimes e^{im\theta}e^{in\varphi}\right)\mathrm{d}\varphi\,,
\end{aligned}
$$

where (θ, φ) are coordinates[26] on the torus with orientation given by the volume form $\mathrm{d}\theta \wedge \mathrm{d}\varphi$ and with only finitely many non-zero coefficients. The space of differential 1-forms $\Omega^1(T^2)$ is a $C^\infty(T^2)$-module. It has a basis given by $\{\mathrm{d}\theta, \mathrm{d}\varphi\}$.

With a slight abuse of notation, we denote the Lie subalgebra by the same symbol and drop the complexification symbol. Now that the vector space $\widehat{\mathcal{G}}_\Lambda(\mathfrak{su}(2))$ has countable dimension, we can choose a suitable basis:

$$
\begin{aligned}
\mathrm{J}_{\mu mn} &:= t_\mu \otimes e^{im\theta}e^{in\varphi}\,,\\
\mathrm{K}_{\mu mn} &:= \left(t_\mu \otimes e^{im\theta}e^{in\varphi}\right)\mathrm{d}\varphi\,,\\
\mathrm{P}_{\mu mn} &:= \left(t_\mu \otimes e^{im\theta}e^{in\varphi}\right)\mathrm{d}\theta\,.
\end{aligned}
$$

The notation will become clear later on when we make the connection to the isochronous Galilei Lie algebra. A quick computation yields the following lemma.

Lemma 3.1. *In the finite Fourier mode basis, the bracket relations are:*

$$
\begin{aligned}
[\mathrm{J}_{\mu kl}, \mathrm{J}_{\nu mn}] &= \varepsilon^{\lambda}_{\mu\nu}\mathrm{J}_{\lambda(k+m)(l+n)}\,, \qquad (3.2)\\
[\mathrm{J}_{\mu kl}, \mathrm{K}_{\nu mn}] &= \varepsilon^{\lambda}_{\mu\nu}\mathrm{K}_{\lambda(k+m)(l+n)} + im\delta_{\mu,\nu}\delta_{k,-m}\delta_{l,-n}\mathrm{Z}\,,\\
[\mathrm{J}_{\mu kl}, \mathrm{P}_{\nu mn}] &= \varepsilon^{\lambda}_{\mu\nu}\mathrm{P}_{\lambda(k+m)(l+n)} - in\delta_{\mu,\nu}\delta_{k,-m}\delta_{l,-n}\mathrm{Z}\,,\\
[\mathrm{K}_{\mu kl}, \mathrm{P}_{\nu mn}] &= \Lambda\delta_{\mu,\nu}\delta_{k,-m}\delta_{l,-n}\mathrm{Z}\,,
\end{aligned}
$$

and all other brackets vanish.

[25] A non-zero reference connection A_0 does not change the Lie algebra structure; the extra term can be reabsorbed by a redefinition of the generators: $\mathrm{P}_{\mu kl} \longrightarrow \mathrm{P}_{\mu kl} - A0^{(\varphi)}_{\mu -k -l}\mathrm{Z}$ and $\mathrm{K}_{\mu kl} \longrightarrow \mathrm{K}_{\mu kl} - A0^{(\theta)}_{\mu -k -l}\mathrm{Z}$.

[26] The coordinate functions θ and φ are not defined globally, but the induced 1-forms $\mathrm{d}\theta$ and $\mathrm{d}\varphi$ are.

Proof. We will compute the second bracket explicitly and leave the rest as an exercise to the reader. Evaluating the bracket on these basis elements yields:

$$\begin{aligned}
[\mathrm{J}_{\mu kl}, \mathrm{J}_{\nu mn}] &= [t_\mu, t_\nu] \otimes e^{i(k+m)\theta} e^{i(l+n)\varphi} \\
&= \varepsilon^{\lambda}_{\mu\nu} \mathrm{J}_{\lambda(k+m)(l+n)} \\
[\mathrm{J}_{\mu kl}, \mathrm{K}_{\nu mn}] &= \mathrm{ad}_{\mathrm{J}_{\mu kl}} \mathrm{K}_{\nu mn} + \frac{1}{(2\pi)^2} \int_{T^2} \mathrm{K}_{\nu mn} \mathrm{d}\mathrm{J}_{\mu kl} \mathrm{Z} \\
&= \left([t_\mu, t_\nu] \otimes e^{i(k+m)\theta} e^{i(l+n)\varphi} \right) \mathrm{d}\varphi \\
&\quad + \frac{1}{(2\pi)^2} \left(\int_{T^2} ik(t_\nu, t_\mu) e^{i(m+k)\theta} e^{i(n+l)\varphi} \mathrm{d}\varphi \wedge \mathrm{d}\theta \right) \mathrm{Z} \\
&= \varepsilon^{\lambda}_{\mu\nu} \mathrm{K}_{\lambda(k+m)(l+n)} + im\delta_{\mu,\nu}\delta_{k,-m}\delta_{l,-n}\mathrm{Z} \\
[\mathrm{J}_{\mu kl}, \mathrm{P}_{\nu mn}] &= \mathrm{ad}_{\mathrm{J}_{\mu kl}} \mathrm{P}_{\nu mn} + \frac{1}{(2\pi)^2} \int_{T^2} \mathrm{P}_{\nu mn}, \mathrm{d}\mathrm{J}_{\mu kl} \mathrm{Z} \\
&= \left([t_\mu, t_\nu] \otimes e^{i(k+m)\theta} e^{i(l+n)\varphi} \right) \mathrm{d}\theta \\
&\quad + \frac{1}{(2\pi)^2} \left(\int_{T^2} il(t_\nu, t_\mu) e^{i(m+k)\theta} e^{i(n+l)\varphi} \mathrm{d}\theta \wedge \mathrm{d}\varphi \right) \mathrm{Z} \\
&= \varepsilon^{\lambda}_{\mu\nu} \mathrm{P}_{\lambda(k+m)(l+n)} - in\delta_{\mu,\nu}\delta_{k,-m}\delta_{l,-n}\mathrm{Z} \\
[\mathrm{K}_{\mu kl}, \mathrm{P}_{\nu mn}] &= -\frac{1}{(2\pi)^2} \int_{T^2} \Lambda \mathrm{K}_{\mu kl} \mathrm{P}_{\nu mn} \mathrm{Z} \\
&= -\frac{1}{(2\pi)^2} \left(\int_{T^2} \Lambda(t_\mu, t_\nu) e^{i(k+m)\theta} e^{i(l+n)\varphi} \mathrm{d}\varphi \wedge \mathrm{d}\theta \right) \mathrm{Z} \\
&= \Lambda \delta_{\mu,\nu}\delta_{k,-m}\delta_{l,-n}\mathrm{Z}\,.
\end{aligned}$$

□

3.2 Classification of $\widehat{\mathcal{G}}(\mathfrak{su}(2))$ & $\widehat{\mathcal{G}}_\Lambda(\mathfrak{su}(2))$

To provide a classification of $\widehat{\mathcal{G}}(\mathfrak{su}(2))$ and $\widehat{\mathcal{G}}_\Lambda(\mathfrak{su}(2))$, we need to define the zeroth-level subalgebras. The zeroth-level Lie algebra for zero cosmological constant is the 9-dim. subalgebra spanned by generators $\{\mathrm{X}_{\mu 00}\}_{1\le\mu\le 3}$ for $\mathrm{X} \in \{\mathrm{J},\mathrm{K},\mathrm{P}\}$ and is denoted $\widehat{\mathcal{G}}(\mathfrak{su}(2))_0$. The zeroth-level Lie algebra for non-zero cosmological constant is the 10-dim. subalgebra spanned by generators Z and $\{\mathrm{X}_{\mu 00}\}_{1\le\mu\le 3}$ for $\mathrm{X} \in \{\mathrm{J},\mathrm{K},\mathrm{P}\}$ and is denoted $\widehat{\mathcal{G}}_\Lambda(\mathfrak{su}(2))_0$. It is easily verified that these actually constitute Lie subalgebras of $\widehat{\mathcal{G}}(\mathfrak{su}(2))$ and $\widehat{\mathcal{G}}_\Lambda(\mathfrak{su}(2))$ respectively.

Before we state the theorem classifying $\widehat{\mathcal{G}}(\mathfrak{su}(2))_0$ and $\widehat{\mathcal{G}}_\Lambda(\mathfrak{su}(2))_0$, we define the isochronous Galilean Lie algebra and its extension.

Definition 3.2. The **isochronous Galilean Lie algebra** $\mathfrak{igal}(3)$ is the 9-dimensional, real vector space with a Lie bracket defined by

$$[\mathrm{J}_\mu, \mathrm{J}_\nu] \; = \varepsilon^{\lambda}_{\mu\nu} \mathrm{J}_\lambda\,,$$

$$[\mathrm{J}_\mu, \mathrm{K}_\nu] = \varepsilon_{\mu\nu}^{\lambda} \mathrm{K}_\lambda \,,$$
$$[\mathrm{J}_\mu, \mathrm{P}_\nu] = \varepsilon_{\mu\nu}^{\lambda} \mathrm{P}_\lambda \,,$$

where the set $\{\mathrm{J}_\mu, \mathrm{K}_\mu, \mathrm{P}_\mu\}_{1\leq\mu\leq 3}$ denotes a basis of the vector space and $\epsilon_{\mu\nu}^{\lambda}$ denotes the Levi-Civita symbol.

The algebra $\mathfrak{igal}(3)$ is the Lie subalgebra of the Galilean Lie algebra in 3 dimensions $\mathfrak{gal}(3)$ (c.f. [Lév71, (2.21a)-(2.21i)]) excluding the time generator H, hence the prefix isochronous. In 3 dimensions, the Galilean Lie algebra has a unique central extension which is denoted $\widehat{\mathfrak{gal}}(3)$ and is characterized by $[\mathrm{K}_\mu, \mathrm{P}_\nu] = \delta_{\mu,\nu} m\mathrm{I}$ [Lév71, (3.26)]. The isochronous part then forms a Lie subalgebra of $\widehat{\mathfrak{gal}}(3)$ again.

Definition 3.3. The **extended isochronous Galilean Lie algebra** $\mathfrak{i}\widehat{\mathfrak{gal}}(3)$ is the 10-dimensional, real vector space with a Lie bracket defined by

$$\begin{aligned}
[\mathrm{J}_\mu, \mathrm{J}_\nu] &= \varepsilon_{\mu\nu}^{\lambda} \mathrm{J}_\lambda \,, \\
[\mathrm{J}_\mu, \mathrm{K}_\nu] &= \varepsilon_{\mu\nu}^{\lambda} \mathrm{K}_\lambda \,, \\
[\mathrm{J}_\mu, \mathrm{P}_\nu] &= \varepsilon_{\mu\nu}^{\lambda} \mathrm{P}_\lambda \,, \\
[\mathrm{K}_\mu, \mathrm{P}_\nu] &= \delta_{\mu,\nu} m\mathrm{I} \,,
\end{aligned}$$

where the set $\{\mathrm{J}_\mu, \mathrm{K}_\mu, \mathrm{P}_\mu, \mathrm{I}\}_{1\leq\mu\leq 3}$ denotes a basis and $\epsilon_{\mu\nu}^{\lambda}$ denotes the Levi-Civita symbol.

Now, we are ready to identify the zeroth-level Lie algebras.

Theorem 3.4. *There are isomorphisms of Lie algebras:*

$$\widehat{\mathcal{G}}(\mathfrak{su}(2))_0 \cong \mathfrak{igal}(3), \quad \widehat{\mathcal{G}}_\Lambda(\mathfrak{su}(2))_0 \cong \mathfrak{i}\widehat{\mathfrak{gal}}(3),$$

where $\mathfrak{igal}(3)$ *is the 9-dim. isochronous Galilean Lie algebra and* $\mathfrak{i}\widehat{\mathfrak{gal}}(3)$ *the 10-dim. isochronous Lie subalgebra of the extended Galilean Lie algebra with mass* $\Lambda = m$.

Proof. The proof follows immediately by writing out the bracket relations of the zeroth-level subalgebras and identifying them as the ones of the respective Galilean Lie subalgebras. To that end, denote the zeroth-level generators by $\mathrm{X}_\mu \coloneqq \mathrm{X}_{\mu 00}$ for $X \in \{J, K, P\}$ and, for $\Lambda \neq 0$, identify $\Lambda\mathrm{Z} = m\mathrm{I}$. Therefore, the cosmological term has the interpretation of a "mass" from the extended-Galilei-algebra point of view. □

The appearance of the Galilei algebra is not so surprising as one might think and is not an indication of the underlying theory breaking Lorentz invariance. The first two summands of the bracket in (3.1) are reminiscent of a semi-direct structure. Furthermore,

$\mathfrak{igal}(3)$ itself is isomorphic to a semi-direct product of $\mathfrak{su}(2)$, namely $\mathfrak{igal}(3) \cong \mathfrak{su}(2) \ltimes (\mathbb{R}^3 \oplus \mathbb{R}^3)$.

We can also reverse the process and reconstruct the original corner Lie algebra from the zeroth-level Lie subalgebra. Take the double-loop algebra[27] over $\widehat{\mathcal{G}}(\mathfrak{su}(2))_0$ and centrally extend by the cocycle defined in Section 1.3.2.[28] Thus, we have the following corollary:

Corollary 3.5. *The Lie algebras* $\widehat{\mathcal{G}}(\mathfrak{su}(2))$ *and* $\widehat{\mathcal{G}}_\Lambda(\mathfrak{su}(2))$ *are isomorphic to central extensions of the double-loop algebra* $\mathfrak{igal}(3)[z, z^{-1}, w, w^{-1}]$.

Remark 3.6. There is a universal central extension for $\mathfrak{igal}(3)[z, z^{-1}, w, w^{-1}]$ because it is perfect, i.e. $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ (see [Kal73, Proposition 1.3]).

Already, one can guess that finding representations of this involved Lie algebra will prove quite tricky.

3.2.1 Constraints $F_A = 0$ & $F_A + \Lambda B = 0$

In the non-abelian theory, the constraints are not linear and thus not represented by elements of the Lie algebra. Instead, they lie in the UEA of the latter, i.e. $\mathcal{I}_{F_A} \subset \mathcal{U}(\widehat{\mathcal{G}}(\mathfrak{su}(2)))$ and $\mathcal{I}_{F_A+\Lambda B} \subset \mathcal{U}(\widehat{\mathcal{G}}_\Lambda(\mathfrak{su}(2)))$ (recall the heuristics in Section 1.3.4). Furthermore, they are not central anymore but constitute a two-sided ideal in the UEA. Heuristically, we can then form the physical quotient UEAs $\mathcal{U}(\widehat{\mathcal{G}}(\mathfrak{su}(2)))|_{F_A=0} := \mathcal{U}(\widehat{\mathcal{G}}(\mathfrak{su}(2)))\big/\mathcal{I}_{F_A}$ and $\mathcal{U}(\widehat{\mathcal{G}}_\Lambda(\mathfrak{su}(2)))|_{F_A+\Lambda B=0} := \mathcal{U}(\widehat{\mathcal{G}}_\Lambda(\mathfrak{su}(2)))\big/\mathcal{I}_{F_A+\Lambda B}$.

In the following, we just treat the $\Lambda \neq 0$ case since it works almost identically for zero cosmological constant. The ideal of constraints $I_{F_A+\Lambda B}$ in $C^\infty(\mathcal{B}_0)$ is generated by the functionals:

$$\widehat{f} := \frac{1}{(2\pi)^2} \int_{T^2} f(F_A + \Lambda B),$$

where $f \in \Omega^0(T^2) \otimes \mathfrak{su}(2)$[29] and the factor is purely for convenience. We can decompose this multilinear form into an infinite sum of products of linear functionals.

Remark 3.7. Clearly, such a sum can be problematic, and we will come back to this issue later. For now, we will assume that the infinite sum truncates.

If we denote $t_{\mu mn} := t_\mu \otimes e^{im\theta} e^{in\varphi}$, then we can decompose $\widehat{f}$ as follows.

[27] Given any Lie algebra $\mathfrak{g}$, we can construct its double loop algebra $\mathfrak{g}[z, z^{-1}, w, w^{-1}] := \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}, w, w^{-1}]$, where the latter factor is the ring of Laurent polynomials in two variables valued in $\mathbb{C}$. Note that only a sum of finitely many powers of the variables are allowed in the ring.

[28] Note that taking $\widehat{\mathcal{G}}_\Lambda(\mathfrak{su}(2))_0$ does not work, because it already contains the central charge.

[29] We will soon limit ourselves to such functions with finite Fourier expansion again.

Proposition 3.8. *The constraint functionals expressed in terms of an infinite sum of linear functionals are given by*

$$\widehat{f} = \sum_{r,s}\sum_{\lambda} f_{\lambda rs}\left(-is\mathrm{K}_{\lambda rs} - ir\mathrm{P}_{\lambda rs} + \Lambda\mathrm{J}_{\lambda rs} + \sum_{m,n}\sum_{\mu,\nu}\varepsilon^{\lambda}_{\mu\nu}\mathrm{P}_{\mu(r+m)(s+n)}\mathrm{K}_{\nu-m-n}\right)$$

Proof.

$$\begin{aligned}
\widehat{f}(A,B) &= \frac{1}{(2\pi)^2}\int_{T^2}(f \overset{\wedge}{,} \mathrm{d}A + \frac{1}{2}[A,A] + \Lambda B)\\
&= \frac{f_{\lambda rs}}{(2\pi)^2}\int_{T^2}\Big(\mathrm{J}_{\lambda rs} \overset{\wedge}{,} (imA^{(\varphi)}_{\mu mn} - inA^{(\theta)}_{\mu mn} + \Lambda B_{\mu mn})t_{\mu mn}\mathrm{d}\theta\wedge\mathrm{d}\varphi\\
&\quad + \varepsilon^{\rho}_{\mu\nu}A^{(\theta)}_{\mu mn}A^{(\varphi)}_{\nu kl}t_{\rho(m+k)(n+l)})\mathrm{d}\theta\wedge\mathrm{d}\varphi\Big)\\
&= f_{\lambda rs}(imA^{(\varphi)}_{\lambda mn} - inA^{(\theta)}_{\lambda mn} + \Lambda B_{\lambda mn})\frac{1}{(2\pi)^2}\int_{T^2}e^{i(r+m)\theta}e^{i(s+n)\varphi}\mathrm{d}\theta\wedge\mathrm{d}\varphi\\
&\quad + f_{\lambda rs}\varepsilon^{\lambda}_{\mu\nu}A^{(\theta)}_{\mu mn}A^{(\varphi)}_{\nu kl}\frac{1}{(2\pi)^2}\int_{T^2}e^{i(r+m+k)\theta}e^{i(s+n+l)\varphi}\mathrm{d}\theta\wedge\mathrm{d}\varphi\\
&= f_{\lambda rs}(-irA^{(\varphi)}_{\lambda-r-s} + isA^{(\theta)}_{\lambda-r-s} + \Lambda B_{\lambda-r-s} + \varepsilon^{\lambda}_{\mu\nu}A^{(\theta)}_{\mu mn}A^{(\varphi)}_{\nu(-r-m)(-s-n)})\\
&= \sum_{r,s}\sum_{\lambda} f_{\lambda rs}\left(-is\mathrm{K}_{\lambda rs} - ir\mathrm{P}_{\lambda rs} + \Lambda\mathrm{J}_{\lambda rs} + \sum_{m,n}\sum_{\mu,\nu}\varepsilon^{\lambda}_{\mu\nu}\mathrm{P}_{\mu(r+m)(s+n)}\mathrm{K}_{\nu-m-n}\right)(A,B)\,.
\end{aligned}$$

The first equal sign follows by definition, and the second by the expansion of the function and the forms in terms of the finite Fourier basis and the Lie algebra basis. The third equal sign is obtained by contracting the inner product. One then integrates over the volume form to obtain the fourth equal sign. Finally, the linear functionals can be restored using the definition in terms of the non-degenerate pairing. □

Next, we introduce a preferred basis of $\mathcal{U}(\widehat{\mathcal{G}}_\Lambda(\mathfrak{su}(2)))$. By the Poincaré-Birkhoff-Witt (PBW) theorem[30], the monomials

$$\{\mathrm{J}_{\mu_1k_1l_1}\cdots\mathrm{J}_{\mu_ak_al_a}\mathrm{K}_{\nu_1m_1n_1}\cdots\mathrm{K}_{\nu_bm_bn_b}\mathrm{P}_{\rho_1r_1s_1}\cdots\mathrm{P}_{\rho_cr_cs_c}\}\,,$$

where $a,b,c\in\mathbb{N}$, $k_1\le\cdots\le k_a,\cdots,s_1\le\cdots\le s_c$ and $\mu_1\le\cdots\le\mu_a,\cdots,\rho_1\le\ldots\le\rho_c$, along with 1, form a basis of the UEA. We are actually considering a quotient of the UEA by the ideal generated by $1-\mathrm{Z}$ since we are only interested in modules where the central charge acts by 1. The expression $\widehat{f}$ descends to a well-defined equivalence class in the UEA. This follows because the generators in the quadratic term of $\widehat{f}$ commute when summing over the totally-antisymmetric structure constants. We will again denote the resulting equivalence class by $\widehat{f}$. These elements represent the quantization of the constraint functionals.

[30]Note that the PBW theorem holds in arbitrary dimension. For more details, see [Hum73, Section 17.3].

Since the sum is infinite, the quantized constraints should formally be part of some appropriate completion of the UEA. Instead of resolving this issue, we will continue assuming that these infinite sums can be made sense of and accept the resulting statements as heuristics.

Next, we want to examine whether the set of constraints also forms an ideal in the UEA. To that end, we have the following proposition

Proposition 3.9. *The set of constraints* $\{\widehat{f} \in \mathcal{U}(\widehat{\mathcal{G}}_\Lambda(\mathfrak{su}(2)))\}$ *satisfies the bracket relations*

$$\begin{aligned}
[\widehat{f}_{\lambda rs}, \mathrm{J}_{\mu kl}] &= \sum_\rho \varepsilon^\rho_{\lambda\mu} \widehat{f}_{\nu(r+k)(s+l)}\,, \\
[\widehat{f}_{\lambda rs}, \mathrm{K}_{\mu kl}] &= 0\,, \\
[\widehat{f}_{\lambda rs}, \mathrm{P}_{\mu kl}] &= 0\,,
\end{aligned}$$

where $\widehat{f}_{\mu kl} \coloneqq \widehat{\mathrm{J}}_{\mu kl}$*, for all* $1 \leq \lambda, \mu \leq 3$ *and* $r, s, k, l \in \mathbb{Z}$*. By the derivation property of the bracket, the set generates a proper, two-sided ideal* $\mathcal{I}_{F_A+\Lambda B}$ *in* $\mathcal{U}(\widehat{\mathcal{G}}_\Lambda(\mathfrak{su}(2))$*.*

Proof. The relations can be obtained from a direct computation using the definition of $\widehat{f}_{\lambda rs}$ and the Equations (3.2). In the following proof, the implicit sums will be displayed for clarity's sake. First, we compute the commutator with $\mathrm{J}_{\mu kl}$:

$$\begin{aligned}
[\widehat{f}_{\lambda rs}, \mathrm{J}_{\mu kl}] &= [-is\mathrm{K}_{\lambda rs} - ir\mathrm{P}_{\lambda rs} + \Lambda\mathrm{J}_{\lambda rs} + \sum_{m,n}\sum_{\rho,\nu} \varepsilon^\lambda_{\rho\nu} \mathrm{P}_{\rho(r+m)(s+n)}\mathrm{K}_{\nu-m-n}, \mathrm{J}_{\mu kl}] \\
&= [-is\mathrm{K}_{\lambda rs} - ir\mathrm{P}_{\lambda rs} + \Lambda\mathrm{J}_{\lambda rs}, \mathrm{J}_{\mu kl}] \\
&\quad + \sum_{m,n}\sum_{\rho,\nu} \varepsilon^\lambda_{\rho\nu} [\mathrm{P}_{\rho(r+m)(s+n)}\mathrm{K}_{\nu-m-n}, \mathrm{J}_{\mu kl}]
\end{aligned}$$

We will calculate the terms on the right-hand side separately. The first term yields:

$$\begin{aligned}
[-is\mathrm{K}_{\lambda rs} - ir\mathrm{P}_{\lambda rs} + \Lambda\mathrm{J}_{\lambda rs}, \mathrm{J}_{\mu kl}] &= -is(\sum_\rho \varepsilon^\rho_{\lambda\mu}\mathrm{K}_{\rho(r+k)(s+l)} + ik\delta_{\lambda\mu}\delta_{r,-k}\delta_{s,-l}\mathrm{Z}) \\
&\quad - ir(\sum_\rho \varepsilon^\rho_{\lambda\mu}\mathrm{P}_{\rho(r+k)(s+l)} - il\delta_{\lambda\mu}\delta_{r,-k}\delta_{s,-l}\mathrm{Z}) \\
&\quad + \Lambda\sum_\rho \varepsilon^\rho_{\lambda\mu}\mathrm{J}_{\rho(r+k)(s+l)} \\
&= \sum_\rho \varepsilon^\rho_{\lambda\mu}(-is\mathrm{K}_{\rho(r+k)(s+l)} - ir\mathrm{P}_{\rho(r+k)(s+l)} + \Lambda\mathrm{J}_{\rho(r+k)(s+l)})
\end{aligned}$$

The second term yields:

$$\begin{aligned}
\sum_{m,n}\sum_{\rho,\nu} \varepsilon^\lambda_{\rho\nu}[\mathrm{P}_{\rho(r+m)(s+n)}\mathrm{K}_{\nu-m-n}, \mathrm{J}_{\mu kl}] &= \sum_{m,n}\sum_{\rho,\nu} \varepsilon^\lambda_{\rho\nu}[\mathrm{P}_{\rho(r+m)(s+n)}, \mathrm{J}_{\mu kl}]\mathrm{K}_{\nu-m-n} \\
&\quad + \sum_{m,n}\sum_{\rho,\nu} \varepsilon^\lambda_{\rho\nu}\mathrm{P}_{\rho(r+m)(s+n)}[\mathrm{K}_{\nu-m-n}, \mathrm{J}_{\mu kl}]
\end{aligned}$$

$$\begin{aligned}
&= \sum_{m,n}\sum_{\rho,\nu} \varepsilon^{\lambda}_{\rho\nu}\Bigg(\sum_{\sigma}\varepsilon^{\sigma}_{\rho\mu}\mathrm{P}_{\sigma(r+m+k)(s+n+l)} \\
&\quad - il\delta_{\rho\mu}\delta_{r+k,-m}\delta_{s+n,-l}\mathrm{Z}\Bigg)\mathrm{K}_{\nu-m-n} \\
&\quad + \sum_{m,n}\sum_{\rho,\nu}\varepsilon^{\lambda}_{\rho\nu}\mathrm{P}_{\rho(r+m)(s+n)}\Bigg(\sum_{\sigma}\varepsilon^{\sigma}_{\nu\mu}\mathrm{K}_{\sigma(k-m)(l-n)} \\
&\quad + ik\delta_{\nu\mu}\delta_{m,k}\delta_{l,n}\mathrm{Z}\Bigg) \\
&= \sum_{m,n}\sum_{\rho,\nu,\sigma}(\varepsilon^{\lambda}_{\rho\nu}\varepsilon^{\sigma}_{\rho\mu}\mathrm{P}_{\sigma(r+m+k)(s+n+l)}\mathrm{K}_{\nu-m-n} \\
&\quad + \varepsilon^{\lambda}_{\rho\nu}\varepsilon^{\sigma}_{\nu\mu}\mathrm{P}_{\rho(r+m)(s+n)}\mathrm{K}_{\sigma(k-m)(l-n)}) \\
&\quad - il\sum_{\nu}\varepsilon^{\lambda}_{\mu\nu}\mathrm{Z}\mathrm{K}_{\nu(r+k)(s+l)} \\
&\quad + ik\sum_{\rho}\varepsilon^{\lambda}_{\rho\mu}\mathrm{P}_{\rho(r+k)(s+l)}\mathrm{Z} \\
&= \sum_{m,n}\sum_{\rho,\nu,\sigma}(\varepsilon^{\lambda}_{\sigma\rho}\varepsilon^{\nu}_{\rho\mu} - \varepsilon^{\lambda}_{\nu\rho}\varepsilon^{\sigma}_{\rho\mu})\mathrm{P}_{\sigma(r+m+k)(s+n+l)}\mathrm{K}_{\nu-m-n} \\
&\quad - \sum_{\rho}\varepsilon^{\rho}_{\lambda\mu}il\mathrm{K}_{\rho(r+k)(s+l)}\mathrm{Z} \\
&\quad - \sum_{\rho}\varepsilon^{\rho}_{\lambda\mu}ik\mathrm{P}_{\rho(r+k)(s+l)}\mathrm{Z} \\
&= \sum_{m,n}\sum_{\rho,\nu,\sigma}\varepsilon^{\rho}_{\lambda\mu}\varepsilon^{\rho}_{\sigma\nu}\mathrm{P}_{\sigma(r+m+k)(s+n+l)}\mathrm{K}_{\nu-m-n} \\
&\quad - \sum_{\rho}\varepsilon^{\rho}_{\lambda\mu}il\mathrm{K}_{\rho(r+k)(s+l)}\mathrm{Z} \\
&\quad - \sum_{\rho}\varepsilon^{\rho}_{\lambda\mu}ik\mathrm{P}_{\rho(r+k)(s+l)}\mathrm{Z}
\end{aligned}$$

Recall that we consider the quotient where $\mathrm{Z} = 1$. Combining the two results leaves us with:

$$\begin{aligned}
[\widehat{f}_{\lambda rs}, \mathrm{J}_{\mu kl}] &= \sum_{\rho}\varepsilon^{\rho}_{\lambda\mu}(-is\mathrm{K}_{\rho(r+k)(s+l)} - ir\mathrm{P}_{\rho(r+k)(s+l)} + \Lambda\mathrm{J}_{\rho(r+k)(s+l)}) \\
&\quad + \sum_{m,n}\sum_{\rho,\nu,\sigma}\varepsilon^{\rho}_{\lambda\mu}\varepsilon^{\rho}_{\sigma\nu}\mathrm{P}_{\sigma(r+m+k)(s+n+l)}\mathrm{K}_{\nu-m-n} \\
&\quad - \sum_{\rho}\varepsilon^{\rho}_{\lambda\mu}il\mathrm{K}_{\rho(r+k)(s+l)}\mathrm{Z} \\
&\quad - \sum_{\rho}\varepsilon^{\rho}_{\lambda\mu}ik\mathrm{P}_{\rho(r+k)(s+l)}\mathrm{Z} \\
&= \sum_{\rho}\varepsilon^{\rho}_{\lambda\mu}\Big(-i(s+l)\mathrm{K}_{\rho(r+k)(s+l)} - i(r+k)\mathrm{P}_{\rho(r+k)(s+l)} + \Lambda\mathrm{J}_{\rho(r+k)(s+l)}
\end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{m,n}\sum_{\rho,\nu,\sigma} \varepsilon^{\rho}_{\sigma\nu} \mathrm{P}_{\sigma(r+m+k)(s+n+l)}\mathrm{K}_{\nu-m-n}\Bigg) \\
&= \sum_{\rho} \varepsilon^{\rho}_{\lambda\mu} \widehat{f}_{\rho(r+k)(s+l)}
\end{aligned}
$$

The next two commutators are more straightforward.

$$
\begin{aligned}
[\widehat{f}_{\lambda rs}, \mathrm{K}_{\mu kl}] &= [-is\mathrm{K}_{\lambda rs} - ir\mathrm{P}_{\lambda rs} + \Lambda \mathrm{J}_{\lambda rs} + \sum_{m,n}\sum_{\rho,\nu} \varepsilon^{\lambda}_{\rho\nu} \mathrm{P}_{\rho(r+m)(s+n)}\mathrm{K}_{\nu-m-n}, \mathrm{K}_{\mu kl}] \\
&= [-ir\mathrm{P}_{\lambda rs} + \Lambda \mathrm{J}_{\lambda rs}, \mathrm{K}_{\mu kl}] \\
&\quad + \sum_{m,n}\sum_{\rho,\nu} \varepsilon^{\lambda}_{\rho\nu} [\mathrm{P}_{\rho(r+m)(s+n)}, \mathrm{K}_{\mu kl}]\mathrm{K}_{\nu-m-n} \\
&= i(r+k)\Lambda \delta_{\lambda\mu}\delta_{r,-k}\delta_{s,-l}\mathrm{Z} + \Lambda \sum_{\rho} \varepsilon^{\rho}_{\lambda\mu} \mathrm{K}_{\rho(r+k)(s+l)} \\
&\quad - \sum_{\nu} \varepsilon^{\lambda}_{\mu\nu} \Lambda \mathrm{K}_{\nu(r+k)(s+l)} \\
&= 0
\end{aligned}
$$

The calculation of the commutator with $\mathrm{P}_{\mu kl}$ is almost identical to the previous computation. □

We can conclude that physical representations should be with respect to the algebra

$$
\mathcal{U}(\widehat{\mathcal{G}}_\Lambda(\mathfrak{su}(2)))|_{F_A+\Lambda B=0} := \mathcal{U}(\widehat{\mathcal{G}}_\Lambda(\mathfrak{su}(2)))\big/\mathcal{I}_{F_A+\Lambda B}.
$$

or equivalently representations of $\mathcal{U}(\widehat{\mathcal{G}}_\Lambda(\mathfrak{su}(2)))$, where the ideal of constraints is represented trivially. A similar statement should hold for $\mathcal{U}(\widehat{\mathcal{G}}(\mathfrak{su}(2)))|_{F_A=0}$ as well.

As a fun exercise, one can examine how the ideal of constraints behaves when restricting to the zeroth-level algebra. In particular, the restricted constraints $\widehat{f}_\lambda := \Lambda \mathrm{J}_\lambda + \varepsilon^{\lambda}_{\mu\nu}\mathrm{P}_\mu \mathrm{K}_\nu$ satisfy the bracket relations:

$$
\begin{aligned}
[\widehat{f}_\mu, \mathrm{J}_\nu] &= \sum_{\lambda} \varepsilon^{\lambda}_{\mu\nu} \widehat{f}_\lambda\,, \\
[\widehat{f}_\mu, \mathrm{K}_\nu] &= 0\,, \\
[\widehat{f}_\mu, \mathrm{P}_\nu] &= 0\,,
\end{aligned}
$$

and lies in $\mathcal{U}(\mathfrak{igal}(3))$ and $\mathcal{U}(\widehat{\mathfrak{igal}}(3))$ for $\Lambda = 0$ and $\Lambda \neq 0$ respectively. One can then consider their squared sum, producing central elements in the UEA. These are precisely the Casimir elements (2.22b) and (3.27c) found in [Lév71] up to a proportionality factor. The second Casimirs can be interpreted as a sort of intrinsic angular momentum of a particle transforming under the extension of the Galilean algebra [Lév71, Section V.3.a.].

3.3 Interlude: Induced Module Construction

There is a vast literature[31] on infinite-dim. Lie algebras. However, there are few theorems that hold in full generality, and often only selective examples are worked out. In the following Section 3.3.1, we will highlight some approaches to construct representations that seemed promising but ultimately did not prove fruitful. The core of the problem is the fact that $\mathfrak{igal}(3)$ is not semisimple. To move forward, we introduce a modified version of the standard triangular decomposition of a Lie algebra in Section 3.3.2. Afterwards, in Section 3.3.4, we apply a similar induced module construction as the one for Verma modules. This modified construction produces two families of representations parametrized by a vacuum representation instead of a vacuum vector. We can then apply these considerations to the Lie algebra describing the corner structure of 4-dim. non-abelian BF theory on the torus. The representation space can be identified with that of polynomials in infinitely many variables. Furthermore, there is a grading induced by the degree of the polynomials and a distinct vacuum-like vector.

3.3.1 Failed Attempts

One can construct representations of $\widehat{\mathcal{G}}(\mathfrak{su}(2))$ and $\widehat{\mathcal{G}}_\Lambda(\mathfrak{su}(2))$ by using representations of the Galilei Lie algebra that restrict to $\mathfrak{igal}(3)$ and then lifting them to the double-loop algebra and finally extending them trivially to the full algebras. However, these representations are undesirable since the central charge is manifestly represented by zero. Another possibility is to try to non-trivially extend representations of the underlying double loop algebra using Lau's construction [Lau05]. However, one cannot choose the central extension arbitrarily, i.e. the cocycle is determined by the construction itself, up to some limited choices. For most reasonable guesses, the resulting cocycle is very different from the one of interest. Other reasonable approaches include trying to adapt the free field constructions introduced by [Wak86] for $\widehat{\mathfrak{sl}}_2$ and generalized by [FF88] and others to general affine Lie algebras. These representations have even been extended to the double-loop setting in [You21] and [Fra24]. Free field realizations are important in CFT, as they embed the complicated symmetry algebra into a much simpler algebra coming from free field modes. However, these constructions crucially use the simplicity of the original, finite-dim. Lie algebra, so it is not clear how to adapt them to the present case. There is also the framework unifying finite-dim. semisimple, affine, toroidal, and other Lie algebras called extended affine Lie algebras (EALAs) (c.f. [All+97; Neh04]). However, it is also not clear how to deal with the non-semisimplicity in this case. A further possibility might be to describe representations of some larger Lie algebra and obtain the one in question by a contraction procedure. For example, there is a contraction

[31] Unfortunately, the terminologies differ wildly and it is difficult to navigate.

from $\mathfrak{so}(5)$ to the Poincaré algebra and further to the Galilei algebra. Representations of the former are well under control even for extended double-loop algebras (see EALAs above). So one could hope to extend the contraction to the infinite-dim. case. However, we were not able to make much progress with this method.

To the best of our knowledge, there are no results about constructing useful representations for the present case. Instead, we have to introduce a new construction.

3.3.2 Modified Triangular Decomposition

The standard Verma module construction, e.g. Chapter 9 in [KR87], starts with a triangular decomposition. Inspired by that decomposition, we define a modified and abstract version thereof.

Definition 3.10. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K} = \mathbb{R}, \mathbb{C}$, potentially infinite dim. A **modified triangular decomposition (MTD)** of $\mathfrak{g}$ is a decomposition of the Lie algebra into a direct sum of vector subspaces

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+,$$

such that the following equations hold

$$\begin{aligned}
&[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}, \\
&[\mathfrak{h}, \mathfrak{n}^\pm] \subseteq \mathfrak{n}^\pm, \\
&[\mathfrak{n}^\pm, \mathfrak{n}^\mp] \subseteq \mathfrak{h}, \\
&[\mathfrak{n}^\pm, \mathfrak{n}^\pm] = \{0\}\,.
\end{aligned}$$

These equations are compatible with the following $\mathbb{Z}$-grading: $\mathfrak{g}_0 \coloneqq \mathfrak{h}, \mathfrak{g}_{-1} \coloneqq \mathfrak{n}^-, \mathfrak{g}_1 \coloneqq \mathfrak{n}^+$ and $\mathfrak{g}_\alpha \coloneqq \{0\}$ for $\alpha \neq -1, 0, 1$. The essential structural difference to the usual triangular decomposition is that $\mathfrak{h}$ is not required to be abelian.

The following example will be the cornerstone of the application to BF theory.

Example 3.11 (Existence of MTD for $\widehat{\mathcal{G}}(\mathfrak{su}(2))$ & $\widehat{\mathcal{G}}_\Lambda(\mathfrak{su}(2))$)**.** *Define new basis generators by* $\mathrm{X}^\pm \coloneqq \pm \mathrm{X}_1 - i\mathrm{X}_2$ *and* $\mathrm{X}^z \coloneqq -2i\mathrm{X}_3$ *for* $\mathrm{X} \in \{\mathrm{J}, \mathrm{K}, \mathrm{P}\}$. *Then, with a little work, one can show the new relations:*

$$\begin{aligned}
[\mathrm{J}^+_{kl}, \mathrm{J}^-_{mn}] &= \mathrm{J}^z_{(k+m)(l+n)}\,, \\
[\mathrm{J}^z_{kl}, \mathrm{J}^\pm_{mn}] &= \pm 2\mathrm{J}^\pm_{(k+m)(l+n)}\,, \\
[\mathrm{J}^+_{kl}, \mathrm{K}^-_{mn}] &= [\mathrm{K}^+_{kl}, \mathrm{J}^-_{mn}] = \mathrm{K}^z_{(k+m)(l+n)} - 2im\delta_{k,-m}\delta_{l,-n}\mathrm{Z}\,, \\
[\mathrm{K}^z_{kl}, \mathrm{J}^\pm_{mn}] &= \pm 2\mathrm{K}^\pm_{(k+m)(l+n)}\,, \\
[\mathrm{J}^z_{kl}, \mathrm{K}^\pm_{mn}] &= \pm 2\mathrm{K}^\pm_{(k+m)(l+n)}\,,
\end{aligned}$$

$$
\begin{aligned}
[\mathrm{J}^z_{kl}, \mathrm{K}^z_{mn}] &= -4im\delta_{k,-m}\delta_{l,-n}\mathrm{Z}\,,\\
[\mathrm{J}^+_{kl}, \mathrm{P}^-_{mn}] &= [\mathrm{P}^+_{kl}, \mathrm{J}^-_{mn}] = \mathrm{P}^z_{(k+m)(l+n)} + 2in\delta_{k,-m}\delta_{l,-n}\mathrm{Z}\,,\\
[\mathrm{P}^z_{kl}, \mathrm{J}^\pm_{mn}] &= \pm 2\mathrm{P}^\pm_{(k+m)(l+n)}\,,\\
[\mathrm{J}^z_{kl}, \mathrm{P}^\pm_{mn}] &= \pm 2\mathrm{P}^\pm_{(k+m)(l+n)}\,,\\
[\mathrm{J}^z_{kl}, \mathrm{P}^z_{mn}] &= 4in\delta_{k,-m}\delta_{l,-n}\mathrm{Z}\,,\\
[\mathrm{K}^+_{kl}, \mathrm{P}^-_{mn}] &= [\mathrm{K}^-_{kl}, \mathrm{P}^+_{mn}] = -2\Lambda\delta_{k,-m}\delta_{l,-n}\mathrm{Z}\,,\\
[\mathrm{K}^z_{kl}, \mathrm{P}^z_{mn}] &= -4\Lambda\delta_{k,-m}\delta_{l,-n}\mathrm{Z}\,.
\end{aligned}
$$

Note that by setting $\Lambda = 0$, we obtain the relations for $\widehat{\mathcal{G}}(\mathfrak{su}(2))$ instead. The relations of the generators are very reminiscent of ordinary $\mathfrak{su}(2)$ on a structural level. This observation can be explained by the fact that $\mathfrak{igal}(3)$ is isomorphic to the Lie algebra $\mathfrak{su}(2) \ltimes (\mathbb{R}^3 \oplus \mathbb{R}^3)$, so much of the structure of $\mathfrak{su}(2)$ persists. From the relations, we can infer that the Lie algebras $\widehat{\mathcal{G}}(\mathfrak{su}(2))$ and $\widehat{\mathcal{G}}_\Lambda(\mathfrak{su}(2))$ allow for a modified triangular decomposition given by

$$
\begin{aligned}
\mathfrak{h} &\coloneqq \mathrm{span}_{\mathbb{K}}(\{\mathrm{J}^z_{kl}, \mathrm{K}^z_{mn}, \mathrm{P}^z_{rs}, \mathrm{Z}\}_{k,l,m,n,r,s\in\mathbb{Z}})\,,\\
\mathfrak{n}^\pm &\coloneqq \mathrm{span}_{\mathbb{K}}(\{\mathrm{J}^\pm_{kl}, \mathrm{K}^\pm_{mn}, \mathrm{P}^\pm_{rs}\}_{k,l,m,n,r,s\in\mathbb{Z}})\,.
\end{aligned}
$$

Next, we mimic the definition of the Verma module.

3.3.3 Induced Module $\mathrm{M}^\pm_\rho$

Let (V, ρ) be a representation of $\mathfrak{h}$, not necessarily 1 dim. Equivalently, V is a $\mathcal{U}(\mathfrak{h})$-module. Note that $\mathfrak{h} \oplus \mathfrak{n}^\pm$ is a subalgebra of $\mathfrak{g}$.[32] Similarly to the usual highest-weight definition, we can extend the module structure on V to a $\mathcal{U}(\mathfrak{h} \oplus \mathfrak{n}^\pm)$-module by defining

$$
\begin{aligned}
h \cdot v &\coloneqq \rho(h)v \quad \forall h \in \mathfrak{h}, \forall v \in V\,,\\
x \cdot v &\coloneqq 0 \qquad\quad \forall x \in \mathfrak{n}^\pm, \forall v \in V\,,
\end{aligned}
$$

and the rest by linearity and the usual composition law.

Lemma 3.12. *This action is well defined.*

Proof. It is sufficient, in fact equivalent, to show that V is a representation of the Lie algebra $\mathfrak{h} \oplus \mathfrak{n}^\pm$. For clarity's sake, we label the $\mathcal{U}(\mathfrak{h} \oplus \mathfrak{n}^\pm)$-action by P. Linearity of the action follows from:

$$
\mathrm{P}(\alpha h + h') = \rho(\alpha h + h') = \alpha\rho(h) + \rho(h') = \alpha\mathrm{P}(h) + \mathrm{P}(h') \quad \forall h, h' \in \mathfrak{h}, \forall \alpha \in \mathbb{K}\,,
$$

[32] We will not keep track of a consistent ordering of the summands unless confusion could arise.

$$\mathrm{P}(\alpha x + x') = 0 = \alpha \mathrm{P}(x) + \mathrm{P}(x') \qquad \forall x, x' \in \mathfrak{n}^{\pm}, \forall \alpha \in \mathbb{K}\,,$$
$$\mathrm{P}(\alpha h + \beta x) = \alpha \rho(h) + 0 = \alpha \mathrm{P}(h) + \beta \mathrm{P}(x) \qquad \forall h \in \mathfrak{h}, \forall x \in \mathfrak{n}^{\pm} \forall \alpha, \beta \in \mathbb{K}\,.$$

and the homomorphism property from:

$$[\mathrm{P}(h), \mathrm{P}(h')] = [\rho(h), \rho(h')] = \rho([h, h']) = \mathrm{P}([h, h']) \quad \forall h, h' \in \mathfrak{h}\,,$$
$$[\mathrm{P}(x), \mathrm{P}(x')] = [0, 0] = 0 = \mathrm{P}([x, x']) \qquad \forall x, x' \in \mathfrak{n}^{\pm}\,,$$
$$[\mathrm{P}(h), \mathrm{P}(x)] = [\rho(h), 0] = 0 = \mathrm{P}([h, x]) \qquad \forall h \in \mathfrak{h}, \forall x \in \mathfrak{n}^{\pm}\,.$$

□

This enhanced module structure allows us to define the Verma-module equivalent.

Definition 3.13. The **induced $\mathcal{U}(\mathfrak{g})$-module** $\mathrm{M}_\rho^{\pm}$ is defined by the induced module construction

$$\mathrm{M}_\rho^{\pm} \coloneqq \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h} \oplus \mathfrak{n}^{\pm})} V\,.$$

Comparing to the usual highest-weight module, the generalization is essentially allowing for an entire vacuum subspace generating the representation rather than a vacuum vector. Next, we work out a more explicit description of the module.

Proposition 3.14 ([Pav, Proposition 2.5.15.])**.** *As a left $\mathcal{U}(\mathfrak{n}^{\mp})$-module* $\mathrm{M}_\rho^{\pm} \cong \mathcal{U}(\mathfrak{n}^{\mp}) \otimes_{\mathbb{K}} V$

Proof. The proof in [Pav, Proposition 2.5.15.] goes as follows

$$\begin{aligned}\mathrm{M}_\rho^{\pm} &= \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h} \oplus \mathfrak{n}^{\pm})} V \\ &\cong (\mathcal{U}(\mathfrak{n}^{\mp}) \otimes_{\mathbb{K}} \mathcal{U}(\mathfrak{h} \oplus \mathfrak{n}^{\pm})) \otimes_{\mathcal{U}(\mathfrak{h} \oplus \mathfrak{n}^{\pm})} V \\ &\cong \mathcal{U}(\mathfrak{n}^{\mp}) \otimes_{\mathbb{K}} \underbrace{(\mathcal{U}(\mathfrak{h} \oplus \mathfrak{n}^{\pm}) \otimes_{\mathcal{U}(\mathfrak{h} \oplus \mathfrak{n}^{\pm})} V)}_{\cong V} \\ &\cong \mathcal{U}(\mathfrak{n}^{\mp}) \otimes_{\mathbb{K}} V\,,\end{aligned}$$

where the isomorphism on the second line is meant in terms of $\mathcal{U}(\mathfrak{n}^{\mp})$-$\mathcal{U}(\mathfrak{h} \oplus \mathfrak{n}^{\pm})$-bimodules. □

This proposition allows us to find a convenient basis for the underlying vector space. For concreteness, let us restrict to at most countably infinite-dimensional Lie algebras and representations. This is technically not necessary but will make the following notation less cumbersome. Let $\{v_i\}_{i \in \mathbb{Z}}$ be a basis of the representation V and $\{\mathrm{X}_j^{\pm}\}_{j \in \mathbb{Z}}$ be bases of the Lie algebras $\mathfrak{n}^{\pm}$. By the PBW theorem, a basis of $\mathcal{U}(\mathfrak{n}^{\mp}) \otimes_{\mathbb{K}} V$ is given by the set

$$\{\mathrm{X}_{j_1}^{\mp} \cdots \mathrm{X}_{j_n}^{\mp} \otimes_{\mathbb{K}} v_i\}_{j_1 \leq \cdots \leq j_n,\, n \in \mathbb{N},\, i \in \mathbb{Z}}\,. \tag{3.3}$$

The basis vectors arising from 1 on the l.h.s. of (3.3) are always understood to be included as well. By the isomorphism above, the set

$$\{\mathrm{X}^{\mp}_{j_1} \cdots \mathrm{X}^{\mp}_{j_n} \otimes_{\mathcal{U}(\mathfrak{h}\oplus\mathfrak{n}^{\pm})} v_i\}_{j_1\leq\cdots\leq j_n,\, n\in\mathbb{N},\, i\in\mathbb{Z}}\,, \tag{3.4}$$

is a basis of the module $\mathrm{M}^{\pm}_{\rho}$.

We can now attempt to give more meaning to this space. The $\mathrm{X}^{\pm}$-operators commute among each other; therefore, they generate bosonic Fock space states. This motivates the notation from physics:

Notation 3.15. We denote the basis states (3.4) of the induced module $\mathrm{M}^{\pm}_{\rho}$ by

$$\left|x^{\mp}_{j_1} \cdots x^{\mp}_{j_n}; v_i\right\rangle_{j_1\leq\cdots\leq j_n,\, n\in\mathbb{N},\, i\in\mathbb{Z}}\,.$$

The generators and their representations will be denoted by the same letter unless it could lead to confusion. We will also use the words generator and operator indistinguishably.

There are many questions raised by this construction. Here, we would like to list some that we feel would be interesting to explore in the future: What types of Lie algebras admit a MTD? Under what conditions are two induced modules equivalent for different vacuum sectors V and V'? How much of the theory of highest-weight representations carries over? For example, one should be able to define singular sectors as a generalization of singular vectors.

Definition 3.16. A **singular sector** W is a non-empty vector subspace of $\mathrm{M}^{\pm}_{\rho}$ such that:

$$\begin{aligned} \mathfrak{h}\cdot W &\subset W\,, \\ \mathfrak{n}^{\pm}\cdot W &= 0\,. \end{aligned}$$

Note that the action $\mathfrak{n}^{\mp}\cdot W$ is not restricted. The vacuum sector V is an example of a singular sector. One might then try to show the equivalence between subrepresentations and singular sectors.

3.3.4 Induced Module Construction for $\widehat{\mathcal{G}}(\mathfrak{su}(2))$ & $\widehat{\mathcal{G}}_{\Lambda}(\mathfrak{su}(2))$

The induced module considerations can now be applied to the Lie algebra describing the corner structure of 4-dim. non-abelian BF theory on the torus. Let $\widehat{\mathcal{G}}_{\Lambda}(\mathfrak{su}(2)) \cong \mathfrak{n}^{-}\oplus\mathfrak{h}\oplus\mathfrak{n}^{+}$ be the MTD of Example 3.11 and similarly for $\Lambda = 0$. First, we need to determine a suitable representation of $\mathfrak{h}$. To find such representations, we can use the following fact.

Lemma 3.17 (Embedding of the Abelian Case)**.** *There is an isomorphism of Lie algebras*

$$\mathfrak{h} \cong \widehat{\mathcal{G}}_\Lambda ,$$

where $\widehat{\mathcal{G}}_\Lambda$ denotes the abelian Lie algebra from Section 2 with brackets (2.2) - (2.4)*.*

Proof. To see this, consider the linear map $\iota\colon \widehat{\mathcal{G}}_\Lambda \longrightarrow \mathfrak{h}$ defined on the basis by: $\mathrm{E}_{kl} \longmapsto \frac{i}{2}\mathrm{J}^z_{kl}, \Phi_{kl} \longmapsto \frac{i}{2}\mathrm{K}^z_{kl}, \Theta_{kl} \longmapsto \frac{i}{2}\mathrm{P}^z_{kl}$ and $\mathrm{Z} \longrightarrow \mathrm{Z}$. This map is an isomorphism of Lie algebras as one can easily check. □

Consequently, the abelian case is "nicely" embedded in the non-abelian case. This embedding is not unique, however, as there is a second choice for the isomorphism, namely $\mathrm{E}_{kl} \longmapsto \frac{-i}{2}\mathrm{J}^z_{kl}, \Phi_{kl} \longmapsto \frac{-i}{2}\mathrm{K}^z_{kl}, \Theta_{kl} \longmapsto \frac{-i}{2}\mathrm{P}^z_{kl}$. If $\Lambda = 0$, there are even uncountably many choices. We will choose $\mathrm{E}_{kl} \longmapsto \frac{i}{2}\mathrm{J}^z_{kl}, \Phi_{kl} \longmapsto \frac{i}{2}\mathrm{K}^z_{kl}, \Theta_{kl} \longmapsto \frac{i}{2}\mathrm{P}^z_{kl}$. In Theorem 2.2, we have shown that the Lie algebra is isomorphic to the oscillator algebra together with the countably infinite-dim. abelian Lie algebra. Therefore, the trivial representation $\mathbb{C}$ or the bosonic Fock space V are possible representations. The former would force the central charge to act trivially, which we want to avoid.

3.3.5 Bosonic Fock Spaces $\mathcal{H}^\pm_\rho$ & $\mathcal{H}^\pm_{\rho,\Lambda}$

Choose the irreducible[33] bosonic Fock space representation (V, ρ) of $\mathfrak{h}$ and define the induced modules in 3.13, which we will denote $\mathcal{H}^\pm_\rho$ and $\mathcal{H}^\pm_{\rho,\Lambda}$ respectively. It turns out that these modules have some nice properties.

Vacuum Vector The module V possesses a unique vacuum vector: the constant polynomial 1, which we shall denote by $|0\rangle$. Similarly, there is a distinguished vector in the induced representations.

Definition 3.18. The **vacuum vector** or **vacuum state** of $\mathcal{H}^\pm_{\rho,\Lambda}$ is defined as

$$|0\rangle_{\mathcal{H}^\pm_{\rho,\Lambda}} := 1 \otimes_{\mathcal{U}(\mathfrak{h}\oplus\mathfrak{n}^\pm)} |0\rangle ,$$

where 1 is the identity element in the unital algebra $\mathcal{U}(\widehat{\mathcal{G}}_\Lambda(\mathfrak{su}(2))$. Similarly, we define the vacuum vector of $\mathcal{H}^\pm_\rho$.

From now on, we will identify vectors in V with their image under the embedding, i.e. $v = 1 \otimes_{\mathcal{U}(\mathfrak{h}\oplus\mathfrak{n}^\pm)} v$ if there are no ambiguities. To refer to V, we also speak of the **vacuum sector** or **abelian sector**, because they get annihilated by the operators in $\mathfrak{n}^\pm$. These operators will be called **lowering** or **annihilation operators** and the ones

[33] The space of polynomials is already irreducible under the action of $\mathcal{A}$ so adding the abelian summand does not change that fact.

in $\mathfrak{n}^{\mp}$ will be called **raising** or **creation operators**. Additionally, we focus on $\mathcal{H}_\rho^+$ and $\mathcal{H}_{\rho,\Lambda}^+$ for concreteness, from now on and drop the respective minus sign in the label of the basis states to reduce the amount of clutter in the notation. Thus the basis states of the induced modules look as follows:

$$|j_{k_1 l_1} \cdots j_{k_a l_a}\; k_{m_1 n_1} \cdots k_{m_b n_b}\; p_{r_1 s_1} \cdots p_{r_c s_c}; v_i\rangle \ ,$$

where $a, b, c \in \mathbb{N}$, $k_1 \leq \cdots \leq k_a, \ldots, s_1 \leq \cdots \leq s_c$ and $\{v_i\}_{i\in\mathbb{Z}}$ are a basis of V. Later, we will also use an explicit basis of V spanned by the monomials in countably many variables.

$\mathbb{Z}$-Grading The analogy to lowering operators only works when acting on states in V. In general, the action of the various generators only remotely resembles that of the usual ladder operators. There is, however, an operator that acts diagonally in the chosen basis and behaves like a number operator. Define the **number operator** $\mathrm{N} := -\frac{1}{2}\mathrm{J}^z_{00} - i\chi_{\widehat{\mathrm{E}}}$ for $\mathcal{H}_\rho^+$ and $\mathcal{H}_{\rho,\Lambda}^+$ respectively, where $\chi_{\widehat{\mathrm{E}}} \in \mathbb{C}$ is determined by the action of $\widehat{\mathrm{E}}$ on V. The Fock space is irreducible; therefore, by Dixmier's Lemma, it must act by a multiple of the identity.

Lemma 3.19. *The number operator satisfies the following properties:*

1. $[\mathrm{N}, \mathrm{X}^{\mp}_{kl}] = \pm \mathrm{X}^{\mp}_{kl}$ *for* $\mathrm{X} = \mathrm{J}, \mathrm{K}, \mathrm{P}$ *and* $k, l \in \mathbb{Z}$ *and zero otherwise*
2. $\mathrm{N}\,|j_{k_1 l_1} \cdots j_{k_a l_a}\; k_{m_1 n_1} \cdots k_{m_b n_b}\; p_{r_1 s_1} \cdots p_{r_c s_c}; v\rangle = (a+b+c)|j_{k_1 l_1} \cdots j_{k_a l_a}\; k_{m_1 n_1} \cdots k_{m_b n_b}\; p_{r_1 s_1} \cdots p_{r_c s_c}; v\rangle \quad \forall v \in V$

Proof. 1. This statement follows directly from Example 3.11. 2. One can use property 1. to obtain the desired result, because the r.h.s. can be written solely in terms of creation operators acting on the state $|v\rangle$. Therefore, the number operator can be commuted through until it hits the state in the vacuum sector. Finally, using $\mathrm{N}\,|v\rangle = 0, \quad \forall v \in V$, the statement follows. The action on the vacuum is worked out in detail in Section 3.4.1. □

From the properties, we can see that the name is indeed justified. As a consequence, $\mathcal{H}_{\rho,\Lambda}^+$ decomposes

$$\mathcal{H}_{\rho,\Lambda}^+ = \bigoplus_{k\in\mathbb{N}} (\mathcal{H}_{\rho,\Lambda}^+)_k,$$

where $(\mathcal{H}_{\rho,\Lambda}^+)_k$ is the eigenspace of N with eigenvalue $k \in \mathbb{N}$. In particular, $(\mathcal{H}_{\rho,\Lambda}^+)_0 = V$. Each eigenspace is infinite dimensional since the number operator just counts the degree of the monomial and there are infinitely many variables. Analogously, we have a $\mathbb{N}$-grading of $\mathcal{H}_\rho^+$. The operators turn into graded maps accordingly. The $(\pm)$-operators are maps of degree ∓ 1 respectively and the z-operators are maps of degree 0. The degree of a homogeneous element $p \in \mathcal{H}_\rho^+$ will be denoted $|p|$. In Section 3.4.2, we will use the degree of the operators to infer irreducibility for a particular choice of ρ.

In the coming sections, we will treat the cases of zero and non-zero cosmological constant separately as the explicit constructions are different.

3.4 Representations of $\widehat{\mathcal{G}}(\mathfrak{su}(2))$

In this section, we work out a family of explicit representations of the generators on the bosonic Fock space $\mathcal{H}_\rho^+$ in terms of differential operators up to second order. We will prove irreducibility of a particular example. Finally, we explore the consequences of imposing the on-shell constraints. To ensure that the action is well-defined, we are forced to choose certain polarizations of V. Unfortunately, the constraints have a non-zero action in the module, which essentially forces the representation to be trivial if it is irreducible.

3.4.1 Action of the Generators on $\mathcal{H}_\rho^+$

It is reasonable to expect that the generators can be realized as differential operators because they are linear maps on a space of polynomials. One can obtain the individual contributions by acting on higher and higher monomials. To this end, we define the $\mathfrak{h}$-action ρ on V explicitly using the embedded abelian case. Recall how the generators of $\mathfrak{h}$ are expressed in terms of the abelian generators and subsequently the ladder operators from the proof of Theorem 2.2. The representation of the ladder operators is defined by:

$$\begin{aligned}
\mathrm{c}_{kl}^\dagger &\longmapsto v_{kl}, && k \neq 0, l \neq 0,\\
\mathrm{c}_{kl} &\longmapsto i\chi_{\mathrm{Z}}\frac{\partial}{\partial v_{kl}}, && k \neq 0, l \neq 0,\\
\mathrm{a}_l^\dagger &\longmapsto v_{0l}, && l \neq 0,\\
\mathrm{a}_l &\longmapsto i\chi_{\mathrm{Z}}\frac{\partial}{\partial v_{0l}}, && l \neq 0,\\
\mathrm{b}_k^\dagger &\longmapsto v_{k0}, && k \neq 0,\\
\mathrm{b}_k &\longmapsto i\chi_{\mathrm{Z}}\frac{\partial}{\partial v_{k0}}, && k \neq 0,\\
\mathrm{Z} &\longmapsto \chi_{\mathrm{Z}}, &&\\
\widehat{\Phi}_l &\longmapsto \chi_{\widehat{\Phi}_l}, && l \in \mathbb{Z},\\
\widehat{\Theta}_k &\longmapsto \chi_{\widehat{\Theta}_k}, && k \in \mathbb{Z},\\
\widehat{\mathrm{F}}_{kl}^- &\longmapsto \chi_{\widehat{\mathrm{F}}_{kl}^-}, && k, l \neq 0,\\
\widehat{\mathrm{E}} &\longmapsto \chi_{\widehat{\mathrm{E}}},
\end{aligned}$$

and for convenience $v_{00} \equiv 1$, $\frac{\partial}{\partial v_{00}} \equiv 1$. By the same argument as in defining the number operator, the central elements are proportional to the identity and thus determined by a complex number. We call these proportionality factors the respective charge of the generators. Note that we could choose a different assignment of multiplication and differentiation. We call this assignment a choice of polarization in V. It turns out that to impose the constraints, we have to change to a different polarization.

We also sometimes denote the states by the respective polynomial.

$$\begin{aligned} 1 &\equiv |0\rangle \,, \\ j_{k_1 l_1} \cdots j_{k_a l_a}\; k_{m_1 n_1} \cdots k_{m_b n_b}\; p_{r_1 s_1} \cdots p_{r_c s_c}\; v_{p_1 q_1} \cdots v_{p_d q_d} &\equiv \\ &\left| j_{k_1 l_1} \cdots j_{k_a l_a}\; k_{m_1 n_1} \cdots k_{m_b n_b} p_{r_1 s_1} \cdots p_{r_c s_c}; v_{p_1 q_1} \cdots v_{p_d q_d} \right\rangle , \end{aligned}$$

where we now also express the basis of V using the ladder operators defined above acting on the vacuum.

Action on the Vacuum Sector By systematically acting on higher and higher monomials, we can deduce the necessary coefficients in front of the differential operators. We just need to commute the operators until they hit the vacuum sector and then apply the explicit formulas from above. Let us illustrate this with an example. Let $\mathrm{X} \in \mathfrak{n}^-$, $v \in V$, and $\mathrm{Y} \in \mathfrak{g}$ then:

$$\begin{aligned} \mathrm{Y}(\mathrm{X} \otimes_{\mathcal{U}(\mathfrak{h}\oplus\mathfrak{n}^\pm)} v) &= \mathrm{YX} \otimes_{\mathcal{U}(\mathfrak{h}\oplus\mathfrak{n}^\pm)} v \\ &= ([\mathrm{Y},\mathrm{X}] + \mathrm{XY}) \otimes_{\mathcal{U}(\mathfrak{h}\oplus\mathfrak{n}^\pm)} v \,, \end{aligned}$$

where the first equality follows by the definition of the module action. If $\mathrm{Y} \in \mathfrak{n}^-$, the first summand vanishes because the operators commute. The rest cannot be reduced further. If $\mathrm{Y} \in \mathfrak{n}^+$, the first summand lies in $\mathfrak{h}$ and the second one vanishes, i.e. $\mathrm{Y}(\mathrm{X}\otimes_{\mathcal{U}(\mathfrak{h}\oplus\mathfrak{n}^\pm)} v) = 1 \otimes_{\mathcal{U}(\mathfrak{h}\oplus\mathfrak{n}^\pm)} [\mathrm{Y},\mathrm{X}]v$. If $\mathrm{Y} \in \mathfrak{h}$, the first summand lies in $\mathfrak{n}^-$ and the second one can be reduced, i.e. $\mathrm{X}(\mathrm{Y} \otimes_{\mathcal{U}(\mathfrak{h}\oplus\mathfrak{n}^\pm)} v) = \mathrm{X} \otimes_{\mathcal{U}(\mathfrak{h}\oplus\mathfrak{n}^\pm)} \rho(\mathrm{Y})v$. Finally, the expressions where the operators act on v can be simplified further using the explicit formulas for the $\mathfrak{h}$-action ρ. Let us now calculate the action of the generators on the vacuum sector.

The central charge:

$$\begin{aligned} \mathrm{Z}\,|v\rangle &= 1 \otimes_{\mathcal{U}(\mathfrak{h}\oplus\mathfrak{n}^\pm)} \mathrm{Z}\,|v\rangle \,, \\ &\coloneqq \chi_{\mathrm{Z}}\,|v\rangle \;. \end{aligned}$$

Notice that the action of Z is proportional to the identity on all of $\mathcal{H}_\rho^+$ and is determined by its action on the vacuum sector V. From now on, we set χ_{Z}=1 and suppress the tensor product notation.

The creation operators:

$$\begin{aligned}
\mathrm{J}^-_{kl} |v\rangle &= |j_{kl}; v\rangle\,, \\
\mathrm{K}^-_{kl} |v\rangle &= |k_{kl}; v\rangle\,, \\
\mathrm{P}^-_{kl} |v\rangle &= |p_{kl}; v\rangle\ .
\end{aligned}$$

The annihilation operators:

$$\begin{aligned}
\mathrm{J}^+_{kl} |v\rangle &= 0, \\
\mathrm{K}^+_{kl} |v\rangle &= 0, \\
\mathrm{P}^+_{kl} |v\rangle &= 0\,.
\end{aligned}$$

The z-operators:

$$\begin{aligned}
\mathrm{J}^z_{kl} |v\rangle &= -2i\mathrm{E}_{kl} |v\rangle \\
&= \begin{cases} -2i\mathrm{c}_{-k-l} |v\rangle\,, & \text{for } k \neq 0, l \neq 0 \\ -2i\mathrm{b}_{-k} |v\rangle\,, & \text{for } k \neq 0, l = 0 \\ -2i\mathrm{a}_{-l} |v\rangle\,, & \text{for } k = 0, l \neq 0 \\ -2i\widehat{\mathrm{E}} |v\rangle\,, & \text{for } k = l = 0 \end{cases} \\
&= \begin{cases} 2\frac{\partial}{\partial v_{-k-l}} |v\rangle\,, & \text{for } k \neq 0, l \neq 0 \\ 2\frac{\partial}{\partial v_{-k0}} |v\rangle\,, & \text{for } k \neq 0, l = 0 \\ 2\frac{\partial}{\partial v_{0-l}} |v\rangle\,, & \text{for } k = 0, l \neq 0 \\ -2i\widehat{\mathrm{E}} |v\rangle\,, & \text{for } k = l = 0 \end{cases} \\
&= (2 + (-2 - 2i\chi_{\widehat{\mathrm{E}}})\delta_{k,0}\delta_{l,0}) \frac{\partial}{\partial v_{-k-l}} |v\rangle \\
\mathrm{K}^z_{kl} |v\rangle &= -2i\Phi_{kl} |v\rangle \\
&= \begin{cases} -2ik\left(\mathrm{c}^\dagger_{kl} - \mathrm{F}^-_{kl}\right) |v\rangle\,, & \text{for } k \neq 0, l \neq 0 \\ -2ik\mathrm{b}^\dagger_k |v\rangle\,, & \text{for } k \neq 0, l = 0 \\ -2i\widehat{\Phi}_l |v\rangle\,, & \text{for } k = 0, l \in \mathbb{Z} \end{cases} \\
&= \begin{cases} -2ik |v_{kl} v\rangle + 2ik\chi_{\mathrm{F}^-_{kl}} |v\rangle\,, & \text{for } k \neq 0, l \neq 0 \\ -2ik |v_{k0} v\rangle\,, & \text{for } k \neq 0, l = 0 \\ -2i\chi_{\widehat{\Phi}_l} |v\rangle\,, & \text{for } k = 0, l \in \mathbb{Z} \end{cases} \\
&= (-2ikv_{kl} - 2i\chi_{\widehat{\Phi}_l}\delta_{k,0} + 2ik\chi_{\mathrm{F}^-_{kl}}(1 - \delta_{l,0})) |v\rangle \\
\mathrm{P}^z_{kl} |v\rangle &= -2i\Theta_{kl} |v\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \begin{cases} 2il\left(\mathrm{c}^\dagger_{kl} + \mathrm{F}^-_{kl}\right)|v\rangle\,, & \text{for } k \neq 0, l \neq 0 \\ 2il\mathrm{a}^\dagger_l\,|v\rangle\,, & \text{for } k = 0, l \neq 0 \\ -2i\widehat{\Theta}_k\,|v\rangle\,, & \text{for } k \in \mathbb{Z}, l = 0 \end{cases} \\
&= \begin{cases} 2il\,|v_{kl}v\rangle + 2il\chi_{\mathrm{F}^-_{kl}}\,|v\rangle\,, & \text{for } k \neq 0, l \neq 0 \\ 2il\,|v_{0l}v\rangle\,, & \text{for } k = 0, l \neq 0 \\ -2i\chi_{\widehat{\Theta}_k}\,|v\rangle\,, & \text{for } k \in \mathbb{Z}, l = 0 \end{cases} \\
&= (2ilv_{kl} - 2i\chi_{\widehat{\Theta}_k}\delta_{l,0} + 2il\chi_{\mathrm{F}^-_{kl}}(1-\delta_{k,0}))\,|v\rangle
\end{aligned}
$$

One can continue with determining the action on first excited states, i.e. monomials of degree one and so on. The following theorem gives an explicit formula for the induced module of $\widehat{\mathcal{G}}(\mathfrak{su}(2))$. These modules resemble the free field realizations of affine algebras and also the polynomial representations of [Mor+22].

Theorem 3.20. *The assignment*

$$
\begin{aligned}
\mathrm{Z} &\longrightarrow 1 \\
\mathrm{J}^-_{kl} &\longrightarrow j_{kl} \\
\mathrm{K}^-_{kl} &\longrightarrow k_{kl} \\
\mathrm{P}^-_{kl} &\longrightarrow p_{kl} \\
\mathrm{J}^z_{kl} &\longrightarrow (2 + (-2 - 2i\chi_{\widehat{\mathrm{E}}})\delta_{k,0}\delta_{l,0})\frac{\partial}{\partial v_{-k-l}} + \sum_{m,n}(-2j_{(k+m)(l+n)})\frac{\partial}{\partial j_{mn}} \\
&\quad + \sum_{m,n}(-2k_{(k+m)(l+n)})\frac{\partial}{\partial k_{mn}} + \sum_{m,n}(-2p_{(k+m)(l+n)})\frac{\partial}{\partial p_{mn}} \\
\mathrm{K}^z_{kl} &\longrightarrow -2ikv_{kl} - 2i\chi_{\widehat{\Phi}_l}\delta_{k,0} + 2ik\chi_{\mathrm{F}^-_{kl}}(1-\delta_{l,0}) + \sum_{m,n}(-2k_{(k+m)(l+n)})\frac{\partial}{\partial j_{mn}} \\
\mathrm{P}^z_{kl} &\longrightarrow 2ilv_{kl} - 2i\chi_{\widehat{\Theta}_k}\delta_{l,0} + 2il\chi_{\mathrm{F}^-_{kl}}(1-\delta_{k,0}) + \sum_{m,n}(-2p_{(k+m)(l+n)})\frac{\partial}{\partial j_{mn}} \\
\mathrm{J}^+_{kl} &\longrightarrow \sum_{m,n}(2 + (-2 - 2i\chi_{\widehat{\mathrm{E}}})\delta_{k+m,0}\delta_{l+n,0})\frac{\partial}{\partial v_{-(k+m)-(l+n)}}\frac{\partial}{\partial j_{mn}} \\
&\quad + \sum_{m,n}(-2i(k+m)v_{(k+m)(l+n)} + 2i(k+m)\chi_{\mathrm{F}^-_{kl}}(1-\delta_{l+n,0}) \\
&\quad\quad - 2i\chi_{\widehat{\Phi}_{l+n}}\delta_{k+m,0} - 2im\delta_{k+m,0}\delta_{l+n,0})\frac{\partial}{\partial k_{mn}} \\
&\quad + \sum_{m,n}(2i(l+n)v_{(k+m)(l+n)} + 2i(l+n)\chi_{\mathrm{F}^-_{kl}}(1-\delta_{k+m,0}) \\
&\quad\quad - 2i\chi_{\widehat{\Theta}_{k+m}}\delta_{l+n,0} + 2in\delta_{k+m,0}\delta_{l+n,0})\frac{\partial}{\partial p_{mn}}
\end{aligned}
$$

$$+\sum_{m,n}\sum_{r,s}(-2k_{(k+r+m)(l+s+n)})\frac{\partial}{\partial k_{rs}}\frac{\partial}{\partial j_{mn}}+\sum_{m,n}\sum_{r,s}(-2p_{(k+r+m)(l+s+n)})\frac{\partial}{\partial p_{rs}}\frac{\partial}{\partial j_{mn}}$$
$$+\sum_{m,n}\sum_{r,s}\frac{1}{2}(-2j_{(k+r+m)(l+s+n)})\frac{\partial}{\partial j_{rs}}\frac{\partial}{\partial j_{mn}}$$

$$\mathrm{K}^+_{kl}\longrightarrow\sum_{m,n}(-2i(k+m)v_{(k+m)(l+n)}+2i(k+m)\chi_{\mathrm{F}^-_{kl}}(1-\delta_{l+n,0})$$
$$-2i\chi_{\widehat{\Phi}_{l+n}}\delta_{k+m,0}-2im\delta_{k+m,0}\delta_{l+n,0})\frac{\partial}{\partial j_{mn}}$$
$$+\sum_{m,n}\sum_{r,s}\frac{1}{2}(-2k_{(k+r+m)(l+s+n)})\frac{\partial}{\partial j_{rs}}\frac{\partial}{\partial j_{mn}}$$

$$\mathrm{P}^+_{kl}\longrightarrow\sum_{m,n}(2i(l+n)v_{(k+m)(l+n)}+2i(l+n)\chi_{\mathrm{F}^-_{kl}}(1-\delta_{k+m,0})$$
$$-2i\chi_{\widehat{\Theta}_{k+m}}\delta_{l+n,0}+2in\delta_{k+m,0}\delta_{l+n,0})\frac{\partial}{\partial j_{mn}}$$
$$+\sum_{m,n}\sum_{r,s}\frac{1}{2}(-2p_{(k+r+m)(l+s+n)})\frac{\partial}{\partial j_{rs}}\frac{\partial}{\partial j_{mn}}$$

constitutes a representation of $\widehat{\mathcal{G}}(\mathfrak{su}(2))$ on the bosonic Fock space $\mathcal{H}^+_\rho$.

Proof. The proof is quite lengthy and can be found in Appendix B, modulo setting most of the charges to zero for convenience and in light of the constraints. □

To make the formulas more compact, we introduce the following notation:

$$\Delta^{\widehat{\mathrm{E}}}_{kl}\coloneqq 2+(-2-2i\chi_{\widehat{\mathrm{E}}})\delta_{k,0}\delta_{l,0}$$
$$\Delta^{\widehat{\Phi}}_{kl}\coloneqq -2i\chi_{\widehat{\Phi}_l}\delta_{k,0}+2ik\chi_{\mathrm{F}^-_{kl}}(1-\delta_{l,0})$$
$$\Delta^{\widehat{\Theta}}_{kl}\coloneqq -2i\chi_{\widehat{\Theta}_k}\delta_{l,0}+2il\chi_{\mathrm{F}^-_{kl}}(1-\delta_{k,0})$$

and

$$\mathcal{E}_{\mathrm{X,Y}}(k,l)\;\coloneqq\sum_{m,n}(-2x_{(k+m)(l+n)})\frac{\partial}{\partial y_{mn}},\quad \text{for } \mathrm{X,Y}\in\{\mathrm{J,K,P}\}$$
$$\mathcal{E}_{\mathrm{X,YZ}}(k,l)\coloneqq\sum_{m,n}\sum_{r,s}(-2x_{(k+m+r)(l+n+s)})\frac{\partial}{\partial y_{mn}}\frac{\partial}{\partial z_{rs}},\quad \text{for } \mathrm{X,Y,Z}\in\{\mathrm{J,K,P}\}\,.$$

It is interesting to compare this with the free field realization of affine $\mathfrak{sl}(2)$ constructed in [Wak86] by Wakimoto.[34] By applying the construction at hand to the usual triangular decomposition of $\mathfrak{sl}(2)$, one obtains a very similar representation as Wakimoto, but without the more involved polarization and thus without normal ordering. As a consequence, the central charge necessarily acts trivially instead of by $\mathrm{Z}=-2$. The representation

[34] The notation above is inspired by that paper.

can be recovered exactly by changing the polarization and choosing a normal ordering. When trying to impose the constraints, we are also led to change the polarization; however, no normal ordering will be required. It would be interesting to explore the precise connection between the module construction herein and the polynomial representations or free field realizations. Next, we prove that the module $\mathcal{H}_\rho^+$ in Theorem 3.20 is actually irreducible.

3.4.2 Irreducibility of $\mathcal{H}_\rho^+$

We want to understand whether the representation of $\widehat{\mathcal{G}}(\mathfrak{su}(2))$ contains non-trivial subrepresentations because they should be connected to imposing the constraints. Ideally, the constraints vanish on some nice subrepresentation. This will not be the case for $\mathcal{H}_\rho^+$, since it is irreducible.

Proposition 3.21. *The representation $\mathcal{H}_\rho^+$ of $\widehat{\mathcal{G}}(\mathfrak{su}(2))$ from Theorem 3.20 is irreducible.*

Proof. The main idea of the proof is to show that a proper subrepresentation $Y \subset \mathcal{H}_\rho^+$ can only exist if there is a vector $y \in Y$ such that $\mathrm{X}_{kl}^+ y = 0 \quad \forall \mathrm{X} \in \{\mathrm{J}, \mathrm{K}, \mathrm{P}\}$ and $k, l \in \mathbb{Z}$. Subsequently, we show that the condition is only fulfilled if $y \in V$. But this is already enough to show that $Y = \mathcal{H}_\rho^+$. We start by proving the last claim.

1. Fix any vector in the vacuum sector $v \in V$ and suppose $v \in Y$. As a consequence, also $V \subset Y$ because the action of $\mathfrak{h}$ on V is irreducible. In other words, any vector in V can be connected with the vacuum $|0\rangle$ and vice versa.[35] The $(-)$-operators then generate all of $\mathcal{H}_\rho^+$ by acting on the vacuum sector V. Now we prove that a proper subrepresentation can only exist if there is a vector $y \in Y$ such that $\mathrm{X}_{kl}^+ y = 0 \quad \forall \mathrm{X} \in \{\mathrm{J}, \mathrm{K}, \mathrm{P}\}$ and $k, l \in \mathbb{Z}$.

2. Suppose $y \in Y$ then one of the following must be true:

(i) $\mathrm{X}_{kl}^+ y = 0 \quad \forall \mathrm{X} \in \{\mathrm{J}, \mathrm{K}, \mathrm{P}\}$ and $k, l \in \mathbb{Z}$

(ii) $\exists \bar{\mathrm{X}} \in \{\mathrm{J}, \mathrm{K}, \mathrm{P}\}$ and $\bar{k}, \bar{l} \in \mathbb{Z}$ such that $y^{(1)} := \bar{\mathrm{X}}_{\bar{k}\bar{l}}^+ y \neq 0$ and $|y^{(1)}| = |y| - 1$

By iterating this argument one arrives at either:

(i) $\exists n \in \{1, \dots, |y| - 1\}$ such that $\mathrm{X}_{kl}^+ y^{(n)} = 0 \quad \forall \mathrm{X} \in \{\mathrm{J}, \mathrm{K}, \mathrm{P}\}$ and $k, l \in \mathbb{Z}$

(ii) $y^{(|y|)} \neq 0$ and $y^{(|y|)} \in V$ (because it has degree 0)

The only thing left to do is to show that any non-zero element $y \in Y$ that satisfies the properties in (i) necessarily lies in V. Equivalently, condition (i) says that y is an element of the intersection of the kernels of all $(+)$-generators of $\widehat{\mathcal{G}}(\mathfrak{su}(2))$.

[35] Using either $\frac{1}{-2ik}\mathrm{K}_{kl}^z$ for $k \neq 0$ or $\frac{1}{2il}\mathrm{P}_{kl}^z$ for $l \neq 0$, every basis vector of V can be generated. Using $\frac{1}{2}\mathrm{J}_{kl}^z$ for $(k, l) \neq (0, 0)$ any vector can be reduced to the vacuum.

3. To prove the claim, we have to exploit the explicit formula of the (+)-operators. First, observe that y cannot be a monomial if it has non-zero degree, as each summand in the differential operators removes different variables and so are linearly independent terms. Second, the annihilation operators are graded linear maps, therefore, the cancellation of terms needs to happen in every degree separately. We start by showing that such a y satisfying (i), cannot contain any j variables. So let us assume the contrary and split $y = m + q$ such that:

- m is a monomial in the j variables, polynomial in k, p and v, and contains j_{kl} for some arbitrary $k, l \in \mathbb{Z}$
- $q = \sum_i q_i$ is a finite sum of monomials in all variables j, k, p and v
- m and q are non-zero, because y is not a monomial
- m is pairwise linearly independent to each summand in q as j-monomials[36]

Next, choose integers $h, f \in \mathbb{Z}$ such that $v_{(h+k)(f+l)} \notin q$. This choice is possible as there are only finitely many v variables in q. But then $\mathrm{K}^+_{hf} m$ will always contain a non-zero term proportional to $v_{(h+k)(f+l)}$ coming from the $\frac{\partial}{\partial j_{kl}}$ derivative which cannot be canceled by any term in $\mathrm{K}^+_{hf} q$. The only possible terms in $\mathrm{K}^+_{hf} q$ that could cancel it must also originate from a derivative with respect to j_{kl}. However, the resulting monomials $\frac{\partial}{\partial j_{kl}} q_i$ cannot be proportional to $\frac{\partial}{\partial j_{kl}} m$ because they were linearly independent in the j variables which persists after differentiating. As a consequence, $\mathrm{K}^+_{hf} y \neq 0$ contradicting (i). The choice of j_{kl} was arbitrary and therefore no j variables are contained in y. The same line of reasoning using J^+-operators proves that y cannot solely consist of k and v variables or p and v variables, so it must contain both. Therefore, assume that y is a polynomial in k, p and v. We again split $y = m' + q'$ such that:

- m' is a monomial in the k variables, polynomial in p and v and contains k_{kl} for some arbitrary $k, l \in \mathbb{Z}$
- $q' = \sum_i q'_i$ is a finite sum of monomials in all variables k, p and v
- m and q are non-zero, because y is not a monomial
- m is linearly independent to each summand in q as k-monomials

Next, choose integers $h, f \in \mathbb{Z}$ such that $v_{(h+k)(f+l)} \notin q'$. Again, $\mathrm{J}^+_{hf} m$ will always contain a non-zero term proportional to $v_{(h+k)(f+l)}$ from the $\frac{\partial}{\partial k_{kl}}$ derivative. However, this time, there are terms proportional to $v_{(h+k)(f+l)}$ that do not come from a derivative

[36] This means, we ignore any terms in the other variables and only compare the j variables.

with respect to k_{kl} but p_{kl} instead, so the previous argument does not apply. Let us assume the worst case, i.e. all terms in $\mathrm{J}^+_{hf}y$ proportional to $v_{(h+k)(f+l)}$ cancel: $-2i(h+k)v_{(h+k)(f+l)}\frac{\partial}{\partial k_{kl}}m + 2i(f+l)\sum_j v_{(h+k)(f+l)}\frac{\partial}{\partial p_{kl}}q'_j = 0$, where q'_j labels the monomials in q' that contain p_{kl}. But this equation has to hold for all possible choices of h and f such that $v_{(h+k)(f+l)} \notin q'$ of which there are infinitely many. Thus, each of the two terms has to vanish individually, contradicting that m and q are non-zero. But the choice of k_{kl} was arbitrary and therefore no k variables are contained in y, which means there are no p variable either. Finally, the only possibility left is that y is a polynomial in v variables, i.e. an element of V. This concludes the proof of the irreducibility of $\mathcal{H}^+_\rho$. □

As a direct consequence of irreducibility, we can conclude that any other irreducible representation where the central charge χ_{Z} is different must be inequivalent. It would be nice to have a similar statement about in/equivalence of representations for different charges: $\chi_{\hat{\Theta}_k}, \chi_{\hat{\Phi}_l}$ and $\chi_{\mathrm{F}^-_{kl}}$ or more generally different choices of (V,ρ). It seems very plausible that for different polarizations, such as the one we will choose shortly, the representation is irreducible. However, we have not been able to work that out so far.

3.4.3 Action of the Constraint Ideal $\mathcal{I}_{F_A}$ on $\mathcal{H}^+_\rho$

As discussed in Section 1.3.4, we are interested in representations that descend to the quotient $\mathcal{U}(\widehat{\mathcal{G}}(\mathfrak{su}(2)))|_{F_A=0}$, i.e. representations where the ideal of constraints acts trivially. Ideally, choosing appropriate values for the undetermined charges suffices to ensure that the constraints act by zero. Otherwise, we are in trouble and can only hope to restrict to a subrepresentation, where the constraints vanish. Such a restriction is only non-trivial if the representation is reducible.

To examine the action of the constraints, it is convenient to transform them into the ladder basis as well.

$$\begin{aligned}
\hat{f}^\pm_{rs} &= \pm\hat{f}_{1rs} - i\hat{f}_{2rs}\\
&= -is\mathrm{K}^\pm_{rs} - ir\mathrm{P}^\pm_{rs} + \frac{1}{2}\sum_{m,n}\left(\mp\mathrm{P}^\pm_{(r+m)(s+n)}\mathrm{K}^z_{-m-n} \pm \mathrm{P}^z_{(r+m)(s+n)}\mathrm{K}^\pm_{-m-n}\right)\\
\hat{f}^z_{rs} &= -2i\hat{f}_{3rs}\\
&= -is\mathrm{K}^z_{rs} - ir\mathrm{P}^z_{rs} + \sum_{m,n}\left(\mathrm{P}^+_{(r+m)(s+n)}\mathrm{K}^-_{-m-n} - \mathrm{P}^-_{(r+m)(s+n)}\mathrm{K}^+_{-m-n}\right)
\end{aligned}$$

The new bracket relations are given by:

$$\begin{aligned}
[\hat{f}^z_{rs}, \mathrm{J}^\pm_{mn}] &= \pm 2\hat{f}^\pm_{(r+m)(s+n)},\\
[\hat{f}^\pm_{rs}, \mathrm{J}^z_{mn}] &= \mp 2\hat{f}^\pm_{rs},\\
[\hat{f}^\pm_{rs}, \mathrm{J}^\mp_{mn}] &= \pm\hat{f}^z_{(r+m)(s+n)}\,,
\end{aligned}$$

and zero otherwise.

Now, we would like to examine the action of the constraints on $\mathcal{H}^+_\rho$ from Theorem 3.20. However, there is a problem with the representation, because the action of the $\widehat{f}^-_{rs}$-constraints are not well-defined. To see this, consider the expression

$$\begin{aligned}\widehat{f}^-_{kl} &\propto \sum_{m,n}\Big(\mathrm{P}^-_{(r+m)(s+n)}\mathrm{K}^z_{-m-n} - \mathrm{P}^z_{(r+m)(s+n)}\mathrm{K}^-_{-m-n}\Big)\\ &\propto \sum_{m,n}\Big(mp_{(r+m)(s+n)}v_{-m-n} - lv_{(r+m)(s+n)}k_{-m-n}\Big)\,. \end{aligned}\tag{3.5}$$

The action leads to an infinite sum over the linearly independent vectors.

Fortunately, there is some freedom in choosing the representation of V. Instead of the representation (ρ, V), we can choose another representation $(\bar{\rho}, V)$:

$$\begin{aligned} \mathrm{c}^\dagger_{kl} &\longmapsto -\frac{\partial}{\partial v_{kl}}, && k\neq 0, l\neq 0,\\ \mathrm{c}_{kl} &\longmapsto iv_{kl}, && k\neq 0, l\neq 0,\\ \mathrm{a}^\dagger_{l} &\longmapsto -\frac{\partial}{\partial v_{0l}}, && l\neq 0,\\ \mathrm{a}_{l} &\longmapsto iv_{0l}, && l\neq 0,\\ \mathrm{b}^\dagger_{l} &\longmapsto -\frac{\partial}{\partial v_{k0}}, && k\neq 0,\\ \mathrm{b}_{l} &\longmapsto iv_{k0}, && k\neq 0, \end{aligned}\tag{3.6}$$

and $v_{00}\equiv 1$, $\frac{\partial}{\partial v_{00}}\equiv 1$.

The terms in the infinite sum (3.5) now contain derivatives instead of pure multiplication with a variable and act in a well-defined manner on any state in $\mathcal{H}^+_{\bar{\rho}}$.

Remark 3.22. We might expect a similar freedom in the other variables j, k and p. However, it is much more difficult to change the polarization of these variables, because they appear in infinite sums and can become ill-defined if one does not take proper care of ordering them suitably.

Claim 3.23. *We conjecture that the representation $\mathcal{H}^+_{\bar{\rho}}$ is irreducible.*

We expect this claim because the representation is very similar on a structural level. However, we have not proven it. Now, let us examine the set of conditions that the constraints impose on $\mathcal{H}^+_{\bar{\rho}}$. It should be sufficient to impose the constraints on the vacuum sector V. This directly follows from the ideal property and that we can systematically move the constraints past the monomials until they hit the vacuum states. We then have the following proposition

Proposition 3.24. *The action of the on-shell constraints on the vacuum sector V in the polarization $\bar{\rho}$ imposes the following restrictions:*

$$\begin{aligned}
&\widehat{f}^{+}_{rs}V \overset{!}{=} 0, \quad \forall r,s\in\mathbb{Z} \implies \textit{no constraint},\\
&\widehat{f}^{z}_{rs}V \overset{!}{=} 0, \quad \forall r,s\in\mathbb{Z} \implies \chi_{\mathrm{F}^{-}_{kl}} = \chi_{\widehat{\Phi}_l} = \chi_{\widehat{\Theta}_k} = 0 \textit{ for } k\neq 0, l\neq 0,\\
&\widehat{f}^{-}_{rs}V \overset{!}{=} 0, \quad \forall r,s\in\mathbb{Z} \qquad \textit{not possible}.
\end{aligned}$$

Proof. We systematically examine the action of the constraints on V for a suitable restriction of the indices $r=0, s=0$, $r\neq 0, s=0$, $r=0, s\neq 0$ and $r\neq 0, s\neq 0$ subsequently.

The (+)-constraints trivially vanish when acting on the vacuum sector, as there is always a derivative with respect to the j, k or p variables, because the K- and P-operators commute. Therefore, they impose no restriction on the representation.

Let $v\in V$, the z-operators act as follows:

$$\begin{aligned}
\widehat{f}^{z}_{00}v &= 0,\\
\widehat{f}^{z}_{r0}v &= -2r\chi_{\widehat{\Theta}_r}v,\\
\widehat{f}^{z}_{0s}v &= -2s\chi_{\widehat{\Phi}_s}v\,.
\end{aligned}$$

For any physically admissible representations, these terms must vanish. Thus, we have that $\chi_{\widehat{\Phi}_s} = \chi_{\widehat{\Theta}_r} = 0$ for $r\neq 0, s\neq 0$. Assuming these conditions have been imposed, the leftover z-constraints act as:

$$\widehat{f}^{z}_{rs}v = 2(r+s)\chi_{\mathrm{F}_{rs}}v\,.$$

Therefore, the only possibility is to set these charges to zero as well.

The (−)-operators act as follows:

$$\begin{aligned}
\widehat{f}^{-}_{00}|_V &= \frac{1}{2}\sum_{m,n}\left(-2imp_{mn}\frac{\partial}{\partial v_{-m-n}} + 2ink_{-m-n}\frac{\partial}{\partial v_{mn}}\right)\\
&= -\sum_{m,n}\left(imp_{mn} + ink_{mn}\right)\frac{\partial}{\partial v_{-m-n}}
\end{aligned}$$

The fact that it does not vanish for any choice of the charges is problematic. It means that the representation does not descend to the quotient algebra. Also if the representation is irreducible, which we claim (3.23), it is not possible to restrict the representation space to a subrepresentation where this operator could vanish. The only thing left is the trivial representation, where $V=\{0\}$.

One can try to check if there is a possible modification to the representation so that these operators vanish, i.e. to require: $im\mathrm{P}^{-}_{mn} + in\mathrm{K}^{-}_{mn} = 0$ for all $m,n\in\mathbb{Z}$. But then, we have:

$$\begin{aligned} \widehat{f_{r0}^{-}}|_V &= -irp_{r0} + \frac{1}{2}\sum_{m,n}\left(-2imp_{(r+m)n}\frac{\partial}{\partial v_{-m-n}} + 2ink_{-m-n}\frac{\partial}{\partial v_{(r+m)n}}\right) \\ &= irp_{r0} + \sum_{m,n}\left(i(r-m)p_{mn} - ink_{mn}\right)\frac{\partial}{\partial v_{(r-m)-n}} \end{aligned}$$

For this to be zero, we must have $-irp_{r0} = 0$ and $i(r-m)\mathrm{P}^{-}_{mn} - in\mathrm{K}^{-}_{mn} = 0$ for $m,n \in \mathbb{Z}$. Combining with the restriction from the previous case, we conclude that $p_{mn} \equiv 0$ for $m,n \in \mathbb{Z}$. Similarly, one can show that also $\mathrm{K}^{-}_{mn} \equiv 0$ for $m,n \in \mathbb{Z}$, using the constraints $\widehat{f_{0s}^{-}}$. The constraints $\widehat{f_{rs}^{-}}$ add no additional restriction. Finally, all of these restrictions are enough to guarantee that $\widehat{f}|_V \equiv 0$, for any $\widehat{f}$ in the set of constraints. However, by observing the commutation relations (3.2), it is clear that the resulting representation is manifestly trivial in the K^{-}- and P^{-}-generators and the central charge. □

The second part of the restriction, coming from the linear terms in the constraints, nicely coincides with the constraints from the abelian case. Unfortunately, the non-linear terms act non-zero on the class of representation and do not descend to the quotient algebra. To understand what step in the process might be problematic, let us recall the relevant steps so far.

1. When performing deformation quantization, we chose to quantize the subspace of symmetric tensors on $\widehat{\mathcal{G}}_\Lambda(\mathfrak{su}(2)))$ instead of the entire space of functions. This choice is usually not too problematic as one can recover the general functions again by constructing a suitable topology and complete the tensor products (see Footnote 15).

2. We then constructed representations of only a part of $\widehat{\mathcal{G}}_\Lambda(\mathfrak{su}(2)))$, namely the finite Fourier mode Lie subalgebra. It is apriori not clear whether these representations can be extended to the full Lie algebra.

3. The classical constraints are elements of the vector space of multilinear forms on $\widehat{\mathcal{G}}_\Lambda(\mathfrak{su}(2)))$ (see Equation 1.5). As such, the constraints are not elements of the symmetric tensors over $\widehat{\mathcal{G}}_\Lambda(\mathfrak{su}(2)))$, meaning we most likely need to recover them as outlined in 1. This would also allow us to rigorously define the physical quotient algebra.

To summarize, it is not yet clear whether the representations can be saved. One could also explore if related induced module constructions suffer the same fate and how exactly different choices of V affect the situation.

3.5 Representations of $\widehat{\mathcal{G}}_\Lambda(\mathfrak{su}(2))$

In this section, we work out an explicit realization of the generators on the bosonic Fock space $\mathcal{H}^+_{\rho,\Lambda}$ in terms of differential operators. The analysis is very similar to the previous section with the added difficulty of the non-trivial coupling of K- and P-generators. After constructing a representation, we run into the problem that the constraints act ill-defined even in the new polarization. It turns out that we have to significantly alter the representation as a consequence. The final implications of the constraints have not been worked out, but it is expected to behave similarly to the case of $\Lambda = 0$, which means that they do not descend.

3.5.1 Action of the Generators on $\mathcal{H}^+_{\rho,\Lambda}$

We proceed analogously to Section 3.4, to derive an explicit representation. For non-zero cosmological constant, the representation of the ladder operators is defined by:

$$
\begin{aligned}
\mathrm{w}^\dagger_{kl} &\longmapsto x_{kl}, && k \neq 0, l \neq 0, \\
\mathrm{w}_{kl} &\longmapsto i\frac{\partial}{\partial x_{kl}}, && k \neq 0, l \neq 0, \\
\mathrm{u}^\dagger_{l} &\longmapsto x_{0l}, && l \neq 0, \\
\mathrm{u}_{l} &\longmapsto i\frac{\partial}{\partial x_{0l}}, && l \neq 0, \\
\mathrm{v}^\dagger_{k} &\longmapsto x_{k0}, && k \neq 0, \\
\mathrm{v}_{k} &\longmapsto i\frac{\partial}{\partial x_{k0}}, && k \neq 0, \\
\bar{\Phi} &\longmapsto x_{00}, \\
\bar{\Theta} &\longmapsto i\frac{\partial}{\partial x_{00}}, \\
\mathrm{Z} &\longmapsto 1, \\
\hat{\mathrm{u}}_{l} &\longmapsto 0, && l \neq 0, \\
\hat{\mathrm{v}}_{k} &\longmapsto 0, && k \neq 0, \\
\widehat{\mathrm{w}}_{kl} &\longmapsto 0, && k, l \neq 0, \\
\widehat{\mathrm{E}} &\longmapsto 0\,.
\end{aligned}
$$

In contrast to the $\Lambda = 0$ case, the expressions x_{00}, $\frac{\partial}{\partial x_{00}}$ are already assigned and cannot be set to 1 for convenience. Additionally, we have set the central charge Z to one and, in anticipation of the constraints, the charges $\chi_{\hat{\mathrm{u}}_l}, \chi_{\hat{\mathrm{v}}_k}, \chi_{\widehat{\mathrm{w}}_{kl}}$ and $\chi_{\widehat{\mathrm{E}}}$ to zero.

Action on the Vacuum Sector The only operators whose action on the vacuum sector are different compared to $\Lambda = 0$ are the z-operators:

$$
\begin{aligned}
\mathrm{J}^z_{kl}\,|x\rangle &= -2i\mathrm{E}_{kl}\,|x\rangle \\
&= \begin{cases} -2i\mathrm{c}_{-k-l}\,|x\rangle\,, & \text{for } k\neq 0, l\neq 0 \\ -2i\mathrm{b}_{-k}\,|x\rangle\,, & \text{for } k\neq 0, l=0 \\ -2i\mathrm{a}_{-l}\,|x\rangle\,, & \text{for } k=0, l\neq 0 \\ -2i\widehat{\mathrm{E}}\,|x\rangle\,, & \text{for } k=l=0 \end{cases} \\
&= \begin{cases} -2i\mathrm{w}_{-k-l}\,|x\rangle\,, & \text{for } k\neq 0, l\neq 0 \\ -2i\mathrm{v}_{-k}\,|x\rangle\,, & \text{for } \neq 0, l=0 \\ -2i\mathrm{u}_{-l}\,|x\rangle\,, & \text{for } k=0, l\neq 0 \\ 0, & \text{for } k=l=0 \end{cases} \\
&= (2-2\delta_{k,0}\delta_{l,0})\frac{\partial}{\partial x_{-k-l}}\,|x\rangle \\
\mathrm{K}^z_{kl}\,|x\rangle &= -2i\Phi_{kl}\,|x\rangle \\
&= \begin{cases} -2ik\left(\mathrm{c}^\dagger_{kl}-\mathrm{F}^-_{kl}\right)|x\rangle\,, & \text{for } k\neq 0, l\neq 0 \\ -2ik\mathrm{b}^\dagger_k\,|x\rangle\,, & \text{for } k\neq 0, l=0 \\ -2i\widehat{\Phi}_l\,|x\rangle\,, & \text{for } k=0, l\in\mathbb{Z} \end{cases} \\
&= \begin{cases} -2ik\mathrm{w}^\dagger_{kl}\,|x\rangle - \frac{\Lambda}{l}\mathrm{w}_{-k-l}\,|x\rangle\,, & \text{for } k\neq 0, l\neq 0 \\ -2ik\mathrm{v}^\dagger_k\,|x\rangle\,, & \text{for } k\neq 0, l=0 \\ -\frac{2\Lambda}{l}\mathrm{u}_{-l}\,|x\rangle\,, & \text{for } k=0, l\neq 0 \\ 2\Lambda\bar{\Phi}\,|x\rangle\,, & \text{for } k=l=0 \end{cases} \\
&= (-2ikx_{kl}+2\Lambda x_{00}\delta_{k,0}\delta_{l,0}+\Lambda\left(-\frac{i}{l}(1-\delta_{k,0})(1-\delta_{l,0})-\frac{2i}{l}\delta_{k,0}(1-\delta_{l,0})\right)\frac{\partial}{\partial x_{-k-l}})\,|x\rangle \\
\mathrm{P}^z_{kl}\,|x\rangle &= -2i\Theta_{kl}\,|x\rangle \\
&= \begin{cases} 2il\left(\mathrm{c}^\dagger_{kl}+\mathrm{F}^-_{kl}\right)|x\rangle\,, & \text{for } k\neq 0, l\neq 0 \\ 2il\mathrm{a}^\dagger_l\,|x\rangle\,, & \text{for } k=0, l\neq 0 \\ -2i\widehat{\Theta}_k\,|x\rangle\,, & \text{for } k\in\mathbb{Z}, l=0 \end{cases} \\
&= \begin{cases} 2il\mathrm{w}^\dagger_{kl}\,|x\rangle - \frac{\Lambda}{k}\mathrm{w}_{-k-l}\,|x\rangle\,, & \text{for } k\neq 0, l\neq 0 \\ 2il\mathrm{u}^\dagger_l\,|x\rangle\,, & \text{for } k=0, l\neq 0 \\ -\frac{2\Lambda}{k}\mathrm{v}_{-k}\,|x\rangle\,, & \text{for } k\neq 0, l=0 \\ -2i\bar{\Theta}\,|x\rangle\,, & \text{for } k=l=0 \end{cases}
\end{aligned}
$$

$$= (2ilx_{kl} + 2\frac{\partial}{\partial x_{00}}\delta_{k,0}\delta_{l,0} + \Lambda\left(-\frac{i}{k}(1-\delta_{k,0})(1-\delta_{l,0}) - \frac{2i}{k}(1-\delta_{k,0})\delta_{l,0}\right)\frac{\partial}{\partial x_{-k-l}})\,|x\rangle$$

One can continue with determining the action on first excited states, i.e. monomials of degree one and so on. The following theorem gives an explicit formula for the representation of $\widehat{\mathcal{G}}_\Lambda(\mathfrak{su}(2))$.

Theorem 3.25. *The assignment*

$$\begin{aligned}
\mathrm{Z} &\longrightarrow 1\\
\mathrm{J}^-_{kl} &\longrightarrow j_{kl}\\
\mathrm{K}^-_{kl} &\longrightarrow k_{kl}\\
\mathrm{P}^-_{kl} &\longrightarrow p_{kl}\\
\mathrm{J}^z_{kl} &\longrightarrow (2-2\delta_{k,0}\delta_{l,0})\frac{\partial}{\partial x_{-k-l}} + \mathcal{E}_{\mathrm{J,J}}(k,l) + \mathcal{E}_{\mathrm{K,K}}(k,l) + \mathcal{E}_{\mathrm{P,P}}(k,l)\\
\mathrm{K}^z_{kl} &\longrightarrow -2ikx_{kl} + 2\Lambda x_{00}\delta_{k,0}\delta_{l,0} + \Lambda\Delta^{\mathrm{K}}_{kl}\frac{\partial}{\partial x_{-k-l}} + \mathcal{E}_{\mathrm{K,J}}(k,l)\\
\mathrm{P}^z_{kl} &\longrightarrow 2ilx_{kl} + 2\frac{\partial}{\partial x_{00}}\delta_{k,0}\delta_{l,0} + \Lambda\Delta^{\mathrm{P}}_{kl}\frac{\partial}{\partial x_{-k-l}} + \mathcal{E}_{\mathrm{P,J}}(k,l)\\
\mathrm{J}^+_{kl} &\longrightarrow \sum_{m,n}(2-2\delta_{k+m,0}\delta_{l+n,0})\frac{\partial}{\partial x_{(-k-m)(-l-n)}}\frac{\partial}{\partial j_{mn}}\\
&\quad + \sum_{m,n}\Big(-2i(k+m)x_{(k+m)(l+n)} + 2\Lambda x_{00}\delta_{k+m,0}\delta_{l+n,0}\\
&\qquad + \Lambda\Delta^{\mathrm{K}}_{(k+m)(l+n)}\frac{\partial}{\partial x_{(-k-m)(-l-n)}} - 2im\delta_{k+m,0}\delta_{l+n,0}\Big)\frac{\partial}{\partial k_{mn}}\\
&\quad + \sum_{m,n}\Big(2i(l+n)x_{(k+m)(l+n)} + 2\frac{\partial}{\partial x_{00}}\delta_{k+m,0}\delta_{l+n,0}\\
&\qquad + \Lambda\Delta^{\mathrm{P}}_{(k+m)(l+n)}\frac{\partial}{\partial x_{(-k-m)(-l-n)}} + 2in\delta_{k+m,0}\delta_{l+n,0}\Big)\frac{\partial}{\partial p_{mn}}\\
&\quad + \frac{1}{2}\mathcal{E}_{\mathrm{J,JJ}}(k,l) + \mathcal{E}_{\mathrm{K,KJ}}(k,l) + \mathcal{E}_{\mathrm{P,PJ}}(k,l)\\
\mathrm{K}^+_{kl} &\longrightarrow -2\Lambda\frac{\partial}{\partial p_{-k-l}} + \sum_{m,n}\Big(-2i(k+m)x_{(k+m)(l+n)} + 2\Lambda x_{00}\delta_{k+m,0}\delta_{l+n,0}\\
&\qquad + \Lambda\Delta^{\mathrm{K}}_{(k+m)(l+n)}\frac{\partial}{\partial x_{(-k-m)(-l-n)}} - 2im\delta_{k+m,0}\delta_{l+n,0}\Big)\frac{\partial}{\partial j_{mn}}\\
&\quad + \frac{1}{2}\mathcal{E}_{\mathrm{K,JJ}}(k,l)\\
\mathrm{P}^+_{kl} &\longrightarrow 2\Lambda\frac{\partial}{\partial k_{-k-l}} + \sum_{m,n}\Big(2i(l+n)x_{(k+m)(l+n)} + 2\frac{\partial}{\partial x_{00}}\delta_{k+m,0}\delta_{l+n,0}\\
&\qquad + \Lambda\Delta^{\mathrm{P}}_{(k+m)(l+n)}\frac{\partial}{\partial x_{(-k-m)(-l-n)}} + 2in\delta_{k+m,0}\delta_{l+n,0}\Big)\frac{\partial}{\partial j_{mn}}
\end{aligned}$$

$$+\frac{1}{2}\mathcal{E}_{\mathrm{P,JJ}}(k,l)\,,$$

where

$$\Delta^{\mathrm{K}}_{kl} \coloneqq -\frac{i}{l}(1-\delta_{k,0})(1-\delta_{l,0}) - \frac{2i}{l}\delta_{k,0}(1-\delta_{l,0}), \quad \forall k,l\in\mathbb{Z},$$
$$\Delta^{\mathrm{P}}_{kl} \coloneqq -\frac{i}{k}(1-\delta_{k,0})(1-\delta_{l,0}) - \frac{2i}{k}(1-\delta_{k,0})\delta_{l,0}, \quad \forall k,l\in\mathbb{Z},$$

constitutes a representation of $\widehat{\mathcal{G}}_\Lambda(\mathfrak{su}(2))$ on the bosonic Fock space $\mathcal{H}^+_{\rho,\Lambda}$.

Proof. The proof is analogous to the proof of Theorem 3.20. □

Claim 3.26. *We conjecture that the representation is irreducible.*

This can be expected since the overall structure of the shift operators is still the same and so the main ideas of Proposition 3.21 should be adaptable. However, we have not proven it.

3.5.2 Action of the Constraint Ideal $\mathcal{I}_{F_A+\Lambda B}$ on $\mathcal{H}^+_{\rho,\Lambda}$

In Section 3.4.3, we described a way to "regularize" the constraints by passing to a different polarization of the underlying $\mathfrak{h}$-representation (V,ρ). However, the new polarization is not sufficient when $\Lambda \neq 0$. There is an additional ordering ambiguity in the $\widehat{f}^z_{00}$-constraint, because the P^+-operator does not commute with the K^+-operator anymore. Potentially, one could define a normal ordering convention to take care of this issue. Another possibility, the one we will pursue, is to try and change the polarization of the k variables analogously to the one of the v variables in Equation (3.6). This change is not straightforward at all, as the k variables appear in sums of the form: $\mathcal{E}_{\mathrm{K,K}}(k,l) = \sum_{m,n}(-2k_{(k+m)(l+n)})\frac{\partial}{\partial k_{mn}}$ for example, and the change thus leads to new divergences. However, one can get around this issue by replacing the problematic operators with such divergences by non-divergent ones with the same algebraic properties. The essential algebraic properties can be inferred from the proof of Theorem 3.20 and mainly involve the degree shifting of the polynomial operators. The price to pay is that the resulting representation behaves quite differently than the original family (see Remark 3.27).

To summarize, the assignment

$$\mathrm{Z} \longrightarrow 1$$
$$\mathrm{J}^-_{kl} \longrightarrow j_{kl}$$
$$\mathrm{K}^-_{kl} \longrightarrow -\frac{\partial}{\partial k_{-k-l}}$$

$$
\begin{aligned}
\mathrm{P}^-_{kl} \longrightarrow & p_{kl} \\
\mathrm{J}^z_{kl} \longrightarrow & (2-2\delta_{k,0}\delta_{l,0})x_{-k-l} + \mathcal{E}_{\mathrm{J},\mathrm{J}}(k,l) - \mathcal{E}_{\mathrm{K},\mathrm{K}}(k,l) + \mathcal{E}_{\mathrm{P},\mathrm{P}}(k,l) \\
\mathrm{K}^z_{kl} \longrightarrow & 2ik\frac{\partial}{\partial x_{kl}} + 2\Lambda x_{00}\delta_{k,0}\delta_{l,0} + \Lambda\Delta^{\mathrm{K}}_{kl}x_{-k-l} + \sum_{m,n} 2\frac{\partial}{\partial k_{(-k-m)(-l-n)}}\frac{\partial}{\partial j_{mn}} \\
\mathrm{P}^z_{kl} \longrightarrow & -2il\frac{\partial}{\partial x_{kl}} + 2\frac{\partial}{\partial x_{00}}\delta_{k,0}\delta_{l,0} + \Lambda\Delta^{\mathrm{P}}_{kl}x_{-k-l} + \mathcal{E}_{\mathrm{P},\mathrm{J}}(k,l) \\
\mathrm{J}^+_{kl} \longrightarrow & \sum_{m,n}(2-2\delta_{k+m,0}\delta_{l+n,0})x_{(-k-m)(-l-n)}\frac{\partial}{\partial j_{mn}} \\
& + \sum_{m,n}\Big(2i(k+m)\frac{\partial}{\partial x_{(k+m)(l+n)}} + 2\Lambda x_{00}\delta_{k+m,0}\delta_{l+n,0} \\
& \quad + \Lambda\Delta^{\mathrm{K}}_{(k+m)(l+n)}x_{(-k-m)(-l-n)} - 2im\delta_{k+m,0}\delta_{l+n,0}\Big)k_{-m-n} \\
& + \sum_{m,n}\Big(-2i(l+n)\frac{\partial}{\partial x_{(k+m)(l+n)}} + 2\frac{\partial}{\partial x_{00}}\delta_{k+m,0}\delta_{l+n,0} \\
& \quad + \Lambda\Delta^{\mathrm{P}}_{(k+m)(l+n)}x_{(-k-m)(-l-n)} + 2in\delta_{k+m,0}\delta_{l+n,0}\Big)\frac{\partial}{\partial p_{mn}} \\
& + \frac{1}{2}\mathcal{E}_{\mathrm{J},\mathrm{JJ}}(k,l) - \mathcal{E}_{\mathrm{K},\mathrm{KJ}}(k,l) + \mathcal{E}_{\mathrm{P},\mathrm{PJ}}(k,l) \\
\mathrm{K}^+_{kl} \longrightarrow & -2\Lambda\frac{\partial}{\partial p_{-k-l}} + \sum_{m,n}\Big(2i(k+m)\frac{\partial}{\partial x_{(k+m)(l+n)}} + 2\Lambda x_{00}\delta_{k+m,0}\delta_{l+n,0} \\
& \quad + \Lambda\Delta^{\mathrm{K}}_{(k+m)(l+n)}x_{(-k-m)(-l-n)} - 2im\delta_{k+m,0}\delta_{l+n,0}\Big)\frac{\partial}{\partial j_{mn}} \\
& + \frac{1}{2}\sum_{m,n}\sum_{r,s} 2\frac{\partial}{\partial k_{(-k-r-m)(-l-s-n)}}\frac{\partial}{\partial j_{rs}}\frac{\partial}{\partial j_{mn}} \\
\mathrm{P}^+_{kl} \longrightarrow & 2\Lambda k_{kl} + \sum_{m,n}\Big(-2i(l+n)\frac{\partial}{\partial x_{(k+m)(l+n)}} + 2\frac{\partial}{\partial x_{00}}\delta_{k+m,0}\delta_{l+n,0} \\
& \quad + \Lambda\Delta^{\mathrm{P}}_{(k+m)(l+n)}x_{(-k-m)(-l-n)} + 2in\delta_{k+m,0}\delta_{l+n,0}\Big)\frac{\partial}{\partial j_{mn}} \\
& + \frac{1}{2}\mathcal{E}_{\mathrm{P},\mathrm{JJ}}(k,l)\,.
\end{aligned}
$$

constitutes a representation of $\widehat{\mathcal{G}}_\Lambda(\mathfrak{su}(2))$ on the bosonic Fock space $\mathcal{H}^+_{\rho,\Lambda}$, where the action of constraints is well defined.[37]

Remark 3.27. Note that this representation is quite different from the original construction as the J^+-operators do not annihilate V anymore. Therefore, one would have to carefully examine its properties again and examine how the constraints act.

[37] The corresponding bracket relations were verified by using a SymPy Phython script and by modifying the $\Lambda = 0$ representation with the same replacement instead. The replacement should either work for both or none of the representation as they share the same structure.

3.6 Connection to Gravity

In [CC23], the authors discuss the connection of the corner structure of 4-dim. BF theory and gravity in the coframe formalism. It turns out that on the corner, the tangent theory of gravity can be obtained as a constrained BF theory with $\mathfrak{g} = \mathfrak{so}(1,3)$ (see Remark 45 in [CC23]). One has to restrict the functionals and enforce that the Pfaffian of B vanishes. Ideally, one can construct possible state spaces of gravity by inferring them from BF theory. We investigated whether the induced module construction can be applied and whether the Pfaffian constraint induces an ideal. While the construction of modules is applicable, it turns out that without the restriction of the functionals, the constraint does not close. However, changing the possible functional leads to entirely new Lie algebra relations which are reminiscent of an Atiyah algebroid. Even though, the analysis did not end up succeeding, we will still include it for completeness.

3.6.1 Representations of $\widehat{\mathcal{G}}_\Lambda(\mathfrak{so}(1,3))$

We begin by investigating the corner Lie algebra for $\mathfrak{so}(1,3)$ in the same setup as Section 3. There is an isomorphism of complexified Lie algebras $\mathfrak{so}(1,3) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. Therefore, let $\{t^\sigma_\mu\}_{1\leq\mu\leq3, 1\leq\sigma\leq2}$ be a basis of $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ such that: $(t^\sigma_\mu, t^\tau_\nu) = \delta_{\mu,\nu}\delta_{\sigma,\tau}$ and $[t^\sigma_\mu, t^\tau_\nu] = \delta_{\sigma,\tau}\varepsilon^\lambda_{\mu\nu} t^\tau_\lambda$. We obtain the bracket relations:

$$
\begin{aligned}
[\mathrm{J}^\sigma_{\mu kl}, \mathrm{J}^\tau_{\nu mn}] &= \delta_{\sigma,\tau}\varepsilon^\lambda_{\mu\nu}\mathrm{J}^\tau_{\lambda(k+m)(l+n)},\\
[\mathrm{J}^\sigma_{\mu kl}, \mathrm{K}^\tau_{\nu mn}] &= \delta_{\sigma,\tau}\varepsilon^\lambda_{\mu\nu}\mathrm{K}^\tau_{\lambda(k+m)(l+n)} + im\delta_{\sigma,\tau}\delta_{\mu,\nu}\delta_{k,-m}\delta_{l,-n}\mathrm{Z},\\
[\mathrm{J}^\sigma_{\mu kl}, \mathrm{P}^\tau_{\nu mn}] &= \delta_{\sigma,\tau}\varepsilon^\lambda_{\mu\nu}\mathrm{P}^\tau_{\lambda(k+m)(l+n)} - in\delta_{\sigma,\tau}\delta_{\mu,\nu}\delta_{k,-m}\delta_{l,-n}\mathrm{Z},\\
[\mathrm{K}^\sigma_{\mu kl}, \mathrm{P}^\tau_{\nu mn}] &= \Lambda\delta_{\sigma,\tau}\delta_{\mu,\nu}\delta_{k,-m}\delta_{l,-n}\mathrm{Z}\,,
\end{aligned}
$$

where all other brackets vanish. By going to the ladder basis, one can check that this algebra also admits a MTD and leads to a very similar representation as $\mathfrak{su}(2)$. Essentially, there is an isomorphism of Lie algebras: $\widehat{\mathcal{G}}_\Lambda(\mathfrak{su}(2)\oplus\mathfrak{su}(2))) \cong (\widehat{\mathcal{G}}_\Lambda(\mathfrak{su}(2)) \oplus \widehat{\mathcal{G}}_\Lambda(\mathfrak{su}(2)))/\langle \mathrm{Z}_1 - \mathrm{Z}_2\rangle$, where the quotient is with respect to the ideal generated by the central charges of the two summands.

3.6.2 Constraint $\mathrm{Pf}(B) = 0$

Next, we would like to investigate the constraint functionals

$$\widehat{g} := \frac{-4i}{(2\pi)^2}\int_{T^2} g\mathrm{Pf}(B)$$

where Pf denotes the Pfaffian and is defined pointwise for the Lie-algebra-valued forms in the 4-dim. matrix representation.

Proposition 3.28. *The functionals describing the Pfaffian constraint decomposed in terms of an infinite sum of linear functionals are given by*

$$\widehat{g} = g_{kl}\left(\mathrm{J}^2_{\mu-r-s}\mathrm{J}^2_{(k+r)(l+s)} - \mathrm{J}^1_{\mu-r-s}\mathrm{J}^1_{(k+r)(l+s)}\right)$$

Proof. The Pfaffian of B can be expressed as

$$\begin{aligned}
\mathrm{Pf}(B) &= \mathrm{Pf}(\bar{B})\mathrm{d}\theta \wedge \mathrm{d}\varphi \\
&= \sum_{\mu} \frac{i}{4}\left(\bar{B}^2_{\mu}\bar{B}^2_{\mu} - \bar{B}^1_{\mu}\bar{B}^1_{\mu}\right)\mathrm{d}\theta \wedge \mathrm{d}\varphi \\
&= \sum_{\mu}\sum_{k,l}\sum_{m,n} \frac{i}{4}\left(\bar{B}^2_{\mu kl}\bar{B}^2_{\mu mn} - \bar{B}^1_{\mu kl}\bar{B}^1_{\mu mn}\right) e^{i(k+m)\theta} e^{i(l+n)\varphi}\mathrm{d}\theta \wedge \mathrm{d}\varphi
\end{aligned}$$

where the first equality is an implicit definition of $\bar{B}$ and the sums are included for clarity. Of course, this is again meant heuristically as the involved sums are not finite. The constraints can now be expressed in terms of a sum of linear functionals as follows:

$$\begin{aligned}
\widehat{g}(A, B) &:= \frac{-4i}{(2\pi)^2}\int_{T^2} g\mathrm{Pf}(\bar{B})\mathrm{d}\theta \wedge \mathrm{d}\varphi \\
&= \frac{g_{kl}}{(2\pi)^2}\int_{T^2}\left(\bar{B}^2_{\mu rs}\bar{B}^2_{\mu mn} - \bar{B}^1_{\mu rs}\bar{B}^1_{\mu mn}\right) e^{i(k+r+m)\theta} e^{i(l+s+n)\varphi}\mathrm{d}\theta \wedge \mathrm{d}\varphi \\
&= g_{kl}\left(\bar{B}^2_{\mu rs}\bar{B}^2_{\mu mn} - \bar{B}^1_{\mu rs}\bar{B}^1_{\mu mn}\right)\delta_{k+r+m,0}\delta_{l+s+n,0} \\
&= g_{kl}\left(\bar{B}^2_{\mu rs}\bar{B}^2_{\mu(-k-r)(-l-n)} - \bar{B}^1_{\mu rs}\bar{B}^1_{\mu(-k-r)(-l-n)}\right) \\
&= g_{kl}\left(\mathrm{J}^2_{\mu-r-s}\mathrm{J}^2_{(k+r)(l+s)} - \mathrm{J}^1_{\mu-r-s}\mathrm{J}^1_{(k+r)(l+s)}\right)(A, B)
\end{aligned}$$

□

By the usual computations, one can compute the bracket relations for the constraints $\widehat{g}_{kl} := \mathrm{J}^2_{\mu-r-s}\mathrm{J}^2_{(k+r)(l+s)} - \mathrm{J}^1_{\mu-r-s}\mathrm{J}^1_{(k+r)(l+s)}$. They turn out to be:

$$\begin{aligned}
[\widehat{g}_{kl}, \mathrm{J}^{\sigma}_{\mu mn}] &= 0 \\
[\widehat{g}_{kl}, \mathrm{K}^{\sigma}_{\mu mn}] &= (\delta_{2,\sigma} - \delta_{1,\sigma})\left(2im\mathrm{J}^{\sigma}_{\mu(k+m)(l+n)} + \varepsilon^{\mu}_{\alpha\nu}\{\mathrm{K}^{\sigma}_{\alpha(m-r)(n-s)}, \mathrm{J}^{\sigma}_{\nu(k+r)(l+s)}\}\right) \\
[\widehat{g}_{kl}, \mathrm{P}^{\sigma}_{\mu mn}] &= (\delta_{2,\sigma} - \delta_{1,\sigma})\left(-2in\mathrm{J}^{\sigma}_{\mu(k+m)(l+n)} + \varepsilon^{\mu}_{\alpha\nu}\{\mathrm{P}^{\sigma}_{\alpha(m-r)(n-s)}, \mathrm{J}^{\sigma}_{\nu(k+r)(l+s)}\}\right)
\end{aligned}$$

The $\{-,-\}$ bracket denotes the anti-commutator. Evidently, the constraints do not close and thus do not form a proper ideal. This is because we did not restrict the space of functionals first. However, in doing so, the entire analysis of the Lie algebra is not applicable anymore, and one has to take a different approach.

4 Abelian BF on $\Gamma \cong S^2$

In this section, we explore the corner structure of 4-dim. abelian BF on a sphere. Since there is no basis of $\Omega^1(S^2)$ as a $C^\infty(S^2)$-module, we will use the Hodge decomposition to generate a suitable basis of the Lie subalgebra. We again classify the Lie algebras and investigate the necessary restrictions. The results are almost identical to Section 2.

4.1 The Lie Algebra $\widehat{\mathcal{G}}_\Lambda$

The Lie algebra describing 4-dim. abelian BF on a sphere is given by the vector space:

$$\widehat{\mathcal{G}}_\Lambda = \Omega^0(S^2) \oplus \Omega^1(S^2) \oplus \mathbb{R},$$

with brackets

$$[f \oplus \alpha \oplus r, g \oplus \beta \oplus s]_{\widehat{\mathcal{G}}_\Lambda} = -\int_{S^2} (\alpha \mathrm{d}g - \beta \mathrm{d}f + \Lambda \alpha\beta)\mathrm{Z}\,,$$

where $\Lambda \in \mathbb{R}$ is the cosmological constant and Z the central charge.

As a consequence of the hairy ball theorem, there are no globally defined 1-forms that we could use as a basis for the $C^\infty(S^2)$-module (i.e. it is not free). However, we are anyway interested in regarding the space of 1-forms as an $\mathbb{R}$-vector space and then choosing a subspace with countable Hamel basis. To do this, we proceed as follows: Using the Hodge decomposition, we can uniquely express any 1-form on the sphere in terms of an exact and coexact form. In other words, for $\omega \in \Omega^1(S^2)$, there exist $g \in \Omega^0(S^2)$ and $\eta \in \Omega^2(S^2)$ such that there is a unique decomposition:

$$\omega = \mathrm{d}g + \delta\eta, \tag{4.1}$$

where $\delta := -\star \mathrm{d}\star$ is the codifferential and $\star$ is the Hodge star operator.

We can express $\eta = \star h$ for a function $h \in \Omega^0(S^2)$ and using $\star^2|_{\Omega^1(S^2)} = -1$, we can re-express Formula (4.1) as follows:

$$\omega = \mathrm{d}g - \star\mathrm{d}h. \tag{4.2}$$

We consider the Lie subalgebra of $(\widehat{\mathcal{G}}_\Lambda)_\mathbb{C}$ that consists of elements of the form (4.2) where the functions can be expanded in spherical harmonics Y as

$$g = \sum_{l\in\mathbb{N}} \sum_{m=-l}^{l} g_{lm} Y_{lm}(\theta, \varphi),$$
$$h = \sum_{l\in\mathbb{N}} \sum_{m=-l}^{l} h_{lm} Y_{lm}(\theta, \varphi)\,,$$

with only finitely many non-zero coefficients. In the formula above, (θ, φ) denote the standard spherical coordinate functions. In fact, we shall choose a different set of coordinates, namely (z, φ), where $z = \cos(\theta)$ with the orientation defined by the volume form $\mathrm{d}z \wedge \mathrm{d}\varphi$. The spherical harmonics in this chart will be denoted by $\tilde{Y}(z, \varphi)$ and define elements

$$\begin{aligned}
\mathrm{A}_{lm} &\coloneqq \tilde{Y}_{lm}(z,\varphi) && \text{for } l \in \mathbb{N}, |m| \leq l\,, \\
\mathrm{G}_{lm} &\coloneqq \mathrm{d}\tilde{Y}_{lm}(z,\varphi) && \text{for } l \in \mathbb{N}_{>0}, |m| \leq l\,, \\
\mathrm{H}_{lm} &\coloneqq \star\mathrm{d}\tilde{Y}_{lm}(z,\varphi) && \text{for } l \in \mathbb{N}_{>0}, |m| \leq l\,.
\end{aligned}$$

The Lie subalgebra has a basis, as a $\mathbb{C}$ vector space, given by these elements, i.e. for any 1-form $\omega = \omega^{(G)}_{lm}\mathrm{G}_{lm} + \omega^{(H)}_{jn}\mathrm{H}_{jn} +$, where $\omega^{(G)}_{lm}, \omega^{(H)}_{jn} \in \mathbb{C}$ and the sum is finite.

Lemma 4.1. *The bracket relations in this basis take the following form:*

$$\begin{aligned}
[\mathrm{A}_{lm}, \mathrm{G}_{jn}] &= 0, \\
[\mathrm{A}_{lm}, \mathrm{H}_{jn}] &= (-1)^{n+1} j(j+1)\delta_{l,j}\delta_{m,-n}\mathrm{Z}, \\
[\mathrm{G}_{lm}, \mathrm{H}_{jn}] &= (-1)^{n+1} j(j+1)\Lambda\delta_{l,j}\delta_{m,-n}\mathrm{Z}\,.
\end{aligned}$$

All other brackets vanish.

Proof. The proof of the bracket relations is straightforward. The Laplacian on the sphere is defined by $\Delta \coloneqq \mathrm{d}\delta + \delta\mathrm{d}$ and acts by $\Delta Y_{lm} = l(l+1)Y_{lm}$ on spherical harmonics.[38] Evaluating the bracket on the basis elements and remembering the chosen orientation on S^2 yields:

$$\begin{aligned}
[\mathrm{A}_{kl}, \mathrm{G}_{jn}] &= \int_{S^2} \mathrm{G}_{jn}\mathrm{d}\mathrm{A}_{lm}\mathrm{Z} \\
&= \int_{S^2} \mathrm{d}\tilde{Y}_{jn} \wedge \mathrm{d}\tilde{Y}_{lm}\mathrm{Z} \\
&= 0 \\
[\mathrm{A}_{lm}, \mathrm{H}_{jn}] &= \int_{S^2} \mathrm{H}_{jn}\mathrm{d}\mathrm{A}_{lm}\mathrm{Z} \\
&= -\int_{S^2} \delta(\tilde{Y}_{jn}\mathrm{d}z \wedge \mathrm{d}\varphi)\mathrm{d}\tilde{Y}_{lm}\mathrm{Z} \\
&= -\int_{S^2} \Delta(\tilde{Y}_{jn}\mathrm{d}z \wedge \mathrm{d}\varphi)\tilde{Y}_{lm}\mathrm{Z} \\
&= -j(j+1)\left(\int_{S^2} \tilde{Y}_{jn}\tilde{Y}_{lm}\mathrm{d}z \wedge \mathrm{d}\varphi\right)\mathrm{Z} \\
&= (-1)^{n+1} j(j+1)\left(\int_{S^2} \tilde{Y}^*_{j-n}\tilde{Y}_{lm}\mathrm{d}z \wedge \mathrm{d}\varphi\right)\mathrm{Z}
\end{aligned}$$

[38] The minus sign difference comes from the definition the Laplacian.

$$
\begin{aligned}
&= (-1)^{n+1} j(j+1)\delta_{l,j}\delta_{m,-n}\mathrm{Z}\,. \\
[\mathrm{G}_{lm}, \mathrm{H}_{jn}] &= -\Lambda \int_{S^2} \mathrm{G}_{lm}\mathrm{H}_{jn}\mathrm{Z} \\
&= -\Lambda \left(\int_{S^2} \tilde{Y}_{jn} \Delta \tilde{Y}_{lm} \mathrm{d}z \wedge \mathrm{d}\varphi \right) \mathrm{Z} \\
&= (-1)^{n+1} j(j+1)\Lambda \delta_{l,j}\delta_{m,-n}\mathrm{Z}
\end{aligned}
$$

□

Next, we would like to classify this Lie algebra for zero and non-zero cosmological constant.

4.2 Classification of $\widehat{\mathcal{G}}$ & $\widehat{\mathcal{G}}_\Lambda$

The following theorem establishes a connection of $\widehat{\mathcal{G}}$ and $\widehat{\mathcal{G}}_\Lambda$ with a known infinite-dim. Lie algebra.

Theorem 4.2. *There is an isomorphism of Lie algebras:*

$$\widehat{\mathcal{G}} \cong \widehat{\mathcal{G}}_\Lambda \cong \mathcal{A} \oplus \mathfrak{a},$$

where $\mathcal{A}$ is the infinite-dim. oscillator algebra and $\mathfrak{a}$ is the countably infinite-dim. abelian Lie algebra.

Proof. The proof is essentially an easier version of the proof of Theorem 2.2.

Case $\Lambda = 0$ Define new generators

$$
\begin{aligned}
\mathrm{c}_{lm} &:= \mathrm{A}_{l,-m}, && l \neq 0, |m| \leq l, \\
\mathrm{c}^\dagger_{lm} &:= \frac{(-1)^{m+1}}{l(l+1)} \mathrm{H}_{l,m}, && l \neq 0, |m| \leq l, \\
\widehat{\mathrm{A}} &:= A_{00}, \\
\widehat{\mathrm{G}}_{lm} &:= G_{lm}, && l \neq 0, |m| \leq l\,.
\end{aligned}
$$

They satisfy $[\mathrm{c}_{lm}, \mathrm{c}^\dagger_{lm}] = \mathrm{Z}$ and zero otherwise. The abelian summand is spanned by the central elements $\widehat{\mathrm{A}}, \widehat{\mathrm{G}}_{lm}$.

Case $\Lambda \neq 0$ Define new generators

$$
\begin{aligned}
\mathrm{w}_{lm} &:= \frac{1}{2}\left(\mathrm{A}_{l,-m} + \frac{1}{2}\mathrm{G}_{l-m} \right), && l \neq 0, |m| \leq l, \\
\mathrm{w}^\dagger_{lm} &:= \frac{(-1)^{m+1}}{l(l+1)} \mathrm{H}_{l,m}, && l \neq 0, |m| \leq l,
\end{aligned}
$$

$$\widehat{\mathrm{A}} \quad := A_{00},$$
$$\widehat{\mathrm{w}}_{lm} := \frac{1}{2}\left(\mathrm{A}_{l,-m} - \frac{1}{2}\mathrm{G}_{l-m}\right), \quad l \neq 0, |m| \leq l\,.$$

They satisfy $[\mathrm{w}_{lm}, \mathrm{w}^\dagger_{lm}] = \mathrm{Z}$ and zero otherwise. The abelian summand is spanned by the central elements $\widehat{\mathrm{A}}, \widehat{\mathrm{w}}_{lm}$.

□

4.3 Constraints $\mathrm{d}A = 0$ & $\mathrm{d}A + \Lambda B = 0$

We have the following proposition:

Proposition 4.3. *There is an isomorphism of Lie algebras:*

$$\widehat{\mathcal{G}}|_{\mathrm{d}A=0} \cong \mathcal{A} \oplus \mathbb{C}, \quad \widehat{\mathcal{G}}_\Lambda|_{\mathrm{d}A+\Lambda B=0} \cong \mathcal{A}\,.$$

Proof.

Case $\Lambda = 0$ The constraint ideal is generated by the functionals:

$$\widehat{f} := \int_\Gamma f \mathrm{d}A = \int_\Gamma (\mathrm{d}f) A\,,$$

for any smooth function $f \in C^\infty(M)$. In the usual basis, these functionals correspond precisely to the elements $\widehat{\mathrm{G}}_{lm}$. Taking the quotient with respect to the span of these elements, one is left with:

$$\widehat{\mathcal{G}}|_{\mathrm{d}A=0} \cong \mathcal{A} \oplus \mathbb{C},$$

which describes the quantization of the degree zero cohomology of corner fields in abelian BF for $\Lambda = 0$. Again, it makes sense that the center (ignoring the extension) is 1-dim. The first two cohomology groups of the sphere are $H^0(S^2, \mathbb{R}) = \mathbb{R}$ and $H^1(S^2, \mathbb{R}) = \{0\}$.

Case $\Lambda \neq 0$ Similarly to the $\Lambda = 0$ case, the constraints amount to setting most charges in $\mathfrak{a}$ to zero. The constraint ideal is generated by the functionals:

$$\widehat{f} := \int_\Gamma f \mathrm{d}A + \Lambda B = -\int_\Gamma (\mathrm{d}f) A + \int_\Gamma \Lambda f B$$

And therefore, we just need to set the combination of $\mathrm{d}f - \Lambda f \in \widehat{\mathcal{G}}_\Lambda$ forms to zero. In the usual basis, the constraints correspond precisely to the entire abelian summand generated by $\widehat{\mathrm{A}}$ and $\widehat{\mathrm{w}}_{lm}$. Taking the quotient with respect to the span of these elements, one is left with:

$$\widehat{\mathcal{G}}|_{\mathrm{d}A+\Lambda B=0} \cong \mathcal{A}\,,$$

which describes the quantization of the degree zero cohomology of corner fields in abelian BF for $\Lambda \neq 0$.

□

4.4 Representations of $\widehat{\mathcal{G}}|_{\mathrm{d}A}$ & $\widehat{\mathcal{G}}_\Lambda|_{\mathrm{d}A+\Lambda B=0}$

A representation of $\mathcal{A}$, e.g. the bosonic Fock space representation, automatically induces representations of $\widehat{\mathcal{G}}|_{\mathrm{d}A}$ & $\widehat{\mathcal{G}}_\Lambda|_{\mathrm{d}A+\Lambda B=0}$. If these representations can be extended to the full Lie algebra (not just finite modes), the modules constitute possible state spaces of 4-dim. abelian BF on surfaces that bound a sphere. It would also be interesting to see whether similar results hold for surfaces with higher genus, i.e. $g > 1$.

5 Outlook

In this thesis, we attempted to construct representations of the quantized corner algebra associated to a torus and a sphere in the BV-BFV formalism of 4-dim. BF theory. While the construction was successful in the abelian case, we only obtained mixed results in the non-abelian case. We have constructed a family of representations for a central extension of a double-loop algebra over a non-semisimple Lie algebra. However, they do not descend to a representation of the physical corner algebra. In the future, one could investigate whether the reduction to the physical corner algebra can be done non-trivially or whether there is a general theorem preventing this construction from working. Regardless, these representations might still be interesting from a mathematical perspective. To conclude the thesis, we present the current standings of the investigation into the corner algebra of 4-dim. non-abelian BF on the sphere.

5.1 Non-Abelian BF on $\Gamma \cong S^2$

The quantized corner structure of 4-dim. non-abelian BF on a sphere is not fully worked out, because it involves difficult integrals of derivatives of spherical harmonics. At least using this particular method. In the following section, we provide the results that were already obtained and state the necessary equations that need to be solved to make progress.

5.1.1 The Lie Algebra $\widehat{\mathcal{G}}_\Lambda(\mathfrak{su}(2))$

The Lie algebra describing 4-dim. non-abelian BF on a sphere with $\mathfrak{g} = \mathfrak{su}(2)$ is given by the vector space:

$$\widehat{\mathcal{G}}_\Lambda(\mathfrak{su}(2)) = \Omega^0(S^2) \otimes \mathfrak{su}(2) \oplus \Omega^1(S^2) \otimes \mathfrak{su}(2) \oplus \mathbb{R}\,,$$

with brackets

$$[f \oplus \alpha \oplus r, g \oplus \beta \oplus s]_{\widehat{\mathcal{G}}_\Lambda(\mathfrak{su}(2))} = [f,g] \oplus (\mathrm{ad}_f\beta - \mathrm{ad}_g\alpha) \oplus -\int_{S^2} (\alpha \mathrm{d}g - \beta \mathrm{d}f + \Lambda\alpha\beta)\mathrm{Z}\,,$$

where we choose the trivial reference connection A_0. Choose the basis of the complex Lie subalgebra of finite modes to be:

$$\begin{aligned}
\mathrm{A}_{\mu lm} &\coloneqq t_\mu \otimes \tilde{Y}_{lm}(z,\varphi) && \text{for } l \in \mathbb{N}, |m| \leq l\,,\\
\mathrm{G}_{\mu lm} &\coloneqq t_\mu \otimes \mathrm{d}\tilde{Y}_{lm}(z,\varphi) && \text{for } l \in \mathbb{N}_{>0}, |m| \leq l\,,\\
\mathrm{H}_{\mu lm} &\coloneqq t_\mu \otimes \star\,\mathrm{d}\tilde{Y}_{lm}(z,\varphi) && \text{for } l \in \mathbb{N}_{>0}, |m| \leq l\,.
\end{aligned}$$

The resulting bracket relations are the following:

$$[\mathrm{A}_{\mu lm}, \mathrm{A}_{\nu jn}] = \varepsilon^{\lambda}_{\mu\nu} c_{lmjnLM} \mathrm{A}_{\lambda LM}\,,$$

$$[\mathrm{A}_{\mu lm}, \mathrm{G}_{\nu jn}] = \varepsilon_{\mu\nu}^{\lambda} a_{lmjnLM} \mathrm{G}_{\lambda LM} + \varepsilon_{\mu\nu}^{\lambda} b_{lmjnLM} \mathrm{H}_{\lambda LM},$$
$$[\mathrm{A}_{\mu lm}, \mathrm{H}_{\nu jn}] = -\varepsilon_{\mu\nu}^{\lambda} b_{lmjnLM} \mathrm{G}_{\lambda LM} + \varepsilon_{\mu\nu}^{\lambda} a_{lmjnLM} \mathrm{H}_{\lambda LM} + (-1)^{n+1} j(j+1) \delta_{\mu,\nu} \delta_{l,j} \delta_{m,-n} \mathrm{Z},$$
$$[\mathrm{G}_{\mu lm}, \mathrm{H}_{\nu jn}] = (-1)^{n+1} j(j+1) \Lambda \delta_{\mu,\nu} \delta_{l,j} \delta_{m,-n} \mathrm{Z}\,,$$

and all other brackets vanish. The coefficients c_{lmjnLM} are defined implicitly via the tensor product decomposition $\tilde{Y}_{lm}\tilde{Y}_{jn} = \sum_{|l-j| \leq L \leq |l+j|, M \leq |L|} c_{lmjnLM} \tilde{Y}_{LM}$ and thus can be computed explicitly by Clebsch-Gordan coefficients. The coefficients a_{lmjnLM} and b_{lmjnLM} are defined via the Hodge decomposition

$$\tilde{Y}_{lm} \mathrm{d}\tilde{Y}_{jn} = a_{lmjnLM} \mathrm{d}\tilde{Y}_{LM} + b_{lmjnLM} \star \mathrm{d}\tilde{Y}_{LM}\,.$$

5.1.2 Coefficients in the Hodge Decomposition

In principle, both coefficients can be computed with the two integrals

$$\begin{aligned}
\int_{S^2} \tilde{Y}_{lm} \mathrm{d}\tilde{Y}_{jn} \wedge \star \mathrm{d}\tilde{Y}_{LM} &= a_{lmjnJN} \int_{S^2} \mathrm{d}\tilde{Y}_{JN} \wedge \star \mathrm{d}\tilde{Y}_{LM} \\
&= a_{lmjnJN} \int_{S^2} \tilde{Y}_{JN} \Delta \tilde{Y}_{LM} \mathrm{d}z \wedge \mathrm{d}\varphi \\
&= a_{lmjnL-M} (-1)^M L(L+1) \\
\implies a_{lmjnLM} &= \frac{(-1)^M}{L(L+1)} \int_{S^2} \tilde{Y}_{lm} \mathrm{d}\tilde{Y}_{jn} \wedge \mathrm{d} \star \tilde{Y}_{L-M} \text{ for } L \neq 0
\end{aligned}$$

and

$$\begin{aligned}
\int_{S^2} \tilde{Y}_{lm} \mathrm{d}\tilde{Y}_{jn} \wedge \mathrm{d}\tilde{Y}_{LM} &= b_{lmjnJN} \int_{S^2} \star \mathrm{d}\tilde{Y}_{JN} \wedge \mathrm{d}\tilde{Y}_{LM} \\
&= -b_{lmjnJN} \int_{S^2} (\Delta \tilde{Y}_{JN}) \tilde{Y}_{LM} \mathrm{d}z \wedge \mathrm{d}\varphi \\
&= -b_{lmjnL-M} (-1)^M L(L+1) \\
\implies b_{lmjnLM} &= \frac{(-1)^{M+1}}{L(L+1)} \int_{S^2} \tilde{Y}_{lm} \mathrm{d}\tilde{Y}_{jn} \wedge \mathrm{d}\tilde{Y}_{L-M} \text{ for } L \neq 0
\end{aligned}$$

The former can be calculated explicitly via the identity $\Delta(fg) = -2\langle \mathrm{d}f, \mathrm{d}g \rangle + f\Delta g + g\Delta f$, leading to the formula:

$$a_{lmjnLM} = (-1)^{M+m} \frac{-l(l+1) + j(j+1) + L(L+1)}{2L(L+1)} c_{jnL-Ml-m} \qquad \text{for } L \neq 0$$

We did not manage to solve the "YdYdY" integral yet. Thus, the b coefficients are left implicit. Integrals of derivatives of spherical harmonics are related to the multipole expansions and might be contained in one of the numerous integral books on these subjects. However, despite an intensive search, we could not find anything useful.

5.1.3 $(m = 0)$-Level Lie Subalgebra

Another possibility is to be content with studying a subalgebra.

Proposition 5.1. *The $(m = 0)$**-level Lie subalgebra** is spanned by generators* $\mathrm{Z}, \mathrm{A}_{l0}, \mathrm{G}_{l0}$ *and* H_{l0} *and denoted by* $\widehat{\mathcal{G}}(\mathfrak{su}(2))_{m=0}$ *and* $\widehat{\mathcal{G}}_\Lambda(\mathfrak{su}(2))_{m=0}$ *respectively.*

Proof. These vector subspaces actually form subalgebras because $\mathrm{d}\tilde{Y}_{l0} \propto \mathrm{d}\theta$ and $\star\mathrm{d}\tilde{Y}_{l0} \propto \mathrm{d}\varphi$, so that $\tilde{Y}_{l0}\mathrm{d}\tilde{Y}_{j0} = a_{l0j0L0}\mathrm{d}\tilde{Y}_{L0}$ and $\tilde{Y}_{l0} \star \mathrm{d}\tilde{Y}_{j0} = a_{l0j0L0} \star \mathrm{d}\tilde{Y}_{L0}$. In other words, the m index can never become non-zero via commutation relations. Furthermore, all coefficients that are involved are known because the b coefficients drop out. □

5.2 Constraint $F_A + \Lambda B = 0$

We just treat the $\Lambda \neq 0$ case since it works almost identically for zero cosmological constant. The ideal of constraints in $C^\infty(\mathcal{B}_0)$ is generated by the functionals:

$$\widehat{f} := \int_{S^2} f(F_A + \Lambda B),$$

where $f \in \Omega^0(S^2) \otimes \mathfrak{su}(2)$. As usual, we consider the subspace of finite modes and decompose this multilinear form into an infinite sum of products of linear functionals. The functional has the following description:

Proposition 5.2. $\widehat{f} = f_{\lambda lm}\Big(\mathrm{G}_{\lambda lm} + \Lambda \mathrm{A}_{\lambda lm} + \frac{(-1)^{n+1}}{j(j+1)}\varepsilon^\lambda_{\mu\nu} a_{lmj-nLM}\mathrm{H}_{\mu j-n}\mathrm{G}_{\nu L-M}\Big)$

Proof. Denote $\tilde{Y}_{\mu mn} := t_\mu \otimes \tilde{Y}_{lm}$, then

$$\begin{aligned}
\widehat{f}(A,B) &= \int_{S^2} (f \overset{\wedge}{,} \mathrm{d}A + \frac{1}{2}[A,A] + \Lambda B) \\
&= f_{\lambda lm} \int_{S^2} \Big(\tilde{Y}_{\lambda lm} \overset{\wedge}{,} A^{(\mathrm{H})}_{\mu jn} \mathrm{d} \star \mathrm{d}\tilde{Y}_{\mu jn} + \Lambda B_{\mu jn} \tilde{Y}_{\mu jn} \mathrm{d}z \wedge \mathrm{d}\varphi \\
&\quad + \varepsilon^\rho_{\mu\nu} t_\rho \otimes A^{(\mathrm{G})}_{\mu jn} A^{(\mathrm{H})}_{\nu LM} \mathrm{d}\tilde{Y}_{jn} \wedge \star \mathrm{d}\tilde{Y}_{LM} \Big) \\
&= f_{\lambda lm} \int_{S^2} \Big(\tilde{Y}_{\lambda lm} \overset{\wedge}{,} - j(j+1) A^{(\mathrm{H})}_{\mu jn} \tilde{Y}_{\mu jn} \mathrm{d}z \wedge \mathrm{d}\varphi + \Lambda B_{\mu jn} \tilde{Y}_{\mu jn} \mathrm{d}z \wedge \mathrm{d}\varphi \\
&\quad + \varepsilon^\rho_{\mu\nu} t_\rho \otimes A^{(\mathrm{G})}_{\mu jn} A^{(\mathrm{H})}_{\nu LM} \mathrm{d}\tilde{Y}_{jn} \wedge \star \mathrm{d}\tilde{Y}_{LM} \Big) \\
&= f_{\lambda lm} \Big(-l(l+1)(-1)^m A^{(\mathrm{H})}_{\lambda l-m} + \Lambda (-1)^m B_{\lambda l-m} \\
&\quad + \varepsilon^\lambda_{\mu\nu} A^{(\mathrm{G})}_{\mu jn} A^{(\mathrm{H})}_{\nu LM} \int_{S^2} \tilde{Y}_{\lambda lm} \mathrm{d}\tilde{Y}_{jn} \wedge \star \mathrm{d}\tilde{Y}_{LM} \Big) \\
&= f_{\lambda lm} \Big(-l(l+1)(-1)^m A^{(\mathrm{H})}_{\lambda l-m} + \Lambda (-1)^m B_{\lambda l-m} \\
&\quad + L(L+1)(-1)^M \varepsilon^\lambda_{\mu\nu} a_{lmjnL-M} A^{(\mathrm{G})}_{\mu jn} A^{(\mathrm{H})}_{\nu LM} \Big) \\
&= f_{\lambda lm} \Big(\mathrm{G}_{\lambda lm} + \Lambda \mathrm{A}_{\lambda lm} + \frac{(-1)^{n+1}}{j(j+1)} \varepsilon^\lambda_{\mu\nu} a_{lmj-nLM} \mathrm{H}_{\mu j-n} \mathrm{G}_{\nu L-M} \Big)(A+B)\,,
\end{aligned}$$

where we used the definition of the linear functionals via the pairing in the last step. □

One can now replace the a coefficients by the c coefficients which admit a closed form expression. These elements descend to the UEA and represent the quantization of the set of constraints. The bracket relations with the generators of the Lie algebra have not been worked out yet. It is expected that the constraints generate a proper, two-sided ideal in the UEA similarly to Proposition 3.9.

A Lemmas for the Proof of Theorem 3.20

In this appendix, we provide a short list of lemmas that are used in the proof of Theorem 3.20. The lemmas deal with the commutation relation between the various shift operators and partial derivative operators.

Lemma A.1.

$$
\begin{aligned}
[\mathcal{E}_{\mathrm{X,Y}}(k,l), \mathcal{E}_{\mathrm{Z,W}}(t,u)] &= \left[\sum_{m,n}(-2x_{(k+m)(l+n)})\frac{\partial}{\partial y_{mn}}, \sum_{r,s}(-2z_{(t+r)(u+s)})\frac{\partial}{\partial w_{rs}}\right] \\
&= \sum_{m,n}\sum_{r,s}(-2x_{(k+m)(l+n)})(-2)\delta_{\mathrm{Y,Z}}\delta_{m,t+r}\delta_{n,u+s}\frac{\partial}{\partial w_{rs}} \\
&\qquad - \sum_{r,s}\sum_{m,n}(-2z_{(t+r)(u+s)})(-2)\delta_{\mathrm{W,X}}\delta_{r,k+m}\delta_{s,l+n}\frac{\partial}{\partial y_{mn}} \\
&= -2\delta_{\mathrm{Y,Z}}\mathcal{E}_{\mathrm{X,W}}(k+t, l+u) + 2\delta_{\mathrm{W,X}}\mathcal{E}_{\mathrm{Z,Y}}(k+t, l+u)\,,
\end{aligned}
$$

for $k,l,t,u \in \mathbb{Z}$ *and* $\mathrm{X,Y,Z,W} \in \{\mathrm{J,K,P}\}$*. In particular*

$$
[\mathcal{E}_{\mathrm{X,X}}(k,l), \mathcal{E}_{\mathrm{Y,Y}}(t,u)] = 0\,,
$$

for $k,l,t,u \in \mathbb{Z}$ *and* $\mathrm{X,Y} \in \{\mathrm{J,K,P}\}$

Lemma A.2.

$$
\begin{aligned}
\left[\mathcal{E}_{\mathrm{X,Y}}(k,l), \frac{\partial}{\partial z_{tu}}\right] &= -\frac{\partial}{\partial z_{tu}}\sum_{m,n}(-2x_{(k+m)(l+n)})\frac{\partial}{\partial y_{mn}} \\
&= \sum_{m,n} 2\delta_{\mathrm{X,Y}}\delta_{t,k+m}\delta_{u,l+n}\frac{\partial}{\partial y_{mn}} \\
&= 2\delta_{\mathrm{X,Z}}\frac{\partial}{\partial y_{(t-k)(u-l)}},
\end{aligned}
$$

for $k,l,t,u \in \mathbb{Z}$ *and* $\mathrm{X,Y,Z} \in \{\mathrm{J,K,P}\}$*.*

Lemma A.3.

$$
\begin{aligned}
\left[\mathcal{E}_{\mathrm{X,X}}(k,l), \sum_{r,s}\mathcal{E}_{\mathrm{Y,Y}}(t+r,u+s)\frac{\partial}{\partial z_{rs}}\right] &= \sum_{r,s}\left[\mathcal{E}_{\mathrm{X,X}}(k,l), \mathcal{E}_{\mathrm{Y,Y}}(t+r,u+s)\right]\frac{\partial}{\partial z_{rs}} \\
&\qquad + \sum_{r,s}\mathcal{E}_{\mathrm{Y,Y}}(t+r,u+s)\left[\mathcal{E}_{\mathrm{X,X}}(k,l), \frac{\partial}{\partial z_{rs}}\right] \\
&= 2\delta_{\mathrm{X,Z}}\sum_{r,s}\mathcal{E}_{\mathrm{Y,Y}}(t+r,u+s)\frac{\partial}{\partial x_{(r-k)(s-l)}},
\end{aligned}
$$

for $k,l,t,u \in \mathbb{Z}$ *and* $\mathrm{X,Y,Z} \in \{\mathrm{J,K,P}\}$*.*

Lemma A.4.

$$\begin{aligned}\left[\mathcal{E}_{\mathrm{X,Y}}(k,l), \sum_{r,s} \mathcal{E}_{\mathrm{Z,U}}(t+r,u+s)\frac{\partial}{\partial w_{rs}}\right] &= \sum_{r,s} \left[\mathcal{E}_{\mathrm{X,Y}}(k,l), \mathcal{E}_{\mathrm{Z,U}}(t+r,u+s)\right] \frac{\partial}{\partial w_{rs}} \\ &+ \sum_{r,s} \mathcal{E}_{\mathrm{Z,U}}(t+r,u+s) \left[\mathcal{E}_{\mathrm{X,Y}}(k,l), \frac{\partial}{\partial w_{rs}}\right] \\ &= \sum_{r,s}(-2\delta_{\mathrm{Y,Z}}\mathcal{E}_{\mathrm{X,U}}(k+t+r,l+u+s) \\ &+ \sum_{r,s} 2\delta_{\mathrm{U,X}}\mathcal{E}_{\mathrm{Z,Y}}(k+t+r,l+u+s))\frac{\partial}{\partial w_{rs}} \\ &+ \sum_{r,s} \mathcal{E}_{\mathrm{Z,U}}(t+r,u+s)2\delta_{\mathrm{X,W}}\frac{\partial}{\partial y_{(r-k)(s-l)}} \\ &= \sum_{r,s}(-2\delta_{\mathrm{Y,Z}}\mathcal{E}_{\mathrm{X,U}}(k+t+r,l+u+s) \\ &+ \sum_{r,s} 2\delta_{\mathrm{U,X}}\mathcal{E}_{\mathrm{Z,Y}}(k+t+r,l+u+s))\frac{\partial}{\partial w_{rs}} \\ &+ \sum_{r,s} 2\delta_{\mathrm{X,W}}\mathcal{E}_{\mathrm{Z,U}}(t+r+k,u+s+l)\frac{\partial}{\partial y_{rs}},\end{aligned}$$

for $k,l,t,u \in \mathbb{Z}$ *and* $\mathrm{X,Y,Z,U,W} \in \{\mathrm{J,K,P}\}$.

Lemma A.5.

$$\begin{aligned}\Bigg[\sum_{m,n} &\left(\mathcal{E}_{\mathrm{K,K}}(k+m,l+n) + \mathcal{E}_{\mathrm{P,P}}(k+m,l+n) + \frac{1}{2}\mathcal{E}_{\mathrm{J,J}}(k+m,l+n)\right)\frac{\partial}{\partial j_{mn}}, \\ &\sum_{r,s}\left(\mathcal{E}_{\mathrm{K,K}}(t+r,u+s) + \mathcal{E}_{\mathrm{P,P}}(t+r,u+s) + \frac{1}{2}\mathcal{E}_{\mathrm{J,J}}(t+r,u+s)\right)\frac{\partial}{\partial j_{rs}}\Bigg] = 0\end{aligned}$$

for $k,l,t,u \in \mathbb{Z}$.

B Proof of Theorem 3.20

Proof. To make proving the bracket relations less involved, we introduce new notation inspired by [Wak86]:

$$\mathcal{E}_{\mathrm{XY}}(k,l) := \sum_{m,n}(-2x_{(k+m)(l+n)})\frac{\partial}{\partial y_{mn}}$$

for $k, l \in \mathbb{Z}$ and $\mathrm{X}, \mathrm{Y} \in \{\mathrm{J}, \mathrm{K}, \mathrm{P}\}$. Furthermore, we set $\chi_{\mathrm{F}^-_{kl}} = \chi_{\hat{\Phi}_l} = \chi_{\hat{\Theta}_k} = 0$ for $k \neq 0, l \neq 0$ for two reasons. Firstly, they are related to the abelian constraints. Secondly, the relevant steps of the proof are unchanged but much more clutter is avoided.

The free field realization simplifies to the following expression:

$$\begin{aligned}
\mathrm{Z} \longrightarrow & 1 \\
\mathrm{J}^-_{kl} \longrightarrow & j_{kl} \\
\mathrm{K}^-_{kl} \longrightarrow & k_{kl} \\
\mathrm{P}^-_{kl} \longrightarrow & p_{kl} \\
\mathrm{J}^z_{kl} \longrightarrow & (2 + (2\chi_{\hat{\mathrm{E}}} - 2)\delta_{k,0}\delta_{l,0})\frac{\partial}{\partial v_{-k,-l}} + \mathcal{E}_{\mathrm{J,J}}(k,l) + \mathcal{E}_{\mathrm{K,K}}(k,l) + \mathcal{E}_{\mathrm{P,P}}(k,l) \\
\mathrm{K}^z_{kl} \longrightarrow & -2ikv_{kl} - 2i\chi_{\hat{\Phi}_0}\delta_{k,0}\delta_{l,0} + \mathcal{E}_{\mathrm{K,J}}(k,l) \\
\mathrm{P}^z_{kl} \longrightarrow & 2ilv_{kl} - 2i\chi_{\hat{\Theta}_0}\delta_{k,0}\delta_{l,0} + \mathcal{E}_{\mathrm{P,J}}(k,l) \\
\mathrm{J}^+_{kl} \longrightarrow & \sum_{m,n}\Big(2 + (2\chi_{\hat{\mathrm{E}}} - 2)\delta_{k+m,0}\delta_{l+n,0}\Big)\frac{\partial}{\partial v_{-(k+m)-(l+n)}}\frac{\partial}{\partial j_{mn}} \\
& + \sum_{m,n}\Big(-2i(k+m)v_{(k+m)(l+n)} - 2i(\chi_{\hat{\Phi}_0} + m)\delta_{k+m,0}\delta_{l+n,0}\Big)\frac{\partial}{\partial k_{mn}} \\
& + \sum_{m,n}\Big(2i(l+n)v_{(k+m)(l+n)} - 2i(\chi_{\hat{\Theta}_0} - n)\delta_{k+m,0}\delta_{l+n,0}\Big)\frac{\partial}{\partial p_{mn}} \\
& + \sum_{m,n}\Big(\mathcal{E}_{\mathrm{K,K}}(k+m,l+n) + \mathcal{E}_{\mathrm{P,P}}(k+m,l+n) + \frac{1}{2}\mathcal{E}_{\mathrm{J,J}}(k+m,l+n)\Big)\frac{\partial}{\partial j_{mn}} \\
\mathrm{K}^+_{kl} \longrightarrow & \sum_{m,n}\Big(-2i(k+m)v_{(k+m)(l+n)} - 2i(\chi_{\hat{\Phi}_0} + m)\delta_{k+m,0}\delta_{l+n,0} + \frac{1}{2}\mathcal{E}_{\mathrm{K,J}}(k+m,l+n)\Big)\frac{\partial}{\partial j_{mn}} \\
\mathrm{P}^+_{kl} \longrightarrow & \sum_{m,n}\Big(2i(l+n)v_{(k+m)(l+n)} - 2i(\chi_{\hat{\Theta}_0} - n)\delta_{k+m,0}\delta_{l+n,0} + \frac{1}{2}\mathcal{E}_{\mathrm{P,J}}(k+m,l+n)\Big)\frac{\partial}{\partial j_{mn}}
\end{aligned}$$

Next, we verify each bracket. Note that the representation of K and P are essentially identical, hence we will only prove the brackets with respect to prior generators. This proof is unfortunately more of an exercise in patience than in skill.

$$[\mathrm{K}^z_{kl}, \mathrm{K}^z_{tu}] = [\mathrm{K}^z_{kl}, \mathrm{K}^\pm_{tu}] = [\mathrm{K}^\pm_{kl}, \mathrm{K}^\pm_{tu}] = 0$$

The operators only involve derivatives with respect to j variables, whereas the coefficient functions do not depend on any j variable; therefore, the operators commute.

$$\begin{aligned}
[\mathrm{J}^z_{kl}, \mathrm{J}^z_{tu}] &= \Bigg[(2 + (2\chi_{\widehat{\mathrm{E}}} - 2)\delta_{k,0}\delta_{l,0})\frac{\partial}{\partial v_{-k,-l}} + \mathcal{E}_{\mathrm{J,J}}(k,l) + \mathcal{E}_{\mathrm{K,K}}(k,l) + \mathcal{E}_{\mathrm{P,P}}(k,l), \\
&\qquad (2 + (2\chi_{\widehat{\mathrm{E}}} - 2)\delta_{t,0}\delta_{u,0})\frac{\partial}{\partial v_{-t,-u}} + \mathcal{E}_{\mathrm{J,J}}(t,u) + \mathcal{E}_{\mathrm{K,K}}(t,u) + \mathcal{E}_{\mathrm{P,P}}(t,u)\Bigg] \\
&= [\mathcal{E}_{\mathrm{J,J}}(k,l) + \mathcal{E}_{\mathrm{K,K}}(k,l) + \mathcal{E}_{\mathrm{P,P}}(k,l), \mathcal{E}_{\mathrm{J,J}}(t,u) + \mathcal{E}_{\mathrm{K,K}}(t,u) + \mathcal{E}_{\mathrm{P,P}}(t,u)] \\
&= 0
\end{aligned}$$

The last equality follows from Lemma $A.1$.

$$\begin{aligned}
[\mathrm{J}^z_{kl}, \mathrm{J}^-_{tu}] &= \left((2 + (2\chi_{\widehat{\mathrm{E}}} - 2)\delta_{k,0}\delta_{l,0})\frac{\partial}{\partial v_{-k,-l}} + \mathcal{E}_{\mathrm{J,J}}(k,l) + \mathcal{E}_{\mathrm{K,K}}(k,l) + \mathcal{E}_{\mathrm{P,P}}(k,l)\right) j_{tu} \\
&= \mathcal{E}_{\mathrm{J,J}}(k,l) j_{tu} \\
&= -2 j_{(k+t)(l+u)} \\
&= -2\mathrm{J}^-_{(k+t)(l+u)}
\end{aligned}$$

$$\begin{aligned}
[\mathrm{J}^z_{kl}, \mathrm{J}^+_{tu}] &= \Bigg[(2 + (2\chi_{\widehat{\mathrm{E}}} - 2)\delta_{k,0}\delta_{l,0})\frac{\partial}{\partial v_{-k,-l}} + \mathcal{E}_{\mathrm{J,J}}(k,l) + \mathcal{E}_{\mathrm{K,K}}(k,l) + \mathcal{E}_{\mathrm{P,P}}(k,l), \\
&\qquad \sum_{r,s} \left(2 + (2\chi_{\widehat{\mathrm{E}}} - 2)\delta_{t+r,0}\delta_{u+s,0}\right) \frac{\partial}{\partial v_{-(t+r)-(u+s)}} \frac{\partial}{\partial j_{rs}} \\
&\qquad + \sum_{r,s} \left(-2i(t+r)v_{(t+r)(u+s)} - 2i(\chi_{\widehat{\Phi}_0} + r)\delta_{t+r,0}\delta_{u+s,0}\right) \frac{\partial}{\partial k_{rs}} \\
&\qquad + \sum_{r,s} \left(2i(u+s)v_{(t+r)(u+s)} - 2i(\chi_{\widehat{\Theta}_0} - s)\delta_{t+r,0}\delta_{u+s,0}\right) \frac{\partial}{\partial p_{rs}} \\
&\qquad + \sum_{r,s} \left(\mathcal{E}_{\mathrm{K,K}}(t+r,u+s) + \mathcal{E}_{\mathrm{P,P}}(t+r,u+s) + \frac{1}{2}\mathcal{E}_{\mathrm{J,J}}(t+r,u+s)\right) \frac{\partial}{\partial j_{rs}}\Bigg] \\
&= \Bigg[(2 + (2\chi_{\widehat{\mathrm{E}}} - 2)\delta_{k,0}\delta_{l,0})\frac{\partial}{\partial v_{-k,-l}}, \\
&\qquad + \sum_{r,s} \left(-2i(t+r)v_{(t+r)(u+s)}\right) \frac{\partial}{\partial k_{rs}} \\
&\qquad + \sum_{r,s} \left(2i(u+s)v_{(t+r)(u+s)}\right) \frac{\partial}{\partial p_{rs}}\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&+ \Bigg[\mathcal{E}_{\mathrm{J,J}}(k,l) + \mathcal{E}_{\mathrm{K,K}}(k,l) + \mathcal{E}_{\mathrm{P,P}}(k,l), \\
&\qquad \sum_{r,s} \left(2 + (2\chi_{\hat{\mathrm{E}}} - 2)\delta_{t+r,0}\delta_{u+s,0}\right) \frac{\partial}{\partial v_{-(t+r)-(u+s)}} \frac{\partial}{\partial j_{rs}} \\
&\qquad + \sum_{r,s} \left(-2i(t+r)v_{(t+r)(u+s)} - 2i(\chi_{\hat{\Phi}_0} + r)\delta_{t+r,0}\delta_{u+s,0}\right) \frac{\partial}{\partial k_{rs}} \\
&\qquad + \sum_{r,s} \left(2i(u+s)v_{(t+r)(u+s)} - 2i(\chi_{\hat{\Theta}_0} - s)\delta_{t+r,0}\delta_{u+s,0}\right) \frac{\partial}{\partial p_{rs}}\Bigg] \\
&+ \left[\mathcal{E}_{\mathrm{J,J}}(k,l), \sum_{r,s} \left(\mathcal{E}_{\mathrm{K,K}}(t+r,u+s) + \mathcal{E}_{\mathrm{P,P}}(t+r,u+s) + \frac{1}{2}\mathcal{E}_{\mathrm{J,J}}(t+r,u+s)\right) \frac{\partial}{\partial j_{rs}}\right]
\end{aligned}
$$

In this step, we used the linearity of the bracket and dropped manifestly vanishing contributions. Now, we can apply the Lemmas A.1 - A.3 to compute the remaining brackets.

$$
\begin{aligned}
[\mathrm{J}^z_{kl}, \mathrm{J}^+_{tu}] &= \sum_{r,s} \left((2 + (2\chi_{\hat{\mathrm{E}}} - 2)\delta_{k,0}\delta_{l,0})(-2i(t+r))\delta_{k+t+r,0}\delta_{l+u+s}\right) \frac{\partial}{\partial k_{rs}} \\
&\qquad + \sum_{r,s} \left((2 + (2\chi_{\hat{\mathrm{E}}} - 2)\delta_{k,0}\delta_{l,0})(2i(u+s))\delta_{k+t+r,0}\delta_{l+u+s}\right) \frac{\partial}{\partial p_{rs}} \\
&+ 2\sum_{r,s} \left(2 + (2\chi_{\hat{\mathrm{E}}} - 2)\delta_{t+r,0}\delta_{u+s,0}\right) \frac{\partial}{\partial v_{-(t+r)-(u+s)}} \frac{\partial}{\partial j_{(r-k)(s-l)}} \\
&\qquad + 2\sum_{r,s} \left(-2i(t+r)v_{(t+r)(u+s)} - 2i(\chi_{\hat{\Phi}_0} + r)\delta_{t+r,0}\delta_{u+s,0}\right) \frac{\partial}{\partial k_{(r-k)(s-l)}} \\
&\qquad + 2\sum_{r,s} \left(2i(u+s)v_{(t+r)(u+s)} - 2i(\chi_{\hat{\Theta}_0} - s)\delta_{t+r,0}\delta_{u+s,0}\right) \frac{\partial}{\partial p_{(r-k)(s-l)}} \\
&+ 2\sum_{r,s} \left(\mathcal{E}_{\mathrm{K,K}}(t+r,u+s) + \mathcal{E}_{\mathrm{P,P}}(t+r,u+s) + \frac{1}{2}\mathcal{E}_{\mathrm{J,J}}(t+r,u+s)\right) \frac{\partial}{\partial j_{(r-k)(s-l)}}
\end{aligned}
$$

So, by relabeling the sums, we obtain:

$$
\begin{aligned}
[\mathrm{J}^z_{kl}, \mathrm{J}^+_{tu}] &= \sum_{r,s} \left((2 + (2\chi_{\hat{\mathrm{E}}} - 2)\delta_{k,0}\delta_{l,0})(-2i(t+r))\delta_{k+t+r,0}\delta_{l+u+s}\right) \frac{\partial}{\partial k_{rs}} \\
&\qquad + \sum_{r,s} \left((2 + (2\chi_{\hat{\mathrm{E}}} - 2)\delta_{k,0}\delta_{l,0})(2i(u+s))\delta_{k+t+r,0}\delta_{l+u+s}\right) \frac{\partial}{\partial p_{rs}} \\
&+ 2\sum_{r,s} \left(2 + (2\chi_{\hat{\mathrm{E}}} - 2)\delta_{t+r+k,0}\delta_{u+s+l,0}\right) \frac{\partial}{\partial v_{-(t+r+k)-(u+s+l)}} \frac{\partial}{\partial j_{rs}} \\
&\qquad + 2\sum_{r,s} \left(-2i(t+r+k)v_{(t+r+k)(u+s+l)} - 2i(\chi_{\hat{\Phi}_0} + r + k)\delta_{t+r+k,0}\delta_{u+s+l,0}\right) \frac{\partial}{\partial k_{rs}} \\
&\qquad + 2\sum_{r,s} \left(2i(u+s+l)v_{(t+r+k)(u+s+l)} - 2i(\chi_{\hat{\Theta}_0} - s - l)\delta_{t+r+k,0}\delta_{u+s+l,0}\right) \frac{\partial}{\partial p_{rs}}
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\sum_{r,s}(\mathcal{E}_{\mathrm{K,K}}(t+r+k,u+s+l) + \mathcal{E}_{\mathrm{P,P}}(t+r+k,u+s+l) \\
&\qquad + \frac{1}{2}\mathcal{E}_{\mathrm{J,J}}(t+r+k,u+s+l))\frac{\partial}{\partial j_{rs}}
\end{aligned}
$$

This is already quite close to the expected result of $2\mathrm{J}^{+}_{(k+t)(l+u)}$. Let us examine the $\frac{\partial}{\partial k_{rs}}$ and $\frac{\partial}{\partial p_{rs}}$ terms more closely.

$$
\begin{aligned}
&\sum_{r,s}\Big((2+(2\chi_{\hat{\mathrm{E}}}-2)\delta_{k,0}\delta_{l,0})(-2i(t+r))\delta_{k+t+r,0}\delta_{l+u+s}\Big)\frac{\partial}{\partial k_{rs}} \\
&\qquad + \sum_{r,s}\Big((2+(2\chi_{\hat{\mathrm{E}}}-2)\delta_{k,0}\delta_{l,0})(2i(u+s))\delta_{k+t+r,0}\delta_{l+u+s}\Big)\frac{\partial}{\partial p_{rs}} \\
&\qquad + 2\sum_{r,s}\Big(-2i(t+r+k)v_{(t+r+k)(u+s+l)} - 2i(\chi_{\hat{\Phi}_0}+r+k)\delta_{t+r+k,0}\delta_{u+s+l,0}\Big)\frac{\partial}{\partial k_{rs}} \\
&\qquad + 2\sum_{r,s}\Big(2i(u+s+l)v_{(t+r+k)(u+s+l)} - 2i(\chi_{\hat{\Theta}_0}-s-l)\delta_{t+r+k,0}\delta_{u+s+l,0}\Big)\frac{\partial}{\partial p_{rs}} \\
&= 2\sum_{r,s}\Big(-2i(t+r+k)v_{(t+r+k)(u+s+l)} - 2i(\chi_{\hat{\Phi}_0}+r+k+t+r)\delta_{t+r+k,0}\delta_{u+s+l,0}\Big)\frac{\partial}{\partial k_{rs}} \\
&\qquad + 2\sum_{r,s}\Big(2i(u+s+l)v_{(t+r+k)(u+s+l)} - 2i(\chi_{\hat{\Theta}_0}-s-l-u-s)\delta_{t+r+k,0}\delta_{u+s+l,0}\Big)\frac{\partial}{\partial p_{rs}} \\
&= 2\sum_{r,s}\Big(-2i(t+r+k)v_{(t+r+k)(u+s+l)} - 2i(\chi_{\hat{\Phi}_0}+r)\delta_{t+r+k,0}\delta_{u+s+l,0}\Big)\frac{\partial}{\partial k_{rs}} \\
&\qquad + 2\sum_{r,s}\Big(2i(u+s+l)v_{(t+r+k)(u+s+l)} - 2i(\chi_{\hat{\Theta}_0}-s)\delta_{t+r+k,0}\delta_{u+s+l,0}\Big)\frac{\partial}{\partial p_{rs}}
\end{aligned}
$$

The delta functions kills the contribution proportional to $(2\chi_{\hat{\mathrm{E}}}-2)$ in the first equality and in the second one the delta function ensure that the necessary factors survive. Finally, we can combine this result with the previous derivation to obtain

$$
\begin{aligned}
[\mathrm{J}^{z}_{kl},\mathrm{J}^{+}_{tu}] &= 2\sum_{r,s}\Big(2+(2\chi_{\hat{\mathrm{E}}}-2)\delta_{t+r+k,0}\delta_{u+s+l,0}\Big)\frac{\partial}{\partial v_{-(t+r+k)-(u+s+l)}}\frac{\partial}{\partial j_{rs}} \\
&+ 2\sum_{r,s}\Big(-2i(t+r+k)v_{(t+r+k)(u+s+l)} - 2i(\chi_{\hat{\Phi}_0}+r)\delta_{t+r+k,0}\delta_{u+s+l,0}\Big)\frac{\partial}{\partial k_{rs}} \\
&+ 2\sum_{r,s}\Big(2i(u+s+l)v_{(t+r+k)(u+s+l)} - 2i(\chi_{\hat{\Theta}_0}-s)\delta_{t+r+k,0}\delta_{u+s+l,0}\Big)\frac{\partial}{\partial p_{rs}} \\
&+ 2\sum_{r,s}(\mathcal{E}_{\mathrm{K,K}}(t+r+k,u+s+l) + \mathcal{E}_{\mathrm{P,P}}(t+r+k,u+s+l) \\
&+ \frac{1}{2}\mathcal{E}_{\mathrm{J,J}}(t+r+k,u+s+l))\frac{\partial}{\partial j_{rs}} \\
&= 2\mathrm{J}^{+}_{(k+t)(l+u)}
\end{aligned}
$$

$$\begin{aligned}[\mathrm{J}^+_{kl}, \mathrm{J}^-_{tu}] &= \Big(2 + (2\chi_{\widehat{\mathrm{E}}} - 2)\delta_{k+t,0}\delta_{l+u,0}\Big) \frac{\partial}{\partial v_{-(k+t)-(l+u)}} \\ &+ \mathcal{E}_{\mathrm{K,K}}(k+t, l+u) + \mathcal{E}_{\mathrm{P,P}}(k+t, l+u) + \frac{1}{2}\mathcal{E}_{\mathrm{J,J}}(k+t, l+u) \\ &= \mathrm{J}^z_{(k+t)(l+u)}\end{aligned}$$

$$[\mathrm{J}^-_{kl}, \mathrm{J}^-_{tu}] = 0$$

$$\begin{aligned}[\mathrm{J}^+_{kl}, \mathrm{J}^+_{tu}] = &\Bigg[\sum_{m,n} \Big(2 + (2\chi_{\widehat{\mathrm{E}}} - 2)\delta_{k+m,0}\delta_{l+n,0}\Big) \frac{\partial}{\partial v_{-(k+m)-(l+n)}} \frac{\partial}{\partial j_{mn}} \\ &+ \sum_{m,n} \Big(-2i(k+m)v_{(k+m)(l+n)} - 2i(\chi_{\widehat{\Phi}_0} + m)\delta_{k+m,0}\delta_{l+n,0}\Big) \frac{\partial}{\partial k_{mn}} \\ &+ \sum_{m,n} \Big(2i(l+n)v_{(k+m)(l+n)} - 2i(\chi_{\widehat{\Theta}_0} - n)\delta_{k+m,0}\delta_{l+n,0}\Big) \frac{\partial}{\partial p_{mn}} \\ &+ \sum_{m,n} \Big(\mathcal{E}_{\mathrm{K,K}}(k+m, l+n) + \mathcal{E}_{\mathrm{P,P}}(k+m, l+n) + \frac{1}{2}\mathcal{E}_{\mathrm{J,J}}(k+m, l+n)\Big) \frac{\partial}{\partial j_{mn}}, \\ &\qquad \sum_{r,s} \Big(2 + (2\chi_{\widehat{\mathrm{E}}} - 2)\delta_{t+r,0}\delta_{u+s,0}\Big) \frac{\partial}{\partial v_{-(t+r)-(u+s)}} \frac{\partial}{\partial j_{rs}} \\ &\qquad + \sum_{r,s} \Big(-2i(t+r)v_{(t+r)(u+s)} - 2i(\chi_{\widehat{\Phi}_0} + r)\delta_{t+r,0}\delta_{u+s,0}\Big) \frac{\partial}{\partial k_{rs}} \\ &\qquad + \sum_{r,s} \Big(2i(u+s)v_{(t+r)(u+s)} - 2i(\chi_{\widehat{\Theta}_0} - s)\delta_{t+r,0}\delta_{u+s,0}\Big) \frac{\partial}{\partial p_{rs}} \\ &\qquad + \sum_{r,s} \Big(\mathcal{E}_{\mathrm{K,K}}(t+r, u+s) + \mathcal{E}_{\mathrm{P,P}}(t+r, u+s) + \frac{1}{2}\mathcal{E}_{\mathrm{J,J}}(t+r, u+s)\Big) \frac{\partial}{\partial j_{rs}}\Bigg] \\ = &\Bigg[\sum_{m,n} \Big(2 + (2\chi_{\widehat{\mathrm{E}}} - 2)\delta_{k+m,0}\delta_{l+n,0}\Big) \frac{\partial}{\partial v_{-(k+m)-(l+n)}} \frac{\partial}{\partial j_{mn}}, \\ &\qquad \sum_{r,s} \Big(-2i(t+r)v_{(t+r)(u+s)}\Big) \frac{\partial}{\partial k_{rs}} \\ &\qquad + \sum_{r,s} \Big(2i(u+s)v_{(t+r)(u+s)}\Big) \frac{\partial}{\partial p_{rs}} \\ &\qquad + \sum_{r,s} \frac{1}{2}\mathcal{E}_{\mathrm{J,J}}(t+r, u+s) \frac{\partial}{\partial j_{rs}}\Bigg] \\ &+ \Bigg[\sum_{m,n} \Big(-2i(k+m)v_{(k+m)(l+n)} - 2i(\chi_{\widehat{\Phi}_0} + m)\delta_{k+m,0}\delta_{l+n,0}\Big) \frac{\partial}{\partial k_{mn}}\end{aligned}$$

$$+\sum_{m,n}\Big(2i(l+n)v_{(k+m)(l+n)}-2i(\chi_{\widehat{\Theta}_0}-n)\delta_{k+m,0}\delta_{l+n,0}\Big)\frac{\partial}{\partial p_{mn}},$$

$$\sum_{r,s}\Big(2+(2\chi_{\widehat{\mathrm{E}}}-2)\delta_{t+r,0}\delta_{u+s,0}\Big)\frac{\partial}{\partial v_{-(t+r)-(u+s)}}\frac{\partial}{\partial j_{rs}}$$

$$+\sum_{r,s}\left(\mathcal{E}_{\mathrm{K,K}}(t+r,u+s)+\mathcal{E}_{\mathrm{P,P}}(t+r,u+s)\right)\frac{\partial}{\partial j_{rs}}\Bigg]$$

$$+\Bigg[\sum_{m,n}\left(\mathcal{E}_{\mathrm{K,K}}(k+m,l+n)+\mathcal{E}_{\mathrm{P,P}}(k+m,l+n)+\frac{1}{2}\mathcal{E}_{\mathrm{J,J}}(k+m,l+n)\right)\frac{\partial}{\partial j_{mn}},$$

$$\sum_{r,s}\Big(2+(2\chi_{\widehat{\mathrm{E}}}-2)\delta_{t+r,0}\delta_{u+s,0}\Big)\frac{\partial}{\partial v_{-(t+r)-(u+s)}}\frac{\partial}{\partial j_{rs}}$$

$$+\sum_{r,s}\Big(-2i(t+r)v_{(t+r)(u+s)}-2i(\chi_{\widehat{\Phi}_0}+r)\delta_{t+r,0}\delta_{u+s,0}\Big)\frac{\partial}{\partial k_{rs}}$$

$$+\sum_{r,s}\Big(2i(u+s)v_{(t+r)(u+s)}-2i(\chi_{\widehat{\Theta}_0}-s)\delta_{t+r,0}\delta_{u+s,0}\Big)\frac{\partial}{\partial p_{rs}}$$

$$+\sum_{r,s}\left(\mathcal{E}_{\mathrm{K,K}}(t+r,u+s)+\mathcal{E}_{\mathrm{P,P}}(t+r,u+s)+\frac{1}{2}\mathcal{E}_{\mathrm{J,J}}(t+r,u+s)\right)\frac{\partial}{\partial j_{rs}}\Bigg]$$

In this step, we used the linearity of the bracket and dropped manifestly vanishing contributions. Next, we label the three large bracket-summands from a)-c) and compute them individually.

a)

$$\Bigg[\sum_{m,n}\Big(2+(2\chi_{\widehat{\mathrm{E}}}-2)\delta_{k+m,0}\delta_{l+n,0}\Big)\frac{\partial}{\partial v_{-(k+m)-(l+n)}}\frac{\partial}{\partial j_{mn}},$$

$$\sum_{r,s}\Big(-2i(t+r)v_{(t+r)(u+s)}\Big)\frac{\partial}{\partial k_{rs}}$$

$$+\sum_{r,s}\Big(2i(u+s)v_{(t+r)(u+s)}\Big)\frac{\partial}{\partial p_{rs}}$$

$$+\sum_{r,s}\frac{1}{2}\mathcal{E}_{\mathrm{J,J}}(t+r,u+s)\frac{\partial}{\partial j_{rs}}\Bigg]$$

$$=\sum_{m,n}\sum_{r,s}\Big(2+(2\chi_{\widehat{\mathrm{E}}}-2)\delta_{k+m,0}\delta_{l+n,0}\Big)\left(-2i(t+r)\right)\delta_{k+m+t+r}\delta_{l+n+u+s}\frac{\partial}{\partial j_{mn}}\frac{\partial}{\partial k_{rs}}$$

$$+\sum_{m,n}\sum_{r,s}\Big(2+(2\chi_{\widehat{\mathrm{E}}}-2)\delta_{k+m,0}\delta_{l+n,0}\Big)\left(2i(u+s)\right)\delta_{k+m+t+r}\delta_{l+n+u+s}\frac{\partial}{\partial j_{mn}}\frac{\partial}{\partial p_{rs}}$$

$$+\sum_{m,n}\sum_{rs}\Big(2+(2\chi_{\widehat{\mathrm{E}}}-2)\delta_{k+m,0}\delta_{l+n,0}\Big)\frac{\partial}{\partial v_{-(k+m)-(l+n)}}\left[\frac{\partial}{\partial j_{mn}},\frac{1}{2}\mathcal{E}_{\mathrm{J,J}}(t+r,u+s)\right]\frac{\partial}{\partial j_{rs}}$$

$$= \sum_{m,n}\sum_{r,s}\left(2+(2\chi_{\hat{\mathrm{E}}}-2)\delta_{k+m,0}\delta_{l+n,0}\right)(-2i(t+r))\delta_{k+m+t+r}\delta_{l+n+u+s}\frac{\partial}{\partial j_{mn}}\frac{\partial}{\partial k_{rs}}$$
$$+\sum_{m,n}\sum_{r,s}\left(2+(2\chi_{\hat{\mathrm{E}}}-2)\delta_{k+m,0}\delta_{l+n,0}\right)(2i(u+s))\delta_{k+m+t+r}\delta_{l+n+u+s}\frac{\partial}{\partial j_{mn}}\frac{\partial}{\partial p_{rs}}$$
$$-\sum_{m,n}\sum_{r,s}(2+(2\chi_{\hat{\mathrm{E}}}-2)\delta_{k+m,0}\delta_{l+n,0})\frac{\partial}{\partial v_{-(k+m)-(l+n)}}\frac{\partial}{\partial j_{(m-t-r)(n-u-s)}}\frac{\partial}{\partial j_{rs}}$$
$$= \sum_{m,n}\sum_{r,s}\left(2+(2\chi_{\hat{\mathrm{E}}}-2)\delta_{k+m,0}\delta_{l+n,0}\right)(-2i(t+r))\delta_{k+m+t+r}\delta_{l+n+u+s}\frac{\partial}{\partial j_{mn}}\frac{\partial}{\partial k_{rs}}$$
$$+\sum_{m,n}\sum_{r,s}\left(2+(2\chi_{\hat{\mathrm{E}}}-2)\delta_{k+m,0}\delta_{l+n,0}\right)(2i(u+s))\delta_{k+m+t+r}\delta_{l+n+u+s}\frac{\partial}{\partial j_{mn}}\frac{\partial}{\partial p_{rs}}$$
$$-\sum_{m,n}\sum_{r,s}(2+(2\chi_{\hat{\mathrm{E}}}-2)\delta_{k+m+t+r,0}\delta_{l+n+u+s,0})\frac{\partial}{\partial v_{-(k+m+t+r)-(l+n+u+s)}}\frac{\partial}{\partial j_{mn}}\frac{\partial}{\partial j_{rs}}$$

b)

$$\Bigg[\sum_{m,n}\left(-2i(k+m)v_{(k+m)(l+n)}-2i(\chi_{\hat{\Phi}_0}+m)\delta_{k+m,0}\delta_{l+n,0}\right)\frac{\partial}{\partial k_{mn}}$$
$$+\sum_{m,n}\left(2i(l+n)v_{(k+m)(l+n)}-2i(\chi_{\hat{\Theta}_0}-n)\delta_{k+m,0}\delta_{l+n,0}\right)\frac{\partial}{\partial p_{mn}},$$
$$\sum_{r,s}\left(2+(2\chi_{\hat{\mathrm{E}}}-2)\delta_{t+r,0}\delta_{u+s,0}\right)\frac{\partial}{\partial v_{-(t+r)-(u+s)}}\frac{\partial}{\partial j_{rs}}$$
$$+\sum_{r,s}(\mathcal{E}_{\mathrm{K,K}}(t+r,u+s)+\mathcal{E}_{\mathrm{P,P}}(t+r,u+s))\frac{\partial}{\partial j_{rs}}\Bigg]$$
$$= -\sum_{m,n}\sum_{r,s}\left(2+(2\chi_{\hat{\mathrm{E}}}-2)\delta_{t+r,0}\delta_{u+s,0}\right)(-2i(k+m))\delta_{t+r+k+m,0}\delta_{u+s+l+n}\frac{\partial}{\partial j_{rs}}\frac{\partial}{\partial k_{mn}}$$
$$-\sum_{m,n}\sum_{r,s}\left(2+(2\chi_{\hat{\mathrm{E}}}-2)\delta_{t+r,0}\delta_{u+s,0}\right)(2i(l+n))\delta_{t+r+k+m,0}\delta_{u+s+l+n}\frac{\partial}{\partial j_{rs}}\frac{\partial}{\partial p_{mn}}$$
$$-2\sum_{m,n}\sum_{r,s}\left(-2i(k+m)v_{(k+m)(l+n)}-2i(\chi_{\hat{\Phi}_0}+m)\delta_{k+m,0}\delta_{l+n,0}\right)\frac{\partial}{\partial k_{(m-t-r)(n-u-s)}}\frac{\partial}{\partial j_{rs}}$$
$$-2\sum_{m,n}\sum_{r,s}\left(2i(l+n)v_{(k+m)(l+n)}-2i(\chi_{\hat{\Theta}_0}-n)\delta_{k+m,0}\delta_{l+n,0}\right)\frac{\partial}{\partial p_{(m-t-r)(n-u-s)}}\frac{\partial}{\partial j_{rs}}$$
$$= -\sum_{m,n}\sum_{r,s}\left(2+(2\chi_{\hat{\mathrm{E}}}-2)\delta_{t+r,0}\delta_{u+s,0}\right)(-2i(k+m))\delta_{t+r+k+m,0}\delta_{u+s+l+n}\frac{\partial}{\partial j_{rs}}\frac{\partial}{\partial k_{mn}}$$
$$-\sum_{m,n}\sum_{r,s}\left(2+(2\chi_{\hat{\mathrm{E}}}-2)\delta_{t+r,0}\delta_{u+s,0}\right)(2i(l+n))\delta_{t+r+k+m,0}\delta_{u+s+l+n}\frac{\partial}{\partial j_{rs}}\frac{\partial}{\partial p_{mn}}$$
$$-2\sum_{m,n}\sum_{r,s}(-2i(k+m+t+r)v_{(k+m+t+r)(l+n+u+s)}$$
$$-2i(\chi_{\hat{\Phi}_0}+m+t+r)\delta_{k+m+t+r,0}\delta_{l+n+u+s,0})\frac{\partial}{\partial k_{mn}}\frac{\partial}{\partial j_{rs}}$$

$$- 2\sum_{m,n}\sum_{r,s}(2i(l+n+u+s)v_{(k+m+t+r)(l+n+u+s)} - 2i(\chi_{\hat{\Theta}_0} - n - u - s)\delta_{k+m+t+r,0}\delta_{l+n+u+s,0})\frac{\partial}{\partial p_{mn}}\frac{\partial}{\partial j_{rs}}$$

c) By Lemma A.5, the last summand in the right hand side of the bracket commutes and is therefore dropped in the following.

$$\Bigg[\sum_{m,n}\Big(\mathcal{E}_{\mathrm{K,K}}(k+m,l+n) + \mathcal{E}_{\mathrm{P,P}}(k+m,l+n) + \frac{1}{2}\mathcal{E}_{\mathrm{J,J}}(k+m,l+n)\Big)\frac{\partial}{\partial j_{mn}},$$
$$\sum_{r,s}\Big(2 + (2\chi_{\hat{\mathrm{E}}} - 2)\delta_{t+r,0}\delta_{u+s,0}\Big)\frac{\partial}{\partial v_{-(t+r)-(u+s)}}\frac{\partial}{\partial j_{rs}}$$
$$+ \sum_{r,s}\Big(-2i(t+r)v_{(t+r)(u+s)} - 2i(\chi_{\hat{\Phi}_0} + r)\delta_{t+r,0}\delta_{u+s,0}\Big)\frac{\partial}{\partial k_{rs}}$$
$$+ \sum_{r,s}\Big(2i(u+s)v_{(t+r)(u+s)} - 2i(\chi_{\hat{\Theta}_0} - s)\delta_{t+r,0}\delta_{u+s,0}\Big)\frac{\partial}{\partial p_{rs}}\Bigg]$$
$$= \sum_{m,n}\sum_{r,s}\Big(2 + (2\chi_{\hat{\mathrm{E}}} - 2)\delta_{t+r,0}\delta_{u+s,0}\Big)\frac{\partial}{\partial v_{-(t+r)-(u+s)}}\frac{\partial}{\partial j_{(r-k-m)(s-l-n)}}\frac{\partial}{\partial j_{mn}}$$
$$+ 2\sum_{m,n}\sum_{r,s}\Big(-2i(t+r)v_{(t+r)(u+s)} - 2i(\chi_{\hat{\Phi}_0} + r)\delta_{t+r,0}\delta_{u+s,0}\Big)\frac{\partial}{\partial k_{(r-k-m)(s-l-n)}}\frac{\partial}{\partial j_{mn}}$$
$$+ 2\sum_{m,n}\sum_{r,s}\Big(2i(u+s)v_{(t+r)(u+s)} - 2i(\chi_{\hat{\Theta}_0} - s)\delta_{t+r,0}\delta_{u+s,0}\Big)\frac{\partial}{\partial p_{(r-k-m)(s-l-n)}}\frac{\partial}{\partial j_{mn}}$$
$$= \sum_{m,n}\sum_{r,s}\Big(2 + (2\chi_{\hat{\mathrm{E}}} - 2)\delta_{t+r+k+m,0}\delta_{u+s+l+n,0}\Big)\frac{\partial}{\partial v_{-(t+r+k+m)-(u+s+l+n)}}\frac{\partial}{\partial j_{rs}}\frac{\partial}{\partial j_{mn}}$$
$$+ 2\sum_{m,n}\sum_{r,s}(-2i(t+r+k+m)v_{(t+r+k+m)(u+s+l+n)} - 2i(\chi_{\hat{\Phi}_0} + r + k + m)\delta_{t+r+k+m,0}\delta_{u+s+l+n,0})\frac{\partial}{\partial k_{rs}}\frac{\partial}{\partial j_{mn}}$$
$$+ 2\sum_{m,n}\sum_{r,s}(2i(u+s+l+n)v_{(t+r+k+m)(u+s+l+n)} - 2i(\chi_{\hat{\Theta}_0} - s - l - n)\delta_{t+r+k+m,0}\delta_{u+s+l+n,0})\frac{\partial}{\partial p_{rs}}\frac{\partial}{\partial j_{mn}}$$

Finally, by collecting all terms, we get:

$$[\mathrm{J}^+_{kl}, \mathrm{J}^+_{tu}] = \sum_{m,n}\sum_{r,s}\Big(2 + (2\chi_{\hat{\mathrm{E}}} - 2)\delta_{k+m,0}\delta_{l+n,0}\Big)\,(-2i(t+r))\delta_{k+m+t+r}\delta_{l+n+u+s}\frac{\partial}{\partial j_{mn}}\frac{\partial}{\partial k_{rs}}$$
$$- 2\sum_{m,n}\sum_{r,s}(-2i(k+m+t+r)v_{(k+m+t+r)(l+n+u+s)} - 2i(\chi_{\hat{\Phi}_0} + m + t + r)\delta_{k+m+t+r,0}\delta_{l+n+u+s,0})\frac{\partial}{\partial k_{mn}}\frac{\partial}{\partial j_{rs}}$$

$$
\begin{aligned}
&+ 2\sum_{m,n}\sum_{r,s}(-2i(t+r+k+m)v_{(t+r+k+m)(u+s+l+n)}\\
&\qquad - 2i(\chi_{\hat{\Phi}_0}+r+k+m)\delta_{t+r+k+m,0}\delta_{u+s+l+n,0})\frac{\partial}{\partial k_{rs}}\frac{\partial}{\partial j_{mn}}\\
&- \sum_{m,n}\sum_{r,s}\left(2+(2\chi_{\hat{\mathrm{E}}}-2)\delta_{t+r,0}\delta_{u+s,0}\right)(-2i(k+m))\delta_{t+r+k+m,0}\delta_{u+s+l+n}\frac{\partial}{\partial j_{rs}}\frac{\partial}{\partial k_{mn}}\\
&- \sum_{m,n}\sum_{r,s}\left(2+(2\chi_{\hat{\mathrm{E}}}-2)\delta_{t+r,0}\delta_{u+s,0}\right)(2i(l+n))\delta_{t+r+k+m,0}\delta_{u+s+l+n}\frac{\partial}{\partial j_{rs}}\frac{\partial}{\partial p_{mn}}\\
&+ \sum_{m,n}\sum_{r,s}\left(2+(2\chi_{\hat{\mathrm{E}}}-2)\delta_{k+m,0}\delta_{l+n,0}\right)(2i(u+s))\delta_{k+m+t+r}\delta_{l+n+u+s}\frac{\partial}{\partial j_{mn}}\frac{\partial}{\partial p_{rs}}\\
&- 2\sum_{m,n}\sum_{r,s}(2i(l+n+u+s)v_{(k+m+t+r)(l+n+u+s)}\\
&\qquad - 2i(\chi_{\hat{\Theta}_0}-n-u-s)\delta_{k+m+t+r,0}\delta_{l+n+u+s,0})\frac{\partial}{\partial p_{mn}}\frac{\partial}{\partial j_{rs}}\\
&+ 2\sum_{m,n}\sum_{r,s}(2i(u+s+l+n)v_{(t+r+k+m)(u+s+l+n)}\\
&\qquad - 2i(\chi_{\hat{\Theta}_0}-s-l-n)\delta_{t+r+k+m,0}\delta_{u+s+l+n,0})\frac{\partial}{\partial p_{rs}}\frac{\partial}{\partial j_{mn}}\\
&- \sum_{m,n}\sum_{r,s}\left(2+(2\chi_{\hat{\mathrm{E}}}-2)\delta_{t+r,0}\delta_{u+s,0}\right)(2i(l+n))\delta_{t+r+k+m,0}\delta_{u+s+l+n}\frac{\partial}{\partial j_{rs}}\frac{\partial}{\partial p_{mn}}\\
&- \sum_{m,n}\sum_{r,s}(2+(2\chi_{\hat{\mathrm{E}}}-2)\delta_{k+m+t+r,0}\delta_{l+n+u+s,0})\frac{\partial}{\partial v_{-(k+m+t+r)-(l+n+u+s)}}\frac{\partial}{\partial j_{mn}}\frac{\partial}{\partial j_{rs}}\\
&+ \sum_{m,n}\sum_{r,s}\left(2+(2\chi_{\hat{\mathrm{E}}}-2)\delta_{t+r+k+m,0}\delta_{u+s+l+n,0}\right)\frac{\partial}{\partial v_{-(t+r+k+m)-(u+s+l+n)}}\frac{\partial}{\partial j_{rs}}\frac{\partial}{\partial j_{mn}}
\end{aligned}
$$

The final two summands clearly cancel, and we can add the rest together to obtain:

$$
\begin{aligned}
[\mathrm{J}^+_{kl},\mathrm{J}^+_{tu}] &= \sum_{m,n}\sum_{r,s}(-4i(t+r))\delta_{k+m+t+r}\delta_{l+n+u+s}\frac{\partial}{\partial j_{mn}}\frac{\partial}{\partial k_{rs}}\\
&+ \sum_{m,n}\sum_{r,s}4i(\chi_{\hat{\Phi}_0}+m+t+r)\delta_{k+m+t+r,0}\delta_{l+n+u+s,0})\frac{\partial}{\partial k_{mn}}\frac{\partial}{\partial j_{rs}}\\
&- \sum_{m,n}\sum_{r,s}4i(\chi_{\hat{\Phi}_0}+r+k+m)\delta_{t+r+k+m,0}\delta_{u+s+l+n,0})\frac{\partial}{\partial k_{rs}}\frac{\partial}{\partial j_{mn}}\\
&- \sum_{m,n}\sum_{r,s}(-4i(k+m))\delta_{t+r+k+m,0}\delta_{u+s+l+n}\frac{\partial}{\partial j_{rs}}\frac{\partial}{\partial k_{mn}}\\
&+ \sum_{m,n}\sum_{r,s}(4i(u+s))\delta_{k+m+t+r}\delta_{l+n+u+s}\frac{\partial}{\partial j_{mn}}\frac{\partial}{\partial p_{rs}}\\
&+ \sum_{m,n}\sum_{r,s}4i(\chi_{\hat{\Theta}_0}-n-u-s)\delta_{k+m+t+r,0}\delta_{l+n+u+s,0})\frac{\partial}{\partial p_{mn}}\frac{\partial}{\partial j_{rs}}
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{m,n}\sum_{r,s} 4i(\chi_{\widehat{\Theta}_0} - s - l - n)\delta_{t+r+k+m,0}\delta_{u+s+l+n,0})\frac{\partial}{\partial p_{rs}}\frac{\partial}{\partial j_{mn}} \\
&-\sum_{m,n}\sum_{r,s}(4i(l+n))\delta_{t+r+k+m,0}\delta_{u+s+l+n}\frac{\partial}{\partial j_{rs}}\frac{\partial}{\partial p_{mn}} \\
&=\sum_{m,n}\sum_{r,s} 4i(t-t+k-k+r-r)\delta_{k+m+t+r}\delta_{l+n+u+s}\frac{\partial}{\partial j_{mn}}\frac{\partial}{\partial k_{rs}} \\
&+\sum_{m,n}\sum_{r,s} 4i(l-l+u-u+l-l)\delta_{k+m+t+r}\delta_{l+n+u+s}\frac{\partial}{\partial j_{mn}}\frac{\partial}{\partial p_{rs}} \\
&=0
\end{aligned}
$$

$$
\begin{aligned}
[\mathrm{J}^z_{kl}, \mathrm{K}^z_{tu}] = \Bigg[&(2+(2\chi_{\widehat{\mathrm{E}}}-2)\delta_{k,0}\delta_{l,0})\frac{\partial}{\partial v_{-k,-l}} + \mathcal{E}_{\mathrm{J,J}}(k,l) + \mathcal{E}_{\mathrm{K,K}}(k,l) + \mathcal{E}_{\mathrm{P,P}}(k,l), \\
&\quad - 2itv_{tu} - 2i\chi_{\widehat{\Phi}_0}\delta_{t,0}\delta_{u,0} + \mathcal{E}_{\mathrm{K,J}}(t,u)\Bigg] \\
&= (2+(2\chi_{\widehat{\mathrm{E}}}-2)\delta_{k,0}\delta_{l,0})(-2it)\delta_{k+t,0}\delta_{l+u} \\
&= -4it\delta_{k+t,0}\delta_{l+u} \\
&= -4it\delta_{k,-t}\delta_{l,-u}\mathrm{Z}
\end{aligned}
$$

$$
\begin{aligned}
[\mathrm{J}^z_{kl}, \mathrm{K}^+_{tu}] = \Bigg[&(2+(2\chi_{\widehat{\mathrm{E}}}-2)\delta_{k,0}\delta_{l,0})\frac{\partial}{\partial v_{-k,-l}} + \mathcal{E}_{\mathrm{J,J}}(k,l) + \mathcal{E}_{\mathrm{K,K}}(k,l) + \mathcal{E}_{\mathrm{P,P}}(k,l), \\
&\sum_{m,n}\Big(-2i(t+m)v_{(t+m)(u+n)} - 2i(\chi_{\widehat{\Phi}_0}+m)\delta_{t+m,0}\delta_{u+n,0} \\
&+\frac{1}{2}\mathcal{E}_{\mathrm{K,J}}(t+m,u+n)\Big)\frac{\partial}{\partial j_{mn}}\Bigg] \\
&= \sum_{m,n}(2+(2\chi_{\widehat{\mathrm{E}}}-2)\delta_{k,0}\delta_{l,0})(-2i(t+m))\delta_{k+t+m,0}\delta_{l+u+n,0}\frac{\partial}{\partial j_{mn}} \\
&+\Bigg[\mathcal{E}_{\mathrm{J,J}}(k,l) + \mathcal{E}_{\mathrm{K,K}}(k,l), \sum_{m,n}\Big(-2i(t+m)v_{(t+m)(u+n)} - 2i(\chi_{\widehat{\Phi}_0}+m)\delta_{t+m,0}\delta_{u+n,0} \\
&+\frac{1}{2}\mathcal{E}_{\mathrm{K,J}}(t+m,u+n)\Big)\frac{\partial}{\partial j_{mn}}\Bigg] \\
&= \sum_{m,n}(-4i(t+m))\delta_{k+t+m,0}\delta_{l+u+n,0}\frac{\partial}{\partial j_{mn}} \\
&+\sum_{m,n} 2\mathcal{E}_{\mathrm{KJ}}(t+k+m,u+l+n)\frac{\partial}{\partial j_{mn}}
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{m,n} \frac{1}{2}(-2\mathcal{E}_{\mathrm{KJ}}(k+t+m, l+u+n)) \frac{\partial}{\partial j_{mn}} \\
&+ \sum_{m,n}(-2i(t+m)v_{(t+m)(u+n)} - 2i(\chi_{\widehat{\Phi}_0} + m)\delta_{t+m,0}\delta_{u+n,0})2\frac{\partial}{\partial j_{(m-k)(n-l)}} \\
= {}& 2\sum_{m,n} \frac{1}{2}\mathcal{E}_{\mathrm{KJ}}(t+k+m, u+l+n)\frac{\partial}{\partial j_{mn}} \\
&+ 2\sum_{m,n}(-2i(t+m+k)v_{(t+m+k)(u+n+l)} \\
&- 2i(\chi_{\widehat{\Phi}_0} + m + k + t + m)\delta_{t+m+k,0}\delta_{u+n+l,0})\frac{\partial}{\partial j_{mn}} \\
= {}& 2\mathrm{K}^+_{(k+t)(l+u)}
\end{aligned}
$$

$$
\begin{aligned}
[\mathrm{J}^z_{kl}, \mathrm{K}^-_{tu}] &= \mathcal{E}_{\mathrm{K,K}}(k,l)k_{tu} \\
&= -2k_{(k+t)(l+u)} \\
&= -2\mathrm{K}_{(k+t)(l+u)}
\end{aligned}
$$

$$
\begin{aligned}
[\mathrm{K}^z_{kl}, \mathrm{J}^+_{tu}] = {}& \Bigg[-2ikv_{kl} - 2i\chi_{\widehat{\Phi}_0}\delta_{k,0}\delta_{l,0} + \mathcal{E}_{\mathrm{K,J}}(k,l), \\
&\sum_{r,s}\left(2 + (2\chi_{\widehat{\mathrm{E}}} - 2)\delta_{t+r,0}\delta_{u+s,0}\right)\frac{\partial}{\partial v_{-(t+r)-(u+s)}}\frac{\partial}{\partial j_{rs}} \\
&+ \sum_{r,s}\left(-2i(t+r)v_{(t+r)(u+s)} - 2i(\chi_{\widehat{\Phi}_0} + r)\delta_{t+r,0}\delta_{u+s,0}\right)\frac{\partial}{\partial k_{rs}} \\
&+ \sum_{r,s}\left(2i(u+s)v_{(t+r)(u+s)} - 2i(\chi_{\widehat{\Theta}_0} - s)\delta_{t+r,0}\delta_{u+s,0}\right)\frac{\partial}{\partial p_{rs}} \\
&+ \sum_{r,s}\left(\mathcal{E}_{\mathrm{K,K}}(t+r,u+s) + \mathcal{E}_{\mathrm{P,P}}(t+r,u+s) + \frac{1}{2}\mathcal{E}_{\mathrm{J,J}}(t+r,u+s)\right)\frac{\partial}{\partial j_{rs}}\Bigg] \\
= {}& -\sum_{r,s}\left(2 + (2\chi_{\widehat{\mathrm{E}}} - 2)\delta_{t+r,0}\delta_{u+s,0}\right)(-2ik)\delta_{t+r+k,0}\delta_{u+s+l}\frac{\partial}{\partial j_{rs}} \\
&+ 2\sum_{r,s}\left(-2i(t+r)v_{(t+r)(u+s)} - 2i(\chi_{\widehat{\Phi}_0} + r)\delta_{t+r,0}\delta_{u+s,0}\right)\frac{\partial}{\partial j_{(r-k)(s-l)}} \\
&+ \sum_{r,s}\left[\mathcal{E}_{\mathrm{K,J}}(k,l), \left(\mathcal{E}_{\mathrm{K,K}}(t+r,u+s) + \frac{1}{2}\mathcal{E}_{\mathrm{J,J}}(t+r,u+s)\right)\frac{\partial}{\partial j_{rs}}\right] \\
= {}& \sum_{r,s} 4ik\delta_{t+r+k,0}\delta_{u+s+l}\frac{\partial}{\partial j_{rs}} \\
&+ 2\sum_{r,s}\left(-2i(t+r+k)v_{(t+r+k)(u+s+l)} - 2i(\chi_{\widehat{\Phi}_0} + r + k)\delta_{t+r+k,0}\delta_{u+s+l,0}\right)\frac{\partial}{\partial j_{rs}}
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \sum_{r,s} 2\mathcal{E}_{\mathrm{KJ}}(k+t+r, l+u+s))\frac{\partial}{\partial j_{rs}} + \sum_{r,s} \frac{1}{2}(-2\mathcal{E}_{\mathrm{KJ}}(k+t+r, l+u+s)\frac{\partial}{\partial j_{rs}} \\
&= 2\sum_{r,s} \Big(-2i(t+r+k)v_{(t+r+k)(u+s+l)} - 2i(\chi_{\hat{\Phi}_0}+r)\delta_{t+r+k,0}\delta_{u+s+l,0}\Big)\,\frac{\partial}{\partial j_{rs}} \\
&\quad + 2\sum_{r,s} \frac{1}{2}\mathcal{E}_{\mathrm{KJ}}(k+t+r, l+u+s))\frac{\partial}{\partial j_{rs}} \\
&= 2\mathrm{K}^+_{(k+t)(l+u)}
\end{aligned}
$$

$$
\begin{aligned}
[\mathrm{J}^+_{kl}, \mathrm{K}^+_{tu}] &= \Bigg[\sum_{m,n} \Big(2 + (2\chi_{\hat{\mathrm{E}}} - 2)\delta_{k+m,0}\delta_{l+n,0}\Big)\,\frac{\partial}{\partial v_{-(k+m)-(l+n)}}\frac{\partial}{\partial j_{mn}} \\
&\quad + \sum_{m,n} \Big(-2i(k+m)v_{(k+m)(l+n)} - 2i(\chi_{\hat{\Phi}_0}+m)\delta_{k+m,0}\delta_{l+n,0}\Big)\,\frac{\partial}{\partial k_{mn}} \\
&\quad + \sum_{m,n} \Big(2i(l+n)v_{(k+m)(l+n)} - 2i(\chi_{\hat{\Theta}_0}-n)\delta_{k+m,0}\delta_{l+n,0}\Big)\,\frac{\partial}{\partial p_{mn}} \\
&\quad + \sum_{m,n} \Big(\mathcal{E}_{\mathrm{K,K}}(k+m, l+n) + \mathcal{E}_{\mathrm{P,P}}(k+m, l+n) + \frac{1}{2}\mathcal{E}_{\mathrm{J,J}}(k+m, l+n)\Big)\,\frac{\partial}{\partial j_{mn}}, \\
&\qquad \sum_{r,s}\Big(-2i(t+r)v_{(t+r)(u+s)} - 2i(\chi_{\hat{\Phi}_0}+r)\delta_{t+r,0}\delta_{u+s,0} \\
&\qquad + \frac{1}{2}\mathcal{E}_{\mathrm{K,J}}(t+r, u+s)\Big)\frac{\partial}{\partial j_{rs}}\Bigg] \\
&= \Bigg[\sum_{m,n} \Big(2 + (2\chi_{\hat{\mathrm{E}}} - 2)\delta_{k+m,0}\delta_{l+n,0}\Big)\,\frac{\partial}{\partial v_{-(k+m)-(l+n)}}\frac{\partial}{\partial j_{mn}}, \\
&\qquad \sum_{r,s}(-2i(t+r)v_{(t+r)(u+s)})\frac{\partial}{\partial j_{rs}}\Bigg] \\
&\quad + \Bigg[\sum_{m,n} \Big(-2i(k+m)v_{(k+m)(l+n)} - 2i(\chi_{\hat{\Phi}_0}+m)\delta_{k+m,0}\delta_{l+n,0}\Big)\,\frac{\partial}{\partial k_{mn}}, \\
&\qquad \sum_{r,s} \frac{1}{2}\mathcal{E}_{\mathrm{K,J}}(t+r, u+s)\frac{\partial}{\partial j_{rs}}\Bigg] \\
&\quad + \Bigg[\sum_{m,n} \Big(\mathcal{E}_{\mathrm{K,K}}(k+m, l+n) + \mathcal{E}_{\mathrm{P,P}}(k+m, l+n) + \frac{1}{2}\mathcal{E}_{\mathrm{J,J}}(k+m, l+n)\Big)\,\frac{\partial}{\partial j_{mn}}, \\
&\qquad \sum_{r,s}\Big(-2i(t+r)v_{(t+r)(u+s)} - 2i(\chi_{\hat{\Phi}_0}+r)\delta_{t+r,0}\delta_{u+s,0} + \frac{1}{2}\mathcal{E}_{\mathrm{K,J}}(t+r, u+s)\Big)\frac{\partial}{\partial j_{rs}}\Bigg] \\
&= \sum_{m,n}\sum_{r,s} \Big(2 + (2\chi_{\hat{\mathrm{E}}} - 2)\delta_{k+m,0}\delta_{l+n,0}\Big)\,(-2i(t+r))\delta_{k+m+t+r,0}\delta_{l+n+u+s,0}\frac{\partial}{\partial j_{mn}}\frac{\partial}{\partial j_{rs}}
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{m,n}\sum_{r,s}\Big(-2i(k+m)v_{(k+m)(l+n)} - 2i(\chi_{\hat{\Phi}_0}+m)\delta_{k+m,0}\delta_{l+n,0}\Big)\frac{\partial}{\partial j_{(m-t-r)(n-u-s)}}\frac{\partial}{\partial j_{rs}}\\
&+ \sum_{m,n}\sum_{r,s}\Big(-2i(t+r)v_{(t+r)(u+s)} - 2i(\chi_{\hat{\Phi}_0}+r)\delta_{t+r,0}\delta_{u+s,0}\Big)\frac{\partial}{\partial j_{(r-k-m)(s-l-n)}}\frac{\partial}{\partial j_{mn}}\\
&\qquad - \frac{1}{2}\sum_{m,n}\sum_{r,s}\mathcal{E}_{\mathrm{K,J}}(k+m+t+r, l+n+u+s)\frac{\partial}{\partial j_{mn}}\frac{\partial}{\partial j_{rs}}\\
&\qquad + \frac{1}{2}\sum_{m,n}\sum_{r,s}\mathcal{E}_{\mathrm{K,J}}(t+r, u+s)\frac{\partial}{\partial j_{(r-k-m)(s-l-n)}}\frac{\partial}{\partial j_{mn}}\\
&= \sum_{m,n}\sum_{r,s}(-2i(2t+2r))\delta_{k+m+t+r,0}\delta_{l+n+u+s,0}\frac{\partial}{\partial j_{mn}}\frac{\partial}{\partial j_{rs}}\\
&- \sum_{m,n}\sum_{r,s}\Big(-2i(k+m+t+r)v_{(k+m+t+r)(l+n+u+s)}\\
&\qquad - 2i(\chi_{\hat{\Phi}_0}+m+t+r)\delta_{k+m+t+r,0}\delta_{l+n+u+s,0}\Big)\frac{\partial}{\partial j_{mn}}\frac{\partial}{\partial j_{rs}}\\
&+ \sum_{m,n}\sum_{r,s}\Big(-2i(t+r+k+m)v_{(t+r+k+m)(u+s+l+n)}\\
&\qquad - 2i(\chi_{\hat{\Phi}_0}+r+k+m)\delta_{t+r+k+m,0}\delta_{u+s+l+n,0}\Big)\frac{\partial}{\partial j_{rs}}\frac{\partial}{\partial j_{mn}}\\
&- \frac{1}{2}\sum_{m,n}\sum_{r,s}\mathcal{E}_{\mathrm{K,J}}(k+m+t+r, l+n+u+s)\frac{\partial}{\partial j_{mn}}\frac{\partial}{\partial j_{rs}}\\
&+ \frac{1}{2}\sum_{m,n}\sum_{r,s}\mathcal{E}_{\mathrm{K,J}}(t+r+k+m, u+s+l+n)\frac{\partial}{\partial j_{rs}}\frac{\partial}{\partial j_{mn}}\\
&= \sum_{m,n}\sum_{r,s}(-2i(t-k))\delta_{k+m+t+r,0}\delta_{l+n+u+s,0}\frac{\partial}{\partial j_{mn}}\frac{\partial}{\partial j_{rs}}\\
&+ \sum_{m,n}\sum_{r,s}(-2i(-m-t-r)\delta_{k+m+t+r,0}\delta_{l+n+u+s,0})\frac{\partial}{\partial j_{mn}}\frac{\partial}{\partial j_{rs}}\\
&+ \sum_{m,n}\sum_{r,s}(-2i(r+k+m)\delta_{t+r+k+m,0}\delta_{u+s+l+n,0})\frac{\partial}{\partial j_{rs}}\frac{\partial}{\partial j_{mn}}\frac{\partial}{\partial j_{rs}}\frac{\partial}{\partial j_{mn}}\\
&= \sum_{m,n}\sum_{r,s}(-2i(t-t+k-k+r-r+m-m)))\delta_{k+m+t+r,0}\delta_{l+n+u+s,0}\frac{\partial}{\partial j_{mn}}\frac{\partial}{\partial j_{rs}}\\
&= 0
\end{aligned}
$$

The first term is symmetric in r and m so we can replace $2r$ by $r+m$.

$$
\begin{aligned}
[\mathrm{J}^+_{kl}, \mathrm{K}^-_{tu}] &= \Bigg(\sum_{m,n}\Big(2 + (2\chi_{\hat{\mathrm{E}}} - 2)\delta_{k+m,0}\delta_{l+n,0}\Big)\frac{\partial}{\partial v_{-(k+m)-(l+n)}}\frac{\partial}{\partial j_{mn}}\\
&+ \sum_{m,n}\Big(-2i(k+m)v_{(k+m)(l+n)} - 2i(\chi_{\hat{\Phi}_0}+m)\delta_{k+m,0}\delta_{l+n,0}\Big)\frac{\partial}{\partial k_{mn}}
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{m,n} \Big(2i(l+n)v_{(k+m)(l+n)} - 2i(\chi_{\widehat{\Theta}_0} - n)\delta_{k+m,0}\delta_{l+n,0}\Big)\frac{\partial}{\partial p_{mn}} \\
&+ \sum_{m,n} \bigg(\Big(\mathcal{E}_{\mathrm{K,K}}(k+m,l+n) + \mathcal{E}_{\mathrm{P,P}}(k+m,l+n) + \frac{1}{2}\mathcal{E}_{\mathrm{J,J}}(k+m,l+n)\Big)\frac{\partial}{\partial j_{mn}}\bigg)k_{tu} \\
&= -2i(k+t)v_{(k+t)(l+u)} - 2i(\chi_{\widehat{\Phi}_0} + t)\delta_{k+t,0}\delta_{l+u,0} \\
&+ \sum_{m,n} \frac{\partial}{\partial j_{mn}}\mathcal{E}_{\mathrm{K,K}}(k+m,l+n)k_{tu} \\
&= -2i(k+t)v_{(k+t)(l+u)} - 2i\chi_{\widehat{\Phi}_0}\delta_{k+t,0}\delta_{l+u,0} - 2it\delta_{k+t,0}\delta_{l+u,0} \\
&+ \sum_{m,n}(-2)k_{(k+t+m,l+u+n)}\frac{\partial}{\partial j_{mn}} \\
&= \mathrm{K}^z_{(k+t)(l+u)} - 2it\delta_{k,-t}\delta_{l,-u}\mathrm{Z}
\end{aligned}
$$

$$
\begin{aligned}
[\mathrm{K}^z_{kl}, \mathrm{J}^-_{tu}] &= \mathcal{E}_{\mathrm{K,J}}(k,l)j_{tu} \\
&= -2k_{(k+t)(l+u)} \\
&= -2\mathrm{K}^-_{(k+t)(l+u)}
\end{aligned}
$$

$$
\begin{aligned}
[\mathrm{K}^+_{kl}, \mathrm{J}^-_{tu}] = \sum_{m,n}\Big(&-2i(k+m)v_{(k+m)(l+n)} - 2i(\chi_{\widehat{\Phi}_0} + m)\delta_{k+m,0}\delta_{l+n,0} \\
&+ \frac{1}{2}\mathcal{E}_{\mathrm{K,J}}(k+m,l+n)\Big)\frac{\partial}{\partial j_{mn}}j_{tu} \\
= -2i(k+t)&v_{(k+t)(l+u)} - 2i(\chi_{\widehat{\Phi}_0} + t)\delta_{k+t,0}\delta_{l+u,0} + \mathcal{E}_{\mathrm{K,J}}(k+t,l+u) \\
= -2i(k+t)&v_{(k+t)(l+u)} - 2i\chi_{\widehat{\Phi}_0}\delta_{k+t,0}\delta_{l+u,0} + \mathcal{E}_{\mathrm{K,J}}(k+t,l+u) - 2it\delta_{k,-t}\delta_{l,-u} \\
= \mathrm{K}^z_{(k+t)(l+u)}& - 2it\delta_{k,-t}\delta_{l,-u}\mathrm{Z}
\end{aligned}
$$

Note that the notation obscures the product rule which is the reason that the factor of $\frac{1}{2}$ vanishes. Finally:

$$[\mathrm{J}^-_{kl}, \mathrm{K}^-_{tu}] = 0\,,$$

which concludes the proof. □

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