Supergravity and Spinors in the BV-BFV Formalism

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[1] Mostly bikes.

Abstract

The purpose of this thesis is to study the classical BV-BFV (Batalin–Fradkin–Vilkovisky) structure of gravity coupled to spinors and, specifically, of the simplest case of supergravity, where only one gravitino is introduced in dimension four.

After a synthetic but thorough introduction on the BV-BFV machinery with some simple examples, this thesis presents a summary of known results on Palatini–Cartan gravity in the BV-BFV formalism, along with minor redefinitions, serving as a starting point for the further developments and leading to an original description of Palatini–Cartan–Dirac gravity on manifolds with boundary, in which, starting from the study of the boundary structure of the classical fields via the Kijowski–Tulczjiew construction, a BFV formulation is first obtained and then linked to its BV bulk counterpart by means of the 1–dimensional AKSZ construction.

The main body of the present work is a thorough BV-BFV alnalysis of $N = 1, D = 4$ supergravity. In particular, after studying constraints of the theory and identifying the relevant gauge symmetries, the existence of a BFV structure is established, but not directly computed due to technical difficulties. Such study, along with the simpler case of PCD gravity, provides enough insights to study the BV structure of SuGra in the bulk, where a complete off-shell BV formulation is obtained, generalizing the results of Baulieu et al.

Finally, the last part of the thesis complements the above findings by constructing a BV-BFV extendable theory of $N = 1, D = 4$ supergravity, which is obtained by eliminating the degrees of freedom which are responsible for the obstruction in the BV-BFV extension of the theory. Such procedure goes by the name of BV–pushforward, a technique that formalizes the concept of "integrating out" certain modes, which is adapted here to the case of classical supergravity.

These results provide a foundational step toward the quantization of supergravity theories in the presence of boundaries.

Original content and self-plagiarism

The contents of appendix A and parts of chapters 3, 5 appeared in [CCF22; Can+24; Cat+24; CF25; Fil25]. These papers contain original work, and were written by myself and by Giovanni Canepa, Alberto S. Cattaneo, Manuel Tecchiolli and Valentino Huang in equal measure.

Furhter parts of this thesis are extracts from the papers listed below, with minor changes.

[CCF22] G. Canepa, A. S. Cattaneo, and F. Fila-Robattino. "Boundary structure of gauge and matter fields coupled to gravity". *Advances in Theoretical and Mathematical Physics* (2022). URL: `https://api.semanticscholar.org/CorpusID:250113854`;

[Can+24] G. Canepa et al. "Boundary Structure of the Standard Model Coupled to Gravity". *Annales Henri Poincaré* (Sept. 2024). DOI: `10.1007/s00023-024-01485-4`;

[Cat+24] A. Cattaneo et al. *GRAVITY COUPLED WITH SCALAR, SU(N), AND SPINOR FIELDS ON MANIFOLDS WITH NULL-BOUNDARY*. Feb. 2024. DOI: `10.48550/arXiv.2401.09337`;

[Fil25] F. Fila-Robattino. *Tools for Supergravity in the spin coframe formalism.* 2025. arXiv: `2503.07355 [math-ph]`. URL: `https://arxiv.org/abs/2503.07355`;

[CF25] A. Cattaneo and F. Fila-Robattino. *BV description of $N = 1$, $D = 4$ Supergravity in the first order formalism.* Mar. 2025. DOI: `10.48550/arXiv.2503.07373`.

Contents

Introduction

In the context of theoretical and mathematical physics, many efforts over the years have been done towards defining a consistent notion of quantization of physical field theories. Historically, this has concretely been obtained by performing a perturbative expansion around critical points of the action functional, a method based on the stationary phase formula, which allows to evaluate oscillatory integrals as an asymptotic power series whose coefficients are computed via Feynman diagrams.

A major problem with such approach was posed in the mid-20th century in regards to the quantization of gauge theories, emerging from the need to properly handle the infinite redundancies introduced by local symmetry groups. Indeed, in such theories there is no isolated critical point of the action functional, but rather a whole "orbit". The earliest technique for addressing this issue was presented by Fadeev and Popov [FP67], who proposed the introduction of additional fields — the Fadeev–Popov ghosts — in order to obtain a well-defined gauge-fixed theory.

Shortly after, the geometrical interpretation of these ghost fields was given by Becchi, Rouet and Stora [BRS76], and independently by Tyutin [Tyu75], in the so-called BRST formalism. In this context, the original classical space of fields is enlarged to encompass the introduction of the ghost fields, unphysical degrees of freedom — typically represented by anti-commuting bosons or, in the case of supersymmetric theories, by commuting spinors, in both cases violating the spin statistics theorem — which are interpreted as the parameters generating the gauge symmetries. On top of this, anti-ghosts are introduced, acting as the canonical momenta associated to the ghosts. Thanks to the odd parity of the newly defined objects, one can assign a grading — given by the difference between ghost and anti-ghost number — to the enlarged space of fields and construct a nilpotent operator of degree 1 — the BRST operator —, which acts as the local gauge symmetry on the classical fields. The introduction of the BRST operator then allows to construct a chain complex, and to obtain the classical observables as its degree-0 cohomology. Algebraically, this equates to extending the space of functionals by the Chevalley-Eilenberg complex, where the CE differential is precisely the BRST operator.

The applications of the BRST formalism are not only limited to the study of field theories in the bulk, as it is also a powerful tool in the reduction of constrained Hamiltonian systems, which arise when dealing with boundary conditions for field theories on Cauchy surfaces. Historically, such analysis was first given by Dirac [Dir58] in terms of first and second class constraints, employing the Poisson structure on the space of boundary fields. However, thanks to a construction due to Kijowski and Tulczjew (KT) [KT79], one can, in favorable cases, associate a symplectic structure to the space of fields on the Cauchy surface. The BRST formalism then turns out to be instrumental in obtaining the reduced phase space of the theory, which is a central object in the context of quantization, as functions on it can be regarded as physical observables and provide the perfect candidate to be promoted to operators on a Hilbert space in the quantized theory. The caveat is that such procedure only works in the case where the constraints can be recast as components of a momentum map. In this context, the Hamiltonian vector fields of the constraints are interpreted as the generators of the gauge symmetries, and the BRST operator is constructed in such a way that its degree-0 cohomology is exactly given by the algebra of functions on the reduced phase space. In the smooth finite-dimentional setting, such construction turns out to be the algebraic equivalent of the Marsden–Weinstein reduction [MW74].

Despite being suitable to describe a large class of gauge theories, the BRST procedure fails in specific cases, among which are gravity and supergravity. The BV and BFV formalisms then arise from the need of generalizing the BRST framework, where the former is suited for the description of a field theory in the bulk and the latter provides its boundary counterpart. In particular, it is sometimes the case — for example when dealing with a supersymmetric theory — that the gauge symmetries close only modulo the equations of motion, while the BRST framework is best suited for gauge algebras that close off-shell. To address these limitations, Batalin and Vilkovisky [BV77; BV81] extended the formalism to include antifields,[2] seen as degree -1 canonical momenta associated to the classical fields and ghosts. Such introduction allows for the definition of a canonical -1–shifted symplectic structure on the BV space of fields, rendering the latter a graded symplectic supermanifold $\mathcal{F}$. On $\mathcal{F}$, the main theorem of BV assures that there exists a degree-1 operator — generalizing the BRST one — which, thanks to the introduction of anti-fields, is now guaranteed to be nilpotent, equating to the requirement that the gauge algebra closes off-shell. Furthermore, such cohomological vector field can be interpreted as the Hamiltonian vector field associated to the BV action, a functional on the BV space of fields extending the classical action. In this context, the gauge fixing is performed by choosing a Lagrangian submanifold of $\mathcal{F}$, while the gauge invariance is given by the independence on the choice of Lagrangian submanifold, in an appropriate sense. For an exhaustive introduction to the formalism, we refer to [Mne17], while [BBH95; BG05] offer alternative descripitions.

Analogously, the BFV (Batalin-Fradkin-Vilkovisky) [BF83] formalism provides a generalization of the BRST analysis of constrained Hamiltonian systems. As already mentioned, thanks to the KT construction one can assign the structure of a symplectic manifold to the space of boundary fields, and define constraints on it. The BRST formalism is particularly useful in the cohomological resolution of the reduced phase space, arising as the symplectic reduction of the zero locus of the constraints on the boundary. However, such procedure is only suitable for this goal when the constraints can be reinterpreted as components of a momentum map, but there are cases, like the one of gravity, where the constraints are just first-class, without any underlying momentum map in the classical sense. This is precisely when the BFV construction comes into play, indeed, thanks to a theorem by Batalin and Fradkin, which has later been mathematically formalised within the context of homological algebra by Stasheff [Sta97] and for general coisotropic submanifolds by [Sch09]: one can always extend the space of boundary fields by introducing ghosts and antighosts in such a way to define a degree–0 symplectic form and an action functional of degree 1, in which the constraints are recast and whose Hamiltonian vector field is cohomological.

In recent years, Cattaneo, Mnev and Reshetikin [CMR11; CMR14; CMR18] have developed the framework in which the BV and BFV formalism become compatible, in such a way to account for cutting and gluing, similarly to what had been previously proposed for the quantization of topological field theories by Atiyah and Segal [Ati88; Seg88], who defined an axiomatization of TQFT's based on the categorical object of a "quantization" functor whose source category is the one of cobordisms, i.e. manifolds with boundary (and possibly corners). The CMR program offers a viable alternative suitable for a general class of gauge field theories, which is still categorical

[2] It is worth mentioning that antifields were present in the original work of BRS with the name of "BRS sources", introduced in the construction of the Slavnov–Taylor identities [Tay71; Sla72], which are generalization of the well-known Ward identities in the non-abelian gauge theory setting.

in nature and is based on the BV technology, providing an algebraic solution to the problem of the path integral evaluation, relating it to the cohomology of a well-defined cochain complex.

The BV-BFV axioms proposed by CMR state that, when a boundary is introduced, the BV data in the bulk naturally induce a BFV structure on the boundary. Specifically, the variation of the BV action gives a boundary term, which is the analog of a Noether one-form on the space of boundary conditions, hence failing to be exactly the Hamiltonian of a cohomological vector field. The variation of the Noether one-form is usually a degenerate closed two form, whose reduction is assumed to be smooth, hence obtaining a graded symplectic structure on the space of boundary fields, which is now promoted to the space of BFV fields. Furthermore, the Classical Master Equation (CME), i.e. the requirement that the bulk BV action Poisson-commutes with itself, is only satisfied up to a boundary term, which can be interpreted as a functional on the space of BFV fields, taking the role of the BFV action. If the induced theory on the boundary fulfills the BFV axioms, then one obtains a genuine BV-BFV theory, which has proved to be the relevant object towards a boundary BV quantisation approach. This program has been applied to a variety of different theories, notably BF theories [CMR20], Chern–Simons [CMW17] and, recently, to the different incarnations of gravity [CS19a; CS19b; CCS21a].

The BV-BFV description of gravity

For decades, efforts have been made towards the quantization of gravity, with varying success, but ultimately without a fully satisfying answer. The earliest attempts at providing a BV-BFV analysis in this context have been performed classically by Cattaneo and Schiavina in the case of Einstein–Hilbert gravity in [CS16], where they showed that the theory is BV-BFV extendable, in the case where one assumes the boundary metric to be lightlike or timelike.

The Einstein-Hilbert formulation is just one of the different ways gravity can be described, indeed a classically equivalent counterpart — which presents the same moduli space of solution as of Einstein–Hilbert gravity — is given by Palatini–Cartan gravity, where the metric is substituted by a coframe and the connection is assumed to be an independent field. In dimension 3, the coframe formulation of gravity was shown [CS19a] to be strongly BV equivalent — a notion that amounts to asking that the two BV theories present the same cohomological data — to BF theory, and henceforth topological. In a recent paper [CS25], the authors obtained the BFV quantization of 3D Palatini–Cartan gravity.

Despite the equivalence at the level of the Euler–Lagrange locus, the EL and PC theories of gravity present many differences off-shell, since, for example, the latter is not BV-BFV extendable in dimension four, as proved in [CS19b], where a BV structure was found, but the induced pre-symplectic form on the space of boundary fields was shown to be singular, hence not providing a smooth quotient.

The advantages of PC gravity are several, among which the main one is that of dealing solely with differential forms, making it significantly less challenging to restrict to the boundary, and allowing for a quick and simple use of Cartan calculus in a coordinate-free notation. In order to resolve the limitations in the BV-BFV extention, such theory has been studied extensively in the literature. In particular, the classical study of the boundary structure of PC gravity was performed by Canepa, Cattaneo and Schiavina in [CS19c] and subsequently in [CCS21a], where, thanks to the Kijowski–Tulczjew construction, the authors were able to find the reduced phase space of the theory, and later embed it in the language of the BFV formalism.

Such description produced many insights on how to resolve the obstruction to the BFV extension of the BV PC theory in the bulk. Notably, this issue has been first resolved by employing the 1-dimensional AKSZ (Alexandrov–Kontsevich–Schwarz–Zaboronsky) construction. Such formalism was first introduced in [Ale+97], providing a canonical method for building solutions to the CME. In this setting, starting from a well-defined BFV theory on the boundary, it allows to induce in the bulk the data of a theory automatically satisfying all the BV axioms, hence defining a BV-BFV extandable theory. The key observation is that the induced BV theory in the bulk differs from the original one, which was not suitable for a BFV extention. In particular, such theory presents a reduced space of BV fields, which are subject to constraints descending down from the boundary and assuring that the boundary symplectic form is well-defined. Such results were found in [CCS21b], where the authors also showed that the AKSZ-induced BV theory can be embedded in the original BV PC theory.

The two theories are actually equivalent as can be formally explained through a construction which goes by the name of BV-pushforward. Such procedure formalizes, in the context of the BV technology, the idea of "integrating out" certain degrees of freedom, similarly to what happens when heavy modes are discarded to obtained an effective field theories. Roughly speaking, one assumes that the original space of fields can be, at least locally, split into the product of two -1-symplectic supermanifolds. Integration along a Lagrangian submanifold of just one of the two factors is guaranteed to produce an "effective" BV theory on the other factor, safisfying all the relevant axioms. In some cases, like the one at hand, the BV-pushforward can be inverted cohomologically, proving the equivalence of the two theories.

In the case of PC gravity, one can locally split the original space of BV fields into the reduced space (obtained also via 1D AKSZ) of fields satisfying some constraints descending from the boundary, and the rest. It was proved in [CC25b] that the latter space is a -1-symplectic supermanifold, whose fields are responsible for the obstruction in the BFV extension of the full BV PC theory. Integrating out the unconstrained fields exactly produces the desired result.

Supersymmetry, Supergravity and the necessity for BV

Parallelly to the development of quantum gauge theories, the mathematical physics scenario was introduced to the concept of supersymmetry. As the prefix “super-“ suggests, this is an extension beyond the normal kind of symmetry against which normal gauge theories are invariant. Mathematically, this amounts to requiring that the Lie algebra generating the symmetry is a superLie algebra, containing an anti commuting set of generators. Physically, transformations with respect to these generators send bosons to fermions and vice versa. One of the main feature of the supersymmetry transformations is that, when squared, they recover the usual translations, which is going to be a crucial observation in supergravity theories.

The case case of supergravity is peculiar, since it is the super symmetric extension of gravity, which can be regarded as the gauge theory of the Poincaré group, including translations, rotations and Lorentz boosts. In particular, the local translation invariance implies general covariance, which is just another name for diffeomorphism invariance. When considering supergravity, one needs to introduce the gauge fields associated to the supersymmetry generators, which take the name of gravitinos. Given the anti-commuting nature of the generators, such gauge fields will be represented by spinors, whose spin is 3/2, hence the necessity to obtain a well-defined theory of gravity coupled with spinors.

The easiest case, which serves as a warm-up for supergravity, is the coupling of a Dirac spinor, which is a spin 1/2 field. The study of such theory requires to introduce the necessary formalism for the definition of spinor fields on manifolds. This is usually done in terms of spin structures, whose existence is guaranteed only under some topological conditions, specifically that the second Stiefel-Whitney class vanishes.

The problem with spin structures, however, is that they are defined on (pseudo)Riemannian manifold with a fixed metric, which, in the case of gravity, is the central dynamical object of the theory. This issue is resolved by the introduction of spin coframes, generalizing the notion of coframes in general relativity, allowing for the coupling of half-integer spin fields, without the introduction of a space-time metric. A theorem in [NF22] guarantees that the existence of spin coframes is equivalent to the existence of spin structures, and one can induce one object from the other and vice versa.

With the above technical background, the coupling of a Dirac spinor to the PC action is just obtained by addition of the Dirac Lagrangian, where the flat derivative is replaced by the covariant derivative with respect to the spin connection.

For the case of supergravity, however, the study of Dirac spinors just provides an insightful toy model. Indeed, one of the requirements of supersymmetry is that the bosonic and fermionic degrees of freedom of the theory should match, which constraints the type of spinors one is allowed to consider, depending on the space—time dimension of the chosen theory. In dimension 4, the correct number of degrees of freedom is matched by Majorana spinors, particular Dirac spinors that satisfy a reality condition given in terms of the charge conjugation matrix.[3]

The case investigated in this thesis is the simpler one: $N = 1, D = 4$ supergravity, namely the one where just one gravitino appears. This theory has been largely studied in the past, employing different techniques, such as the superspace formalism, where fields are described as sections of certain (vector or principal) bundles over a supermanifold $\mathcal{M}$ of type $(D|N)$, where D is the space–time dimension and N is the number of supercharges. Such formalism has found a geometrical interpretation in the so-called "rheonomy approach" [CDF91a; Cas18; DAu20], where $\mathcal{M}$ is obtained as the coset space of two (super)groups G/H.[4] However, in this thesis the supermanifold language appears only in relation to BV/BFV formalism, and the rheonomy approach is not contemplated. Indeed one of the insights of this work is that supersymmetry emerges from the minimal coupling of gravity to a Majorana spinor. This is clear in the KT approach to this theory, where the supersymmetry transformations arise as the Hamiltonian vector fields of the constraints.

Indeed, the KT analysis of $N = 1, D = 4$ supegravity is performed in chapter 4, starting by considering the induced pre-symplectic form on the space of boundary fields.

The main difference from the pure gravity case is that the gravitino interaction introduces a torsion term quadratic in the gravitino, which implies that the connection satisfying the equations of motion is not the usual Levi–Civita one. In general, a torsion term is introduced whenever spinors are coupled to gravity via the spin connection, as it is also the case for the spin $\frac{1}{2}$ Dirac field. Furthermore, the gravitino satisfies the (massless) Rarita–Schwinger equation, which, contrary to the simpler Dirac equation, descends to the boundary as a constraint.

After the symplectic reduction is obtained on the space of boundary fields, yielding a smooth space called the "geometric phase space", the study of the constraints of $N = 1, D = 4$ supergravity is performed in 4.1, showing that they form a first-class set. This implies that the zero-locus of the constraints, the actual phase space of the theory, is a coisotropic submanifold. Such space is of great importance as it represents, on a Cauchy surface, the space of boundary conditions of the theory, which is in one-to-one correspondence with the space of solutions to the Euler–Lagrange equations in the bulk. However, the phase space includes gauge-equivalent boundary conditions, related by the action of the symmetries of the theory. In order to obtained the reduced phase space, one needs to compute its coistropic reduction, which in most cases

[3]In principle, Weyl spinors would be suitable candidates for describing the gravitino as they also match the required degrees of freedom. However, the only consistent solution in the $3+1$ signature is provided by the choice of Majorana spinors.

[4]This idea stems from the fact the superspace $\mathbb{R}^{(D-1,1)|N}$ is obtained as the quotient of the superPoincaré group $\mathrm{Iso}(\mathbb{R}^{(D-1,1)|N})$ with respect to the spin group $\mathrm{Spin}(D-1,1)$.

turns out to be non-smooth. Ultimately, this is why the BFV formalism provides such a useful tool, as it allows to cohomologically resolve the reduced phase space, obtaining the functions on it as the degree 0 cohomology of the BFV operator.

Having shown that the constraints form a first-class set is enough to guarantee the existence of a BFV structure. However, in the case of supergravity, the derivation of an explicit BFV action provides very hard computational challenges. Indeed, in the simpler cases one recovers the BFV action as the twisting of the Koszul and of the Chevalley–Eilenberg differentials, rephrased in the language of supergeometry, obtaining an expression that is at most linear in the antighosts. In the case of supergravity, one needs to take into account the fact that the supersymmetry only closes modulo the equations of motion, which implies the introduction of rank-2 terms (i.e. quadratic in the antighosts) in the action.

In light of such remarks, it is more convenient to first study the BV structure of $N = 1, D = 4$ supergravity in the bulk. As already mentioned, the naive definition of the BV operator as the sum of the infinitesimal gauge symmetries, which is the solution proposed in the BRST formalism, does not guarantee the off shell nilpotency property. In particular, the BRST procedure is not sufficient to treat theories whose symmetry algebra closes only on shell, however, the main theorem of BV guarantees that it is always possible to add terms to the BV action forming a polynomial in the antifields of increasing degree, such that its Hamiltonian vector field — the BV operator — is cohomological. This makes the BV formalism the only option for the study of many supersymmetric theories, including supergravity, by means of cohomological methods.

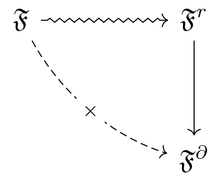
In the case of $N = 1, D = 4$ supergravity, already in the half-shell case, where the torsion equation is imposed, it was proved in [Bau+90] that rank-2 terms are necessary in the BV action. In chapter 5, the theory is studied in its off shell formulation, leaving the spin connection unconstrained since, when restrcited to the boundary, it plays the role of the conjugate momentum of the vielbein.

Dropping the torsion constraint in the bulk introduces many computational difficulties, which are resolved by employing many technical results involving properties of the vielbein, gamma matrices and Fierz identities, which are regrouped in A.3. Thanks to these tools, a fully covariant fully off-shell BV formulation of $N = 1, D = 4$ supergravity is obtained in 5.1.1.

However, as it is the case for pure PC gravity, the theory is shown to be not 1-extendable. In particular, the same obstructions to the definition of a regular BFV symplectic forms are found, where the introduction of spinors only plays a marginal role. Indeed, the problem arises in the boundary components of the spin connection, which is where the singularity of the kernel of the induced boundary pre-symplectic form arises. Unfortunately, the problem cannot be solved as in the case of the pure PC gravity, where the BV theory that produces the correct BFV extension was found by means of the 1-D AKSZ construction.

In the case at hand, it is however possible to employ the methods of the BV pushforward, eliminating the problematic degrees of freedom of the spin connection which are responsible for the obstruction to the BFV extension, obtaining the restricted space of BV fields subject to the appropriate constraints, which are retrieved from the classical structure of the theory on the boundary. Specifically, the spin connection and its antifield are constrained in such a way as to make the constraint set invariant under the BV operator, which amounts to require that they are gauge invariant. With such method, it is then possible to obtain a well-defined BFV theory, which in this thesis is only computed implicitly.

The above considerations are summarised in the following diagram



where $\mathfrak{F}$ and $\mathfrak{F}^r$ are respectively the full BV and restricted BV theory of $N = 1, D = 4$ supergravity, $\mathfrak{F}^\partial$ is the BFV theory on the boundary, while the squiggly arrow represents the BV pushforward and the straight one the usual BV-BFV extension.

Outline of the thesis

The thesis is structured as follows

- Chapter 1 provides an overview of classical field theory on manifolds with boundary, introducing the the KT construction, and the BV-BFV technology;
- Chapter 2 reviews some known results in the theory of Palatini–Cartan gravity on manifolds with boundary, as well as its BV-BFV formulation, providing the starting point for the coupling of spinors to gravity;
- Chapter 3 couples Dirac fermions to Palatini–Cartan gravity, yielding a full BV-BFV theory of Palatini–Cartan–Dirac gravity, with analysis of second-class constraints and the reduced phase space, as well as the 1-dimensional AKSZ construction;
- Chapter 4 contains the constraint analysis of $N = 1, D = 4$ supergravity within the KT construction, laying the groundwork for a BFV formulation;
- Chapter 5 presents the complete BV construction of supergravity and shows the existence of a compatible BFV theory via the methods of the (classical) BV pushforward, applied to the specific case of supergravity and generalized the known results for the PC theory;
- Appendix A is a self-contained review of Clifford algebras, spin groups, and spin coframes, culminating in the construction of Majorana spinors and Fierz identities;
- Appendix B contains all the lengthy computations of the main results of this thesis, which were too long and cumbersome to include in the main chapters.

Outlook and Future Directions

This thesis, while rooted in classical field theory, lays the groundwork for future studies in the context of supergravity and gravity with spinors. Indeed, several insights can be extracted from this work. Firstly, the BV-BFV formalism offers a powerful language which allows to rigorously relate classical and quantum field theories, particularly when extended to systems with boundaries (and possibly corners), allowing the description of supersymmetric theories which require the introduction of higher order corrections. Secondly, the Palatini–Cartan formulation of gravity, while naturally allowing the coupling of spinors and torsion, proves indispensable for a geometric understanding of supergravity. Lastly, the BV pushforward technique and the AKSZ construction provide not only formal solutions to the master equation but also concrete tools for building boundary theories.

In conclusion, this thesis contributes to the mathematical formalism required for the consistent treatment of gauge theories on manifolds with boundary and provides a rigorous foundation for understanding supergravity from a geometric and cohomological perspective. It invites further investigations into the role of boundary conditions in supersymmetric models, and the interplay between classical geometry and quantum field theory in the BV-BFV framework.

Chapter 1

Preliminaries

This chapter contains a description of the mathematical tools underlying the main results of the thesis.

1.1 Classical Lagrangian field theories with boundary

In this section we introduce the necessary notions to define a field theory on a manifold with boundary in a short and systematic way.

Let M be a D–dimensional manifold with boundary $\partial M =: \Sigma$ and let F be a vector bundle on M. For a large variety of theories—and in particular the ones at hand—the space of fields F_M is in general defined a (an open subspace of) an affine space modeled on the space of smooth local sections ϕ on F, i.e. $F_M := \Gamma(M, F)$, which is in general an infinite-dimensional manifold (inheriting the structure of a Fréchet space) on which we assume that Cartan calculus is defined.

To define precisely the objects employed in the context of Lagrangian field theory, one first needs to define the local calculus on $M \times F_M$. Let us consider the infinite jet bundle $J^\infty F$. The smooth local sections of the infinite jet bundle $\Gamma(M, J^\infty F)$, can also be obtained by the jet prolongation $j^\infty \colon \Gamma(M, F) \to \Gamma(M, J^\infty F)$. We can define a map e_∞ by precomposing j^∞ with the evaluation map $\mathrm{ev} \colon M \times F_M \to F : (x, \phi) \mapsto \phi(x)$, i.e.

$$e_\infty \colon M \times F_M \xrightarrow{(\mathrm{id}, j^\infty)} M \times \Gamma(M, J^\infty F) \xrightarrow{\mathrm{ev}} J^\infty F$$

It is a well known fact [Del18; And] that differential forms on $J^\infty F$ carry a double degree, defining a bicomplex with respect to a vertical differential d_V and a horizontal differential d_H, such that $d = d_V + d_H$ is the usual de Rham differential. In particular, this implies that $d_V^2 = 0$, $d_H^2 = 0$ and $d_V d_H + d_H d_V = 0$. It is then possible to define local forms on $M \times F_M$ by pulling back forms on $J^\infty F$ along e_∞. This produces a double complex of local forms defined by

$$\Omega^{(p,q)}_{\mathrm{loc}}(M \times F_M) := e_\infty^* \Omega^{(p,q)}(J^\infty F), \tag{1.1}$$

where p is the vertical degree and q the horizontal one. The differentials are defined by $d := e_\infty^* d_H$ and $\delta := e_\infty^* d_V$, representing respectively the de Rham differential on differential forms on M and the “variational differential” on forms on F. In particular, d measures variations of fields at the space–time level, while δ measures variations of the field configuration at a given space–time point. A Lagrangian L_M is defined to be a $(D, 0)$ local form which, when evaluated at a field configuration ϕ, is called Lagrangian density $L_M(\phi)$.

Definition 1.1. A field theory on M is given by a space of fields F_M and an action functional S_M, seen as the integral of the Lagrangian L_M.

The physical content of the theory is encapsulated by the action (or equivalently by the Lagrangian), whose variation produces the Euler-Lagrange equations. In particular, in presence of a boundary, one has

$$\delta S_M = el_M + \int_\Sigma a_M,$$

where el_M is the Euler-Lagrange (integrated) local 1–form and a_M is the boundary term arising after integration by parts, which is a local 1-form depending on the fields and their jets at ∂M. The critical locus is defined to be the set of solutions to the equations of motion

$$EL_M := \{\psi \in F_M \mid el_M|_\psi = 0\}.$$

In this context, the symmetries of the theory are defined by vector fields on F_M leaving the action invariant, i.e. $\mathbb{X} \in \mathfrak{X}(F_M)$ such that $\mathrm{L}_{\mathbb{X}}(S_M) = 0$, where $\mathrm{L}_{\mathbb{X}} := \iota_{\mathbb{X}}\delta + \delta\iota_{\mathbb{X}}$ is the Lie derivative on the space of fields.

1.1.1 The Kijowski–Tulczijew construction

Defining $\alpha_M := \int_{\partial M} a_M$, one notices

- α_M is δ–exact on EL_M, as $\alpha_M = \delta S_M|_{EL_M}$
- defining $\varpi_M := \delta\alpha_M$, we have $\varpi_M|_{EL_M} = 0$ and that ϖ_M is invariant under the following transformation of the Lagrangian L_M

$$\begin{aligned} \mathrm{L}_M &\longmapsto \tilde{\mathrm{L}}_M := \mathrm{L}_M + \phi_M \\ \alpha_M &\longmapsto \tilde{\alpha}_M := \alpha_M + \delta\Phi_M, \end{aligned}$$

  where $\Phi_M := \int_{\partial M} \phi_M$ is a boundary term.

What this tells us is that α_M can be regarded as a one-form connection on a line–bundle over F_M, and the property above is just a form of gauge–invariance of the curvature two-form ϖ_M. Indeed, when one considers e^{iS_M} as a section of the line–bundle over F_M, $L_M \mapsto \mathrm{L}_M + \phi$ is just the infinitesimal action of the gauge transformation $e^{i\Phi_M}$.

Furthermore, we notice that ϖ_M is degenerate, i.e. $\ker \varpi_M := \{\mathbb{X} \in \mathfrak{X}(F_M) \mid \iota_{\mathbb{X}}\varpi_M = 0\} \neq \{0\}$. In particular, vector fields on F_M that preserve the fields at the boundary are by definition in the kernel of ϖ_M, since ϖ_M only depends on the values of the fields (and their jets) on Σ. This allows us to define the space of pre–boundary fields $\tilde{F}_\Sigma$ as the leaf space of the distribution of such vector fields. This amounts to restricting the fields and their transversal jets to the boundary, which in turn defines a surjective submersion

$$\tilde{\pi} : F_M \to \tilde{F}_\Sigma.$$

Additionally, $\tilde{\pi}$ uniquely induces the forms $\tilde{\alpha}_\Sigma$ and $\tilde{\varpi}^\partial$ on $\tilde{F}_\Sigma$. In particular, $\tilde{\varpi}^\partial = \delta\tilde{\alpha}_\Sigma$ is still a closed two–form and

$$\delta S_M = el_M + \tilde{\pi}^*\tilde{\alpha}_\Sigma.$$

It is also convenient to define the subspace $\tilde{L}_M := \tilde{\pi}(EL_M)$ of pre–boundary fields that can be extended to a solution of the E–L equations in the bulk. Such space is isotropic with respect to $\tilde{\varpi}^\partial$, since as before $\tilde{\varpi}^\partial|_{\tilde{L}_M} = 0$.[1].

[1] In most cases, $\tilde{L}_M$ is a submanifold. Here we assume that it is the case

It is possible that $\tilde{\varpi}^\partial$ is still degenerate, hence pre-symplectic, on the space of pre-boundary fields $\tilde{F}_\Sigma$. In this case, one needs to perform another quotient with respect to the kernel of the pre-symplectic form $\tilde{\varpi}^\partial$, assuming that it defines a regular integrable distribution. The quotient space F_Σ is called "geometric phase space" and is obtained as the leaf space of the characteristic distribution of $\ker \tilde{\varpi}^\partial$, with quotient map $p : \tilde{F}_\Sigma \to F_\Sigma$. If F_Σ is smooth, then $\pi = p \circ \tilde{\pi}$ is a surjective submersion.

By construction, it is clear that $(F_\Sigma, \varpi_\Sigma)$ defines a symplectic manifold, but as such it is not yet the physical phase space of the theory, as the latter is seen as the set of "Cauchy data" of the theory (i.e. the space of boundary conditions of the theory).[2]

To better understand this statement, we notice that the Euler-Lagrange equations split into evolution equations that contain derivatives transversal to the boundary, and constraints which only contain derivatives of the fields in the directions tangential to the boundary. The constraints need to be imposed on the space of pre–boundary fields, which usually enlarges the kernel of the pre–symplectic form. The corresponding reduction leads to the reduced phase space.

To be precise, one works with the cylindrical manifold $M_\epsilon := \Sigma \times [0, \epsilon]$, for some positive ϵ. The boundary ∂M is then given by $(\Sigma \times \{0\}) \sqcup (\Sigma \times \{\epsilon\})$, while $\tilde{L}_{M_\epsilon}$ can be seen as a relation[3] between $\tilde{F}_\Sigma \simeq \tilde{F}_{\Sigma \times \{0\}}$ and $\tilde{F}_{\Sigma \times \{\epsilon\}}$. The space of Cauchy data $\tilde{C}_\Sigma$ is then defined to be the subset of pre–boundary fields at Σ that can be extended to solutions to the E–L equations in a cylindrical neighborhood of Σ, i.e.

$$\tilde{C}_\Sigma := \{c \in \tilde{F}_\Sigma \simeq \tilde{F}_{\Sigma \times \{0\}} \mid \exists u \in \tilde{F}_{\Sigma \times \{\epsilon\}} \text{ s.t. } (c, u) \in \tilde{L}_{M_\epsilon}\}.$$

The induced 2–form $\tilde{\varpi}^C_\Sigma$ is generally degenerate on $\tilde{C}_\Sigma$, and the quotient $\underline{C}_\Sigma$ is finally the reduced phase space of the theory. Such space is often non–smooth, but in the context of field theory one is interested in the algebra of functions on it, i.e. the physical observables of the theory. Under certain assumptions, we will see in section 1.2.1 how such an algebra can be obtained cohomologically within the BFV formalism.

Example 1.1. *The easiest example is given by classical mechanics. The "space–time" is simply given by an interval $I := [a.b]$, while fields are in general curves in $\mathbb{R}^n$, denoted by*

$$q\colon [a, b] \to \mathbb{R}^n : t \mapsto (q^i(t)),$$

with $i = 1, \cdots, n$. The action functional is simply

$$S_M[q] := \int_I \left(\frac{1}{2} m \delta_{ij} \dot{q}^i \dot{q}^j - V(q)\right) dt,$$

where $V \in \mathcal{C}^\infty(\mathbb{R}^n))$ is the potential function. The variation of S_M yields the Euler-Lagrange 1-form and a boundary term

$$\delta S_M = \int_I \left(\delta_{ij} m \ddot{q}^j - \partial_i V\right) \delta q^i dt \; - (\delta_{ij} m \dot{q}^j \delta q^i)|_a^b.$$

Clearly, the bulk term contains the equations of motion, while the boundary term is the difference of the so–called Noether 1–form at times b and a. In particular, defining the momenta $p_j := m\delta_{ij}\dot{q}^i$, the Noether one–form is just given by

$$\alpha = p_i \delta q^i.$$

[2]Indeed, assuming the Cauchy problem is well-posed, to any boundary condition one can associate a unique solution to the E-L equations in the bulk. Functions on the space of Cauchy data are then in one-to-one correspondence with classical observables.

[3]A relation between two sets A and B is a subset of $A \times B$.

In the context of the KT construction, the boundary of the interval I is given by two disjoint components $\{a\}$ and $\{b\}$. Choosing one component, e.g. $\{a\}$, one sees that the space of pre–boundary fields described in the section above is nothing but the set of all possible initial positions and momenta $(q^i, p_i) \ni F_a = T^*\mathbb{R}^n$, while the variation of the Noether one–form is just the canonical symplectic form on $\mathbb{R}^{2n}$, given by

$$\omega = \delta p_i \delta q^i,$$

where we have omitted the $\wedge$ symbol.[4]

In this case, no further reduction is required and $T^*\mathbb{R}^n$ is already identified with the space of Cauchy data of the theory, i.e. the (reduced) phase space.

The classical dynamics of the system is then encoded in the Hamiltonian function $H := \frac{1}{2m}\delta^{ij}p_ip_j + V(q)$ defined on the phase space, since the flow of Hamiltonian vector field X of H (i.e such that $\iota_X\omega + dH = 0$) governs the time evolution of the system. In particular, one obtains the first order differential equation

$$\begin{cases} \dot{p}_i = -\frac{1}{m}\partial_i V \\ \dot{q}^i = \frac{1}{m}\delta^{ij}p_j. \end{cases}$$

One can then show that the graph $\mathrm{L}_{[a,b]}$ of the flow of the Hamiltonian vector field X is a Lagrangian submanifold on $F_a \times F_b = T^*\mathbb{R}^n \times T^*\mathbb{R}^n$, being careful to take $-\omega$ to be the symplectic form on F_b.[5]

Example 1.2. *Consider the simple case of electromagnetism over a D-dimensional Lorentzian manifold (M, g). The dynamical fields of the theory are given by $U(1)$–connections on a line–bundle over M, modeled by 1–forms $A \in \Omega^1(M)$. Letting $\{x^\mu\}$, $\mu = 1, \cdots, \dim M$ be local coordinates on M the classical action is*

$$S_M[A] = \int_M g^{\mu\nu}g^{\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\sqrt{|\det g|}d^Dx,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the curvature of A. However, it is more convenient to work in the so–called first order formalism, as that provides the correct starting point for the discussion in the next section.

We introduce a 2–form field $B \in \Omega^2(M)$, which acts as a Lagrange multiplier and has the advantage of eliminating second order derivatives in the Lagrangian. Indeed one can define

$$S_M[A, B] = \int_M B \wedge F_A + \frac{1}{2}B \wedge \star B.$$

The equations of motion are simply $B = \star F_A$ and $dB = 0$, which, after substituting the first into the second, yield Maxwell's equation for the connection A. The variation of $S_M[A, B]$ immediately gives the boundary term

$$\alpha_\Sigma = \int_\Sigma B \wedge \delta A,$$

where $B \in \Omega^2(\Sigma)$ and $A \in \Omega^1(\Sigma)$. It is a quick computation to show that $\varpi_\Sigma = \delta\alpha_\Sigma$ has a trivial kernel, hence it is symplectic and we can define the space of boundary fields as $F_\Sigma = \Omega^1(\Sigma) \times \Omega^2(\Sigma)$, with

$$\varpi_\Sigma = \int_\Sigma \delta B \wedge \delta A.$$

[4] Notice that, since the space F_a is finite-dimensional, the variational differential δ coincides with the de-Rham differential d on $\Omega(T^*\mathbb{R}^n)$.

[5] Indeed from the KT construction, one would in principle have $\varpi := \delta\alpha = \delta p_i \delta q^i|_a - \delta p_i \delta q^i|_b$.

Letting x^i, $i = 1, 2, 3$ be the local coordinates along Σ and x^0 the one transversal to Σ, we see that $dB = 0$ in the bulk splits into $\partial_0 B = \dot{B} = 0$, which is an evolution equation, and $\partial_i B = 0$,[6] which is a constraint that needs to be imposed on the space of boundary fields and defines the space of Cauchy data C_Σ. It is convenient to define C_Σ as the zero-locus of the following functional

$$J_\mu = \int_\Sigma \mu dB,$$

where $\mu \in \mathcal{C}^\infty(\Sigma)$ is a Lagrange multiplier. As it turns out, the Hamiltonian vector field of J_μ is given by

$$\mathbb{J}_A = d\mu \qquad \mathbb{J}_B = 0, \qquad \text{with} \qquad \delta J_\mu = \iota_{\mathbb{J}_\mu} \varpi_\Sigma,$$

which tells us that B is gauge–invariant (which tells us that F_A is too) and that $A \mapsto A + d\mu$ is the infinitesimal $U(1)$ gauge transformation of the connection. Furthermore, by definition of C_Σ, we have that $\mathrm{Ker}(\varpi^\partial|_{C_\Sigma}) = \mathrm{span}(\mathbb{J}_\mu)$, since $0 = \delta J_\mu|_{C_\Sigma}$ and $\delta J_\mu = \iota_{\mathbb{J}_\mu} \varpi^\partial$

The reduced phase space $\underline{C}_\Sigma$ is then given by the quotient of the space of solutions of Gauss' law by the gauge transformations.

For the interested reader, more examples and the theoretical background of the KT construction can be found in a modern notation in [Cat25].

1.2 The BV-BFV formalism on manifolds with boundary

The BV-BFV formalism, first introduced by Batalin, Fradkin and Vilkovisky [BV77; BV81; BF83], is a general framework for the treatment of gauge field theories on manifolds with boundary. The main construction requires the space of fields to be enlarged to a $\mathbb{Z}$-graded[7] supermanifold, and to be endowed with a symplectic form and a cohomological Hamiltonian vector field encoding the classical symmetries of the system.

Definition 1.2. A BV manifold on M is the assignment of data $(\mathcal{F}_M, \mathcal{S}_M, Q, \varpi_M)$, where $(\mathcal{F}_M, \varpi_M)$ is a $\mathbb{Z}$-graded manifold endowed with a -1-symplectic form ϖ_M, and $\mathcal{S}_M$ and Q are respectively a degree 0 functional (called BV action) and a degree 1 vector field on $\mathcal{F}_M$ such that

- $\iota_Q \varpi_M = \delta \mathcal{S}_M$, i.e. Q is the Hamiltonian vector field of $\mathcal{S}_M$;
- $Q^2 = \frac{1}{2}[Q, Q] = 0$, i.e. Q is cohomological.

Remark 1.1. As a consequence of Q being cohomological, the BV action satisfies the classical master equation

$$(S, S) = 0, \tag{1.2}$$

where $(\cdot, \cdot)$ is the Poisson bracket induced by the symplectic form ϖ_M.

Definition 1.3. A BFV manifold on Σ is the assignment of data $(\mathcal{F}_\Sigma, \mathcal{S}_\Sigma, Q_\partial, \varpi_\Sigma)$,[8]where $(\mathcal{F}_\Sigma, \varpi_\Sigma)$ is a $\mathbb{Z}$-graded manifold endowed with a 0-symplectic form ϖ_Σ, and $\mathcal{S}_\Sigma$ and Q_∂ are respectively a degree 1 functional (called BFV action) and a degree 1 vector field on $\mathcal{F}_\Sigma$ such that

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[6] Applying $B = \star F_A$ and defining $E^i := g_\partial^{ij} F_{0j}$ as the electric field, we see $dB = 0$ on the boundary is equivalent to Gauss' law $\mathrm{div} E = 0$.

[7] The grading is commonly referred to as "ghost degree", but here we consider for simplicity the total grading, i.e. the sum of all the degrees of a field belonging to various graded vector fields.

[8] Notice the distinction between ϖ^∂ arising from the KT construction and the BFV symplectic form ϖ_Σ

- $\iota_Q \varpi_\Sigma = \delta \mathcal{S}_\Sigma$, i.e. Q_∂ is the Hamiltonian vector field of $\mathcal{S}_\Sigma$;
- $Q_\partial^2 = 0$, i.e. Q_∂ is cohomological.

If the symplectic form ϖ_Σ is exact, then $(\mathcal{F}_\Sigma, \mathcal{S}_\Sigma, Q_\Sigma, \varpi_\Sigma)$ is called an exact BFV manifold.

We then see that the notions of BV and BFV manifolds only differ by the grading of the symplectic form and the action. Typically the space of bulk fields will be given by a BV manifold, while the boundary fields are modeled by a BFV manifold, hence the grading can be related to the codimension of the "space–time" manifold.

To see how the theory on the bulk is related to the boundary one we notice that, in the presence of a boundary, the condition $\delta S_M = \iota_Q \varpi_M$ will be only satisfied up to a boundary term, i.e.

$$\delta S_M = \iota_Q \varpi_M - \vartheta_M.$$

This suggests generalizing the KT construction to define $\mathcal{F}_\Sigma$ as the symplectic graded (super)manifold on which $\varpi_\Sigma := \delta\vartheta_\Sigma$ defines a symplectic form, where $\vartheta_M = \pi^*(\vartheta_\Sigma)$ and $\pi : \mathcal{F}_M \to \mathcal{F}_\Sigma$.

Definition 1.4. An exact BV-BFV pair is given by the data $(\mathcal{F}_M, \mathcal{S}_M, Q_M, \varpi_M; \pi)$, where $(\mathcal{F}_M, \mathcal{S}_M, Q_M, \varpi_M)$ is a broken BV manifold and $\pi : \mathcal{F}_M \to \mathcal{F}_\Sigma$ is a surjective submersion, such that

$$\iota_{Q_M} \varpi_M = \delta \mathcal{S}_M - \pi^* \vartheta_\Sigma, \tag{1.3}$$

together with an exact BFV manifold $(\mathcal{F}_\Sigma = \pi(\mathcal{F}_M), \mathcal{S}_\Sigma, Q_\Sigma, \delta\vartheta_\Sigma)$, where one requires that $\pi_*(Q_M) = Q_\Sigma$.

Remark 1.2. Notice that, in the presence of boundary, the CME in the bulk is not satisfied anymore, indeed one has

$$Q(\mathcal{S}) = \iota_Q \delta \mathcal{S}_M = \iota_Q \iota_Q \varpi_M - \iota_Q \pi^* \vartheta_\Sigma$$

One can define

$$\mathcal{S}_\Sigma \coloneqq \frac{1}{2} \iota_Q \iota_Q \varpi_M, \tag{1.4}$$

as in general, by the non–degeneracy of ϖ_M, one only obtains a boundary term from $\iota_Q \iota_Q \varpi_M$. Furthermore, by the generalized CME

$$\delta \mathcal{S}_\Sigma = \iota_{\delta\pi Q} \delta\vartheta_\Sigma = \iota_{Q_\Sigma} \varpi_\theta,$$

where $\delta\pi$ is the differential of π and $Q_\Sigma \coloneqq \delta\pi Q$.

1.2.1 The BFV formalism and the reduced phase space

As it turns out, the BFV formalism is helpful in the cohomological resolution of the reduced phase space defined in the previous section. We start by studying the finite dimensional setting. Here (F, ω) is a finite dimensional symplectic manifold, with functions $\psi_i \in \mathcal{C}^\infty(F)$, $i = 1, \cdots, n$, whose differentials are independent and such that their zero locus C defines a coisotropic submanifold, i.e. such that there exist functions $f^\gamma_{\alpha\beta} \in \mathcal{C}^\infty(F)$ such that

$$\{\psi_i, \psi_j\} = f^k_{ij} \psi_k.$$

If we let $\mathcal{I}$ be the ideal generated by the functions ψ_i's, then the functions on C are simply given by $\mathcal{C}^\infty(F)/\mathcal{I}$, as functions differing by a combination of the ψ_i's will coincide on C. Now, letting X_i's be the Hamiltonian vector fields associated to the ψ_i's, they span the characteristic foliation, hence the coisotropic reduction $\underline{C}$ is given as $C/\{X_i\}$.[9] Furthermore, if $\underline{C}$ is smooth, then

$$\mathcal{C}^\infty(\underline{C}) \simeq (\mathcal{C}^\infty(F)/\mathcal{I})^{(X_1 \cdots X_n)},$$

meaning that the functions on $\underline{C}$ are equivalent to the X_i–invariant functions on C.

Now, we can introduce odd coordinates c^i (the ghosts) and $c_i^\dashv$ (the antighosts) respectively of degree 1 and -1 (seen as coordinates on $T^*R^n[-1]$, extending F to a graded symplectic manifold $F \times T^*R^n[-1]$ with 0–symplectic form given by $\omega + dc^i dc_i^\dashv$. We can furthermore introduce a cohomological vector field Q on $F \times T^*R^n[-1]$ given by $Q(f) = c^i X_i(f)$, $Q(c_i^\dashv) = \psi_i$ and $Q(c^i) = 0$, for all $f \in \mathcal{C}^\infty(F)$. As it turns out, this is the Hamiltonian vector field of $S = c^\alpha \psi_\alpha$, and its degree zero cohomology gives exactly the functions on $\underline{C}$, as $H_Q^0 \simeq \{f \in \mathcal{C}^\infty(F) \mid X_\alpha(f) = 0\}/\mathcal{I}$.

Remark 1.3. It can happen that Q defined as above is not cohomological, i.e. it does not square to zero. However, a result from Stasheff [Sta97] proves that one can always deform the Hamiltonian S in such a way that Q is cohomological.

In practice one can always start by defining

$$S = c^i \psi_i + \frac{1}{2} f_{ij}^k c_k^\dashv c^i c^j + R, \tag{1.5}$$

where R is determined degree by degree by requiring $\{S, S\} = 0$.

Theorem 1.1 (CMR12). *Let (F, ω) be a symplectic manifold and C a coisotropic submanifold, then there exist a BFV manifold $(\mathcal{F}, Q, S, \varpi)$ whose body is given by F and such that*

$$H_Q^0 \simeq \mathcal{C}^\infty(\underline{C}).$$

In the field theory setting, the BFV data $(\mathcal{F}_\Sigma, \varpi_\Sigma, Q_\partial, \mathcal{S}_\Sigma)$ on the boundary is enough to obtain the algebra of functions on the reduced phase space $\underline{C}_\Sigma \simeq H^0_{Q_\partial}(\mathcal{F}_\Sigma)$.

1.3 A paradigmatic example of classical BV theory

After having seen how the BFV formalism is linked to the boundary structure of a field theory, we shed some light on the BV formalism and how it relates to the bulk structure of a field theory. In this context, our goal is to embed the classical space of bulk fields F_M as the body of a graded supermanifold $\mathcal{F}_M$, on which a functional $\mathcal{S}_M$ is defined, generalizing the classical action S_M – to which it reduces on the body F_M – and containing terms depending on all the other fields in $\mathcal{F}_M$, in such a way that its Hamiltonian vector field Q encodes all the symmetries of the classical theory and is cohomological.

The graded part of $\mathcal{F}_M$ will consist of the ghosts, which arise naturally from the boundary structure as the Lagrange multipliers of the constraints,[10] and of antifields, comprising the field momenta and the ghost momenta. In particular, contrary to the boundary case where a symplectic form is organically obtained via the KT construction, here we need to introduce field momenta, which allow to obtain a symplectic form in a Darboux chart.

In most cases, the symmetries of the system form a distribution $D \subset \mathfrak{X}(F_M)$, which in general is only required to be involutive on the Euler-Lagrange locus $EL_M := \{\phi \in F_M \mid \delta S|_\phi = 0\}$.

We start by reviewing a less general case, and see how it can provide a starting point for generalizations.

[9] Indeed the kernel of the restriciton to C of the symplectic form ω_C is spanned precisely by the X_i's, as $\iota_{X_i}\omega = \delta\psi_i = 0$ on C.

[10] Seen here as the gauge parameters related to the gauge symmetries of the system

1.3.1 The BRST case

In the simpler case where the distribution D is given by a Lie algebra action, we might employ the BRST formalism. It offers a lower degree of generality compared to the BV procedure, but, under certain assumptions, it yields the same result.

Essentially, one still obtains a cohomological resolution of the space of observables of the theory, however it only involves the extention of the space of functionals by the Chevalley-Eilenberg complex, without the twisting with the Koszul complex.

Definition 1.5. A BRST manifold on M is the assignment of data $(\mathcal{F}_{BRST}, \mathcal{S}_{BRST}, Q_{BRST})$, where

- $\mathcal{F}_{BRST}$ is a $\mathbb{Z}$–graded supermanifold;
- $\mathcal{S}_{BRST}$ is a degree 0 functional on $\mathcal{F}_{BRST}$;
- Q_{BRST} is a cohomological vector field.

Letting G be a Lie group with Lie algebra $\mathfrak{g}$ and $\rho\colon \mathfrak{g} \to \mathfrak{X}(F_M)$ the Lie algebra action, we have $D = \rho(\mathfrak{g})$ and we can define

$$\mathcal{F}_{BRST} := F_M \times \Omega^0(M, \mathfrak{g}[1]) \ni (\phi^a, c^i) =: \Phi^\alpha,$$

where $\phi \in F_M$ are the classical fields of the theory and the $c \in \Omega^0(M, \mathfrak{g}[1])$ are the ghosts. Considering a basis $\{v_i\}$ of $\mathfrak{g}[1]$, we define the $\mathbb{X}_i := \rho(v_i) \in \mathfrak{X}[1](F_M)$ as $\mathbb{X}^i = \int_M \mathbb{X}_i(\phi^a)\frac{\delta}{\delta\phi^a}$

$$Q_{BRST} := \int_M c^i \mathbb{X}_i(\phi^a)\frac{\delta}{\delta\phi^a} - \frac{1}{2} f^k_{ij} c^i c^j \frac{\delta}{\delta c^k},$$

which is cohomological[11] and defines the Chevalley-Eilenberg differential on the complex $\wedge^\bullet \mathfrak{g}^* \otimes \mathcal{C}^\infty(F_M) \simeq \mathcal{C}^\infty(F_M \times \mathfrak{g}[1])$. Then it is immediate to notice that

$$H^0_{Q_{BRST}} = \{f \in \mathcal{C}^\infty(F_M) \mid Q(F) = c^i \mathbb{X}_i(F) = 0\} = \mathcal{C}^\infty(F_M)^{\mathfrak{g}},$$

corresponding to the gauge–invariant functionals on the classical space of fields. The observant reader will notice that the form of the cohomological vector field defined above is similar to the expression (1.5). To see how the two equation relate, we embed the BRST formalism within the BV one. We start by defining

$$\mathcal{F}_M := T^*[-1]\mathcal{F}_{BRST} = T^*[-1]D[1] \ni (\Phi^\alpha, \Phi^{\dashv}_\alpha)$$

where the odd cotangent fibers define the anti-fields of degree -1 and -2. Such graded spaces of fields can now be endowed with the canonical -1–symplectic form defined on a -1–shifted cotangent bundle. In this setting, denoting as before by $\Phi = (\Phi^\alpha)$ the multiplet containing fields and ghosts in $D[1]$ and by $\Phi^\dagger = (\Phi^\dagger_\alpha)$ its canonical conjugate containing the anti-fields in the fiber of $T^*[-1]D[1]$, one can define the BV symplectic form as

$$\varpi_M = \int_M \delta\Phi^{\dashv}_\alpha \wedge \delta\Phi^\alpha.$$

Notice that functions on $T^*[-1]D[1]$ are in one–to–one correspondence with vector fields on $D[1] \simeq F_M \times \Omega^0(M, \mathfrak{g}[1])$, hence we can lift Q_{BRST} to the functional

$$S_{BRST} = \int_M c^i \mathbb{X}_i(\phi^a)\phi^{\dashv}_a - \frac{1}{2} f^k_{ij} c^i c^j c^{\dashv}_k,$$

[11] Because the structure constant f^k_{ij} of $\mathfrak{g}$ satisfy the Jacobi identity

which, after setting $Q_0(\phi^a) := \mathbb{X}_i(\phi^a)$ and $Q_o(c^k) := -\frac{1}{2}f^k_{ij}c^i c^j$, can be rewritten as $S_{BRST} = \int_M Q_0(\Phi^\alpha)\Phi^\dashv_\alpha$. The full BV action then becomes

$$\mathcal{S}_M = S_M + S_{BRST}.$$

It automatically satisfies the CME because its Hamiltonian vector field is given by Q_0 on Φ and one can prove that $Q_0^2(\Phi^\alpha) = 0$ is enough to obtain a cohomological vector field on the whole space $\mathcal{F}_M$. In particular, one has

$$Q(\Phi^\alpha) = Q_0(\Phi^\alpha), \qquad \text{and} \qquad Q(\Phi^\dashv_\alpha) = \frac{\delta L_M}{\delta \Phi^\alpha} - (-1)^\beta \Phi^\dashv_\beta \frac{\delta(Q_0\phi^\beta)}{\delta\Phi^\alpha},$$

Remark 1.4. Notice that Q on the anti-fields contains a term $\frac{\delta L_M}{\delta\Phi^\alpha}$ which, on the body, defines the equations of motion related to the field Φ^α. Therefore, when computing the degree–0 cohomology of Q, one can intuitively see how this is related to the gauge-invariant functions on the Euler-Lagrange locus, as they are given by

$$\frac{\ker Q : C^0 \to C^1}{\mathrm{Im} Q : C^{-1} \to C^0} \simeq C^\infty \left\{ \Phi^\alpha \mid \frac{\delta L_M}{\delta\Phi^\alpha} = 0 \right\} / \{\text{gauge transformation}\},$$

where C^k are the functions of ghost degree k on $\mathcal{F}_M$.

We provide a simple example which is instructive for the remainder of the thesis: BF theory.

Example 1.3 (Yang–Mills theory)**.** *Consider a D-dimensional manifold M with boundary $\partial M =: \Sigma$ and a semisimple Lie group G with Lie algebra $\mathfrak{g}$, endowed with the pairing $< -, - > := \mathrm{tr}(--)$. In the first order formalism, the fields of a D-dimensional Yang–Mills theory are just a connection 1–form, seen as 1–form on M with values in the Lie algebra $A \in \Omega^1(M, \mathfrak{g})$ and a $D-2$ form $B \in \Omega^{D-2}(M, \mathfrak{g})$, hence*

$$F_{YM} = \Omega^1(M, \mathfrak{g}) \times \Omega^{D-2}(M, \mathfrak{g}).$$

The action functional is

$$S_{YM} := \int_M \mathrm{tr}\left(B \wedge F_A + \frac{1}{2} B \wedge \star B \right).$$

We notice immediately that this action is invariant under the gauge transformation

$$A \longmapsto A + d_A c \qquad\qquad B \longmapsto B + [c, B],$$

for any $c \in \Gamma(M, \mathfrak{g})$. According to the procedure explained above, we can promote c to a ghost field by shifting its grading by one and construct the following spaces of fields

$$\mathcal{F}^{\mathrm{BRST}}_{YM} := \Omega^1(M, \mathfrak{g}) \times \Omega^{D-2}(M, \mathfrak{g}) \times \Gamma[1](M, \mathfrak{g}),$$
$$\mathcal{F}^{\mathrm{BV}}_{YM} := T^*[-1](\Omega^1(M, \mathfrak{g}) \times \Omega^{D-2}(M, \mathfrak{g})\Gamma(\mathfrak{g})[1]).$$

On the latter space, we can furthermore define the canonical -1–symplectic form

$$\varpi_{YM} = \int_M tr\left(\delta A \wedge \delta A^\dagger + \delta B \wedge \delta B^\dagger + \delta c \wedge \delta c^\dagger\right).$$

The action of the gayge transformation on c is given, according to the Chevalley-Eilenberg differential, by $c \mapsto c + \frac{1}{2}[c, c]$. One can then check that the vector field on $\mathcal{F}^{\mathrm{BRST}}_{YM}$ defined by the gauge transformations is cohomological, hence we obtain the BV action as

$$\mathcal{S}_{YM} = \int_M tr\left(BF_A + \frac{1}{2} B \star B + A^\dagger d_A c + B^\dagger [B, c] + \frac{1}{2} c^\dagger [c, c] \right),$$

with cohomological vector field defined by

$$Q_M = tr\int_M d_A c\frac{\delta}{\delta A} + [B,c]\frac{\delta}{\delta B} + \frac{1}{2}[c,c]\frac{\delta}{\delta c} + (d_A B + [A^\dagger, c])\frac{\delta}{\delta A^\dagger} + (F_A + \star B + [B^\dagger, c])\frac{\delta}{\delta B^\dagger} + (d_A A^\dagger + [B, B^\dagger] + [c, c^\dagger])\frac{\delta}{\delta c^\dagger}.$$

In the presence of a boundary, the variation of the action produces a boundary term

$$\delta \mathcal{S}_{YM} = \iota_{Q_M}\omega_{YM} - \int_\Sigma tr(B\delta A + A^\dagger \delta c),$$

which gives a boundary pre–symplectic two form $\tilde{\varpi}^\partial_{YM}$

$$\tilde{\varpi}^\partial_{YM} = \mathrm{tr}\int_\Sigma \delta B \delta A + \delta A^\dagger \delta c,$$

defined on the space of pre–boundary fields $\tilde{\mathcal{F}}^\Sigma_{YM}$*, which is given by the restriction of the BV fields to the boundary. Notice that* $c^{\dashv}$ *is a top-form on* M*, hence it cannot be restricted to* Σ*. Furthermore, when computing the kernel of* $\tilde{\varpi}^\partial_{YM}$*, one finds that it contains every vector field of the form* $\mathbb{X}_{B^\dashv}\frac{\delta}{\delta B^\dashv}$*. The corresponding quotient is then given by*

$$\mathcal{F}^{\mathrm{BFV}}_{YM} = \Omega^1(\Sigma,\mathfrak{g}) \times \Omega^{n-2}(\Sigma,\mathfrak{g}) \times \Omega^0[1](\Sigma,\mathfrak{g}) \times \Omega^{n-1}[-1](\Sigma,\mathfrak{g}).$$

As it is clear by the symplectic form being in Darboux form, we have that (A, B) *and* $(c, A^\dashv)$ *are pairs of conjugate momenta. Lastly, one finds the BFV action to be*

$$\mathcal{S}^\Sigma_{YM} = \int_\Sigma tr\left(B d_A c + \frac{1}{2}A^\dagger[c,c]\right),$$

giving the same cohomological vector field as in the bulk.

1.3.2 Beyond BRST

The power of the BV formalism becomes clear when one considers systems with gauge symmetries that close only on shell, which is the case of supergravity. The general solution to the problem of finding an appropriate cohomological vector field is given by the BV algorithm. We define a generic BV action by adding terms which are polynomials in the antifields, i.e.

$$\mathcal{S}_M = S_M + \int_M Q_0(\Phi^\alpha)\Phi^\dashv_\alpha + \sum_{I=2}^P M^{\alpha_1\cdots\alpha_I}\Phi^\dashv_{\alpha 1}\cdots\Phi^\dashv_{\alpha_I},$$

where P is called rank of the BV action. It is a result of BV [BV77; BV81] that the rank is finite, analogously to the BFV case.

In most cases, it is enough to stop at rank 2. Explicitly, we have

$$\mathcal{S}_M = S_M + \int_M \Phi^\dashv_\alpha Q_0(\phi^\alpha) + \frac{1}{2}\Phi^\dashv_\alpha\Phi^\dashv_\beta M^{\alpha\beta}(\Phi),$$

which modifies the Hamiltonian vector field as

$$Q(\Phi^\alpha) = Q_0(\Phi^\alpha) + \Phi^\dashv_\beta M^{\alpha\beta},$$
$$Q(\Phi^\dashv_\alpha) = \frac{\delta L_M}{\delta\Phi^\alpha} - (-1)^\beta\Phi^\dashv_\beta\frac{\delta(Q_0\phi^\beta)}{\delta\Phi^\alpha} + \frac{(-1)^{\beta+\gamma}}{2}\Phi^\dashv_\beta\Phi^\dashv_\gamma\frac{\delta M^{\beta\gamma}}{\delta\Phi^\alpha}.$$

In this case, the classical master equation reads

$$(\mathcal{S}_M, \mathcal{S}_M) = \int_M Q_0(L_M) - (-1)^\beta \Phi^\dashv_\beta \left(Q_0^2 \Phi^\alpha - (-1)^{\beta(\alpha+1)} \frac{\delta L_M}{\delta \Phi^\alpha} M^{\alpha\beta} \right) + \frac{(-1)^{\beta+\gamma}}{2} \Phi^\dashv_\gamma \Phi^\dashv_\beta \left(Q_0(M^{\beta\gamma}) - (-1)^{\gamma+\beta\alpha} \frac{\delta Q_0 \Phi^\beta}{\delta \Phi^\alpha} M^{\alpha\gamma} - (-1)^{\beta+\gamma\alpha} \frac{\delta Q_0 \Phi^\gamma}{\delta \Phi^\alpha} M^{\alpha\beta} \right) + \frac{(-1)^{\alpha\beta}}{2} \Phi^\dashv_\beta \Phi^\dashv_\rho \Phi^\dashv_\gamma \frac{\delta M^{\rho\gamma}}{\delta \Phi^\alpha} M^{\alpha\beta}.$$

We know that $Q_0(L_M) = 0$[12] by definition of Q_0, while the remaining terms at each order in the anti-fields(ghost) must be imposed separately,

$$Q_0^2 \Phi^\alpha - (-1)^{\beta(\alpha+1)} \tfrac{\delta L_M}{\delta \Phi^\alpha} M^{\alpha\beta} = 0, \tag{1.6}$$

$$Q_0(M^{\beta\gamma}) - (-1)^{\gamma+\beta\alpha} \tfrac{\delta Q_0 \Phi^\beta}{\delta \Phi^\alpha} M^{\alpha\gamma} - (-1)^{\beta+\gamma\alpha} \tfrac{\delta Q_0 \Phi^\gamma}{\delta \Phi^\alpha} M^{\alpha\beta} = 0, \tag{1.7}$$

$$\tfrac{\delta M^{\rho\gamma}}{\delta \Phi^\alpha} M^{\alpha\beta} = 0. \tag{1.8}$$

We then see from (1.6) that Q_0 squares to a linear combination of the equations of motion, hence computing Q_0^2 defines the next terms in the BV action, which depend on the $M^{\alpha\beta}$. One then needs to check that (1.7) and (1.8) hold, which amounts to show that $Q^2 = 0$. If also Q squares to zero only on shell, one can continue this procedure inductively and compute the next terms in the BV action.

Remark 1.5. It is just a matter of computations to show that $Q^2(\Phi^\alpha) = 0$ implies (1.6), (1.7) and (1.8). Hence it is not needed to show $Q^2(\Phi^\dashv_\alpha) = 0$, as it follows naturally.

1.4 The AKSZ construction

Another useful realization of the BV-BFV formalism can be obtained through a construction due to Aleksandrov, Kontsevich, Schwartz and Zaboronsky (AKSZ). It is also particularly useful when tackling the problem of inducing a compatible BV structure in the bulk from a well defined BFV one on the boundary, in the case of cylindrical spacetimes.

We start by considering the finite-dimensional setting. Let $\mathcal{M}$ be a $\mathbb{Z}$–graded supermanifold, and N be a regular manifold. The parity of the supermanifold $\mathcal{M}$ is just set to be the grading modulo 2. We assume there exist a degree n function S on $\mathcal{M}$ and a non degenerate exact 2-form (hence symplectic) $\omega = d\alpha$, where α is of degree $n-1$, assuming $n > 0$. With these assumptions, $\mathcal{M}$ is called a dg-Hamiltonian manifold if $\{S, S\} = 0$, i.e. if and only if $Q \coloneqq \{S, \cdot\}$ is cohomological.

Definition 1.6. Given the following diagram

$$\begin{array}{ccc} \mathrm{Map}(T[1]N, \mathcal{M}) \times T[1]N & \xrightarrow{\mathrm{ev}} & \mathcal{M} \\ \big\downarrow{\scriptstyle p} & & \\ \mathrm{Map}(T[1]N, \mathcal{M}) & & \end{array}$$

we define the transgression map

$$\mathfrak{T}_N^{(\bullet)} \colon \Omega^\bullet(\mathcal{M}) \longrightarrow \Omega^\bullet(\mathrm{Map}(T[1]N, \mathcal{M}))$$

as $\mathfrak{T}_N^{(\bullet)} \coloneqq p_* \mathrm{ev}^*$, setting $p_* = \int_N \mu_N$ where μ_N is the canonical Berezinian on $T[1]N$.

[12]Under certain assumptions it could also be a boundary term.

Remark 1.6. Notice that the differential on N can be reinterpreted as a vector field on $\mathrm{Map}(T[1]N, \mathcal{M})$. Indeed, considering coordinates (u^i, ξ^i) on $T[1]N$, we map $\xi^i \longmapsto du^i$ provides provides the isomorphism $\mathcal{C}^\infty(T[1]N) \simeq \Omega^\bullet(N)$, and $d = du^i\partial_i$ is the image of $\xi^i\partial_i \in \mathfrak{X}(T[1]N)$.

Theorem 1.2 ([Ale+97]). *Letting* $\mathcal{F}^{AKSZ} := \mathrm{Map}(T[1]N, \mathcal{M})$, $\varpi^{AKSZ} := \mathfrak{T}^2_N(\omega)$ *and* $\mathcal{S}_{AKSZ} := \mathfrak{T}^0_N(S) + \iota_{d_N}\mathfrak{T}^1_N(\alpha)$, *the data*

$$\mathfrak{F}_{AKSZ} := (\mathcal{F}^{AKSZ}, \varpi^{AKSZ}, \mathcal{S}_{AKSZ}, Q_{AKSZ})$$

define a BV theory.

To better understand the content of the above theorem, let us consider $f \in \mathcal{C}^\infty(\mathcal{M})$ and a homogeneous degree map $X \in \mathcal{F}_{AKSZ}$. Then we can define the composition $X_f = f \circ X \in \mathcal{C}^\infty(T[1]N) \simeq \Omega^\bullet(N)$. In particular, if (u^i, ξ^i) are coordinates over $T[1]N$, we can see X_f can be ralized as a differential form on N as

$$X_f(u) = X_f(u)_{i_1\cdots i_k} du^{i_1} \wedge \cdots \wedge du^{i_k}. \tag{1.9}$$

If we let (x^a) be coordinates on $\mathcal{M}$, we can compose the coordinate fucntions x^a with X and see $X^a(u) = X^a(u)_{i_1\cdots i_k} du^{i_1} \wedge \cdots \wedge du^{i_k}$. Then the AKSZ action and symplectic form take the form

$$\mathcal{S}_{AKSZ} = \int_N S[X^a(u)] + \alpha_a(X(u)) \wedge dX^a(u)$$

$$\varpi_{AKSZ} = \int_N \omega_{ab}(X(u))\delta X^a(u) \wedge \delta X^b(u).$$

1.4.1 The 1–dimensional case and its relation to the boundary

In this section we investigate how a BFV theory on Σ can induce a BV theory on $I \times \Sigma$, where I is an interval.

Letting $\mathfrak{F}_\Sigma$ be the data of an exact BFV theory, as it appears after the KT construction, we notice that it automatically satisfies the definition of Hamiltonian dg manifold with $n = 1$. We can therefore apply theorem 1.2 to see that the data $\mathfrak{F}_{AKSZ}$ given by

$$\begin{aligned}
&\mathcal{F}^{AKSZ} := \mathrm{Map}(T[1]I, \mathcal{F}_\Sigma) \\
&\varpi^{AKSZ} := \mathfrak{T}^2_I(\varpi_\Sigma) \\
&\mathcal{S}_{AKSZ} := \mathfrak{T}^0_I(\mathcal{S}_\Sigma) + \iota_{d_I}\mathfrak{T}^1_I(\alpha_\Sigma) \\
&Q_{AKSZ} \quad \text{s.t.} \quad \iota_{Q_{AKSZ}}\varpi^{AKSZ} = \delta\mathcal{S}_{AKSZ}
\end{aligned}$$

yield a BV theory on $\Sigma \times I$.

The setting gets simplified when we notice that $\mathcal{F}^{AKSZ} \simeq \Omega^\bullet(I) \times \mathcal{F}_\Sigma$, i.e. we see that AKSZ fields are just boundary fields times sections of a graded vector space, where in particular

$$\Omega^\bullet(I) = \mathcal{C}^\infty(I) \oplus \Omega^1(I)[-1], \tag{1.10}$$

we obtain two fields for every boundary field in $\mathcal{F}_\Sigma$. In particular, if ϕ^I are fields on $\mathcal{F}_\Sigma$, we obtain that the AKSZ fields are $\Phi^I := \phi^I(t) + \psi^I(t)dt$, where we point out that $\phi^I(t) \in \mathcal{C}^\infty(I) \times \mathcal{F}_\Sigma$. With this definition, the AKSZ action and symplectic form become

$$\varpi_{AKSZ} = \int_{I\times\Sigma} (\varpi_\Sigma[\Phi])_{IJ}\delta\phi^I(t) \wedge \delta\psi^J(t)dt \tag{1.11}$$

$$\mathcal{S}_{AKSZ} = \int_{I\times\Sigma} (\alpha_\Sigma[\Phi])_I\delta\psi^i(t)dt + (\mathcal{S}_\Sigma[\Phi])^{\mathrm{top}} \tag{1.12}$$

Furthermore, one can easily see that the BV theory obtained in this way automatically gives rise to a a BV-BFV extendible theory, where the BFV one is given by the data on the target of the AKSZ.

1.5 The quantum picture and the BV-pushforward

Given a classical BV-BFV theory as defined in the previous sections, in order to obtain a quantum description we need to introduce the so–called BV laplacian Δ, an operator of degree -1 on the space of BV fields and such that $\Delta^2 = 0$ and the following (almost Leibniz) rule holds

$$\Delta(fg) = (\Delta f)g + (-1)^{|f|} f\Delta g + (-1)^{|f|}\{f, g\}_M,$$

where $\{\cdot,\cdot\}_M$ is the canonical Poisson bracket induced by the symplectic form ϖ_M.

To better illustrate the construction, we work in the finite–dimensional case, where the existence of a BV laplacian is always guaranteed. We start by considering a $\mathbb{Z}$–graded manifold $\mathcal{F}$ endowed with a -1–symplectic form ϖ. We denote by $\mathrm{Dens}^{\frac{1}{2}}(\mathcal{F})$ the space of half–densities on $\mathcal{F}$. A theorem by [Khu00] [Sev06] allows to define the following.

Definition 1.7. The BV laplacian Δ is a degree 1 coboundary operator, acting on $\mathrm{Dens}^{\frac{1}{2}}(\mathcal{F})$. In a Darboux chart (x^i, ξ_i), it reads

$$\Delta := \sum_i \frac{\partial}{\partial x^i}\frac{\partial}{\partial \xi_i}.$$

Furthermore, if μ is a Δ-close, never vanishing element of $\mathrm{Dens}^{\frac{1}{2}}(\mathcal{F})$, we can construct a μ–dependent BV laplacian Δ_μ acting on functions on $\mathcal{F}$ as $\mu^{\frac{1}{2}}\Delta_\mu f = \Delta(\mu^{\frac{1}{2}} f) \in \mathcal{C}^\infty(\mathcal{F})$. Letting $\mathcal{L} \subset \mathcal{F}$ be a Lagrangian submanifold, we define the BV integral to be the following composition

$$\mathrm{Dens}^{\frac{1}{2}}(\mathcal{F}) \xrightarrow{\cdot|_{\mathcal{L}}} \mathrm{Dens}^{\frac{1}{2}}(\mathcal{L}) \xrightarrow{\int_{\mathcal{L}}} \mathbb{C} \qquad \xi \longmapsto \int_{\mathcal{L}} \xi|_{\mathcal{L}}.$$

The main theorem of BV [BV77; BV81] shows that for all half densities ξ on $\mathcal{F}$ and for all Lagrangian submanifolds $\mathcal{L}$,

$$\int_{\mathcal{L}} \Delta\xi = 0.$$

Furthermore, if ξ satisfies $\Delta\xi = 0$, and $\mathcal{L}_t$ is a smooth family of Lagrangians parametrized by $t \in [0, 1]$, we have

$$\frac{d}{dt}\int_{\mathcal{L}_t} \xi = 0 \quad \Rightarrow \int_{\mathcal{L}_0} \xi = \int_{\mathcal{L}_1} \xi.$$

Remark 1.7. In the context of field theory, the invariance under the choice of Lagrangian submanifold expresses the invariance under deformations of gauge fixing, while the choice of a Lagrangian submanifold equates to gauge fixing. The relevant object is the "path integral measure" $\mu e^{\frac{i}{\hbar}\mathcal{S}}$, where $\mathcal{S}$ is the BV action. The quantum master equation is then given by the condition

$$\Delta_\mu(e^{\frac{i}{\hbar}\mathcal{S}}) = 0 \quad \Leftrightarrow \quad \frac{1}{2}(\mathcal{S}, \mathcal{S}) - i\hbar\Delta\mathcal{S} = 0,$$

which provides a modification to the CME.[13]

[13]If we assume we can expand $\mathcal{S}$ in powers of $\hbar$, we see that at order zero we retrieve the CME.

Definition 1.8. Assume that $\mathcal{F}$ factorises as a direct product of -1–symplectic graded manifolds $\mathcal{F} = \mathcal{F}' \times \mathcal{F}''$, with $\varpi = \varpi' + \varpi''$. Then

$$\mathrm{Dens}^{\frac{1}{2}}(\mathcal{F}) = \mathrm{Dens}^{\frac{1}{2}}(\mathcal{F}') \otimes \mathrm{Dens}^{\frac{1}{2}}(\mathcal{F}'').$$

The BV pushforward is a map between half densities $\mathrm{Dens}^{\frac{1}{2}}(\mathcal{F}) \to \mathrm{Dens}^{\frac{1}{2}}(\mathcal{F}')$ defined by BV integration over the second factor. In particular, choosing a Lagrangian submanifold, the BV pushforward is given by the map

$$\mathcal{P}_{\mathcal{L}} \colon \mathrm{Dens}^{\frac{1}{2}}(\mathcal{F}) \xrightarrow{\mathrm{id}\otimes\int_{\mathcal{L}}} \mathrm{Dens}^{\frac{1}{2}}(\mathcal{F}'),$$

sending a half density $\phi' \otimes \phi''$ to $\mathcal{P}_{\mathcal{L}}(\phi' \otimes \phi'') := \phi' \int_{\mathcal{L}} \phi''$.

Remark 1.8. In the context of field theory, the BV pushforward is a consistent way to eliminate "heavy modes", resulting in an effective theory. The main feature of this construction is that the effective action obtained after integrating out the heavy modes (that lie in $\mathcal{F}''$) still satisfies the QME, as showed in the following theorem.

Theorem 1.3. *Letting Δ, Δ' and Δ'' be the canonical BV laplacians respectively on $\mathcal{F}$, $\mathcal{F}'$ and $\mathcal{F}''$, we have:*

1. *$\mathcal{P}_{\mathcal{L}}$ is a chain map, i.e.*
$$\int_{\mathcal{L}} \Delta\xi = \Delta' \int_{\mathcal{L}} \xi.$$
2. *if $\mathcal{L}_0$ and $\mathcal{L}_1$ are homotopic Lagrangian submanifolds in $\mathcal{F}''$, then*
$$\left(\int_{\mathcal{L}_0} \xi - \int_{\mathcal{L}_1} \xi\right) = \Delta'\eta$$
for some η.
3. *if $\mathcal{S} \in \mathcal{C}^\infty(\mathcal{F})[\![\hbar]\!]$ satisfies the QME, there is a unique $\mathcal{S}' \in \mathcal{C}^\infty(\mathcal{F}')[\![\hbar]\!]$ is such that*
$$(\mu')^{\frac{1}{2}} e^{\frac{i}{\hbar}\mathcal{S}'} = \mathcal{P}_{\mathcal{L}}\left(\mu^{\frac{1}{2}} e^{\frac{i}{\hbar}\mathcal{S}}\right),$$
and $\mathcal{S}'$ satisfies the QME.

In the case we are interested in, the space of fields does not factor as a direct product of odd symplectic manifolds, but rather it involves the more general case where we only have a fiber bundle $\mathcal{F} \to \mathcal{F}'$, that is locally given by $\mathcal{F}' \times \mathcal{F}''$.

Definition 1.9. Letting $\mathcal{F}''$ be a -1–symplectic graded manifold, a BV hedgehog is given by an odd symplectic fiber bundle $\mathcal{F} \to \mathcal{F}'$ with fiber given by $\mathcal{F}''$, endowed with a surjective submersion $\pi \colon \mathcal{F} \to \mathcal{F}'$ such that for every point $p \in \mathcal{F}'$, there exists a neighborhood $\mathcal{U} \ni p$ and a symplectomorphism $\varphi_{\mathcal{U}} : \pi^{-1}(\mathcal{U}) \to \mathcal{U} \times \mathcal{F}''$. Furthermore, on the overlap of two patches $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ the transition functions $\varphi_{\alpha\beta} \colon \pi^{-1}(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \to \pi^{-1}(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$ are by construction symplectomorphisms of $\mathcal{F}$ constant over $\mathcal{F}'$, which we require to be connected to the identity.

Then we can see that, for any Lagrangian submanifold $\tilde{\mathcal{L}} \subset \tilde{\mathcal{F}}''$, setting $\mathcal{L} := \Phi^{-1}(\tilde{\mathcal{L}})$ we obtain

$$\mathcal{P}_{\mathcal{L}} := (\phi^{-1})_* \circ P_{\tilde{\mathcal{L}}} \circ \Phi_*$$

for which all the results of theorem 1.3 hold.

Chapter 2

Known results in pure gravity

In this chapter we give a review of the Palatini-Cartan theory of gravity in the context of the BV and BFV formalisms, as it provides the starting point for the study of Supergravity. The main advantage of the PC formalism, also known as first order formulation of gravity, is that it yields a neat boundary structure, where the spin connection morally acts as the conjugated momentum of the vielbein, which is the field replacing the metric in the Einstein-Hilbert formulation of gravity. The fact that we only have to deal with differential forms allows to avoid the use of coordinates, which is particularly useful when applying Stokes' theorem and computing boundary terms.

The following review is based on the results of the works of Canepa, Cattaneo, Schiavina and Tecchiolli [CCS21a; CCT21], appearing in a series of papers in which PC gravity was studied as a classical theory on manifolds with boundary within the BV-BFV framework, employing the KT construction discussed in chapter 1.

2.1 Palatini–Cartan gravity in various dimensions

Let M be an D-dimensional manifold and let P_{SO} be an $SO(D-1,1)$-principal bundle on it. We consider a D-dimensional vector space (V,η) with the Minkowski metric, on which we let the Lie group $SO(D-1,1)$ act via the fundamental representation $\rho\colon SO(D-1,1)\to \mathrm{End}(V)$. Next we consider the adjoint vector bundle $\mathcal{V} := P_{SO}\times_\rho V$. Finally, we require that there is an isomorphism $e\colon TM\to\mathcal{V}$. The first field of the theory is then an explicit choice of isomorphism $e\colon TM\to\mathcal{V}$, a.k.a. a vielbein (the Lorentzian metric in the classically equivalent Einstein–Hilbert formalism will be recovered by pull back: $g=\eta(e,e)$).[1][2]

The other field that we consider is a connection on P_{SO}. Let $\omega\in\Omega^1(P_{SO},\mathfrak{so}(D-1,1))$ be the associated connection 1-form. We want to consider the gauge field as a dynamical field of the theory. The following proposition gives a useful way to include it in this setting.

Proposition 2.1. *The space of principal connections on P_{SO} over M is an affine space modeled on $\mathcal{A}(M)=\Omega^1(M,\wedge^2\mathcal{V})$.*

Proof. It is well known that it is possible to identify the affine space of principal connections as the space of one forms with values in the corresponding Lie algebra $\mathfrak{so}(D-1,1)$. Furthermore, it is possible to identify $\mathfrak{so}(D-1,1)$ with $\wedge^2\mathcal{V}$ by means of η. □

[1] Note that we can pull back the fiber metric η and this defines a Lorentzian metric on M, so the setting described above assumes that M admits a Lorentzian structure.

[2] The interest reader can learn more about the vielbein field in chapter A.2.1

We define the space of (i,j)-forms to be the differential i-forms with values in the j-th exterior power of V, namely

$$\Omega^{(i,j)}(M) := \Omega^i(M, \wedge^j \mathcal{V}).$$

The space of fields of our theory is then defined to be

$$F_{PC} := \Omega^{(1,1)}_{nd} \times \mathcal{A}(M),$$

where $\Omega^{(1,1)}_{nd}$ is the space of vielbeins as nondegenerate one-forms with values in V. This formalism has the further advantage that all the fields are expressed as differential forms and hence can easily be restricted to a suitable submanifold of M (e.g. its boundary, if it has one).

Classical action

We are looking for an action functional that gives the same Euler–Lagrange locus modulo symmetries as Einstein–Hilbert theory. The Palatini–Cartan action is

$$S_{PC} := \int_M \frac{1}{(D-2)!} e^{D-2} \wedge F_\omega + \frac{\Lambda}{D!} e^D, \tag{2.1}$$

where $e^k := \underbrace{e \wedge e \wedge \cdots \wedge e}_{\text{k times}}$ and $F_\omega := d\omega + \frac{1}{2}[\omega, \omega]$ is the curvature associated to ω which we regard as a $(2,2)$ form. We can find equations of motion by varying the action

$$\begin{aligned} \delta S_{PC} &= \int_M \frac{1}{(D-3)!} e^{D-3} \delta e F_\omega - \frac{1}{(D-2)!} e^{D-2} d_\omega(\delta\omega) + \frac{\Lambda}{(D-1)!} e^{D-1} \delta e \\ &= \int_M \left[\frac{1}{(D-3)!} e^{D-3} F_\omega + \frac{\Lambda}{(D-1)!} e^{D-1} \right] \delta e + \frac{1}{(D-2)!} d_\omega(e^{D-2}) \delta\omega \\ &\quad - \frac{1}{(D-2)!} d(e^{D-2} \delta\omega), \end{aligned} \tag{2.2}$$

where we used integration by parts and the fact that $\delta_\omega F_\omega = -d_\omega(\delta\omega)$.[3] The last term in (2.2) will produce a boundary term if $\partial M \neq \emptyset$, due to Stokes theorem.

Then we find equations of motion

$$e^{D-3} d_\omega e = 0; \tag{2.3}$$

$$\frac{1}{(D-3)!} e^{D-3} F_\omega + \frac{\Lambda}{(D-1)!} e^{D-1} = 0. \tag{2.4}$$

Equation (2.3) is equivalent to $d_\omega e = 0$ because of the non-degeneracy condition (and because e^{D-3} is injective in this case [CCS21a]). Furthermore, it fixes ω to be torsionless, and since it is compatible with η, then $d_\omega e = 0$ implies the metricity condition $d_{e^*(\omega)} g = 0$, which is uniquely solved by the Levi-Civita metric connection.

After imposing (2.3), we find that (2.4) is equivalent to Einstein's field equation, with the addition of a cosmological constant Λ.

Remark 2.1. It is important to notice that, even if e is an isomorphism, $e \wedge \cdot$ might not be, indeed $e^{D-3} \wedge F_\omega = 0$ is not equivalent to the flatness condition $F_\omega = 0$

[3] $\delta\omega F_\omega = \delta_\omega(d\omega + \frac{1}{2}[\omega,\omega]) = -d\delta\omega + \frac{1}{2}[\delta\omega,\omega] - \frac{1}{2}[\omega,\delta\omega] = -d(\delta\omega) - [\omega,\delta\omega] = -d_\omega(\delta\omega)$.

Remark 2.2. There are two ways of showing that the PC and EH theories are equivalent. The first one is to rewrite equation (2.4) after imposing (2.3) and see that it actually yields Einstein's field equation. The other way is to use (2.3) and rewrite the action S_{PC} in terms of the metric tensor, to see that it is equivalent to the Einstein–Hilbert action. This is seen by noticing that

$$\frac{e^D}{D!} = \sqrt{-\det(g)}d^D x = \mathrm{Vol}_g, \qquad \frac{e^{D-2}}{(D-2)!}F_\omega = R\mathrm{Vol}_g, \tag{2.5}$$

where R is the Ricci scalar. In this case, the theory we obtain differs from the original Einstein–Hilbert theory just by the fact that the connection is a free field.[4] However, as already remarked, the EoM's admit the unique solution given by the Levi-Civita connection.

2.2 The reduced phase space of PC gravity and its BFV formulation

We present here the results of [CCS21a] concerning the structure of the reduced phase space of Palatini–Cartan gravity for $D \geq 4$, keeping in mind that we are particularly interested in the $D = 4$ case, which is the starting point for the $N = 1$ Supergravity. The results of this section have been obtained through the Kijowski–Tulcjiev (KT) construction and are the background construction that we will adapt when adding spinor fields in the following chapters.

The starting point of the KT analysis is the boundary term that we get when varying the action (2.2):

$$\widetilde{\alpha}_{PC} = \frac{1}{(D-2)!}\int_\Sigma e^{D-2}\delta\omega.$$

Assumption 2.1. *We further assume that the bulk vielbein satisfies the extra nondegeneracy condition that the induced boundary metric g^∂, defined by $g^\partial := \iota_\Sigma^* e^*(\eta)$, is nondegenerate.*[5] *This is an open condition on the space of bulk field that ensures that the constrained submanifold C_Σ is coisotropic.*

The classical fields on the boundary will again be indicated by (e, ω). The inclusion $\iota : \Sigma \hookrightarrow M$ of Σ in M induces the bundles $P|_\Sigma := \iota^*(P)$ and $\mathcal{V}|_\Sigma := \iota^*(\mathcal{V})$. The (pre-)boundary fields are respectively defined as

- e is a nondegenerate section of $T^*\Sigma \otimes \mathcal{V}|_\Sigma$, meaning that (i) at each point the three components are linearly independent and (ii) the underlying metric g, defined by $g := e^*(\eta)$, is nondegenerate (because of Assumption 2.1);
- ω is an element of the space of connections $\mathcal{A}(\Sigma)$, locally modeled by $\Gamma(T^*\Sigma \otimes \bigwedge^2 \mathcal{V}|_\Sigma)$.

We denote the space of preboundary fields as $\tilde{F}^\partial_{PC} = \Omega^{(1,1)}_{\partial,\mathrm{n.d.}} \times \mathcal{A}(\Sigma)$, having defined

$$\Omega^{(i,j)}_\partial := \Omega^i(\Sigma, \wedge^j \mathcal{V}|_\Sigma)$$

We note that $\tilde{\alpha}_{PC}$ is the integral of a local $(\mathrm{top}, 1)$ form on $\tilde{F}^\partial_{PC} \times \Sigma$ as defined in (1.1) and therefore a 1-form on $\tilde{F}^\partial_{PC}$. By taking its variation (the variational vertical differential), we obtain a two-form on $\tilde{F}^\partial_{PC}$

$$\tilde{\varpi}^\partial_{PC} := \delta\alpha = \frac{1}{(D-3)!}\int_\Sigma e^{D-3}\delta e\delta\omega. \tag{2.6}$$

[4] That is the Palatini formulation of EH gravity.

[5] One might also consider the stronger condition that the induced boundary metric is space-like, but this is not needed for the following considerations.

By construction, $\tilde{\varpi}^\partial_{PC}$ is closed on $\tilde{F}^\partial_{PC}$ and so satisfies the first requirement to be a symplectic form on $\tilde{F}^\partial_{PC}$. However, it is degenerate, namely $\ker(\tilde{\varpi}^\partial_{PC}) := \{X \in T\tilde{F}^\partial_{PC} \mid \iota_X \tilde{\varpi}^\partial_{PC} = 0\} \neq \{0\}$. In [CCS21a] it was proven that ther kernel is regular. Hence, in order to get rid of this degeneracy, we can perform a symplectic reduction.[6] The quotient space F^∂_{PC} will be called the geometric phase space of the theory

$$F^\partial_{PC} := \frac{\tilde{F}^\partial_{PC}}{\ker(\tilde{\varpi}^\partial_{PC})}, \tag{2.7}$$

with the canonical projection $\pi_\partial \colon \tilde{F}^\partial_{PC} \to F^\partial_{PC}$. Hence the space of boundary fields is a bundle $F^\partial \to \Omega^1_{nd}(\Sigma, \mathcal{V})$ with local trivialization on an open $\mathcal{U}_\Sigma \subset \Omega^1_{nd}(\Sigma, \mathcal{V})$

$$F^\partial \simeq \mathcal{U}_\Sigma \times \mathcal{A}^{red}(\Sigma),$$

where $\mathcal{A}^{red}(\Sigma)$ is the space of equivalence classes of connections $\omega \in \mathcal{A}(\Sigma)$ under the equivalence relation $\omega \sim \omega + v$ for every $v \in \Omega^{1,2}(\Sigma)$ such that $e^{D-3}v = 0$. The corresponding symplectic form is

$$\varpi^\partial_{PC} = \frac{1}{(D-3)!} \int_\Sigma e^{D-3} \delta e \delta[\omega]. \tag{2.8}$$

In order to define the constraints on this quotient space, and to give an explicit description of the reduced phase space, it is better to fix a representative of the equivalence relation described above, since the restriction of the EL equations to the boundary are not invariant under the equivalence relation. A convenient choice is given by the following construction.

Definition 2.1. We choose a nowhere vanishing section ϵ_n of $\mathcal{V}|_\Sigma$ and we restrict the space of fields by the conditions that $e_1, e_2, e_3, \epsilon_n$ form a basis, where $e_i := e(\partial_i)$.[7] Then, $F^{\epsilon_n}_{PC}$ is defined to be the space of pre-boundary fields $\tilde{F}^\partial_{PC}$ together with $\epsilon_n \in \mathcal{V}$.

On this space we have the following theorem:

Theorem 2.1 ([CCS21a])**.** *Suppose that g^∂, the metric induced on the boundary, is nondegenerate. Given any $\widetilde{\omega} \in \Omega^{1,2}$, there is a unique decomposition*

$$\widetilde{\omega} = \omega + v \tag{2.9}$$

with ω and v satisfying

$$e^{D-3} v = 0 \quad \text{and} \quad \epsilon_n e^{D-4} d_\omega e \in \operatorname{Im} W_1^{\partial,(1,1)}. \tag{2.10}$$

Let us denote by $F'^{\epsilon_n}_{PC}$ the subspace of $F^{\epsilon_n}_{PC}$ of the fields satisfying (2.10).

Corollary 2.1 ([CCS21a])**.** *$F'^{\epsilon_n}_{PC}$ is symplectomorphic to F^∂_{PC}.*

Hence from now on we will require (2.10) and work on $F'^{\epsilon_n}_{PC}$. The space of coframes and connections satisfying this last equation is the geometric phase space of the PC gravity theory.

[6] The vector fields in the kernel of the presymplectic form span a smooth involutive distribution. The quotient space $\tilde{F}^\partial_{PC}/\ker(\tilde{\varpi}^\partial_{PC})$ is the set of leaves in the foliation induced by $\ker(\tilde{\varpi}^\partial_{PC})$. In our case, the vector fields in the kernel only act, at fixed e, as translations of the connection ω, therefore it is easy to see that the quotient space is still a smooth manifold.

[7] There is actually no restriction in the space-like case; otherwise, one has to work on charts of the space of fields and pick an ϵ_n for each chart

Remark 2.3. One notices that, since the map $e^{D-3}\wedge\cdot\colon \Omega^{(1,2)} \to \Omega^{(D-2,D-1)}$ is an isomorphism,[8] then in the bulk $e^{D-3}d_\omega e = 0$ must give rise to the same solution space of $d_\omega e = 0$. However, when moving to the boundary, $e_\partial^{D-3}\wedge\cdot\colon \Omega_\partial^{(1,2)} \to \Omega_\partial^{(D-2,D-1)}$ is not injective anymore. The constraint (2.10), which we call structural constraint, not only allows to fix the representative of $[\omega] \in \mathcal{A}_{\text{red}}(\Sigma)$, but also ensures that the equivalence between $e^{D-3}d_\omega e = 0$ and $d_\omega e = 0$ is respected also on the boundary.

We can now analyze the restriction of the Euler–Lagrange equations on the boundary to see which further constraints they impose on the geometric phase space. In order to simplify the computation of their Hamiltonian vector fields, it is convenient to rewrite the constraints on F'_{ϵ_n} as discussed in [CCS21a]:

$$
\begin{aligned}
L_c &= \int_\Sigma c e^{D-3} d_\omega e, \\
P_\xi &= \int_\Sigma \iota_\xi e e^{D-3} F_\omega + \iota_\xi(\omega-\omega_0) e^{D-3} d_\omega e, \\
H_\lambda &= \int_\Sigma \lambda \epsilon_n \left(\frac{1}{(D-3)!} e^{D-3} F_\omega + \frac{1}{(D-1)!} \Lambda e^{D-1} \right),
\end{aligned}
$$

where ω_0 is a reference connection and $c \in \Omega_\partial^{0,2}$, $\xi \in \mathfrak{X}(\Sigma)$ and $\lambda \in \Omega_\partial^{0,0}$ are Lagrange multipliers.

Remark 2.4. To be precise, the exact form of P_ξ is originally given by the constraint

$$
P_\xi = \int_\Sigma \iota_\xi e e^{D-3} F_\omega,
$$

but in order to simplify the computation of the Hamiltonian vector field, we redefine it via the transformation $P_\xi \mapsto P_\xi + L_{\iota_\xi(\omega-\omega_0)}$, which does not change the zero–locus of the theory.

From now on we are going to consider the fields c, ξ and λ to be odd fields (shifted by 1 in a suitable supermanifold). This will be useful later for the BFV formalism. For more details we refer to [CCS21a].

The constraints above are of first class, hence defining a coisotropic submanifold of the geometric phase space. The structure is specified by the following

Theorem 2.2 ([CCS21a])**.** *Under Assumption 2.1, the functions L_c, P_ξ, H_λ define a coisotropic submanifold of F_{ϵ_n} with respect to the symplectic structure ϖ^∂_{PC}. In particular they satisfy the relations*

$$
\{L_c, L_c\} = -\frac{1}{2} L_{[c,c]} \qquad \{P_\xi, P_\xi\} = \tfrac{1}{2} P_{[\xi,\xi]} - \tfrac{1}{2} L_{\iota_\xi \iota_\xi F_{\omega_0}} \tag{2.11a}
$$

$$
\{L_c, P_\xi\} = L_{\mathrm{L}_\xi^{\omega_0} c} \quad \{L_c, H_\lambda\} = -P_{X^{(i)}} + L_{X^{(i)}(\omega-\omega_0)_a} - H_{X^{(n)}} \tag{2.11b}
$$

$$
\{H_\lambda, H_\lambda\} = 0 \qquad \{P_\xi, H_\lambda\} = P_{Y^{(i)}} - L_{Y^{(i)}(\omega-\omega_0)_a} + H_{Y^{(n)}} \tag{2.11c}
$$

where $X = [c, \lambda\epsilon_n]$, $Y = \mathrm{L}_\xi^{\omega_0}(\lambda\epsilon_n)$ and $Z^{(i)}$, $Z^{(n)}$ are the components of $Z \in \{X, Y\}$ with respect to the frame (e_i, ϵ_n).

Furthermore, the notation L_ξ^ω denotes the covariant Lie derivative along the odd vector field ξ with respect to a connection ω:

$$
\mathrm{L}_\xi^\omega A = \iota_\xi d_\omega A - d_\omega \iota_\xi A \qquad A \in \Omega_\partial^{i,j}.
$$

[8] See [Can24]

2.2.1 The BFV PC structure

The data of theorem 2.2 can be translated into the BFV formalism as explained in Section 1.2.1. The result is the following theorem.

Theorem 2.3 ([CCS21a]). *Under Assumption 2.1, let $\mathcal{F}_{PC}$ be the bundle*

$$\mathcal{F}_{\Sigma}^{PC} \longrightarrow \Omega^1_{nd}(\Sigma, \mathcal{V}), \tag{2.12}$$

with local trivialisation on an open $\mathcal{U}_\Sigma \subset \Omega^1_{nd}(\Sigma, \mathcal{V})$

$$\mathcal{F}_{\Sigma}^{PC} \simeq \mathcal{U}_\Sigma \times \mathcal{A}(\Sigma) \oplus T^*\left(\Omega_\partial^{0,2}[1] \oplus \mathfrak{X}[1](\Sigma) \oplus C^\infty[1](\Sigma)\right) =: \mathcal{U}_\Sigma \times \mathcal{T}_{PC}, \tag{2.13}$$

and fields denoted by $e \in \mathcal{U}_\Sigma$ and $\omega \in \mathcal{A}(\Sigma)$ in degree zero such that they satisfy the structural constraint $\epsilon_n e^{D-4} d_\omega e \in \operatorname{Im} W_1^{\partial,(1,1)}$, ghost fields $c \in \Omega_\partial^{0,2}[1]$, $\xi \in \mathfrak{X}[1](\Sigma)$ and $\lambda \in \Omega^{0,0}[1]$ in degree one, $k^\daleth \in \Omega_\partial^{D-1,D-2}[-1]$, $\lambda^\daleth \in \Omega_\partial^{D-1,D}[-1]$ and $\zeta^\daleth \in \Omega_\partial^{1,0}[-1] \otimes \Omega_\partial^{D-1,D}$ in degree minus one, together with a fixed $\epsilon_n \in \Gamma(\mathcal{V})$, completing the image of elements $e \in \mathcal{U}_\Sigma$ to a basis of $\mathcal{V}$; define a symplectic form and an action functional on $\mathcal{F}_{PC}^{\Sigma}$ respectively by

$$\begin{aligned}
\varpi_{PC}^{\Sigma} &= \int_\Sigma \frac{1}{(D-3)!} e^{D-3} \delta e \delta \omega + \delta c \delta k^\daleth + \delta \lambda \delta \lambda^\daleth + \iota_{\delta \xi} \delta \zeta^\daleth, \\
\mathcal{S}_\Sigma^{PC} &= \int_\Sigma \frac{1}{(D-3)!} c e^{D-3} d_\omega e + \frac{1}{(D-3)!} \iota_\xi e e^{D-3} F_\omega + \frac{1}{(D-3)!} \iota_\xi (\omega - \omega_0) e^{D-3} d_\omega e \\
&\quad + \lambda \epsilon_n \left(\frac{1}{(D-3)!} e^{D-3} F_\omega + \frac{1}{(D-1)!} \Lambda e^{D-1} \right) + \frac{1}{2}[c,c] k^\daleth \\
&\quad - \mathrm{L}_\xi^{\omega_0} c k^\daleth + \frac{1}{2} \iota_\xi \iota_\xi F_{\omega_0} k^\daleth + [c, \lambda \epsilon_n]^{(i)} (\zeta_i^\daleth - (\omega - \omega_0)_i k^\daleth) + [c, \lambda \epsilon_n]^{(n)} \lambda^\daleth \\
&\quad - \mathrm{L}_\xi^{\omega_0} (\lambda \epsilon_n)^{(i)} (\zeta_i^\daleth - (\omega - \omega_0)_i k^\daleth) - \mathrm{L}_\xi^{\omega_0} (\lambda \epsilon_n)^{(n)} \lambda^\daleth - \frac{1}{2} \iota_{[\xi,\xi]} \zeta^\daleth.
\end{aligned}$$

Then the triple $(\mathcal{F}_{PC}^{\Sigma}, \varpi_\Sigma^{PC}, \mathcal{S}_\Sigma^{PC})$ defines a BFV structure on Σ.

Following [CCS21a] we can change variables to get rid of the redundancies introduced in remark 2.4 in $D = 4$. In particular, we define

$$c' = c + \iota_\xi(\omega - \omega_0) \qquad \zeta_\bullet^{\daleth'} = \zeta_\bullet^\daleth - (\omega - \omega_0)_\bullet k^\daleth, \tag{2.14}$$

where the bullet $\bullet$ indicates any component in the one-form factor in $\Omega_\partial^{1,0}[-1] \otimes \Omega_\partial^{D-1,D}$. In other words, $\zeta_\bullet^{\daleth'} \in \Omega_\partial^{D-1,D}$, with $\bullet = 1,2,3$.

Lastly, we can define the new variable $y^\daleth \in \Omega_\partial^{(3,3)}[-1]$ such that $e_i y^\daleth = \zeta_i^{\daleth'}$ and $\epsilon_n y^\daleth = \lambda^\daleth$, which, omitting the $'$ apex, yields

$$\varpi_{PC}^{\Sigma} = \int_\Sigma e \delta e \delta \omega + \delta c \delta k^\daleth + \delta \omega \delta(\iota_\xi k^\daleth) - \delta \lambda \epsilon_n \delta y^\daleth + \iota_{\delta \xi} \delta(e y^\daleth), \tag{2.15}$$

$$\begin{aligned}
\mathcal{S}_\Sigma^{PC} &= \int_\Sigma c e d_\omega e + \iota_\xi e e F_\omega + \lambda \epsilon_n \left(e F_\omega + \frac{1}{3!} \Lambda e^3 \right) + \frac{1}{2}[c,c] k^\daleth - \mathrm{L}_\xi^\omega c k^\daleth \\
&\quad + \frac{1}{2} \iota_\xi \iota_\xi F_\omega k^\daleth - [c, \lambda \epsilon_n] y^\daleth + \mathrm{L}_\xi^{\omega_0} (\lambda \epsilon_n) y^\daleth + \frac{1}{2} \iota_{[\xi,\xi]} \zeta^\daleth.
\end{aligned} \tag{2.16}$$

2.3 The degenerate boundary case in $D = 4$

So for we have been assuming the induced boundary metric g^∂ to be non–degenerate, which is an open condition on the set of vielbeins on the boundary. This section is dedicated to investigating the boundary structure of PC gravity in the case of a degenerate boundary metric.

The problem of fixing the representative of the connection in the pre-symplecitc form $\tilde{\varpi}^\partial_{PC}$ is present in both the non-degenerate and degenerate cases; however, the form of the structural constraint strictly depends on the nature of the boundary (null or non-null). In fact, in the non-degenerate case, the structural constraint (2.10) alone is sufficient to fix the representative of $[\omega] \in \mathbb{A}_{\text{red}}(\Sigma)$. On the other hand, on a null boundary, due to the degeneracy of e we also have to impose an additional degeneracy constraint. We will see that, from a different perspective, the structural constraint of the non-degenerate case is just a specific characterization of the structural and the degeneracy constraint where the latter is trivial.

Definition 2.2. Let $e \in \Omega^{1,1}_\partial$ and $e^k \in \Omega^{k,k}_\partial$ be the wedge product of k elements e. Then, we define the following maps:

$$\begin{aligned} W_k^{\Sigma,(i,j)} \colon \Omega^{i,j}_\partial &\longrightarrow \Omega^{i+k,j+k}_\partial \\ \alpha &\longmapsto e^k \wedge \alpha \end{aligned}$$

$$\begin{aligned} \varrho^{(i,j)} \colon \Omega^{i,j}_\partial &\longrightarrow \Omega^{i+1,j-1}_\partial \\ \alpha &\longmapsto [e, \alpha] \end{aligned} \tag{2.17}$$

$$\begin{aligned} \tilde{\varrho}^{(i,j)} \colon \Omega^{i,j}_\partial &\longrightarrow \Omega^{i+1,j-1}_\partial \\ \alpha &\longmapsto [\tilde{e}, \alpha], \end{aligned} \tag{2.18}$$

with $\tilde{e} \in \tilde{\Omega}^{1,1}_\partial$ being a degenerate vielbein, namely $\tilde{e}^*\eta = 0$.

We also give the definitions of three geometrical objects that we will require in the following theorems.

Definition 2.3. Let J be a complement[9] in $\Omega^{2,1}_\partial$ of the space $\text{Im}\, \varrho^{(1,2)}|_{\text{Ker} W_1^{\Sigma,(1,2)}}$. Then, we define the following subspaces:

$$\mathcal{T} := \text{Ker} W_1^{\Sigma(2,1)} \cap J \subset \Omega^{2,1}_\partial \tag{2.19}$$

$$\mathcal{S} := \text{Ker} W_1^{\Sigma,(1,3)} \cap \text{Ker} \tilde{\varrho}^{(1,3)} \subset \Omega^{1,3}_\Sigma \tag{2.20}$$

$$\mathcal{K} := \text{Ker} W_1^{\Sigma,(1,2)} \cap \text{Ker} \varrho^{(1,2)} \subset \Omega^{1,2}_\partial. \tag{2.21}$$

We present the initial key result for the degenerate theory, which will ensure the equivalence between $d_\omega e = 0$ and $e d_\omega e = 0$ at the boundary. While it may appear initially quite redundant with respect to lemma A.9, it will have profound implications for the geometry of the theory, as highlighted in 2.6.

[9]To obtain an explicit expression for the complement, one can follow these steps. Start by selecting an arbitrary Riemannian metric on the boundary manifold Σ and extend it to the space $\Omega^{2,1}$. Then, the orthogonal complement of the image of the map $\varrho^{(1,2)}|\text{Ker} W_1^{\Sigma,(1,2)}$ in $\Omega^{2,1}_\partial$ can be identified as the space J, with respect to the chosen Riemannian metric.

Lemma 2.1 (Corollary of A.9). *Let $\epsilon_n \in \Omega_\partial^{0,1}$ be fixed such that, for a chosen vielbein $e \in \tilde{\Omega}_\partial^{1,1}$, $\{e(v_1), e(v_2), e(v_3), \epsilon_n\}$*[10] *is a basis of $i^*\mathcal{V}$, where $\{v_1, v_2, v_3\}$ is a basis of $T\Sigma$. Moreover, let $\alpha \in \Omega_\partial^{2,1}$. Then, we have that*

$$\alpha = 0$$

if and only if

$$\begin{cases} \alpha \in \mathrm{Ker} W_1^{\Sigma,(2,1)} \\ \epsilon_n(\alpha - p_\mathcal{T}\alpha) \in \mathrm{Im}\, W_1^{\Sigma,(1,1)} \\ p_\mathcal{T}\alpha = 0, \end{cases} \tag{2.22}$$

where $p_\mathcal{T}$ is the projector onto $\mathcal{T}$. We call the second and third conditions in (2.22) *respectively the structural and the degeneracy constraints.*

The next lemma provides a formulation of the degeneracy constraint in terms of an integral functional.

Lemma 2.2 ([CCT21]). *Let $\alpha \in \Omega_\partial^{2,1}$. Then, we have the following equivalence*

$$p_\mathcal{T}\alpha = 0 \quad \Longleftrightarrow \quad \int_\Sigma \tau\alpha = 0 \quad \forall \tau \in \mathcal{S}. \tag{2.23}$$

Remark 2.5. As long as we do not specify any α, these two lemmas remain purely geometrical and do not depend on the properties of the field equations. We will then be able to use these results for the interactive theories where the equivalence condition on the boundary will differ from $d_\omega e = 0$ and $ed_\omega e = 0$ (since the field equations will be different themselves). Therefore, in general, we need to specify α for each different theory. In particular, for the Palatini–Cartan theory, $\alpha = d_\omega e$ and the structural and the degeneracy constraints read

$$\begin{cases} \epsilon_n(d_\omega e - p_\mathcal{T} d_\omega e) \in \mathrm{Im}\, W_1^{\Sigma,(1,1)} \\ p_\mathcal{T} d_\omega e = 0. \end{cases} \tag{2.24}$$

Remark 2.6. It is important to emphasize that Eq.s (2.22) are trivially equivalent to the structural constraint

$$\epsilon_n\alpha \in \mathrm{Im}\, W_1^{\Sigma,(1,1)} \tag{2.25}$$

in the non-degenerate case. Nonetheless, the introduction of this split plays a crucial role in the analysis of the degenerate theory. More specifically, apart from $p_\mathcal{T}$ not being trivial, 2.25 alone will not be sufficient to uniquely fix a representative of the equivalence class defining the symplectic space (see 2.4). In other words, since in the non-degenerate case $p_\mathcal{T}\alpha = 0$ holds trivially, we can infer that the second equation in (2.22) is the most general form of the structural constraint of the theory, whose geometrical implications are only visible in the degenerate case. In fact, the peculiar integral condition of the degenerate case, introduced in 2.2, carries significant consequences. It can be interpreted as a modification of the set of constraints of the theory by incorporating a new functional constraint. For $\alpha = d_\omega e$ (the case of the Palatini–Cartan theory), this is denoted as

$$R_\tau = \int_\Sigma \tau d_\omega e. \tag{2.26}$$

Further discussions of this matter will be presented in the next section.

[10] Notice in particular that, in any neighborhood of e of the space of boundary fields, we are allowed to pick ϵ_n independently of the dynamics of the vielbein e. In other words, we can state that ϵ_n is constant in the field e. This trivially implies that ϵ_n has no variation along e.

Fixing the representative

The reduction by the kernel of the presymplectic form, as shown in [CS19b], is equivalent to a quotient space with an equivalence relation on the connection form, as stated in the following theorem.

Theorem 2.4 ([CS19b])**.** *The geometric phase space for the Palatini–Cartan theory is the symplectic manifold $(\mathcal{F}_\Sigma, \varpi)$ given by the following equivalence relation on the space of pre-boundary fields $\tilde{\mathcal{F}}_\Sigma$*

$$\omega' \sim \omega \quad \Longleftrightarrow \quad \omega' - \omega \in \mathrm{Ker} W_1^{\Sigma,(1,2)} \tag{2.27}$$

and the symplectic form

$$\varpi = \int_\Sigma e\delta e\delta[\omega]. \tag{2.28}$$

We refer to this equivalence class as $\mathcal{A}(\Sigma)_{\mathrm{red}}$.

Remark 2.7. To study the reduced phase space of the theory, we make use of representatives for the equivalence classes defined in (2.27). In the non-degenerate case, these representatives are uniquely determined by the structural constraint itself. In other words, ensuring the equivalence of $d_\omega e = 0$ and $ed_\omega e = 0$ on the boundary, is enough to determine uniquely the representatives of the equivalence classes defined in (2.27). However, in the degenerate case, the structural constraint and the degenerate constraint (or its integral form R_τ), despite the fact that they indeed ensure on the boundary the equivalence mentioned above, are not sufficient to uniquely assign a representative to each equivalence class. Therefore, it is necessary to seek an alternative way to guarantee the unambiguous determination of these representatives.

We can accomplish that through the following lemma.

Lemma 2.3 ([CCT21])**.** *Let g^∂ be degenerate. Then, given $\omega \in \Omega_\partial^{1,2}$ and $\epsilon_n \in \Omega_\partial^{0,1}$ as in 2.1, the conditions*

$$\begin{cases} \epsilon_n(d_\omega e - p_{\mathcal{T}}(d_\omega e)) \in \mathrm{Im}\, W_1^{\Sigma,(1,1)} \\ p_{\mathcal{K}}\omega = 0 \end{cases} \tag{2.29}$$

uniquely define a representative of the equivalence class $[\omega] \in \mathcal{A}(\Sigma)_{\mathrm{red}}$.

Remark 2.8. In [CCT21], it has been proved that the analysis is independent of the choice of the representative of the equivalence class (2.27). In more rigorous terms, for each choice of the representatives there is a canonical symplectomorphism between the symplectic space defined by representatives and the geometric phase space of the theory.

Remark 2.9. It is important to highlight that, in the non-degenerate case, the subspaces $\mathcal{T}$, $\mathcal{S}$, and $\mathcal{K}$ of definition 2.3 are trivial. It follows that the projectors $p_{\mathcal{K}}$ and $p_{\mathcal{T}}$ are also trivial. Once again, this means that, in the non-degenerate theory, the structural constraint alone serves the purpose of establishing the equivalence between $d_\omega e = 0$ and $ed_\omega e = 0$ on the boundary, as well as uniquely determining the representatives of the equivalence classes defined in (2.27).

We have seen that, on a null-boundary, we need both the structural and the degeneracy constraints together with the additional equation $p_{\mathcal{K}}\omega = 0$ in order to guarantee the equivalence between $d_\omega e = 0$ and $ed_\omega e = 0$ on the boundary and uniquely fix the representative of the equivalence class $[\omega] \in \mathcal{A}(\Sigma)_{\mathrm{red}}$.
More specifically, the role of the structural constraint together with the integral constraint R_τ is

the one of ensuring the aforementioned equivalence condition, whereas, the structural constraint together with $p_{\mathcal{K}}\omega = 0$ will uniquely fix the representatives.

We display now the constraints of the theory.

Definition 2.4. Let[11] $c \in \Omega_\partial^{0,2}[1]$, $\xi \in \mathfrak{X}(\Sigma)[1]$, $\lambda \in C^\infty(\Sigma)[1]$ and $\tau \in \mathcal{S}[1]$. Then, we define the following functionals

$$L_c = \int_\Sigma c e d_\omega e \tag{2.30}$$

$$P_\xi = \int_\Sigma \frac{1}{2}\iota_\xi(e^2)F_\omega + \iota_\xi(\omega - \omega_0) e d_\omega e \tag{2.31}$$

$$H_\lambda = \int_\Sigma \lambda\epsilon_n \Big(eF_\omega + \frac{\Lambda}{3!}e^3\Big) \tag{2.32}$$

$$R_\tau = \int_\Sigma \tau d_\omega e. \tag{2.33}$$

We refer to these as the constraints of the Palatini–Cartan (degenerate) theory.

We are now able to determine the algebra of the constraints of the theory. This differs from the one of the non-degenerate theory, since the new constraint R_τ changes the nature of the Poisson brackets, which are no longer first-class.

Theorem 2.5 ([CCT21])**.** *Let i^*g be degenerate. Then the structure of the Poisson brackets of the constraints L_c, P_ξ, H_λ and R_τ is given by the following expressions*

$$\begin{aligned}
&\{L_c, L_c\} = -\frac{1}{2}L_{[c,c]} && \{P_\xi, P_\xi\} = \frac{1}{2}P_{[\xi,\xi]} - \frac{1}{2}L_{\iota_\xi\iota_\xi F_{\omega_0}}\\
&\{L_c, P_\xi\} = L_{L_\xi^{\omega_0}c} && \{H_\lambda, H_\lambda\} \approx 0\\
&\{L_c, R_\tau\} = -R_{p_{\mathcal{S}}[c,\tau]} && \{P_\xi, R_\tau\} = R_{p_{\mathcal{S}}L_\xi^{\omega_0}\tau}.\\
&\{R_\tau, H_\lambda\} \approx G_{\lambda\tau} && \{R_\tau, R_\tau\} \approx F_{\tau\tau}\\
&\{L_c, H_\lambda\} = -P_{X^{(i)}} + L_{X^{(i)}(\omega-\omega_0)_a} - H_{X^{(n)}}\\
&\{P_\xi, H_\lambda\} = P_{Y^{(i)}} - L_{Y^{(i)}(\omega-\omega_0)_a} + H_{Y^{(n)}}
\end{aligned}$$

with $X = [c, \lambda\epsilon_n]$ and $Y = L_\xi^{\omega_0}(\lambda\epsilon_n)$ and where the superscripts (i) and (n) describe their components with respect to e_a, ϵ_n. Furthermore $F_{\tau\tau}$ and $G_{\lambda\tau}$ are functionals of e, ω, τ and λ that are not proportional to any other constraint.

Remark 2.10. The symbol $\approx$ indicates the identity on the zero locus of the constraints. In particular, this means that those brackets written with this symbol are not a linear combination of the constraints themselves. On the other hand, all the brackets written with an $=$ vanish on the zero locus, for example $\{L_c, L_c\} \approx 0$.

Remark 2.11. The distinctive feature of the degenerate theory, highlighted in [CCT21], is that the additional constraint R_τ turns out to be second-class, for τ not constant.

2.4 The BV PC theory

The previous sections gave us insight on the action of the infinitesimal gauge symmetries, seen as the Hamiltonian vector fields associated to the constraints on the geometric phase space. In

[11] The notation [1] indicates a shift in parity.

this section we implement them as components of the cohomological vector field Q_{PC}, which, despite the fact that the diffeomorphism symmetry cannot be canonically realized as the action of a proper Lie algebra of a (finite dimensional) Lie group, turns out to be of BRST type. This formulation was first found in [Pig00] and later refined within the BV formalism in [CS19b].

The classical infinitesimal symmetries correspond to the internal Lorentz gauge symmetry, whose gauge parameter is given by the ghost $c \in \Omega^{(0,2)}[1]$, and the diffeomorphism symmetry, whose gauge parameter is $\xi \in \mathfrak{X}[1](M)$. Explicitly, one can infer

$$\delta_\xi e = \mathrm{L}_\xi^\omega e \qquad \delta_\xi \omega = \iota_\xi F_\omega$$
$$\delta_c e = [c, e] \qquad \delta_c \omega = d_\omega c.$$

Theorem 2.6 ([CS19b]). *The collection $(\mathcal{F}_{PC}^M, \varpi_{PC}^M, Q_{PC}, \mathcal{S}_{PC}^M)$ defines a BV structure, where $\mathcal{F}_{PC}^M := T^*[-1]F_{PC}^M$ and*

$$F_{PC}^M = \Omega_{\text{n.d.}}^{(1,1)} \oplus \mathcal{A}_M \oplus \Omega^{(0,2)}[1] \oplus \mathfrak{X}[1](M) \ni (e, \omega, c, \xi).$$

The symplectic form is canonically defined as

$$\varpi_{PC}^M = \int_M \delta e \delta e^\dagger + \delta\omega\delta\omega^\dagger + \delta c \delta c^\dagger + \iota_{\delta\xi}\delta\xi^\dagger,$$

while the BV action reads

$$\mathcal{S}_{PC}^M = \int_M \frac{e^2}{2} F_\omega - (L_\xi^\omega e - [c, e])e^\dagger + (\iota_\xi F_\omega - d_\omega c)\omega^\dagger + \frac{1}{2}(\iota_\xi\iota_\xi F_\omega - [c, c])c^\dagger + \frac{1}{2}\iota_{[\xi,\xi]}\xi^\dagger.$$

Lastly, one easily recovers the cohomological vector field as the Hamiltonian vector field of $\mathcal{S}_{PC}^M$

$$Q_{PC} e = L_\xi^\omega e - [c, e] \qquad Q_{PC}\omega = \iota_\xi F_\omega - d_\omega c$$
$$Q_{PC} c = \frac{1}{2}(\iota_\xi\iota_\xi F_\omega - [c.c]) \qquad Q_{PC}\xi = \frac{1}{2}[\xi, \xi]$$
$$Q_{PC} e^\dagger = eF_\omega + \mathrm{L}_\xi^\omega e^\dagger - [c, e^\dagger]$$
$$Q_{PC}\omega^\dagger = e d_\omega e - d_\omega \iota_\xi \omega^\dagger - [c, \omega^\dagger] + \iota_\xi[e, e^\dagger] - \frac{1}{2} d_\omega \iota_\xi \iota_\xi c^\dagger$$
$$Q_{PC} c^\dagger = -d_\omega \omega^\dagger - [e, e^\dagger] - [c, c^\dagger]$$
$$Q_{PC}\xi_\bullet^\dagger = (F_\omega)_\bullet \omega^\dagger - d_{\omega_\bullet} e e^\dagger + \iota_\xi (F_\omega)_\bullet c^\dagger + \mathrm{L}_\xi^\omega(\xi_\bullet^\dagger) + (d_\omega \iota_\xi \xi^\dagger)_\bullet$$

2.4.1 Obstruction to the BV-BFV extension of the PC theory

After obtaining the BV description of PC gravity in $D \geq 4$, we could be tempted to apply the construction in 1.2 with the hope to recover the BFV theory obtained from the KT construction. However, it was proved in [CS19b] that the classical BV PC theory is not extendible to the boundary. In particular, in $D = 4$ the induced exact two-form on the boundary is given by

$$\check{\varpi}_\Sigma^{PC} = \int_\Sigma - e\delta e \delta\omega + \delta(e_n^\dagger \xi^n)\delta e + \delta e^\dagger \delta(e_n \xi^n) + \delta e^\dagger \iota_{\delta\xi} e + e^\dagger \iota_{\delta\xi}\delta e$$
$$\delta(\iota_\xi \omega^\dagger)\delta\omega + \delta(\omega_n^\dagger \xi^n)\delta\omega + \delta\omega^\dagger \delta c - \delta(\iota_\xi c_n^\dagger \xi^n)\delta\omega$$
$$\delta(\xi^n \iota_{\delta\xi}\chi)\mathrm{vol}_{g^\partial} - \delta\xi^n \delta(\xi^n \chi)\mathrm{vol}_{g^\partial}$$

The kernel of $\check{\varpi}_{\Sigma}^{PC}$ is given by the following set of equations

$$\begin{aligned} e\mathbb{X}_e &= \iota_{\mathbb{X}_\xi}\omega^\dashv - \mathbb{X}_{\xi^n}\omega_n^\dashv + \iota_\xi \mathbb{X}_{\omega^\dashv} + \mathbb{X}_{\omega_n^\dashv}\xi^n - \mathbb{X}_{\iota_\xi c_n^\dashv \xi^n} \\ e\mathbb{X}_\omega &= \iota_{\mathbb{X}_\xi} e^\dashv + \mathbb{X}_{\xi^n} e_n^\dashv + \mathbb{X}_{e_n^\dashv}\xi^n, \end{aligned} \tag{2.34}$$

where in general a vector field $\mathbb{X} \in \mathfrak{X}(\check{\mathcal{F}}_{PC}^{\Sigma})$ is given by $\mathbb{X} = \int_\Sigma \mathbb{X}_\alpha \frac{\delta}{\delta \Phi^\alpha} + \mathbb{X}^\alpha \frac{\delta}{\delta \Phi_\alpha^\dashv}$, where $(\Phi, \Phi^\dashv)$ correspond to the field/antifield pair.[12]

The main problem with equation (2.34) is that it is singular, hence the symplectic structure on the boundary fields is not well defined, since the corresponding quotient is not smooth.

2.5 The full BV-BFV description of PC gravity

In order to tackle the problem of finding a BV-BFV extendible PC theory in $D = 4$, we consider the particular case of a cylindrical space–time $M = I \times \Sigma$, where $I = [0, 1]$. In the following, we will see how to achieve this with two different constructions: the AKSZ construction and the BV–pushforward.

Remark 2.12. It was proved in [CS19a] that, in $D = 3$, Palatini–Cartan gravity is BV-BFV extendible, and that it is BV–equivalent to BF theory, a topological field theory. In dimension 4, this is spoiled by an extra e factor in the action[13], which generates a non–trivial kernel in the boundary 1–form.

To solve this issue, we employ two seemingly unrelated strategies, which turn out to provide the same result.

2.5.1 PC BV-BFV from the AKSZ construction

We denote by $\mathfrak{F}_{PC}^{\Sigma} = (\mathcal{F}_{PC}^{\Sigma}, \mathcal{S}_{PC}^{\Sigma}, \varpi_{PC}^{\Sigma})$ the BFV theory of Palatini–Cartan gravity developed in the case of non–degenerate boundary in section 2.2.1. We employ the construction in 1.4.1. We promote the fields in $\mathcal{F}_{PC}^{\Sigma}$ to fields in $\mathcal{F}_{PC}^{AKSZ}$ by considering [CCS21b]

$$\begin{aligned} \mathfrak{e} &= e + f^\dashv & \mathfrak{w} &= \omega + u^\dashv \\ \mathfrak{c} &= c + w & \mathfrak{z} &= \xi + z \\ \mathfrak{l} &= \lambda + \mu & \mathfrak{c}^\dashv &= k^\dashv + c^\dashv \\ \mathfrak{y}^\dashv &= e^\dashv + y^\dashv \end{aligned} \tag{2.35}$$

where we used the same letters for the boundary fields which are now promoted to fields in $\mathcal{C}^\infty(I) \otimes \mathcal{F}_{PC}^{\Sigma}$. In particular, if $\phi \in \mathcal{F}_{PC}^{\Sigma}$, the corresponding AKSZ field becomes

$$\mathfrak{P} = \phi + \psi^\dashv, \qquad \text{where} \qquad \phi \in \mathcal{C}^\infty(I) \otimes \mathcal{F}_{PC}^{\Sigma}, \qquad \text{and} \qquad \psi^\dashv \in \Omega^1[-1](I) \otimes \mathcal{F}_{PC}^{\Sigma}$$

Theorem 2.7 ([CCS21b]). *The AKSZ data $\mathfrak{F}_{PC}^{AKSZ}$ on $M = I \times \Sigma$ are given by*

$$\mathcal{F}_{PC}^{AKSZ} = T^*[-1](\mathrm{Map}(I, \mathcal{F}_{PC}^{\Sigma}),$$

$$\varpi_{PC}^{AKSZ} = \int_{I\times\Sigma} \mathfrak{e}\delta\mathfrak{e}\delta\mathfrak{w} + \delta\mathfrak{c}\delta\mathfrak{c}^\dashv + \delta\mathfrak{w}\delta(\iota_{\mathfrak{z}}\mathfrak{c}^\dashv) - \delta\mathfrak{l}\epsilon_n\delta\mathfrak{y}^\dashv + \iota_{\delta\mathfrak{z}}\delta(\mathfrak{e}\mathfrak{y}^\dashv),$$

[12]In our specific case, we have that $\mathbb{X}_e$ is the component of $\mathbb{X}$ along $\frac{\delta}{\delta e}$, and so on.

[13]Specifically, in the term $\frac{e^2}{2}F_\omega$.

$$\begin{aligned}\mathcal{S}_{PC}^{AKSZ} = \int_{I\times\Sigma} &\mathfrak{e}^2 d_i \mathfrak{w} + \mathfrak{c} d_I \mathfrak{c}^{\dashv} + d_I \mathfrak{w} \iota_{\mathfrak{z}} \mathfrak{c}^{\dashv} - \iota_{d_I \mathfrak{z}} \mathfrak{e}\mathfrak{y}^{\dashv} + d_I \mathfrak{l} \epsilon_n \mathfrak{y}^{\dashv} \\ &+ \mathfrak{c}\mathfrak{e} d_{\mathfrak{w}} \mathfrak{e} + \iota_{\mathfrak{z}} \mathfrak{e}\mathfrak{e} F_{\mathfrak{w}} + \epsilon_n \mathfrak{l}\mathfrak{e} F_{\mathfrak{w}} + \frac{1}{2}[\mathfrak{c}, \mathfrak{c}]\mathfrak{c}^{\dashv} - \mathrm{L}_{\mathfrak{z}}^{\mathfrak{w}} \mathfrak{c}\mathfrak{c}^{\dashv} + \frac{1}{2}\iota_{\mathfrak{z}}\iota_{\mathfrak{z}} F_{\mathfrak{w}} \mathfrak{c}^{\dashv} \\ &- [\mathfrak{c}, \epsilon_n \mathfrak{l}]\mathfrak{y}^{\dashv} + \mathrm{L}_{\mathfrak{z}}^{\mathfrak{w}}(\epsilon_n \mathfrak{l})\mathfrak{y} + \frac{1}{2}\iota_{[\mathfrak{z},\mathfrak{z}]}(\mathfrak{e}\mathfrak{y}^{\dashv}),\end{aligned}$$

where it is understood that only the terms containing fields in $\Omega^1[-1](I)$ should be selected in the above expressions, to obtain a top form on $I \times \Sigma$.

The above theorem provides a compatible BV-BFV theory of gravity in the first order formalism. In the next section, following [CCS21b], we see how it relates to the full BV theory in the bulk

The reduced PC BV theory from AKSZ

The question arises on how to relate the AKSZ PC theory $\mathfrak{F}_{PC}^{AKSZ}$ with the full BV PC theory in the bulk $\mathfrak{F}_{PC}$. The main obvious difference is that the former is BV-BFV extendible, while the latter is not, and the reason for that is that in $\mathfrak{F}_{PC}^{AKSZ}$ the connection is constrained by the structural constraint (4.11) of the BFV PC theory $\mathfrak{F}_{PC}^{\partial}$. Such constraint restricts the space AKSZ of fields in the bulk in such a way that the induced boundary symplectic form is non-degenerate. This motivates the following theorem

Theorem 2.8 ([CCS21b]). *There exists a map $\Phi\colon \mathfrak{F}_{PC}^{AKSZ} \to \mathfrak{F}_{PC}$ such that $\Phi\colon \mathcal{F}_{PC}^{AKSZ} \to \mathcal{F}_{PC}$ is an embedding and $\Phi^*(\varpi_{PC}^{M}) = \varpi_{PC}^{AKSZ}$. Such map is called a BV embedding.*

Remark 2.13. The map Φ actually splits as the composition of two maps. In particular, one can first define a restricted space of BV fields $\mathcal{F}_{PC}^{r}$, in which the connection and its antifield are constrained, and establish a symplectomorphism φ between $\mathcal{F}_{PC}^{AKSZ}$ and $\mathcal{F}_{PC}^{r}$. Then one can simply embed the restricted BV PC theory $\mathfrak{F}_{PC}^{r} \hookrightarrow \mathfrak{F}_{PC}$ via the BV inclusion $\iota_r\colon \mathcal{F}_{PC}^{r} \hookrightarrow \mathcal{F}_{PC}$ and obtain $\Phi = \iota_r \circ \varphi$

Remark 2.14. In principle, one would need to show that $\mathfrak{F}_{PC}^{r}$ defines a BV theory, however to find a symplectomorphism φ between $\mathcal{F}_{PC}^{AKSZ}$ and $\mathcal{F}_{PC}^{r}$ such that $\varphi^*(\mathcal{S}_{PC}^{r}) = \mathcal{S}_{PC}^{AKSZ}$ is enough, as the CME will automatically be satisfied.

From now and for the remainder of the chapter, we indicate any bulk field ϕ with the bold character $\boldsymbol{\phi}$. Furthermore, letting $\boldsymbol{\phi} \in \Omega^k(I \times \Sigma)$, we set

$$\boldsymbol{\phi} = \tilde{\phi} + \underline{\tilde{\phi}_n}, \qquad \text{with} \quad \tilde{\phi} \in \Omega^k(\Sigma) \otimes \mathcal{C}^\infty(I), \quad \underline{\tilde{\phi}_n} \in \Omega^{k-1}(\Sigma) \otimes \Omega^1(I),$$

assuming x^n to be the coordinate along I, then $\underline{\tilde{\phi}_n} = \tilde{\phi}_n dx^n$, with $\tilde{\phi}_n \in \mathcal{C}^\infty(I) \otimes \Omega^{k-1}(\Sigma)$. In the same way a vector field $\boldsymbol{\zeta} \in \mathfrak{X}(I \times \Sigma)$ is going to be split as

$$\boldsymbol{\zeta} = \tilde{\zeta} + \overline{\overline{\tilde{\zeta}}}^{n}, \qquad \text{with} \quad \tilde{\zeta} \in \mathfrak{X}(\Sigma) \otimes \mathcal{C}^\infty(I), \quad \overline{\overline{\tilde{\zeta}}}^{n} \in \mathcal{C}^\infty(\Sigma) \otimes \mathfrak{X}(I),$$

with $\overline{\overline{\tilde{\zeta}}}^{n} = \tilde{\zeta}^n \partial_n$.

Definition 2.5. The restricted space of BV fields is given by the subspace of $\mathcal{F}_{PC}$ satisfying the following structural constraints

$$\underline{\mathfrak{W}}^{\dashv} := \underline{\tilde{\omega}}_n^{\dashv} - \iota_{\underline{z}} \tilde{\omega}^{\dashv} - \iota_{\tilde{\xi}} \underline{\tilde{c}}_n^{\dashv} + \iota_{\underline{z}} \tilde{c}_n^{\dashv} \tilde{\xi}^n \in \mathrm{Im}(W_{\tilde{e}}^{(1,1)}) \tag{2.36}$$

$$\epsilon_n d_{\tilde{\omega}} \tilde{e} - \epsilon_n W_{\tilde{e}}^{-1}(\underline{\mathfrak{W}}) d\tilde{\xi}^n + \iota_{\hat{X}}(\tilde{\omega}_n^{\dashv} - \tilde{c}_n^{\dashv} \tilde{\xi}^n) \in \mathrm{Im}(W_{\tilde{e}}^{(1,1)}), \tag{2.37}$$

where $W_{\tilde{e}^k}^{(i,j)}\colon \Omega^{(i,j)} \to \Omega^{(i+k,j+k)}\colon \alpha \mapsto \tilde{e}^k \wedge \alpha$ shares the same properties of $W_k^{\partial,(i,j)}$ and

$$\tilde{X} = \mathrm{L}_{\tilde{\xi}}^{\tilde{\omega}}(\epsilon_n) - d_{\tilde{\omega}_n}(\epsilon_n)\tilde{\xi}^n - [\tilde{c}, \epsilon_n]; \qquad \hat{X} = \tilde{e}_a^i \tilde{X}^a \partial_i \tag{2.38}$$

- $\varpi_{PC}^r := \varpi_{PC}(I \times \Sigma)|_{\mathcal{F}_{PC}^r}$;
- $\mathcal{S}_{PC}^r := \mathcal{S}_{PC}|_{\mathcal{F}_{PC}^r}$;
- $Q_{PC}^r = Q_{PC}$.

We will make sense of the constraints defined above in the next section. We can now reinterpret the content of theorem 2.8 as saying that the following diagram commutes

$$\begin{array}{ccc} & & \mathfrak{F}_{PC} \\ & \overset{\Phi}{\nearrow} & \uparrow{\scriptstyle \iota_r} \\ \mathfrak{F}_{PC}^{AKSZ} & \xrightarrow{\varphi} & \mathfrak{F}_{PC}^r \end{array}$$

We will provide an explicit expression for φ in chapter 3.3, adapted to the presence of a Dirac spinor ψ. The free PC theory symplectomorphism φ is then recovered simply by 'turning off' all the terms containing spinors.

Remark 2.15. The constraints (2.36) and (2.37) defining the reduced theory might seem arbitrarily defined, however, we remark that they arise from the definition of the fields in $\mathfrak{F}_{PC}^{AKSZ}$, which are constrained by definition, since $\mathfrak{F}_{PC}^{AKSZ}$ is defined from the BFV PC theory obtained through the KT construction, which required a choice of representative of the connection ω, given by the structural constraint. In the next section, we will see how such constraints arise in a more systematic way, without the introduction of the PC AKSZ theory.

2.5.2 PC BV-BFV from the BV-pushforward

In this section we will see how the restricted BV theory of PC gravity can be recovered via the BV-pushforward [CC25b], explained in section 1.5. However, it is first worth investigating the source of the constraints (2.36) and (2.37) in a more direct way, as we will want to generalize them to the case of supergravity.

The constraints of the reduced BV PC theory

First of all, we see that the BFV-PC theory of section 2.2.1 found in [CCS21a] is

$$\begin{aligned} \mathcal{S}_{PC}^{\Sigma} = \int_{\Sigma} (\iota_\xi e + \lambda \epsilon_n) e F_\omega + c e d_\omega e + \left(\frac{1}{2}[c,c] - \frac{1}{2}\iota_\xi \iota_\xi F_\omega - \mathrm{L}_\xi^\omega c\right) k^{\lrcorner} \\ - \frac{1}{2}\iota_{[\xi,\xi]} e y^{\lrcorner} + \big([c, \lambda\epsilon_n] - \mathrm{L}_\xi^\omega(\lambda\epsilon_n)\big)\, y^{\lrcorner} \end{aligned} \tag{2.39}$$

$$\varpi_{PC}^{\Sigma} = \int_{\Sigma} e\delta e \delta\omega + \delta c \delta k^{\lrcorner} + \delta\omega\delta\big(\iota_\xi(k^{\lrcorner})\big) - \delta\big(\iota_{\delta\xi}(e) y^{\lrcorner}\big) - \delta\lambda\epsilon_n \delta y^{\lrcorner}. \tag{2.40}$$

As previously remarked, in order to have a well-defined phase space, we need to carefully fix some components of the boundary connection ω, in such a way that the term $e \wedge \delta e \wedge \delta\omega$ in the boundary symplectic form does not give rise to any degeneracy. This is the content of theorem

4.1 (or equivalently of Theorem 33 in [CCS21a]). We might therefore be tempted to impose the same constraint for the bulk fields, i.e.

$$\epsilon_n d_{\tilde{\omega}} \tilde{e} \in \mathrm{Im}(W_{\tilde{e}}^{(1,1)}),$$

which amounts to imposing only some parts of the condition $d_{\boldsymbol{\omega}} \boldsymbol{e} = 0$. Unfortunately, such constraint is not invariant under the action of Q_{PC}, hence not defining a suitable restricted BV theory. As it turns out, the correct one is given by (2.37). In particular, a direct (and immediate) generalization of lemma A.6 tells us that there must exist unique $\sigma \in \mathcal{C}^\infty(I) \otimes \Omega_\partial^{(1,1)}$ and $\rho \in \mathrm{Ker}(W_{\tilde{e}}^{(1,2)})$ such that

$$\epsilon_n d_{\tilde{\omega}} \tilde{e} - \epsilon_n W_{\tilde{e}}^{-1}(\underline{\mathfrak{W}}) d\tilde{\xi}^n + \iota_{\hat{X}}(\tilde{\omega}_n^\dashv - \tilde{c}_n^\dashv \tilde{\xi}^n) = \tilde{e}\sigma + \epsilon_n[\rho, \tilde{e}].$$

In the same way, we can generalize theorem 4.1 and see that we can split $\tilde{\omega}$ as $\tilde{\omega} = \hat{\omega} + \tilde{v}$, with $\tilde{v} \in \mathrm{Ker}(W_{\tilde{e}}^{(2,1)})$ and $\hat{\omega}$ satisfying

$$\epsilon_n d_{\hat{\omega}} \tilde{e} - \epsilon_n W_{\tilde{e}}^{-1}(\underline{\mathfrak{W}}) d\tilde{\xi}^n + \iota_{\hat{X}}(\tilde{\omega}_n^\dashv - \tilde{c}_n^\dashv \tilde{\xi}^n) \in \mathrm{Im}(W_{\tilde{e}}^{(1,1)}).$$

Then, since $\epsilon_n d_{\tilde{\omega}} \tilde{e} = \epsilon_n d_{\hat{\omega}} \tilde{e} + \epsilon_n[\tilde{v}, \tilde{e}]$. Constraint (2.37) is satisfied if and only if $\rho = \tilde{v} = 0$, implying that only $\hat{\omega}$ survives in $\mathcal{F}_{PC}^r$, taking the role of the "reduced boundary connection", which (thanks to theorem 4.1) can be seen as a representative living in $\mathcal{A}_{\mathrm{red}}(I \times \Sigma) := \mathcal{A}(I \times \Sigma)/\ker(W_{\tilde{e}}^{(1,2)})$.

The other structural constraint (2.36) is interpreted as fixing some components of the canonical antifield $\underline{\tilde{\omega}}_n^\dashv$[14]. In particular, since $\underline{\tilde{\omega}}_n^\dashv \in \Omega^1(I) \otimes \Omega_\partial^{(2,2)}$, we can apply lemma A.6 to show there exists a splitting

$$\underline{\omega}_n^\dashv = \tilde{e}\underline{\rho}^\dashv + \epsilon_n[\tilde{e}, \underline{\sigma}^\dashv].$$

Then $\underline{\rho}^\dashv$ is exactly in the "dual" of $\mathcal{C}^\infty(I) \otimes \Omega_\partial^{(1,2)}/\mathrm{Ker}(W_{\tilde{e}}^{(1,2)})$, providing a perfect candidate for the antifield of $\tilde{\omega}$ in $\mathcal{F}_{PC}^r$. However, the constraint $\underline{\omega}_n^\dashv \in \mathrm{Im}(W_{\tilde{e}}^{(1,1)})$ is not Q–invariant, and (2.36) turns out to be the correct one.

In order to shed some more light into the choice of constraints, we consider the "more covariant" combination of fields $\boldsymbol{\omega}^\dashv - \iota_{\boldsymbol{\xi}} \boldsymbol{c}^\dashv$, as such expression appears repeatedly in computations. Specifically, when computing the variation of the BV-PC action, one obtains a total derivative term ϑ_{PC}, whose variation (having used Stokes' theorem on $I \times \Sigma$) produces a term on the boundary

$$\delta\tilde{\vartheta}_{PC}^\partial = \int_\Sigma \cdots + \delta\tilde{\omega}\delta\big(\iota_{\tilde{\xi}}(\tilde{\omega}^\dashv - \tilde{c}_n^\dashv \tilde{\xi}^n)\big) + \cdots .$$

Confronting it with the BFV symplectic form (2.40), one notices that the expression $\tilde{\omega}^\dashv - \tilde{c}_n^\dashv \tilde{\xi}^n$ is a good candidate for the boundary field $k^\dashv$, representing the antighost of $c \in \Omega_\partial^{(0,2)}[1]$. The same expression also appears inside (2.36).

At this point we know from diagram (A.31) that we can redefine $\boldsymbol{\omega}^\dashv = \boldsymbol{e}\check{\boldsymbol{\omega}}$ and $\boldsymbol{c}^\dashv = \frac{\boldsymbol{e}^2}{2}\check{\boldsymbol{c}}$, so we have $\boldsymbol{\omega}^\dashv - \iota_{\boldsymbol{\xi}} \boldsymbol{c}^\dashv = \boldsymbol{e}\left(\check{\boldsymbol{\omega}} - \iota_{\boldsymbol{\xi}} \boldsymbol{e}\check{\boldsymbol{c}} - \frac{1}{2}\boldsymbol{e}\iota_{\boldsymbol{\xi}}\check{\boldsymbol{c}}\right)$, and since $\boldsymbol{\omega}^\dashv = \tilde{\omega}^\dashv + \underline{\tilde{\omega}}_n^\dashv$ and $\boldsymbol{c}^\dashv = \underline{\tilde{c}}_n^\dashv$, unpacking the expression yields

$$\begin{aligned}\boldsymbol{\omega}^\dashv - \iota_{\boldsymbol{\xi}} \boldsymbol{c}^\dashv =& \tilde{\omega}^\dashv - \tilde{c}_n^\dashv \tilde{\xi}^n + \underline{\tilde{\omega}}_n^\dashv - \iota_{\tilde{\xi}} \underline{\tilde{c}}_n^\dashv \\ \boldsymbol{e}\left(\check{\boldsymbol{\omega}} - \iota_{\boldsymbol{\xi}} \boldsymbol{e}\check{\boldsymbol{c}} - \frac{\boldsymbol{e}}{2}\iota_{\boldsymbol{\xi}}\check{\boldsymbol{c}}\right) =& \tilde{e}\left(\check{\tilde{\omega}} + \underline{\check{\tilde{\omega}}}_n - \iota_{\tilde{\xi}}\tilde{e}\check{\tilde{c}} - \tilde{e}_n\tilde{\xi}^n\check{\tilde{c}} - \frac{\tilde{e}}{2}(\iota_{\tilde{\xi}}\check{\tilde{c}} - \check{\tilde{c}}_n\tilde{\xi}^n) - \iota_{\tilde{\xi}}\tilde{e}\underline{\check{\tilde{c}}}_n - \tilde{e}_n\tilde{\xi}^n\underline{\check{\tilde{c}}}_n\right) \\ &+ \underline{\tilde{e}}_n\left(\check{\tilde{\omega}} - \iota_{\tilde{\xi}}\tilde{e}\check{\tilde{c}} - \tilde{e}_n\tilde{\xi}^n\check{\tilde{c}} - \frac{\tilde{e}}{2}(\iota_{\tilde{\xi}}\check{\tilde{c}} - \check{\tilde{c}}_n\tilde{\xi}^n)\right).\end{aligned}$$

[14] Indeed in the symplectic form the relevant term is $\delta\tilde{\omega}\delta\underline{\tilde{\omega}}_n^\dashv$.

By inspection (discarding the terms proportional to dx^n) one could then infer

$$\tilde{\omega}^{\lrcorner} - \tilde{c}_n^{\lrcorner}\tilde{\xi}^n = \tilde{e}\left(\tilde{\check{\omega}} - \iota_{\tilde{\xi}}\tilde{e}\tilde{\check{c}} - \tilde{e}_n\tilde{\xi}^n\tilde{\check{c}} - \frac{\tilde{e}}{2}(\iota_{\tilde{\xi}}\tilde{\check{c}} - \tilde{\check{c}}_n\tilde{\xi}^n)\right) =: \tilde{e}\tilde{\check{k}}, \tag{2.41}$$

having defined $\boldsymbol{k}^{\lrcorner} := \boldsymbol{\omega}^{\lrcorner} - \iota_{\boldsymbol{\xi}}\boldsymbol{c}^{\lrcorner} = \boldsymbol{e}\check{\boldsymbol{k}} = \tilde{e}\tilde{\check{k}} + \underline{\tilde{e}}_n\tilde{\check{k}} + \tilde{e}\tilde{\underline{\check{k}}}_n$. However, we know from diagram $A.32$ that $W_{\tilde{e}}^{(2,1)}$ is surjective but not injective, hence the right hand side of (2.41) is defined up to a term in $\ker W_{\tilde{e}}^{(2,1)}$[15]. In order to fix the representative of the equivalence class in $\mathcal{C}^\infty(I) \otimes \left(\Omega_\partial^{(2,1)}/\ker W_{\tilde{e}}^{(2,1)}\right)$, we can generalize theorem 4.3 to see that we have to impose the constraint

$$\epsilon_n\tilde{\check{k}} = \epsilon_n\left(\tilde{\check{\omega}} - \iota_{\tilde{\xi}}\tilde{e}\tilde{\check{c}} - \tilde{e}_n\tilde{\check{c}}\tilde{\xi}^n - \frac{e}{2}(\iota_{\tilde{\xi}}\tilde{\check{c}} - \tilde{\check{c}}_n\tilde{\xi}^n)\right) \in \mathrm{Im}(W_{\tilde{e}}^{(1,1)}),$$

which, thanks to the following proposition, turns out to be equivalent to (2.36).

Proposition 2.2. *Constraint* (2.36) *is equivalent to*

$$\epsilon_n\tilde{\check{k}} \in \mathrm{Im}(W_{\tilde{e}}^{(1,1)}). \tag{2.42}$$

Furthermore, constraint (2.37) *is obtained by applying the cohomological vector field Q_{PC} to* (2.42)*.*

Remark 2.16. As an immediate corollary, since $Q_{PC}^2 = 0$, one obtains that the structural constraints on $\mathcal{F}_{PC}^r$ are invariant with respect to the action of Q_{PC}.

Proof. We begin by unpacking the terms inside $\underline{\mathfrak{W}}^{\lrcorner} = \underline{\tilde{\omega}}_n^{\lrcorner} - \iota_{\underline{z}}\tilde{\omega}^{\lrcorner} - \iota_{\tilde{\xi}}\underline{\tilde{c}}_n^{\lrcorner} + \iota_{\underline{z}}\tilde{c}_n^{\lrcorner}\tilde{\xi}^n$, starting from the definition $\boldsymbol{k}^{\lrcorner} := \boldsymbol{\omega}^{\lrcorner} - \iota_{\boldsymbol{\xi}}\boldsymbol{c}^{\lrcorner} = \boldsymbol{e}\check{\boldsymbol{k}} = \tilde{e}\tilde{\check{k}} + \underline{\tilde{e}}_n\tilde{\check{k}} + \tilde{e}\tilde{\underline{\check{k}}}_n$. In particular, one sees

$$\begin{aligned}\underline{\tilde{\omega}}_n^{\lrcorner} - \iota_{\tilde{\xi}}\underline{\tilde{c}}_n^{\lrcorner} &= \underline{\tilde{k}}_n^{\lrcorner} = \tilde{e}\tilde{\underline{\check{k}}}_n + \underline{\mu}\epsilon_n\tilde{\check{k}} + \iota_{\underline{z}}\tilde{e}\tilde{\check{k}} \\ \iota_{\underline{z}}\left(\tilde{\omega}^{\lrcorner} - \tilde{c}_n^{\lrcorner}\tilde{\xi}^n\right) &= \iota_{\underline{z}}\tilde{k}^{\lrcorner} = \iota_{\underline{z}}\tilde{e}\tilde{\check{k}} + \tilde{e}\iota_{\underline{z}}\tilde{\check{k}},\end{aligned}$$

hence, noticing $\underline{\mathfrak{W}}^{\lrcorner} = \underline{\tilde{k}}_n^{\lrcorner} - \iota_{\underline{z}}\tilde{k}^{\lrcorner}$, we have

$$\begin{aligned}\underline{\mathfrak{W}}^{\lrcorner} =& \tilde{e}\tilde{\underline{\check{k}}}_n + \underline{\mu}\epsilon_n\tilde{\check{k}} + \iota_{\underline{z}}\tilde{e}\tilde{\check{k}} - \iota_{\underline{z}}\tilde{e}\tilde{\check{k}} - \tilde{e}\iota_{\underline{z}}\tilde{\check{k}} \\ =& \tilde{e}\tilde{\underline{\check{k}}}_n + \underline{\mu}\epsilon_n\tilde{\check{k}} - \tilde{e}\iota_{\underline{z}}\tilde{\check{k}} \in \mathrm{Im}(W_{\tilde{e}}^{(1,1)}) \\ \Leftrightarrow \quad & \epsilon_n\tilde{\check{k}} \in \mathrm{Im}(W_{\tilde{e}}^{(1,1)})\end{aligned}$$

We can see that if $\underline{\mathfrak{W}}^{\lrcorner} = \tilde{e}\underline{\tilde{\tau}}^{\lrcorner}$ for some $\underline{\tilde{\tau}}^{\lrcorner} \in \Omega^1(I) \otimes \Omega_\partial^{(1,1)}[-1]$, and $\epsilon_n\tilde{\check{k}} = \tilde{e}\tilde{\check{a}}$ for some $\tilde{\check{a}} \in \mathcal{C}^\infty(I) \otimes \Omega_\partial^{(1,1)}[-1]$, then

$$\underline{\tilde{\tau}}^{\lrcorner} = \tilde{\underline{\check{k}}}_n + \underline{\mu}\tilde{\check{a}} - \iota_{\underline{z}}\check{k} \tag{2.43}$$

For the second part of the proposition, we first apply Q_{PC} to $\check{\boldsymbol{k}} = \check{\boldsymbol{\omega}} - \iota_{\boldsymbol{\xi}}\boldsymbol{e}\check{\boldsymbol{c}} - \frac{\boldsymbol{e}}{2}\iota_{\boldsymbol{\xi}}\check{\boldsymbol{c}}$, obtaining,

$$Q_{PC}\left(\check{\boldsymbol{k}}\right) = d_{\boldsymbol{\omega}}\boldsymbol{e} + \mathrm{L}_{\boldsymbol{\xi}}^{\boldsymbol{\omega}}(\check{\boldsymbol{k}}) - \left[\boldsymbol{c}, \check{\boldsymbol{k}}\right].$$

[15] Defining $\boldsymbol{k}^{\lrcorner} = \tilde{k}^{\lrcorner} + \underline{\tilde{k}}_n^{\lrcorner}$, we have that $\underline{\tilde{k}}_n^{\lrcorner} = \underline{\tilde{e}}_n\tilde{\check{k}} + \tilde{e}\tilde{\underline{\check{k}}}_n$, which is ill-defined since $\tilde{\check{k}}$ is unique only up to elements in $\mathrm{Ker}(W_{\tilde{e}}^{(2,1)})$. Such ambiguity is resolved by imposing the structural constrain (2.36), as is shown in the next proposition and in remark 2.17

From $Q_{PC}(\check{\boldsymbol{k}})$ we can extract $Q_{PC}(\tilde{\check{k}})$, and since (2.42) is equivalent to $\epsilon_n\tilde{\check{k}} - \tilde{e}\tilde{\check{a}} = 0$, for some $\tilde{\check{a}} \in \mathcal{C}^\infty(I) \otimes \Omega_\partial^{(1,1)}[-1]$, yielding

$$\begin{aligned} Q_{PC}(\epsilon_n\tilde{\check{k}} - \tilde{e}\tilde{\check{a}}) =& - \epsilon_n \left(d_{\tilde{\omega}}\tilde{e} + \mathrm{L}_{\tilde{\xi}}^{\tilde{\omega}}(\tilde{\check{k}}) + d\tilde{\xi}^n\tilde{\check{k}}_n - \tilde{\xi}^n d_{\tilde{\omega}_n}\tilde{\check{k}} - [\tilde{c}, \tilde{\check{k}}]\right) \\ &- (\mathrm{L}_{\tilde{\xi}}^{\tilde{\omega}}\tilde{e} - d\tilde{\xi}^n\tilde{e}_n - d_{\tilde{\omega}_n}\tilde{e}\tilde{\xi}^n - [\tilde{c}, \tilde{e}])\tilde{\check{a}} - \tilde{e}Q_{PC}\tilde{\check{a}} = 0 \\ \Leftrightarrow \quad \epsilon_n d_{\tilde{\omega}}\tilde{e} + d\tilde{\xi}^n(\epsilon_n\tilde{\check{k}}_n - \tilde{e}_n\tilde{\check{a}}) &+ (\mathrm{L}_{\tilde{\xi}}^{\tilde{\omega}}\epsilon_n - d_{\tilde{\omega}_n}(\epsilon_n)\tilde{\xi}^n - [\tilde{c}, \epsilon_n])\tilde{\check{k}} \in \mathrm{Im}(W_{\tilde{e}}^{(1,1)}). \end{aligned}$$

Using (2.43) we see

$$\begin{aligned} \tilde{\check{k}}_n - \tilde{e}_n\tilde{\check{a}} &= \epsilon_n\tilde{\tau}^{\lrcorner} + \mu\epsilon_n\tilde{\check{a}} + \iota_z(\epsilon_n\tilde{\check{k}}) - (\mu\epsilon_n + \iota_z\tilde{e})\tilde{\check{a}} \\ &= \epsilon_n\tilde{\tau}^{\lrcorner} + \iota_z(\tilde{e}\tilde{\check{a}}) - \iota_z\tilde{e}\tilde{\check{a}} \\ &= \epsilon_n\tilde{\tau}^{\lrcorner} + \tilde{e}\iota_z\tilde{a}, \end{aligned}$$

obtaining

$$\epsilon_n d_{\tilde{\omega}}\tilde{e} + \epsilon_n d\tilde{\xi}^n\tilde{\tau}^{\lrcorner} + (\mathrm{L}_{\tilde{\xi}}^{\tilde{\omega}}\epsilon_n - d_{\tilde{\omega}_n}(\epsilon_n)\tilde{\xi}^n - [\tilde{c}, \epsilon_n])\tilde{\check{k}} \in \mathrm{Im}(W_{\tilde{e}}^{(1,1)}).$$

□

Having fixed such constraint, we recall there exist $\underline{\tilde{\tau}}^{\lrcorner} \in \Omega^1(I) \otimes \Omega_\partial^{(1,1)}[-1]$ and $\underline{\tilde{\mu}}^{\lrcorner} \in \mathrm{Ker}(W_{\tilde{e}}^{(1,2)})[-1]$ such that

$$\underline{\mathfrak{W}}^{\lrcorner} = \tilde{e}\underline{\tilde{\tau}}^{\lrcorner} + \epsilon_n[\underline{\tilde{\mu}}^{\lrcorner}, \tilde{e}]. \tag{2.44}$$

Constraint (2.42) becomes $\underline{\mathfrak{W}}^{\lrcorner} = \tilde{e}\underline{\tilde{\tau}}^{\lrcorner}$, telling us that $\underline{\tilde{\tau}}^{\lrcorner}$ is isomorphic to a field in $\mathcal{F}_{PC}^r$ and $\underline{\tilde{\mu}}^{\lrcorner}$ defines its complement in $\mathcal{F}_{PC}$.

Explicitly, setting $\underline{\tilde{\omega}}_n^{\lrcorner} = \underline{\hat{\omega}}_n^{\lrcorner} + \underline{\tilde{v}}^{\lrcorner}$, we can rewrite

$$\underline{\hat{\omega}}_n := \tilde{e}\underline{\tilde{\tau}}^{\lrcorner} + \iota_{\underline{z}}\tilde{\omega}^{\lrcorner} + \iota_{\tilde{\xi}}\underline{\tilde{c}}_n^{\lrcorner} - \iota_{\underline{z}}\tilde{c}_n^{\lrcorner}\tilde{\xi}^n \tag{2.45}$$

$$\underline{\tilde{v}}^{\lrcorner} = \epsilon_n[\underline{\tilde{\mu}}^{\lrcorner}, \tilde{e}]. \tag{2.46}$$

Remark 2.17. Imposing (2.42) is equivalent to set $\underline{\mu}^{\lrcorner} = 0$, implying $\underline{\tilde{v}}^{\lrcorner} = 0$, which is consistent with the fact that the other constraint (2.37) fixes $\tilde{v} = 0$. Furthermore, from (2.43) and (2.45) we see that $\underline{\tilde{\omega}}_n^{\lrcorner} - \iota_{\tilde{\xi}}\underline{\tilde{c}}_n^{\lrcorner} = \tilde{e}\big(\underline{\tilde{k}}_n + \underline{\mu}\tilde{\check{a}}\big) + \iota_{\underline{z}}\tilde{e}\tilde{\check{k}} + \underline{\tilde{v}}^{\lrcorner}$, which, after imposing (2.42), becomes

$$\underline{\tilde{\omega}}_n^{\lrcorner} - \iota_{\tilde{\xi}}\underline{\tilde{c}}_n^{\lrcorner} = \tilde{e}\underline{\tilde{\check{k}}}_n + \underline{\tilde{e}}_n\tilde{\check{k}},$$

which solves any possible ambiguity in the definition of $\underline{\tilde{k}}_n^{\lrcorner}$.

Lastly, we can explicitly write the symplectic form on $\mathcal{F}_{PC}^r$ as

$$\varpi_{PC}^r = \int_{I\times\Sigma} \delta\tilde{e}\delta\underline{\tilde{e}}_n^{\lrcorner} + \delta\underline{\tilde{e}}_n\delta\tilde{e}^{\lrcorner} + \delta\hat{\omega}\delta\underline{\hat{\omega}}_n^{\lrcorner} + \delta\underline{\tilde{\omega}}_n\delta\tilde{\omega}^{\lrcorner} + \delta\tilde{c}\delta\underline{\tilde{c}}_n^{\lrcorner} + \iota_{\delta\tilde{\xi}}\delta\underline{\tilde{\xi}}^{\lrcorner} + \delta\underline{\tilde{\xi}}_n^{\lrcorner}\delta\tilde{\xi}^n.$$

The BV PC pushforward

Proposition 2.3. *[CC25b] There exists a symplectomorphism Φ between the graded -1–symplectic manifolds $\mathfrak{F}_{PC} := (\mathcal{F}_{PC}, \varpi_{PC})$ and $\mathfrak{F}_{PC}^H := (\mathcal{F}_{PC}, \varpi_{PC}^H)$, where*

$$\varpi_{PC}^H := \varpi_{PC}^r + \int_{I\times\Sigma} \delta\tilde{v}\delta\underline{\tilde{v}}^{\lrcorner}, \quad \textit{with} \quad \tilde{v} \in \mathrm{Ker}(W_{\tilde{e}}^{(1,2)}) \quad \textit{and} \quad \underline{\tilde{v}}^{\lrcorner} = \epsilon_n[\underline{\tilde{\mu}}^{\lrcorner}, \tilde{e}],$$

for some $\underline{\tilde{\mu}}^{\dashv} \in \mathcal{C}^\infty(I)\otimes\Omega_\partial^{(1,2)}[-1]$*. Furthermore, fixing a reference tetrad* $\tilde{e}_0 \in \mathcal{C}^\infty(I)\otimes\Omega_\partial^{(1,1)}$ *and considering* $\mathcal{F}_f := T^*[-1](\mathrm{Ker}(W_{e_0}^{(1,2)}))$ *together with its canonical symplectic form* ϖ_f*, there is a surjective submersion* $\pi\colon \mathfrak{F}^H_{PC} \to \mathfrak{F}^r_{PC}$ *such that the quadruple*

$$(\mathfrak{F}^H_{PC}, \mathfrak{F}^r_{PC}, \mathfrak{F}_f, \pi)$$

is a BV hedgehog.

The strategy employed by the author of [CC25b] to prove the above proposition relies on factorising the symplectomorphism between $\mathfrak{F}_{PC}$ and $\mathfrak{F}^H_{PC}$ into $\Phi = \phi_1 \circ \phi_2$. In particular, the splitting occurs as follows

$$\phi_2^*\left(\varpi^r_{PC} + \int_{I\times\Sigma} \delta\tilde{v}\delta\underline{\tilde{v}}^{\dashv}\right) = \varpi^r_{PC} + \int_{I\times\Sigma} \delta\hat{\omega}\delta\underline{\tilde{v}}^{\dashv} + \delta\tilde{v}\delta\underline{\tilde{v}}^{\dashv} \tag{2.47}$$

$$\phi_1^*\left(\varpi^r_{PC} + \int_{I\times\Sigma} \delta\hat{\omega}\delta\underline{\tilde{v}}^{\dashv} + \delta\tilde{v}\delta\underline{\tilde{v}}^{\dashv}\right) = \varpi_{PC}, \tag{2.48}$$

where we remark that

$$\varpi_{PC} = \varpi^r_{PC} + \int_{I\times\Sigma} \delta\tilde{v}\delta\underline{\hat{\omega}}_n^{\dashv} + \delta\hat{\omega}\delta\underline{\tilde{v}}^{\dashv} + \delta\tilde{v}\delta\underline{\tilde{v}}^{\dashv}$$

We will provide the full expression of the above symplectomorphisms in section 5.2 in the case of supergravity. The current maps are recovered by discarding the terms that depend on the spinor fields.

Proposition 2.4. *[CC25b] The symplectomorphism* $\Phi = \phi_1 \circ \phi_2$ *defined above is such that*

$$\Phi^*\left(\mathcal{S}^H_{PC}\right) = \mathcal{S}_{PC}, \tag{2.49}$$

$$\mathcal{S}^H_{PC} = \mathcal{S}^r_{PC} + \int_{I\times\Sigma} \frac{1}{2}\tilde{e}\underline{\tilde{e}}_n[\tilde{v},\tilde{v}] + f(\underline{\tilde{v}}^{\dashv}) \tag{2.50}$$

where

$$f(\underline{\tilde{v}}^{\dashv}) := \left(\mathrm{L}^{\tilde{\omega}}_{\tilde{\xi}}\tilde{v} + d_{\tilde{\omega}_n}\tilde{v}\tilde{\xi}^n + d\tilde{\xi}^n\iota_z\tilde{v} - [\tilde{c},\tilde{v}]\right)\underline{\tilde{v}}^{\dashv} \tag{2.51}$$

Remark 2.18. We notice that, since we have a fiber bundle $\mathcal{F}^H_{PC} \to \mathcal{F}^r_{PC}$ whose fiber is locally modelled by $\mathcal{F}_f := T^*[-1](\mathrm{Ker}(W_{\tilde{e}_0}^{(1,2)})) \ni (\tilde{v}, \underline{\tilde{v}}^{\dashv})$, and since the subspace

$$\mathcal{L}_f := \{(\tilde{v}, \underline{\tilde{v}}^{\dashv}) \in \mathcal{F}_f \mid \underline{\tilde{v}}^{\dashv} = 0\} \subset \mathcal{F}_f$$

is a Lagrangian submanifold with respect to the canonical symplectic form $\varpi_f = \int_{I\times\Sigma}\delta\tilde{v}\delta\underline{\tilde{v}}^{\dashv}$, we can perform the BV pushforward by integrating $\mathcal{S}^H_{PC}$ along $\mathcal{L}_f$ as defined in section 1.5.

Theorem 2.9. *[CC25b] The restricted BV PC theory* $\mathfrak{F}^r_{PC}$ *is the BV pushforward of* $\mathfrak{F}^H_{PC}$ *obtained by integrating along the Lagrangian submanifold* $\mathcal{L}_f := \{(\tilde{v}, \underline{\tilde{v}}^{\dashv}) \in \mathcal{F}_f \mid \underline{\tilde{v}}^{\dashv} = 0\} \subset \mathcal{F}_f$

Remark 2.19. In particular, the above theorem holds because, setting every expression containing $\tilde{v}^{\dashv}$ to zero, the symplectomorphism Φ of proposition 2.3 is such that

$$\Phi^*\left(\mathcal{S}^r_{PC} + \int_{I\times\Sigma}\frac{1}{2}\underline{\tilde{e}}_n\tilde{e}[\tilde{v},\tilde{v}]\right) = \mathcal{S}_{PC}\big|_{\mathcal{L}_f},$$

with $\int_{I\times\Sigma}\frac{1}{2}\underline{\tilde{e}}_n\tilde{e}[\tilde{v},\tilde{v}]$ taking the role of a Gaussian integral[16] inside $\int_{\mathcal{L}_f} e^{\frac{i}{\hbar}\mathcal{S}_{PC}}$, decoupling from the expression and contributing to the partition function just as a multiplying constant.

[16]Indeed it can be proved [CC25b] that the quadratic form $\int_{I\times\Sigma}\frac{1}{2}\underline{\tilde{e}}_n\tilde{e}[\tilde{v},\tilde{v}]$ is non–degenerate.

Chapter 3

Palatini-Cartan-Dirac gravity on manifolds with boundary

In the following Chapter we will describe the structure of Platini–Cartan gravity coupled to a spin $\frac{1}{2}$ spinor field via the Dirac Lagrangian, in which the differential has been replaced by the exterior covariant derivative.

We rely heavily on the objects defined in appendix A and the results therin. In particular, in the following we assume M to be a 4–dimensional spin manifold, and $P_{\text{spin}} \to M$ a $\text{Spin}(3,1)$ principal bundle on it. Defining spin coframes is equivalent to the usual coframes defined on $P_{SO} = l(P_{\text{spin}})$, where l is the bundle morphism given by the double cover $\text{Spin}(3,1) \to SO(3,1)$. It is however necessary to introduce spin bundles in order to define spinors, which are here seen as sections of the associated vector bundle to P_{spin} with respect to a half-integer spin representation.

A spin $\frac{1}{2}$ spinor is just a section of the Dirac spinor bundle $\mathbb{S}_D$, whose parity has been shifted to account for the fermionic nature of the fields. Denoting it by ψ,[1] we have

$$\psi \in \Gamma(M, \Pi\mathbb{S}_D).$$

The other fields of the theory are the vielbein $e \in \Omega^{(1,1)}_{\text{n.d.}}$ and the spin connection $\omega \in \mathcal{A}(M) \simeq \Omega^{(1,2)}$.

3.1 Coupling the Dirac Lagrangian to PC gravity

In the coupling of the spinor field, we apply the principle of covariance and substitute any derivative with a covariant derivative, which in this case amounts to sending $d\psi$ in the free Dirac Lagrangian to $d_\omega\psi := d\psi + [\omega, \psi]$. However, we first have to define what the action of a Lie algebra–valued element $\alpha \in \mathfrak{spin}(3,1)$ is on Dirac spinors.

Since the Dirac spinor bundle $\mathbb{S}_D\colon P_{\text{spin}} \times_\gamma \mathbb{C}^4$ is the associated vector bundle to P_{spin} with respect to the gamma representation, we just need to compute the image of an element $\alpha \in \mathfrak{spin}(3,1) \simeq \wedge^2 V$ under γ. We saw in Proposition A.5 that, if $\{v_a\}$ is a basis of V, the infinitesimal

[1] In the next chapter, ψ will denote the gravitino, but there will be no ambiguities since it will only be denoting Dirac spinors in the current chapter.

action of the double cover l on the generators of the Lie algebra is given by

$$\begin{aligned} \dot{l} \ : \ \mathfrak{so}(3,1) \simeq \wedge^2 V &\longrightarrow \mathfrak{spin}(3,1) \\ v_a \wedge v_b &\longmapsto \frac{1}{4}[v_a, v_b], \end{aligned}$$

which under the gamma representation are sent to $\frac{1}{4}[\gamma_a, \gamma_b]$, where now $[\cdot, \cdot]$ indicates the commutator of gamma matrices. Therefore one has

$$\wedge^2 V \ni \alpha = \frac{1}{2}\alpha^{ab} v_a \wedge v_b \longmapsto \frac{1}{4}\alpha^{ab}\gamma_{ab},$$

where we recall $\gamma_{ab} := \frac{1}{2}[\gamma_a, \gamma_b]$.

At this point it is easy to define the covariant derivative of a Dirac spinor field ψ:

$$d_\omega \psi := d\psi + [\omega, \psi] = d\psi - \frac{1}{4}\omega^{ab}\gamma_{ab}\psi. \tag{3.1}$$

For the sake of consistency, we briefly check that it transforms well under a gauge transformation $\psi \mapsto \psi' = S(x)\psi$, where for each x, $S(x) \in \mathrm{Spin}(N-1,1)$

$$\begin{aligned} d_{\omega'}\psi' &= d_{\omega'}(S\psi) = (d_{\omega'}S)\psi + S d_{\omega'}\psi \\ &= (dS)\psi + [\omega', S]\psi + S d\psi + S[\omega', \psi] \\ &= (dS)\psi + \omega'(S\psi) - S\omega'(\psi) + S d\psi + S\omega'(\psi) \\ &= (dS)\psi + S\omega(\psi) - (dS\psi) + S d\psi = S\{d\psi + \omega(\psi)\} \\ &= S\{d\psi + [\omega, \psi]\} = S d_\omega \psi = (d_\omega \psi)', \end{aligned}$$

where we used $\omega' = S\omega S^{-1} - (dS)S^{-1}$.

The invariant Dirac Lagrangian is constructed via the Dirac pairing defined in Proposition A.6, where we had defined $\bar{\psi} := \psi^\dagger \gamma_0$, and $< \psi, \psi > := \bar{\psi}\psi$. We extend the definition of the covariant derivative to the hermitian conjugate of ψ by requiring that $\overline{d_\omega \psi} = d_\omega \overline{\psi}$,[2] hence obtaining

$$d_\omega \overline{\psi} = d\overline{\psi} + [\omega, \overline{\psi}] = d\overline{\psi} - \frac{1}{4}\omega^{ab}\overline{\psi}\gamma_a\gamma_b. \tag{3.2}$$

The definition of covariant derivative extends also to the gamma matrices, where one must be particularly careful. In particular, throughout the thesis we will be considering $\gamma := \gamma^a v_a$, which has values in $V \otimes \mathcal{C}(V)$, which means that it transforms as a Lorentz vector and via the action of the gamma representation on gamma matrices: indeed, for all $\alpha \in \wedge^2 V$, one obtains the following splitting

$$[\alpha, \gamma] = [\alpha, \gamma]_V + [\alpha, \gamma]_S, \tag{3.3}$$

where $[\alpha, \gamma]_V := \alpha^{ab}\eta_{bc}\gamma^c v_a$ is the action on the vector part of γ and $[\alpha, \gamma]_S := \alpha^{ab}[\gamma(v_a \wedge v_b), \gamma^c]v_c = -\frac{1}{4}\alpha^{ab}(\gamma_{ab}\gamma^c - \gamma^c\gamma_{ab})v_c$ is the adjoint action of the Lie algebra of $\mathrm{Spin}(3,1)$ in the gamma representation.

Lemma 3.1. *Let $\gamma := \gamma^a v_a \in V \otimes \mathbb{C}(4)$ be an element of the vector space V with values in the Clifford algebra (seen as endomorphisms of the spinor bundle $\mathbb{S}_D$). Then*

$$d_\omega \gamma = 0.$$

[2]In other words, the quantity $\bar{\psi}\psi$ should be invariant under gauge transformations, which is consistent with the fact that $\bar{\psi}\psi$ is a scalar.

Proof. γ is a section of $\mathcal{V}$ and an endomorphism of the spin bundle E_λ. Hence its covariant derivative reads

$$(d_\omega\gamma)^b = (d\gamma)^b + \omega^{bc}\gamma_c - \frac{1}{4}\omega^{ac}(\gamma_a\gamma_c\gamma^b - \gamma^b\gamma_a\gamma_c).$$

Note that this formula implies the correct Leibniz rule for $d_\omega(\gamma\psi)$. Using the anti-commutation relation of gamma matrices, we can show that $\omega^{bc}\gamma_c - \frac{1}{4}\omega^{ac}\eta^{bd}(\gamma_a\gamma_c\gamma_d - \gamma_d\gamma_a\gamma_c) = 0$ and conclude the proof by noticing that γ is constant. $\square$

Finally the fully covariant Dirac Lagrangian is given by

$$S_{\text{Dirac}} := \int_M \frac{i}{2\cdot 3!}e^3\left(\overline{\psi}\gamma d_\omega\psi - d_\omega\overline{\psi}\gamma\psi\right) = \int_M \frac{i}{2}\left(\overline{\psi}\gamma^a\nabla_a\psi - \nabla_a\overline{\psi}\gamma^a\psi\right)\text{Vol}_g, \tag{3.4}$$

with $\nabla_a\psi := e_a^\mu\left(\partial_\mu\psi - \frac{1}{4}\omega_\mu^{ab}\gamma_a\gamma_b\psi\right)$. The Palatini-Cartan-Dirac action is then

$$S_{PCD} := \int_M \frac{e^2}{2}F_\omega + \frac{e^4}{4!}\Lambda + \frac{i}{2\cdot 3!}e^3\left(\overline{\psi}\gamma d_\omega\psi - d_\omega\overline{\psi}\gamma\psi\right).$$

Its variation yields

$$\begin{aligned}\delta S = &\delta_\omega S + \int_M \left[eF_\omega + i\frac{e^2}{4}\left(\overline{\psi}\gamma d_\omega\psi - d_\omega\overline{\psi}\gamma\psi\right)\right]\delta e\\ &+ \frac{i}{3!}\delta\overline{\psi}\left[e^3\gamma d_\omega\psi - \frac{1}{2}d_\omega\left(e^3\right)\gamma\psi\right]\\ &+ \frac{i}{3!}\left[e^3 d_\omega\overline{\psi}\gamma + \frac{1}{2}d_\omega\left(e^3\right)\overline{\psi}\gamma\right]\delta\psi,\end{aligned} \tag{3.5}$$

together with a boundary term

$$\tilde{\alpha}^\partial_{PCD} = \int_{\partial M} \frac{e^{N-2}}{(N-2)!}\delta\omega + i\frac{e^{N-1}}{2(N-1)!}\left(\overline{\psi}\gamma\delta\psi - \delta\overline{\psi}\gamma\psi\right). \tag{3.6}$$

To compute $\delta_\omega S$, first we define the internal contraction on $\mathcal{V}$. In particular, for any $X \in V$ and for all $\alpha \in \wedge^k V$, we define for all $\alpha = \frac{1}{k!}\alpha^{i_1\cdots i_k}v_{i_1}\wedge\cdots\wedge v_{i_k}$

$$j_X\alpha := \frac{\eta_{ab}}{(k-1)!}X^a\alpha^{bi_2\cdots i_k}v_{i_2}\wedge\cdots\wedge v_{i_k}. \tag{3.7}$$

With this definition, we obtain

$$[\alpha,\psi] = \frac{1}{4}j_\gamma j_\gamma\alpha\psi, \qquad \text{and} \qquad [\alpha,\overline{\psi}] = -\frac{(-1)^{|\alpha||\psi|}}{4}\overline{\psi}j_\gamma j_\gamma\alpha. \tag{3.8}$$

We know $\delta_\omega S_{PCD} = \delta_\omega S_{PC} + \delta_\omega S_{\text{Dirac}}$, with

$$\begin{aligned}\delta_\omega S_{\text{Dirac}} &= \int_M \frac{i}{2\cdot 3!}e^3\left(\bar{\psi}\gamma[\delta\omega.\psi] - [\delta\omega.\bar{\psi}]\gamma\psi\right)\\ &= \int_M \frac{i}{8\cdot 3!}e^3\bar{\psi}\left[\gamma j_\gamma j_\gamma\delta\omega + j_\gamma j_\gamma\delta\omega\gamma\right]\psi\\ &= \int_M \frac{i}{8\cdot 3!}\bar{\psi}\left[\gamma j_\gamma j_\gamma e^3 + j_\gamma j_\gamma e^3\gamma\right]\psi\delta\omega,\end{aligned}$$

The equations of motion become

$$eF_\omega + i\frac{e^2}{4}\left(\overline{\psi}\gamma d_\omega\psi - d_\omega\overline{\psi}\gamma\psi\right) = 0, \tag{3.9}$$

$$e\left[d_\omega e + \frac{i}{4}(\bar{\psi}\gamma[e^2,\psi] - [e^2,\bar{\psi}]\gamma\psi)\right] = 0, \tag{3.10}$$

$$\frac{e^3}{3!}\gamma d_\omega\psi - d_\omega\left(\frac{e^3}{2\cdot 3!}\right)\gamma\psi = 0, \tag{3.11}$$

$$\frac{e^3}{3!}d_\omega\overline{\psi}\gamma + d_\omega\left(\frac{e^3}{2\cdot 3!}\right)\overline{\psi}\gamma = 0. \tag{3.12}$$

Remark 3.1. First of all, notice that , once we impose $\overline{\psi} = \psi^\dagger\gamma_0$, then equations (3.11) and (3.12) are one the Hermitian conjugate of the other, representing Dirac equation on a curved background.

Secondly, contrary to the case of pure gravity, the coupling of a spinor field introduces a torsion term, given by $\frac{i}{4}(\bar{\psi}\gamma[e^2,\psi] - [e^2,\bar{\psi}]\gamma\psi)$, hence the connection which solves the equation of motion is not Levi-Civita.

3.1.1 Non–degenerate case

In this section we study the boundary structure of the PCD theory in the case where the induced metric g^∂ on Σ is non–degenerate. As usual, we employ the KT construction and restrict the space of fields to the boundary, obtaining $\tilde{F}^{PCD}_\Sigma = \mathcal{A}_\Sigma \times \Omega^{(1,1)}_{\partial,\text{n.d.}} \times \Gamma(\Sigma, \mathbb{S}|_\Sigma) \times \Gamma(\Sigma, \overline{\mathbb{S}}|_\Sigma)$.

The presymplectic form on the space of preboundary fields is given as usual by the variation of the boundary 1-form resulting from the variation of the action. We obtain

$$\tilde{\varpi}^{PCD}_\Sigma = \int_\Sigma e\delta e\delta\omega + i\frac{e^2}{4}\left(\overline{\psi}\gamma\delta\psi - \delta\overline{\psi}\gamma\psi\right)\delta e + i\frac{e^3}{3!}\delta\overline{\psi}\gamma\delta\psi, \tag{3.13}$$

while

$$\begin{aligned}\iota_{\mathbb{X}}\tilde{\varpi}^{PCD}_\Sigma = \int_\Sigma e\mathbb{X}_e\delta\omega + &\left[e\mathbb{X}_\omega + \frac{i}{4}e^2(\overline{\psi}\gamma\mathbb{X}_\psi - \mathbb{X}_{\overline{\psi}}\gamma\psi)\right]\delta e\\ &+ i\delta\overline{\psi}\left(-\frac{e^2}{4}\gamma\psi\mathbb{X}_e + \frac{e^3}{3!}\gamma\mathbb{X}_\gamma\right) + i\left(\frac{e^2}{4}\overline{\psi}\gamma\mathbb{X}_e + \frac{e^3}{3!}\mathbb{X}_{\overline{\psi}}\gamma\right)\delta\psi.\end{aligned}$$

The kernel of the presymplectic form is hence given by the following system of equations:

$$\begin{aligned} &e\mathbb{X}_e = 0 && e\mathbb{X}_\omega + i\frac{e^2}{4}\left(\overline{\psi}\gamma\mathbb{X}_\psi + -\mathbb{X}_{\overline{\psi}}\gamma\psi\right) = 0\\ &-\frac{e^2}{4}\gamma\psi\mathbb{X}_e + \frac{e^3}{3!}\gamma\mathbb{X}_\psi = 0 && \frac{e^2}{4}\overline{\psi}\gamma\mathbb{X}_e + \frac{e^3}{3!}\mathbb{X}_{\overline{\psi}}\gamma = 0.\end{aligned}$$

We can first solve the last two equations, using that γ is invertible and that $W_3^{\partial,(0,0)}$ is injective. We then find $\mathbb{X}_e = \mathbb{X}_\psi = \mathbb{X}_{\overline{\psi}} = 0$ and $e\mathbb{X}_\omega = 0$. The geometric phase space is a bundle over $\Omega^{(1,1)}_{\partial,\text{n.d.}}$ with local trivialization $F^{PCD}_\Sigma \simeq F^{PC}_\Sigma \times \Gamma(\Sigma, \mathbb{S}|_\Sigma) \times \Gamma(\Sigma, \overline{\mathbb{S}}|_\Sigma)$.

To fix the representative of the connection, we generalize theorem 2.1 to the following.

Theorem 3.1. *Suppose that g^∂, the metric induced on the boundary, is nondegenerate. Given any $\widetilde{\omega} \in \Omega^{1,2}$, there is a unique decomposition*

$$\widetilde{\omega} = \omega + v, \tag{3.14}$$

with ω and v satisfying

$$ev = 0 \quad \text{and} \quad \epsilon_n \left[d_\omega e + \frac{i}{4}(\bar{\psi}\gamma[e^2, \psi] - [e^2, \bar{\psi}]\gamma\psi) \right] \in \operatorname{Im} W_1^{\partial,(1,1)}. \tag{3.15}$$

Proof. Let $\widetilde{\omega} \in \Omega_\partial^{1,2}$. From Lemma A.6 we deduce that there exist unique $\sigma \in \Omega_\partial^{1,1}$ and $v \in \mathrm{Ker} W_1^{\partial,(1,2)}$ such that

$$\epsilon_n \left[d_\omega e + \frac{i}{4}(\bar{\psi}\gamma[e^2, \psi] - [e^2, \bar{\psi}]\gamma\psi) \right] = e\sigma + \epsilon_n[v, e].$$

We define $\omega := \widetilde{\omega} - v$. Then ω and v satisfy (3.14) and (3.15).

For uniqueness, suppose that $\widetilde{\omega} = \omega_1 + v_1 = \omega_2 + v_2$ with $ev_i = 0$ and $\epsilon_n d_{\omega_i} e \in \operatorname{Im} W_1^{\partial,(1,1)}$ for $i = 1, 2$. Hence

$$\epsilon_n d_{\omega_1} e - \epsilon_n d_{\omega_2} e = \epsilon_n [v_2 - v_1, e] \in \operatorname{Im} W_1^{\partial,(1,1)}.$$

Hence from Lemma A.5 and Lemma A.6 (for which we need nondegeneracy of g^∂), we deduce $v_2 - v_1 = 0$, since $v_2 - v_1 \in Ker W_1^{\partial,(1,2)}$. □

With the addition of a Dirac spinor, the constraints of the boundary PC become

$$\begin{aligned}
L_c^{PCD} &= \int_\Sigma c \left(ed_\omega e + \frac{i}{(8 \cdot 3!)} \overline{\psi} \left(j_\gamma j_\gamma e^3 \gamma + \gamma j_\gamma j_\gamma e^3 \right) \psi \right), \\
P_\xi^{PCD} &= \int_\Sigma \frac{1}{2} \iota_\xi e^2 F_\omega + \iota_\xi(\omega - \omega_0) \left(ed_\omega e - \frac{i}{8 \cdot 3!} \overline{\psi} \left(j_\gamma j_\gamma e^3 \gamma + \gamma j_\gamma j_\gamma e^3 \right) \psi \right) \\
&\quad + \frac{i}{2 \cdot 3!} \iota_\xi e^3 (\overline{\psi} \gamma d_\omega \psi - d_\omega \overline{\psi} \gamma \psi), \\
H_\lambda^{PCD} &= \int_\Sigma \lambda \epsilon_n \left[eF_\omega + \frac{e^3}{3!} \Lambda + i \frac{e^2}{4} \left(\overline{\psi} \gamma d_\omega \psi - d_\omega \overline{\psi} \gamma \psi \right) \right].
\end{aligned}$$

Remark 3.2. We can rewrite L_c^{PCD} to make the action of the internal symmetry group on the fields more evident. In particular we obtain

$$L_c^{PCD} = \int_\Sigma ced_\omega e - i \frac{e^3}{2 \cdot 3!} \left([c, \overline{\psi}] \gamma \psi - \overline{\psi} \gamma [c, \psi] \right), \tag{3.16}$$

while P_ξ^{PCD} becomes

$$\begin{aligned}
P_\xi^{PCD} &= \int_\Sigma \frac{1}{2} \iota_\xi e^2 F_\omega + \iota_\xi(\omega - \omega_0) ed_\omega e - \frac{i}{8 \cdot 3!} \iota_\xi e^3 \overline{\psi} \left(-[\omega - \omega_0, \overline{\psi} \gamma \psi + \overline{\psi} \gamma [\omega - \omega_0, \psi] \right) \psi \\
&\quad + \frac{i}{2 \cdot 3!} \iota_\xi e^3 (\overline{\psi} \gamma d_\omega \psi - d_\omega \overline{\psi} \gamma \psi) \\
&= \int_\Sigma \frac{1}{2} \iota_\xi e^2 F_\omega + \iota_\xi(\omega - \omega_0) ed_\omega e - i \frac{e^3}{2 \cdot 3!} \left(\overline{\psi} \gamma \iota_\xi d_{\omega_0}(\psi) - \iota_\xi d_{\omega_0}(\overline{\psi}) \gamma \psi \right) \\
&= \int_\Sigma \frac{1}{2} \iota_\xi e^2 F_\omega + \iota_\xi(\omega - \omega_0) ed_\omega e - i \frac{e^3}{2 \cdot 3!} \left(\overline{\psi} \gamma \mathrm{L}_\xi^{\omega_0}(\psi) - \mathrm{L}_\xi^{\omega_0}(\overline{\psi}) \gamma \psi \right).
\end{aligned} \tag{3.17}$$

Theorem 3.2. *The constraints L_c^{PCD}, P_ξ^{PCD}, H_λ^{PCD} define a coisotropic submanifold with respect to the symplectic structure ϖ_s. Their Poisson brackets*[3] *read*

[3]We point out that one should not confuse L with L, which respectively indicate the constraint and the Lie derivative

$$\{P_\xi^{PCD}, P_\xi^{PCD}\} = \frac{1}{2}P_{[\xi,\xi]} - \frac{1}{2}L^{PCD}_{\iota_\xi\iota_\xi F_{\omega_0}} \qquad \{H_\lambda^{PCD}, H_\lambda^{PCD}\} = 0$$
$$\{L_c^{PCD}, P_\xi^{PCD}\} = L^{PCD}_{\mathrm{L}_\xi^{\omega_0}c} \qquad \{L_c^{PCD}, L_c^{PCD}\} = -\frac{1}{2}L^{PCD}_{[c,c]}$$
$$\{L_c^{PCD}, H_\lambda^{PCD}\} = -P^{PCD}_{X^{(a)}} + L^{PCD}_{X^{(a)}(\omega-\omega_0)_a} - H^{PCD}_{X^{(n)}}$$
$$\{P_\xi^{PCD}, H_\lambda^{PCD}\} = P^{PCD}_{Y^{(a)}} - L^{PCD}_{Y^{(a)}(\omega-\omega_0)_a} + H^{PCD}_{Y^{(n)}},$$

where $X = [c, \lambda\epsilon_n]$, $Y = \mathrm{L}_\xi^{\omega_0}(\lambda\epsilon_n)$ *and* $Z^{(a)}$, $Z^{(n)}$ *are the components of* $Z \in \{X, Y\}$ *with respect to the frame* (e_a, ϵ_n).

Proof. The full proof is found in B.1.1 □

3.1.2 The BFV PCD structure

Again, the data of theorem 2.2 can be translated into the BFV formalism as explained in Section 1.2.1.

Theorem 3.3. *Under Assumption 2.1, let* $\mathcal{F}_{PCD}$ *be the bundle*

$$\mathcal{F}_\Sigma^{PCD} \longrightarrow \Omega^1_{nd}(\Sigma, \mathcal{V}), \tag{3.18}$$

with local trivialisation on an open $\mathcal{U}_\Sigma \subset \Omega^1_{nd}(\Sigma, \mathcal{V}) \times \Gamma(\Sigma, \mathbb{S}_\Sigma) \times \Gamma(\Sigma, \overline{\mathbb{S}}_\Sigma)$

$$\mathcal{F}_\Sigma^{PDC} \simeq \mathcal{U}_\Sigma \times \mathcal{A}(\Sigma) \times \Gamma(\Sigma, \mathbb{S}_\Sigma) \times \Gamma(\Sigma, \overline{\mathbb{S}}_\Sigma) \oplus T^*\left(\Omega_\partial^{0,2}[1] \oplus \mathfrak{X}[1](\Sigma) \oplus C^\infty[1](\Sigma)\right) =: \mathcal{U}_\Sigma \times \mathcal{T}_{PCD}, \tag{3.19}$$

and fields denoted by $e \in \mathcal{U}_\Sigma$, $\omega \in \mathcal{A}(\Sigma)$ *in degree zero such that they satisfy the structural constraint* $\epsilon_n \left[d_\omega e + \frac{i}{4}(\bar\psi\gamma[e^2,\psi] - [e^2,\bar\psi]\gamma\psi)\right] \in \operatorname{Im} W_1^{\partial,(1,1)}$ *and* $\psi \in \Gamma(\Sigma, \mathbb{S}_\Sigma)$, *ghost fields* $c \in \Omega_\partial^{0,2}[1]$, $\xi \in \mathfrak{X}[1](\Sigma)$ *and* $\lambda \in \Omega^{0,0}[1]$ *in degree one,* $k^\daleth \in \Omega_\partial^{D-1,D-2}[-1]$, $\lambda^\daleth \in \Omega_\partial^{D-1,D}[-1]$ *and* $\zeta^\daleth \in \Omega_\partial^{1,0}[-1] \otimes \Omega_\partial^{D-1,D}$ *in degree minus one, together with a fixed* $\epsilon_n \in \Gamma(\mathcal{V})$, *completing the image of elements* $e \in \mathcal{U}_\Sigma$ *to a basis of* $\mathcal{V}$; *define a symplectic form and an action functional on* $\mathcal{F}_{PC}^\Sigma$ *respectively by*

$$\varpi_{PCD}^\Sigma = \varpi_{PC}^\Sigma + \int_\Sigma i\frac{e^2}{4}\left(\overline{\psi}\gamma\delta\psi - \delta\overline{\psi}\gamma\psi\right)\delta e + i\frac{e^3}{3!}\delta\overline{\psi}\gamma\delta\psi,$$
$$\mathcal{S}_\Sigma^{PCD} = \mathcal{S}_\Sigma^{PCD} + \int_\Sigma \frac{i}{2\cdot 3!}e^3\left(\overline{\psi}\gamma d_\omega\psi - d_\omega\overline{\psi}\gamma\psi\right).$$

Then the triple $(\mathcal{F}^\Sigma_{PCD}, \varpi_\Sigma^{PC}, \mathcal{S}_\Sigma^{PC})$ *defines a BFV structure on* Σ.

Proof. We follow the same strategy of [CCS21a], from which we also borrow the notation. The only bit that we need to prove, is that the new BFV action $\mathcal{S}_\Sigma^{PCD}$ still satisfies the classical master equation

$$\{\mathcal{S}_\Sigma^{PCD}, \mathcal{S}_\Sigma^{PCD}\} = \iota_{Q^\Sigma_{PCD}}\iota_{Q^\Sigma_{PCD}}\varpi^\Sigma_{PCD} = 0, \tag{3.20}$$

where Q_S is the Hamiltonian vector field of $\mathcal{S}_\Sigma^{PCD}$, defined by $\iota_{Q_S}\varpi_S = \delta\mathcal{S}_\Sigma^{PCD}$. In order to do so, we can exploit the results of [CCS21a] and, noticing that $\mathcal{S}_\Sigma^{PCD} = \mathcal{S}_\Sigma^{PC} * S_{\text{Dirac}}$, by linearity we get

$$\{\mathcal{S}_\Sigma^{PCD}, \mathcal{S}_\Sigma^{PCD}\} = \{\mathcal{S}_\Sigma^{PC}, \mathcal{S}_\Sigma^{PC}\} + 2\{\mathcal{S}_\Sigma^{PC}, S_{\text{Dirac}}\} + \{S_{\text{Dirac}}, S_{\text{Dirac}}\}.$$

We have that $\{\mathcal{S}_{\Sigma}^{PC}, \mathcal{S}_{\Sigma}^{PC}\} = 0$ from Theorem 2.3. The remaining part $2\{\mathcal{S}_{\Sigma}^{PC}, S_{\text{Dirac}}\} + \{S_{\text{Dirac}}, S_{\text{Dirac}}\} = 0$ is instead a consequence of Theorem 3.2. Indeed, the explicit computation of the second bracket follows verbatim the computation of the brackets between the constraints in the proof of the aforementioned theorem by just considering only the terms containing ψ. Nonetheless, the first bracket produces in a trivial way exactly the results of these brackets, since S_{Dirac} does not depend on the ghost momenta. □

3.1.3 Degenerate case

In this section, we study the case where the boundary is null–like, i.e. the induced metric g^{∂} on Σ is degenerate. The same consideration as in the free gravity case hold, with the caveat of the introduction of a torsion term in the structural constraint. This also affects the degeneracy constraint. Explicitly, the structural and the degeneracy constraints take the form

$$\begin{cases} \epsilon_n(\alpha_\psi - p_{\mathcal{T}}\alpha_\psi) \in \operatorname{Im} W_1^{\Sigma,(1,1)} \\ p_{\mathcal{T}}\alpha_\psi = 0. \end{cases} \tag{3.21}$$

with

$$\alpha_\psi := d_\omega e + \frac{i}{4}(\overline{\psi}\gamma[e^2, \psi] - [e^2, \overline{\psi}]\gamma\psi). \tag{3.22}$$

We also recall the definition of subspaces

$$\mathcal{T} := \operatorname{Ker} W_1^{\Sigma(2,1)} \cap J \subset \Omega_{\partial}^{2,1} \tag{3.23}$$

$$\mathcal{S} := \operatorname{Ker} W_1^{\Sigma,(1,3)} \cap \operatorname{Ker} \tilde{\varrho}^{(1,3)} \subset \Omega_{\Sigma}^{1,3} \tag{3.24}$$

$$\mathcal{K} := \operatorname{Ker} W_1^{\Sigma,(1,2)} \cap \operatorname{Ker} \varrho^{(1,2)} \subset \Omega_{\partial}^{1,2}. \tag{3.25}$$

3.1.4 Some non–recurring technical results

Before we continue, we introduce some useful lemmata containing important identities and results.

Proposition 3.1. *Let $\tau \in \mathcal{S}$. Then, $\tau = \epsilon_n\beta$ with $\beta \in \Omega_{\Sigma}^{1,2}[1]$ such that $\epsilon_n\beta \in \operatorname{Ker}\tilde{\varrho}^{1,3}$ and ϵ_n defined as above.*

Proof. From 2.2, in particular, we have that

$$p_{\mathcal{T}}\alpha = 0 \quad \Longrightarrow \quad \int_{\Sigma} \tau\alpha = 0 \quad \forall \tau \in \mathcal{S}, \tag{3.26}$$

for $\alpha \in \Omega_{\Sigma}^{2,1}$. Now, consider an $\alpha \in \Omega_{\Sigma}^{2,1}$ such that $p_{\mathcal{T}}\alpha = 0$ holds together with the structural constraint $\epsilon_n(\alpha - p_{\mathcal{T}}\alpha) = e\sigma$ (notice that this subset of $\Omega_{\Sigma}^{2,1}$ is in general non-trivial because we do not require the condition $\alpha \in \operatorname{Ker} W_1^{\Sigma,(2,1)}$ as in 2.1), then it follows that

$$\int_{\Sigma} \tau\alpha = \int_{\Sigma} ec\alpha + \epsilon_n\beta\alpha = \int_{\Sigma} ecp_{\mathcal{T}^C}\alpha + \beta e\sigma = \int_{\Sigma} ecp_{\mathcal{T}^C}\alpha, \tag{3.27}$$

where $p_{\mathcal{T}^C}$ is the projection onto a complement of $\mathcal{T}$. Since the right hand side of (3.26) must hold for all $\tau \in \mathcal{S}$, if the intersection $\mathcal{S} \cap \operatorname{Im} W_1^{\Sigma,(0,2)}$ were not trivial, we would have an absurdum. This implies $c \in \operatorname{Ker} W_1^{\Sigma,(0,2)}$ for all $\tau \in \mathcal{S}$, which, thanks to the injectivity of $W_1^{\Sigma,(0,2)}$, is equivalent to $c = 0$.

Lastly, the fact that $\epsilon_n\beta \in \operatorname{Ker}\tilde{\varrho}^{1,3}$ follows immediately from the definition of $\mathcal{S}$. □

Lemma 3.2. *Given $A \in \Omega_{\Sigma}^{k,i}$ and $B \in \Omega_{\Sigma}^{l,j}$ with $i, j = 2, 3$ such that $i + j < 6$, then we have*

$$B(\overline{\psi}\gamma[A,\psi] - [A,\overline{\psi}]\gamma\psi) = (-1)^{|A||B|}A(\overline{\psi}\gamma[B,\psi] - [B,\overline{\psi}]\gamma\psi). \tag{3.28}$$

Proof. The proof goes by direct computation of

$$\begin{aligned}
B\gamma\iota_\gamma\iota_\gamma A &= B\gamma\gamma^a\gamma^b[v_a,[v_b,A]] \\
&= (-1)^{|B|}([v_a,B]\gamma + (-1)^{|B|}B\gamma_a)\gamma^a\gamma^b[v_b,A] \\
&= (-1)^{|B|}([v_a,B]\gamma\gamma^a - 4(-1)^{|B|}B)\gamma^b[v_b,A] \\
&= -\big([v_b,[v_a,A]]\gamma\gamma^a\gamma^b - (-1)^{|B|}([v_a,B]\gamma_b\gamma^a\gamma^b + 4[\gamma,B])\big)A \\
&= -(-(-1)^{|B|}\gamma\iota_\gamma\iota_\gamma B - 6(-1)^{|B|}\iota_\gamma B)A \\
&= (-1)^{|B|}(-1)^{|A|(|B|+1)}A(\gamma\iota_\gamma\iota_\gamma B + 6\iota_\gamma B)
\end{aligned}$$

and

$$\begin{aligned}
B\iota_\gamma\iota_\gamma A\gamma &= (-1)^{|A||B|}\gamma^a\gamma^b[v_a,[v_b,A]]B\gamma \\
&= (-1)^{|A||B|}(-1)^{|A}\gamma^a\gamma^b[v_b,A]([v_a,B]\gamma + (-1)^{|B|}\gamma_a B) \\
&= -(-1)^{|A||B|}A\gamma^a\gamma^b\big([v_b,[v_a,B]]\gamma - (-1)^{|B|}([v_a,B]\gamma_b - \gamma_a[v_b,B])\big) \\
&= -(-1)^{|A||B|}A\big(-\iota_\gamma\iota_\gamma B\gamma + (-1)^{|B|}(4[\gamma,B] + \gamma^a\gamma^b\gamma_a[v_b,B])\big) \\
&= (-1)^{|A||B|}A(\iota_\gamma\iota_\gamma B\gamma - (-1)^{|B|}6\iota_\gamma B).
\end{aligned}$$

Then, we can conclude the proof by considering the four possible parities of A and B. □

Proposition 3.2. *Let $\tau \in \mathcal{S}$ and e be a diagonal degenerate boundary vielbein, i.e. $e^*\eta = i^*\tilde{g}$ with $\eta = \mathrm{diag}(1,1,1-1)$ and $i^*\tilde{g} = \mathrm{diag}(1,1,0)$. Then, we have*

$$\epsilon_n[\tau, e] = 0. \tag{3.29}$$

Proof. Given $a = 1,2,3,4$ and let $\mu = 1,2,+$ be the coordinates on the boundary Σ such that we can write the diagonal degenerate boundary vielbein e as

$$\hat{e}^a = \begin{cases} e_1^a & = \delta_1^a \\ e_2^a & = \delta_2^a \end{cases}$$

$$e_+^a = \delta_3^a - \delta_4^a$$

$$\epsilon_n^a = \delta_3^a + \delta_4^a.$$

Then, the definition of $\tau \in \mathcal{S}$ implies the following relations

$$\begin{aligned}
\tau_+^{abc} &= 0 \quad \forall a,b,c \\
\tau_\mu^{123} &= 0 \quad \mu = 1,2 \\
\tau_\mu^{124} &= 0 \quad \mu = 1,2 \\
\tau_1^{234} &= \tau_2^{134} \\
\tau_1^{134} &= -\tau_2^{234}.
\end{aligned}$$

The proof follows simply by computing $\epsilon_n[\tau, e]$ in components implementing the explicit form of the diagonal vielbein above[4]. □

Lemma 3.3. *Let*[5] $A \in \Omega_\Sigma^{k,i}$ *with* $2 \leq i \leq 4$. *Then, it holds*

$$\gamma \iota_\gamma \iota_\gamma A = (-1)^{|A|}(\iota_\gamma \iota_\gamma A\gamma + 4(i-1)[\gamma, A]). \tag{3.30}$$

Proof.

$$\begin{aligned}\gamma \iota_\gamma \iota_\gamma A &= (i-2)!\gamma^a\gamma^b\gamma^c v_a \iota_{v_b}\iota_{v_c} A \\ &= -(i-2)!(\gamma^b\gamma^a\gamma^c + 2\eta^{ab}\gamma^c)v_a \iota_{v_b}\iota_{v_c} A \\ &= -(i-2)!(-\gamma^b\gamma^c\gamma^a + 4\eta^{ab}\gamma^c)v_a \iota_{v_b}\iota_{v_c} A \\ &= (-1)^{|A|}(\iota_\gamma \iota_\gamma A\gamma + 4(i-1)[\gamma, A]).\end{aligned}$$

□

Remark 3.3. This lemma introduces a relation between the action of the brackets over the Clifford algebra and $\mathcal{V}$-algebra. In particular, it is consistent a triviality condition on the bracket in the Clifford algebra, i.e.

$$[A, \overline{\psi}\gamma\psi] = (-1)^{|A|}\overline{\psi}[A, \gamma]_{\mathcal{V}}\psi = \overline{\psi}\gamma[A, \psi]_{Cl} + [\overline{\psi}, A]_{Cl}\gamma\psi,$$

where we occasionally added some redundancy with the labels of the specific algebras, even if we will not use them in general.

3.1.5 The second class constraints

Now, perhaps unsurprisingly, we prove that the constraints form a second class set, under the assumption of degenerate boundary. We first notice that we can get rid of some extra term thanks to the following Proposition.

Proposition 3.3. *Let* $\tau \in \mathcal{S}$. *Then, we have the following identity*

$$\tau(\overline{\psi}\gamma[e^2, \psi] - [e^2, \overline{\psi}]\gamma\psi) = 0. \tag{3.31}$$

Proof. The proof comes by applying twice 3.2. Therefore, by means of 3.1, we have

$$\begin{aligned}\tau(\overline{\psi}\gamma[e^2, \psi] - [e^2, \overline{\psi}]\gamma\psi) &= \epsilon_n\beta(\overline{\psi}\gamma[e^2, \psi] - [e^2, \overline{\psi}]\gamma\psi) \\ &= \epsilon_n e^2(\overline{\psi}\gamma[\beta, \psi] - [\beta, \overline{\psi}]\gamma\psi) \\ &= e\beta(\overline{\psi}\gamma[\epsilon_n e, \psi] - [\epsilon_n e, \overline{\psi}]\gamma\psi) \\ &= 0,\end{aligned}$$

since $\beta \in \mathrm{Ker} W_1^{\Sigma,(1,2)}$. □

[4]We refer to [CCT21] for further details about this kind of computations.
[5]Notice that this may be also a *shifted* variable, like τ for example.

With $\tau \in \mathcal{S}[1]$, the constraints are given by

$$L_c^{PCD} = \int_\Sigma ced_\omega e - i\frac{e^3}{2\cdot 3!}\left([c,\overline{\psi}]\gamma\psi - \overline{\psi}\gamma[c,\psi]\right) \tag{3.32}$$

$$P_\xi^{PCD} = \int_\Sigma \frac{1}{2}\iota_\xi(e^2)F_\omega + \iota_\xi(\omega-\omega_0)ed_\omega e - i\frac{e^3}{2\cdot 3!}\left(\overline{\psi}\gamma \mathrm{L}_\xi^{\omega_0}(\psi) - \mathrm{L}_\xi^{\omega_0}(\overline{\psi})\gamma\psi\right) \tag{3.33}$$

$$H_\lambda^{PCD} = \int_\Sigma \lambda\epsilon_n\Big(eF_\omega + \frac{\Lambda}{3!}e^3 + i\frac{e^2}{4}\left(\overline{\psi}\gamma d_\omega\psi - d_\omega\overline{\psi}\gamma\psi\right)\Big) \tag{3.34}$$

$$R_\tau^{PCD} = \int_\Sigma \tau d_\omega e. \tag{3.35}$$

Theorem 3.4. *Let i^*g be degenerate. Then, the Poisson brackets of the constraints read*

$$\{L_c^{PCD}, L_c^{PCD}\} = -\frac{1}{2}L_{[c,c]} \qquad \{P_\xi^{PCD}, P_\xi^{PCD}\} = \frac{1}{2}P_{[\xi,\xi]}^{PCD} - \frac{1}{2}L_{\iota_\xi\iota_\xi F_{\omega_0}}$$

$$\{L_c^{PCD}, P_\xi^{PCD}\} = L_{\mathcal{L}_\xi^{\omega_0}c}^{PCD} \qquad \{H_\lambda^{PCD}, H_\lambda^{PCD}\} \approx 0$$

$$\{L_c^{PCD}, R_\tau^{PCD}\} = -R_{p_\mathcal{S}[c,\tau]}^{PCD} \qquad \{R_\tau^{PCD}, P_\xi^{PCD}\} = R_{p_\mathcal{S}\mathcal{L}_\xi^{\omega_0}\tau}^{PCD}$$

$$\{R_\tau^{PCD}, H_\lambda^{PCD}\} \approx G_{\lambda\tau} + K_{\lambda\tau}^{PCD} \qquad \{R_\tau^{PCD}, R_\tau^{PCD}\} \approx F_{\tau\tau}$$

$$\{L_c^{PCD}, H_\lambda^{PCD}\} = -P_{X^{(a)}}^{PCD} + L_{X^{(a)}(\omega-\omega_0)_a}^{PCD} - H_{X^{(n)}}^{PCD}$$

$$\{P_\xi^{PCD}, H_\lambda^{PCD}\} = P_{Y^{(a)}}^{PCD} - L_{Y^{(a)}(\omega-\omega_0)_a}^{PCD} + H_{Y^{(n)}}^{PCD},$$

with $X = [c, \lambda\epsilon_n]$, $Y = \mathcal{L}_\xi^{\omega_0}(\lambda\epsilon_n)$ and where the superscripts (a) and (n) describe their components with respect to e_a, ϵ_n. Furthermore, $F_{\tau\tau}$, $G_{\lambda\tau}$ and $K_{\lambda\tau}^{PCD}$ are functionals of $e, \omega, \psi, \overline{\psi}, \tau$ and λ defined in the proof which are not proportional to any other constraint.

Proof. The proof is found in B.1.2 □

3.2 The BV PCD theory

We are now interested in investigating the BV structure of the Palatini-Cartan-Dirac theory in the bulk. As in Chapter 2.4 we look at the Hamiltonian vector fields of the constraints and their Poisson brackets, as they provide insights in the definition of the cohomological vector field of the BV theory, and subsequently of the BV PCD action.

We start by defining the space of fields as usual, i.e.

$$\mathcal{F}_{PCD}^M := T^*[-1](F_{PCD} \times F_{\text{ghosts})} = T^*[-1](\Omega_{\text{n.d.}}^{(1,1)} \times \mathcal{A}(M) \times \Omega^0(M, \Pi\mathbb{S}_D) \times \Omega^{(0,2)}[1] \times \mathfrak{X}[1](M)).$$

Remark 3.4. To make computations slightly easier, we will omit considering the Dirac conjugate field $\bar{\psi}$ as an independent field, and we will fix it as $\bar{\psi} = \psi^\dagger\gamma_0$.

We therefore obtain antifields $e^\dashv \in \Omega^{(3,3)}[-1]$, $\omega^\dashv \in \Omega^{(3,2)}[-1]$, $\psi^\dashv \in \Omega^{(4,4)}[-1](\Pi\mathbb{S}_D)$, $c^\dashv \in \Omega^(4,2)[-2]$ and $\xi^\dashv \in \Omega^{(1,0)}[-2] \otimes \Omega^{(4,4)}$. The BV PCD symplectic form is the canonical one,

$$\varpi_{PCD}^M = \varpi_{PC}^M + \int_M \frac{i}{2}(\delta\bar{\psi}^\dashv\delta\psi + \delta\bar{\psi}\delta\psi^\dashv).$$

Theorem 3.5. *The BV PCD is given by the data $(\mathcal{F}_{PCD}^M, \mathcal{S}_{PCD}^M, \varpi_{PCD}^M, Q_{PCD}^M)$, with*

$$\mathcal{S}_{PCD}^M = \mathcal{S}_{PC}^M + \int_M \frac{i}{2\cdot 3!}e^3\left(\bar{\psi}\gamma d_\omega\psi - d_\omega\bar{\psi}\gamma\psi\right) + \frac{i}{2}\bar{\psi}^\dashv(\mathrm{L}_\xi^\omega(\psi) - [c,\psi]) - \frac{i}{2}(\mathrm{L}_\xi^\omega(\bar{\psi}) - [c,\bar{\psi}])\psi^\dashv.$$

Proof. A very quick computation gives, for the Hamiltonian vector field of $\mathcal{S}^M_{PCD}$,

$$
\begin{aligned}
&Q_{PCD}e = Q_{PC}e && Q_{PCD}\omega = Q_{PC}\omega\\
&Q_{PCD}\psi = \mathrm{L}^\omega_\xi(\psi) + [c,\psi] && Q_{PCD}\bar{\psi} = \mathrm{L}^\omega_\xi(\bar{\psi}) + [c,\bar{\psi}]\\
&Q_{PCD}c = Q_{PC}c && Q_{PCD}\xi = Q_{PC}\xi
\end{aligned}
$$

$$
\begin{aligned}
&Q_{PCD}e^{\dagger} = Q_{PC}e + \frac{i}{4}e^2\left(\bar{\psi}\gamma d_\omega\psi - d_\omega\bar{\psi}\gamma\psi\right)\\
&Q_{PCD}\omega^{\dagger} = Q_{PC}\omega^{\dagger} + \frac{i}{4}e(\bar{\psi}\gamma[e^2,\psi] - [e^2,\bar{\psi}]\gamma\psi) - \frac{i}{2}\left([\iota_\xi\bar{\psi}^{\dagger},\psi] - [\bar{\psi},\iota_\xi\psi^{\dagger}]\right)\\
&Q_{PCD}\psi^{\dagger} = \frac{i}{3!}e^3\gamma d_\omega\psi - \frac{i}{4}e^2 d_\omega e\gamma\psi + \mathrm{L}^\omega_\xi(\psi^{\dagger}) - [c,\psi^{\dagger}]\\
&Q_{PCD}\bar{\psi}^{\dagger} = \frac{i}{3!}e^3 d_\omega\bar{\psi}\gamma + \frac{i}{4}e^2 d_\omega e\bar{\psi}\gamma + \mathrm{L}^\omega_\xi(\bar{\psi}^{\dagger}) - [c,\bar{\psi}^{\dagger}]\\
&Q_{PCD}c^{\dagger} = Q_{PC}c^{\dagger} + \frac{i}{2}\left([\bar{\psi}^{\dagger},\psi] - [\bar{\psi},\psi^{\dagger}]\right)\\
&Q_{PCD}\xi^{\dagger}_{\bullet} = Q_{PC}\xi^{\dagger}_{\bullet} - \frac{i}{2}\left(\bar{\psi}(d_\omega\psi)_{\bullet} - (d_\omega\bar{\psi})_{\bullet}\psi^{\dagger}\right).
\end{aligned}
$$

At this point it is enough to check $Q^2_{PCD} = 0$ on the fields and ghosts, disregarding the antifields. This will ensure the CME is satisfied. Setting $Q_{PCD} = Q_{PC} + Q_D$, we first see $Q^2_{PCD} = Q^2_{PC} + [Q_{PC}, Q_D] + Q^2_D$. We notice that on e, ω, ξ and c one has $Q_{PCD} = Q_{PC}$, and since $Q^2_{PC} = 0$, we only need to check $Q^2_{PCD}\psi = 0$.

$$Q^2_{PCD}\psi = Q_{PCD}(\mathrm{L}^\omega_\xi(\psi) + [c,\psi]) = 0,$$

following directly from theorem 21 in [CS19b]. □

Remark 3.5. Recall PC theory is not BV BFV extendible, the PCD theory exhibits the same behaviour, as the introduction of the spinor fields does not cure the singularity of the presymplectic form induced on the boundary.

3.3 The AKSZ PCD theory

Once again, we consider the cylindrical manifold $M = I \times \Sigma$ and apply the construction of section 1.4.1 to the BFV theory of theorem 3.3. As in section 2.5.1, we promote the fields in $\mathcal{F}^\Sigma_{PCD}$ to fields in $\mathcal{F}^{AKSZ}_{PCD}$ by considering

$$
\begin{aligned}
\mathfrak{e} &= e + f^{\dagger} & \mathfrak{w} &= \omega + u^{\dagger}\\
\mathfrak{p} &= \psi + \theta^{\dagger} & \bar{\mathfrak{p}} &= \bar{\psi} + \bar{\theta}^{\dagger}\\
\mathfrak{c} &= c + w & \mathfrak{z} &= \xi + z\\
\mathfrak{l} &= \lambda + \mu & \mathfrak{c}^{\dagger} &= k^{\dagger} + c^{\dagger}\\
\mathfrak{y}^{\dagger} &= e^{\dagger} + y^{\dagger}
\end{aligned}
\tag{3.36}
$$

where we used the same letters for the boundary fields which are now promoted to fields in $\Omega^\bullet(I) \otimes \mathcal{F}^\Sigma_{PC}$. In particular, if $\phi \in \mathcal{F}^\Sigma_{PC}$, the corresponding AKSZ field becomes

$$\mathfrak{P} = \phi + \varphi^{\dagger}, \qquad \text{where} \qquad \phi \in \mathcal{C}^\infty(I) \otimes \mathcal{F}^\Sigma_{PC}, \qquad \text{and} \qquad \varphi^{\dagger} \in \Omega^1[-1](I) \otimes \mathcal{F}^\Sigma_{PC}.$$

Then, applying theorem 1.2, we obtain

Theorem 3.6. *The AKSZ data* $\mathfrak{F}^{AKSZ}_{PCD}$ *on* $M = I \times \Sigma$ *are given by*

$$\mathcal{F}^{AKSZ}_{PCD} = T^*[-1](\mathrm{Map}(I, \mathcal{F}^{\Sigma}_{PCD}),$$

$$\varpi^{AKSZ}_{PCD} = \varpi^{AKSZ}_{PCD} + \int_{I\times\Sigma} \frac{i}{4}\mathfrak{e}^2(\bar{\mathfrak{p}}\gamma\delta\mathfrak{p} - \delta\bar{\mathfrak{p}}\gamma\mathfrak{p})\delta\mathfrak{p} + \frac{i}{3!}\mathfrak{e}^3\delta\bar{\mathfrak{p}}\gamma\delta\mathfrak{p},$$

$$\mathcal{S}^{AKSZ}_{PCD} = \mathcal{S}^{AKSZ}_{PC} + \int_{I\times\Sigma} \frac{i}{2\cdot 3!}\mathfrak{e}^3\bar{\mathfrak{p}}\gamma(d_I\mathfrak{p} + \mathrm{L}^{\mathfrak{w}}_{\mathfrak{z}}\mathfrak{p} - [c,\mathfrak{p}]) + \frac{i}{4}\epsilon_n \mathfrak{l}\mathfrak{e}^2\bar{\mathfrak{p}}\gamma d_{\mathfrak{w}}\mathfrak{p} + \text{c.c.},$$

where it is understood that only the terms containing fields in $\Omega^1[-1](I)$ *should be selected in the above expressions, to obtain a top form on* $I \times \Sigma$.

We unravel the above expressions, to find

$$\begin{aligned}\varpi^{AKSZ}_{PCD} = \varpi^{AKSZ}_{PCD} + \int_{I\times\Sigma} &\frac{i}{2}ef^{\urcorner}(\bar{\psi}\gamma\delta\psi)\delta e + \frac{i}{4}e^2(\bar{\psi}\gamma\delta\psi)\delta f^{\urcorner} + \frac{i}{4}e^2(\bar{\theta}^{\urcorner}\gamma\delta\psi \\ &+ \bar{\psi}\gamma\delta\theta^{\urcorner})\delta e + \frac{i}{2}e^2 f^{\urcorner}\delta\bar{\psi}\gamma\delta\psi + \frac{i}{3!}e^3\left(\delta\bar{\theta}^{\urcorner}\gamma\delta\psi\right) + \text{c.c.},\end{aligned} \tag{3.37}$$

and

$$\begin{aligned}\mathcal{S}^{AKSZ}_{PCD} = \mathcal{S}^{AKSZ}_{PC} + \int_{I\times\Sigma} &\frac{i}{4}e^2 f^{\urcorner}\bar{\psi}\gamma(\mathrm{L}^{\omega}_{\xi}\psi - [c,\psi]) + \frac{i}{2\cdot 3!}e^3\bar{\theta}^{\urcorner}\gamma(\mathrm{L}^{\omega}_{\xi}\psi - [c,\psi]) \\ &+ \frac{i}{2\cdot 3!}e^3\bar{\psi}\gamma(\mathrm{L}^{\omega}_{z}\psi + [\iota_{\xi}u^{\urcorner},\psi] + \mathrm{L}^{\omega}_{\xi}\theta^{\urcorner} - [w,\psi] - [c,\theta^{\urcorner}]) \\ &+ \frac{i}{4}\mu\epsilon_n e^2\bar{\psi}\gamma d_{\omega}\psi + \frac{i}{2}\lambda\epsilon_n e f^{\urcorner}\bar{\psi}\gamma d_{\omega}\psi + \frac{i}{2\cdot 3!}e^3\bar{\psi}\gamma\partial_n\psi \\ &+ \frac{i}{4}\lambda\epsilon_n e^2(\bar{\theta}^{\urcorner}\gamma d_{\omega}\psi + \bar{\psi}\gamma[u^{\urcorner},\psi] + \bar{\psi}\gamma d_{\omega}\theta^{\urcorner}) + \text{c.c.}\end{aligned} \tag{3.38}$$

Now we want to compare the AKSZ PCD theory with the BV PCD theory in the bulk. To do so, we proceed as in section 2.5.1 and find a map $\Phi : \mathcal{F}^{AKSZ}_{PCD} \to \mathcal{F}_{PCD}$ such that $\Phi^*(\varpi^M_{PCD}) = \varpi^{AKSZ}_{PCD}$ and such that the image of $\mathcal{F}^{AKSZ}_{PC}$ under Φ coincides with the restricted space of BV fields $\mathcal{F}^r_{PCD}$, defined in the following.

Definition 3.1. The restricted space of BV PCD fields is given by the subspace of $\mathcal{F}_{PCD}$ satisfying the following structural constraints

$$\underline{\mathfrak{W}}^{\urcorner} := \underline{\tilde{\omega}}^{\urcorner}_n - \iota_{\underline{z}}\tilde{\omega}^{\urcorner} - \iota_{\tilde{\xi}}\underline{\tilde{c}}^{\urcorner}_n + \iota_{\underline{z}}\tilde{c}^{\urcorner}_n\tilde{\xi}^n \in \mathrm{Im}(W^{(1,1)}_{\tilde{e}}) \tag{3.39}$$

$$\epsilon_n\left(d_{\tilde{\omega}}\tilde{e} + \frac{i}{4}(\bar{\psi}\gamma[e^2,\psi] - [e^2,\bar{\psi}]\gamma\psi)\right) - \epsilon_n W^{-1}_{\tilde{e}}(\underline{\mathfrak{W}})d\tilde{\xi}^n + \iota_{\hat{X}}(\tilde{\omega}^{\urcorner}_n - \tilde{c}^{\urcorner}_n\tilde{\xi}^n) \in \mathrm{Im}(W^{(1,1)}_{\tilde{e}}), \tag{3.40}$$

where $W^{(i,j)}_{\tilde{e}^k} : \Omega^{(i,j)} \to \Omega^{(i+k,j+k)} : \alpha \mapsto \tilde{e}^k \wedge \alpha$ shares the same properties of $W^{\partial,(i,j)}_k$ and

$$\tilde{X} = \mathrm{L}^{\tilde{\omega}}_{\tilde{\xi}}(\epsilon_n) - d_{\tilde{\omega}_n}(\epsilon_n)\tilde{\xi}^n - [\tilde{c},\epsilon_n]; \qquad \hat{X} = \tilde{e}^i_a\tilde{X}^a\partial_i \tag{3.41}$$

- $\varpi^r_{PCD} := \varpi^M_{PCD}(I\times\Sigma)|_{\mathcal{F}^r_{PCD}}$;
- $\mathcal{S}^r_{PCD} := \mathcal{S}_{PCD}|_{\mathcal{F}^r_{PCD}}$;
- $Q^r_{PCD} = Q_{PCD}$.

In the following, we indicate any bulk field ϕ with the bold character $\boldsymbol{\phi}$. Furthermore, letting $\boldsymbol{\phi} \in \Omega^k(I \times \Sigma)$, we set

$$\boldsymbol{\phi} = \tilde{\phi} + \underline{\tilde{\phi}_n}, \qquad \text{with} \quad \tilde{\phi} \in \Omega^k(\Sigma) \otimes \mathcal{C}^\infty(I), \quad \underline{\tilde{\phi}_n} \in \Omega^{k-1}(\Sigma) \otimes \Omega^1(I),$$

assuming x^n to be the coordinate along I, then $\underline{\tilde{\phi}_n} = \tilde{\phi}_n dx^n$, with $\tilde{\phi}_n \in \mathcal{C}^\infty(I) \otimes \Omega^{k-1}(\Sigma)$. In the same way a vector field $\boldsymbol{\zeta} \in \mathfrak{X}(I \times \Sigma)$ is going to be split as

$$\boldsymbol{\zeta} = \tilde{\zeta} + \overline{\tilde{\zeta}}^n, \qquad \text{with} \quad \tilde{\zeta} \in \mathfrak{X}(\Sigma) \otimes \mathcal{C}^\infty(I), \quad \overline{\tilde{\zeta}}^n \in \mathcal{C}^\infty(\Sigma) \otimes \mathfrak{X}(I),$$

with $\overline{\tilde{\zeta}}^n = \tilde{\zeta}^n \partial_n$. Furthermore, we fix $\epsilon_n \in \Gamma(M, \mathcal{V})$ such that $\delta\epsilon_n = \partial_n \epsilon_n = 0$ and such that $\{\tilde{e}_i, \epsilon_n\}$ form a basis of $\mathcal{V}$.

Remark 3.6. In principle one would need to show that the restricted BV PCD theory $\mathfrak{F}^r_{PCD}$ is a genuine BV theory, which is achieved by checking the CME. However, we will obtain it automatically as a corollary of the next theorem.

Theorem 3.7. *There exists a symplectomorphism* $\varpi\colon (\mathcal{F}^{AKSZ}_{PCD}, \varpi^{AKSZ}_{PCD}) \to (\mathcal{F}^r_{PCD}, \varpi^r_{PCD})$ *such that* $\varphi^*(\mathcal{S}^r_{PCD}) = \mathcal{S}^{AKSZ}_{PCD}$*. This data, together with the naive BV embedding* $\iota_r : \mathfrak{F}^r_{PCD} \hookrightarrow \mathfrak{F}_{PCD}$ *gives a BV embedding* $\Phi := \iota_r \circ \varphi$*.*

Proof. The full proof is found in B.1.3 □

Chapter 4

The Reduced Phase Space of $N = 1$, $D = 4$ Supergravity

Supergravity is defined as the supersymmetry theory containing gravity, in which the SUSY is realised locally (the spinor parameter χ is a function of the spacetime coordinates $\chi(x)$. We investigate here the $N = 1$ case, namely the case in which only one supersymmetry generator is introduced, in 4 dimensions, as it is the starting point for further generalizations. We start with pure gravity, and subsequently couple it with a Majorana-type spinor, which will act as the gravitino, the superpartner of the graviton.

Let M be a spin manifold and let P_{Spin} be a principal Spin$(3,1)$ bundle over M. We introduce a 4-dimensional real vector space V with a Lorentz-type metric η of signature $(-,+,+,+)$. Without loss of generality we can assume that $\eta =$diag$(-1,1,1,1)$ is the Minkowski metric and define the associated bundle (called 'Minkowski bundle') $\mathcal{V} := P_{\text{Spin}} \times_\Lambda V$, where Λ is the spin 1 representation of Spin$(3,1)$.

Remark 4.1. Notice that the double cover $l\colon \text{Spin}(3,1) \to SO(3,1)$ induces a bundle morphism to a $SO(3,1)$ bundle $\hat{l}\colon P_{\text{Spin}} \to P_{SO}$, hence $\mathcal{V} \simeq P_{SO} \times_{\Lambda_0} V$, where Λ_0 is the vector representation of $SO(3,1)$, such that $\Lambda = \Lambda_0 \circ l$. Furthermore, one can identify elements of the Lie algebra of Spin$(3,1)$ with the second wedge power of V, as it defines 4×4 antisymmetric matrices: $\mathfrak{spin}(3,1) = \mathfrak{so}(3,1) \simeq \wedge^2 V$.

The last ingredient we need in our setting is what is commonly known as Dirac spinor bundle, namely the following associated vector bundle $\mathbb{S}_D := P_{\text{spin}} \times_\gamma \mathbb{C}^4$, where γ is the gamma representation of the Clifford algebra $\mathcal{C}(V)$ restricted to its spin subgroup Spin(V) $\simeq$Spin$(3,1)$.[1]

The independent fields of the theory are:

- The coframe e (also known as vielbein or tetrad in $D = 4$) defined as an isomorphism $e : TM \to \mathcal{V}$, inducing a metric on spacetime as $g := e^*(\eta)$, i.e. such that $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$, where $\mu = 1,2,3,4$ are curved indices on M while $a = 0,1,2,3$ are flat indices on V.[2] The coframe has the advantage of being expressed as a differential form, indeed $e = e^a_\mu dx^\mu v_a \in \Omega^1(M,\mathcal{V})$, where x are coordinates on M and $\{v_a\}$ is a basis of V.
- The spin connection ω. The space of connections is denoted by $\mathcal{A}_M$, and is locally modeled by 1-forms on M with values in the Lie algebra $\mathfrak{so}(3,1) = \mathfrak{spin}(3,1)$, in our notation $\omega = \omega_\mu dx^\mu v_a \wedge v_b \in \Omega^1(M, \wedge^2\mathcal{V})$.

[1] For more details about the notations and the convention see [Fil25]

[2] Note that e enjoys an internal Lorentz symmetry (acting on the flat indices) on top of the usual diffeomorphisms.

- The gravitino ψ, a spin-$\frac{3}{2}$ Majorana spinor, i.e. a 1-form on M with values in the subbundle of Majorana spinors $\mathbb{S}_M := \{\chi \in \mathbb{S}_D \mid \bar{\chi} := \chi^\dagger\gamma_0 = \chi^t C\}$, where C is the charge conjugation matrix. Furthermore, as we are dealing with a fermion, we need to reverse the parity[3] of $\mathbb{S}_M$, obtaining $\psi = \psi_\mu dx^\mu \in \Omega^1(M, \Pi\mathbb{S}_M)$.

The theory is described by the following action functional[4]

$$S_{SG} = \int_M \frac{e^2}{2} F_\omega + \frac{1}{3!} e\bar{\psi}\gamma^3 d_\omega \psi, \tag{4.1}$$

where $F_\omega = d\omega + \frac{1}{2}[\omega, \omega]$ is the curvature of the connection, γ is an element of $V \otimes \mathcal{C}(V)$ defined by $\gamma = \gamma^a v_a$,[5] and $\{v_a\}$ is a basis of V. Lastly, we define $d_\omega\psi := d\psi - \frac{1}{4}\omega^{ab}\gamma_{ab}\psi$,[6] having set $\gamma_{ab} = \gamma_{[a}\gamma_{b]} = \frac{1}{2}[\gamma_a, \gamma_b]$.

Remark 4.2. In general, when dealing with gamma matrices, we will omit the wedge symbol in products of gamma matrices, so that $\gamma^k := \gamma \wedge \cdots \wedge \gamma = \gamma^{a_1\cdots a_k} v_{a_1} \wedge \cdots \wedge v_{a_k}$ will automatically select the anti-symmetrized product of k gamma matrices. The same hold for the wedge multiplication of k coframes.

Remark 4.3. The bracket $[\cdot, \cdot]$ is defined to encode any (possibly graded[7]) Lie algebra action.[8] In the general case, if a field ϕ transforms in a representation ρ of the Spin group, then we have $[\omega, \phi] := \rho(\omega)(\phi)$. In the case of the gravitino field, transforming in the gamma representation, we obtain

$$[\omega, \psi] = \gamma(\omega^{ab} v_a \wedge v_b)(\psi) = \omega^{ab}\gamma(v_a \wedge v_b)(\psi) = -\frac{1}{4}\omega^{ab}\gamma_{ab}\psi,$$

where $\gamma(v_a \wedge v_b) = -\frac{1}{4}\gamma_{ab}$ is the image under the gamma representation of the generators of the Lie algebra $\mathfrak{spin}(3,1)$[9].

The variation of the $\mathcal{N} = 1, D = 4$ supergravity action produces a boundary term and a bulk term containing the Euler-Lagrange equations

$$\delta S_{SG} = \int_M \left(eF_\omega + \frac{1}{3!}\bar{\psi}\gamma^3 d_\omega\psi\right)\delta e + e\left(d_\omega e - \frac{1}{2}\bar{\psi}\gamma\psi\right)\delta\omega + \frac{1}{3}\left(\frac{1}{2}d_\omega e\bar{\psi}\gamma^3 + e d_\omega\bar{\psi}\gamma^3\right)\delta\psi$$
$$- \int_{\partial M} \frac{e^2}{2}\delta\omega + \frac{1}{3!}e\bar{\psi}\gamma^3\delta\psi,$$

having used the fact that[10]

$$-\frac{1}{3!}e\bar{\psi}\gamma^3[\delta\omega, \psi] = -\frac{1}{2}e\bar{\psi}\gamma\psi\delta\omega. \tag{4.2}$$

[3]The parity reversed Majorana spinor bundle is defined as $\Pi\mathbb{S}_M$ and simply given by $\mathbb{S}_M$ with the requirement that the components of each spinor are Grassmann-odd.

[4]We omit the symbol $\wedge$ when multiplying differential forms and sections of the exterior algebra of V, but the wedge product is assumed in both. Parity in the algebra is defined as the sum of the fermionic parity, the form degree modulo 2, the degree in ΛV modulo 2, and the ghost number (to be introduced below) modulo 2.

[5]Notice in our notation we have the following relations

$$\{\gamma_a, \gamma_b\} = -2\eta_{ab} \qquad \{\gamma_\mu, \gamma_\nu\} = -2g_{\mu\nu},$$

having set $\gamma_\mu = e^a_\mu\gamma_a$.

[6]Alternatively, one can define for all $\alpha \in \wedge^2 V$, $[\alpha, \psi] := \frac{1}{4}\gamma^{ab}\iota_{v_a}\iota_{v_b}\alpha\psi = -\frac{1}{4}\gamma^{ab}\alpha_{ab}\psi$, having set $\iota_{v_a} v_c := \eta_{ac}$.

[7]In our convention, the parity of an element $\alpha \in \Omega^i(M, \wedge^j V)$ is defined to be $|\alpha| = i + j \mod 2$. In the same way, a pure Majorana spinor has parity 1, so that in the case of the gravitino, $|\psi| = 1 + 1 \mod 2 = 0$.

[8]The bracket $[\cdot, \cdot]$ on $\wedge^\bullet V$ (encoding the action of the Lorentz group) can also be induced from the pairing in V, indeed if for any $A, B \in V$ we define $[A, B] := -(-1)^{|B|}\eta(A, B) = -(-1)^{|B|}A^a B^b \eta_{ab}$, then one can extend the action bi-linearly to $\wedge^k V$ requiring that the graded Leibniz rule holds. Furthermore, notice that the bracket defined above is graded, i.e. $[A, B] = -(-1)^{|A||B|}[B, A]$, where $|\cdot|$ denotes the parity.

[9]One can show $-\frac{1}{4}\gamma_{ab}$ are generators of $\mathfrak{spin}(3,1)$

[10]This identity is quickly obtained by applying formula (A.50).

We then obtain the following equations of motion:

$$eF_\omega + \frac{1}{3!}\bar{\psi}\gamma^3 d_\omega\psi = 0, \tag{4.3}$$

$$e\left(d_\omega e - \frac{1}{2}\bar{\psi}\gamma\psi\right) = 0, \tag{4.4}$$

$$ed_\omega\bar{\psi}\gamma^3 + \frac{1}{2}d_\omega e\bar{\psi}\gamma^3 = 0. \tag{4.5}$$

Remark 4.4. In the bulk, eq. (4.4) is equivalent to $d_\omega e - \frac{1}{2}\bar{\psi}\gamma\psi = 0$[11], implying that the background connection has torsion, while eq. 4.5 is equivalent to its complex conjugate, and can be re-interpreted (after imposing (4.4)) as the Rarita-Schwinger equation for a massless Majorana spinor in a curved background

$$e\gamma^3 d_\omega\psi - \frac{1}{4}(\bar{\psi}\gamma\psi)\gamma^3\psi = e\gamma^3 d_\omega\psi = 0.$$

On-shell vs off-shell supersymmetry invariance

So far we have been considering the connection as a dynamical field, in what is called the Palatini-Cartan formalism, also known as the first-order formulation of (super)gravity, referring to the fact that only first order derivatives appear in the Lagrangian. If we impose (4.4) in the absence of the gravitino, we obtain the torsionless condition, which, coupled with the metricity condition, gives the Levi-Civita connection as the pullback of ω by e. Applying the torsionless condition, one obtains the Einstein–Hilbert Lagrangian, which describes the second order formulation of gravity.

In the case of supergravity, (4.4) implies the non-vanishing of torsion, which will be quadratically dependent on the Majorana field ψ. Historically, the formulation of supergravity has been performed in the second–order formalism (the so called 'half-shell' case), i.e. after imposing the kinematical constraint (4.4).

In this setting, introducing a spinorial gauge parameter $\chi = \chi(x)$, defined to be an even[12] section of the Majorana spinor bundle, the infinitesimal supersymmetry transformations on the fields read

$$\delta_\chi e = -\bar{\chi}\gamma\psi, \qquad \delta_\chi\psi = d_\omega\chi,$$

with no need of specifying the variation of ω as it is constrained and can be obtained as a function of ψ and e from (4.4). It is indeed very quick to check the invariance of the action under these transformations

$$\begin{aligned}\delta\chi S_{SG} &= \int_\Sigma -ie\bar{\chi}\gamma\psi F_\omega + \frac{1}{3!}(-i\bar{\chi}\gamma\psi)(\bar{\psi}\gamma^3 d_\omega\psi) + \frac{1}{3!}e(d_\omega\bar{\chi}\gamma^3 d_\omega\psi + \bar{\psi}\gamma^3[F_\omega,\chi]) \\ &= \int_\Sigma -ie\bar{\chi}\gamma\psi F_\omega + \frac{1}{3}(e\bar{\psi}\gamma^3[F_\omega,\chi] - [F_\omega,\bar{\psi}]\gamma^3\chi]) - \frac{1}{3!}\left(d_\omega e - \frac{1}{2}\bar{\psi}\gamma\psi\right)\bar{\chi}\gamma^3 d_\omega\psi = 0,\end{aligned}$$

having used the constraint (4.4), identity (A.50), integration by parts, the Bianchi identity $d_\omega d_\omega(\cdot) = [F_\omega,\cdot]$ and the Fierz identity (A.62) together with the flip relation (A.54) to show $\bar{\psi}\gamma^3 d_\omega\psi\bar{\chi}\gamma\psi = d_\omega\bar{\psi}\gamma^3\psi\bar{\chi}\gamma\psi = -d_\omega\bar{\psi}\gamma^3\chi\bar{\psi}\gamma\psi - d_\omega\bar{\psi}\gamma^3\psi\bar{\chi}\gamma\psi$, implying $\bar{\psi}\gamma^3 d_\omega\psi\bar{\chi}\gamma\psi = -\frac{1}{2}\bar{\psi}\gamma\psi\bar{\chi}\gamma^3 d_\omega\psi$.

[11] That is because $e\wedge\cdot$ is an injective map when acting on $\Omega^2(M,\mathcal{V})$, and it is in fact an isomorphism.

[12] The reason we consider an unphysical Grassmann even fermion will be clear in the following section, as it will represent the ghost field associated to the gravitino

If instead one keeps ω unconstrained, it is necessary to introduce the corresponding local SUSY transformation, which can be either derived by the requirement that the action remains invariant under the local supersymmetry (postulating the same transformations for e and ψ), or by the analysis of the symplectic structure of the fields on the boundary $\Sigma = \partial M$. We use here the first method, discarding the vanishing terms from the previous computations

$$\begin{aligned}\delta\chi S_{SG} &= \int_M -\frac{e^2}{2} d_\omega(\delta_\chi \omega) - \frac{1}{3!} e\bar{\psi}\gamma^3[\delta_\chi\omega, \psi] - \frac{1}{3!}\left(d_\omega e - \frac{1}{2}\bar{\psi}\gamma\psi\right)\bar{\chi}\gamma^3 d_\omega\psi \\ &= \int_M \left(d_\omega e - \frac{1}{2}\bar{\psi}\gamma\psi\right)\left(e\delta_\chi\omega - \frac{1}{3!}\bar{\chi}\gamma^3 d_\omega\psi\right),\end{aligned}$$

from which we obtain

$$\delta_\chi e = -\bar{\chi}\gamma\psi, \tag{4.6}$$

$$e\delta_\chi\omega = \frac{1}{3!}\bar{\chi}\gamma^3 d_\omega\psi, \tag{4.7}$$

$$\delta_\chi\psi = d_\omega\chi \tag{4.8}$$

Notice that we have given only $e\delta_\chi\omega$ and not the explicit expression of $\delta_\chi\omega$ because it is not strictly necessary, since we are sure that $e\delta_\chi\omega$ uniquely determines the expression for $\delta_\chi\omega$. Indeed it suffices to notice that $W_e^{(1,2)} := e \wedge \cdot : \Omega^1(M, \wedge^2\mathcal{V}) \to \Omega^2(M, \wedge^3\mathcal{V})$ provides an isomorphism,[13] hence $\delta_\chi\omega$ is uniquely defined by the above equation.

In the following, it is convenient to recall the following notation, setting $\Sigma := \partial M$,

$$\Omega^{(k,l)} := \Omega^k(M, \wedge^l\mathcal{V}) \qquad \Omega_\partial^{(k,l)} := \Omega^k(\Sigma, \wedge^l\mathcal{V}),$$

furthermore, we define the coframes e as those elements in $\Omega^{1,1}$ non-degenerate, hence $e \in \Omega_{\text{n.d.}}^{(1,1)}$.

4.1 Constraint analysis of $\mathcal{N} = 1$, $D = 4$ SUGRA

We briefly recall the form of the action of $\mathcal{N} = 1, D = 4$ Supergravity:

$$S_{SG} = \int_M \frac{e^2}{2} F_\omega + \frac{1}{3!} e\bar{\psi}\gamma^3 d_\omega\psi. \tag{4.9}$$

The boundary term in the variation of the action depends only on the value of the fields at the boundary. In particular, we consider only those tetrads defining non-degenerate metrics on the boundary.[14]

We have

$$\delta S_{SG} = \int_M \text{EL}_M - \int_\Sigma \frac{e^2}{2}\delta\omega + \frac{1}{3!} e\bar{\psi}\gamma^3\delta\psi,$$

hence obtaining

$$\tilde{\varpi}_{SG}^\partial = \int_\Sigma e\delta e\delta\omega + \frac{1}{3!}\bar{\psi}\gamma^3\delta\psi\delta e + \frac{1}{3!} e\delta\bar{\psi}\gamma^3\delta\psi. \tag{4.10}$$

---

[13] A proof of this statement is found in [Can24]

[14] Specifically, we require $g_{ij}^\partial := (e_i, e_j)$ to be non denegenerate on Σ, i.e. either time–like or space–like.

First reduction and structural constraint

Many of the following tools have been discusses in the previous chapters, notably in 1 and in 2.

As it turns out, $\tilde{\varpi}^{\partial}_{SG}$ is closed but not non–degenerate, its kernel is given by $\mathrm{Ker}(\tilde{\varpi}^{\partial}_{SG}) = \{\int_\Sigma \mathbb{X}_\omega \frac{\delta}{\delta\omega} \in \mathfrak{X}(\tilde{F}_\Sigma) \mid e\mathbb{X}_\omega = 0,\ \mathbb{X}_\omega \in \Omega^{(1,2)}_\partial\}$.[15] In other words, any vector field acting on the boundary connections as $\omega \mapsto \omega + v$, such that $ev = 0$, is in the kernel of the pre–symplectic form. We then define the geometric phase space as the quotient of the space of preboundary fields with respect to the action of the vector fields in $\mathrm{Ker}(\tilde{\varpi}^{\partial}_{SG})$ (i.e. consider the charachteristic foliation of this distribution), obtaining

$$F^{\partial}_{SG} := \left(\Omega^1_{n.d.}(\Sigma,\mathcal{V}) \times \mathcal{A}(\Sigma) \times \Omega^1(\Sigma,\mathbb{S}_M)\right)/\ker(\tilde{\varpi}^{\partial}_{SG}) = \Omega^1_{n.d.}(\Sigma,\mathcal{V}) \times \mathcal{A}_{\Sigma,\mathrm{red}} \times \Omega^1(\Sigma,\mathbb{S}_M),$$

where $\mathcal{A}_{\Sigma,\mathrm{red}} := \mathcal{A}_\Sigma/\{\omega \sim \omega + v,\ ev = 0\}$.

At this point, the constraints are simply obtained by restricting the equations of motion (4.3), (4.4) and (4.5) to the boundary, defining functions on $\tilde{F}_\Sigma$. However as pointed out in [CCS21a], the constraints are generally not invariant under the distribution defined by $\ker(\tilde{\varpi}^{\partial}_{SG})$[16], hence they cannot be naively extended to functions on F_Σ. In order to do so, we cleverly fix a representative of $[\omega] \in \mathcal{A}_{\Sigma,\mathrm{red}}$ (i.e. choose a v-section), such that it imposes the non-invariant part of the constraint.

Remark 4.5. Notice that, in the bulk, the torsion equation $d_\omega e - \frac{1}{2}\bar{\psi}\gamma\psi = 0$ is equivalent to $e\left(d_\omega e - \frac{1}{2}\bar{\psi}\gamma\psi\right) = 0$, but it is not the case when e is the vielbein restricted to the boundary, as $W_1^{\partial(2,1)}$ is not an isomorphism and in particular not injective. Indeed one finds that $e\left(d_\omega e - \frac{1}{2}\bar{\psi}\gamma\psi\right)$ has 6 local components, and is invariant under the action of $\ker(\tilde{\varpi}^{\partial}_{SG})$, while the remaining part of $d_\omega e - \frac{1}{2}\bar{\psi}\gamma\psi$ has another 6 local components, which will be used to fix $v \in \ker(W_e^{\partial(1,2)})$.

Definition 4.1. In the following, we will denote by ϵ_n a nowhere vanishing section in $\Gamma(\Sigma,\mathcal{V})$ such that, if we let $e = e_i dx^i$, with $i = 1,2,3$, $\{e_1, e_2, e_3, \epsilon_n\}$ is a local basis of $\mathcal{V}$.

Remark 4.6. Notice that, in a neighborhood $\mathcal{U} \subset \Omega^{1,1}_{\partial,n.d.}$ of a given tetrad e, we can choose ϵ_n independently of the e's (implying that $\delta_{\mathrm{fields}}\epsilon_n = 0$). In particular, we choose ϵ_n such that its Lie derivative with respect to the vector field normal to the boundary vanishes.

Theorem 4.1. *Assume the metric g^∂ induced by the boundary vielbein e is non-degenerate. Then for any $\tilde{\omega} \in \Omega^{(1,2)}_\partial$ there exists a unique decomposition $\tilde{\omega} = \omega + v$ such that*

$$ev = 0 \qquad \text{and} \qquad \epsilon_n\left(d_\omega e - \frac{1}{2}\bar{\psi}\gamma\psi\right) = e\sigma, \tag{4.11}$$

[15] To be precise, any tangent vector field $\mathbb{X} = \int_\Sigma \mathbb{X}_e \frac{\delta}{\delta e} + \mathbb{X}_\omega \frac{\delta}{\delta\omega} + \mathbb{X}_\psi \frac{\delta}{\delta\psi}$ is in the kernel of $\tilde{\varpi}^{1,4}_{SG}$ iff $e\mathbb{X}_e = 0$, $e\gamma^3\mathbb{X}_\psi = 0$ and $e\mathbb{X}_\omega = 0$, but the first two conditions imply $\mathbb{X}_e = 0$ and $\mathbb{X}_\psi = 0$.

[16] One can check that, for any $v \in \Omega^{(1,2)}_\partial$ such that $ev = 0$, equations (4.3) and (4.5) are invariant under $\omega \mapsto \omega + v$, after applying (4.4), hence they will only depend on $[\omega] \in \mathcal{A}_{\Sigma,\mathrm{red}}$. In particular, one sees

$$\delta_v\left(eF_{\omega+v} + \frac{1}{3!}\bar{\psi}\gamma^3 d_{\omega+v}\psi\right) = -ed_\omega v - \frac{1}{2}e\bar{\psi}\gamma\psi v = -d_\omega(ev) + e\left(d_\omega e - \frac{1}{2}\bar{\psi}\gamma\psi\right)v \approx 0,$$

where the symbol $\approx$ is used to indicate an equality modulo equations of motion, i.e. an equality holding on–shell. We also see

$$\delta_v\left(e\gamma^3 d_\omega\psi - \frac{1}{2}d_\omega e\gamma^3\psi\right) = -e\gamma^3[v,\psi] - \frac{1}{2}[v,e]\gamma^3\psi \overset{(A.50)}{=} -3ev\gamma\psi - \frac{1}{2}e[v,\gamma^3]_V\psi - \frac{1}{2}v[e,\gamma^3]\psi = 0,$$

having used $ev = 0$ and $e[v,\gamma^3]_V = [ev,\gamma^3] - v[e,\gamma^3]$.

for some $\sigma \in \Omega_\partial^{(1,1)}$. Furthermore, the constraint $d_\omega e - \frac{1}{2}\bar{\psi}\gamma\psi = 0$ splits as

$$d_\omega e - \frac{1}{2}\bar{\psi}\gamma\psi = 0 \quad \Leftrightarrow \quad \begin{cases} e\left(d_\omega e - \frac{1}{2}\bar{\psi}\gamma\psi\right) = 0 \\ \epsilon_n\left(d_\omega e - \frac{1}{2}\bar{\psi}\gamma\psi\right) \in \mathrm{Im}(W_1^{\partial(1,1)}) \end{cases} \tag{4.12}$$

We call $\epsilon_n\left(d_\omega e - \frac{1}{2}\bar{\psi}\gamma\psi\right) = e\sigma$ structural constraint and $e\left(d_\omega e - \frac{1}{2}\bar{\psi}\gamma\psi\right) = 0$ invariant constraint.

Proof. We start by noticing that the splitting of the torsion constraints into structural and invariant part is a simple consequence of lemma A.5. Now, from lemma A.6, we know that there exist $\sigma \in \Omega_\partial^{(1,1)}$ and $v \in \mathrm{Ker}(W_1^{\partial(1,2)})$ such that

$$\epsilon_n\left(d_{\tilde{\omega}} e - \frac{1}{2}\bar{\psi}\gamma\psi\right) = e\sigma + \epsilon_n[v, e].$$

Then one fixes $\omega := \tilde{\omega} - v$, obtaining the desired result. The uniqueness is simply shown by assuming there exist different splitting $\tilde{\omega} = \omega_1 + v_1 = \omega_2 + v_2$ as above, implying $[e, v_1 - v_2] \in \mathrm{Im}(W_1^{\partial(1,1)})$, which, by lemma A.6 and A.5 shows $v_1 = v_2$. □

Constraints and the first class condition

Now we can finally define the constraints on F_{SG}^∂ simply as the restriction of (4.5) and (4.3) to the boundary plus the invariant torsion constraint. In order to readily have them as functionals over F_{SG}^∂, we make use of Lagrange multipliers[17] $\mu \in \Omega_\partial^{(1,1)}[1]$, $c \in \Omega_\partial^{(0,2)}$ and $\chi \in \Gamma(\mathbb{S}_M|_\Sigma)$, obtaining

$$\begin{aligned} J_\mu &= \int_\Sigma \mu\left(eF_\omega + \frac{1}{3!}\bar{\psi}\gamma^3 d_\omega\psi\right) \\ L_c &= \int_\Sigma c\left(ed_\omega e - \frac{1}{2}e\bar{\psi}\gamma\psi\right) \\ M_\chi &= \int_\Sigma \frac{1}{3}\bar{\chi}\left(e\gamma^3 d_\omega\psi - \frac{1}{2}d_\omega e\gamma^3\psi\right). \end{aligned}$$

Remark 4.7. As we will see later, the Hamiltonian vector fields are related to the gauge symmetries of the theory (i.e. they define the infinitesimal gauge transformations). In particular, L_c generates the internal Lorentz symmetry, M_χ is the generator of the supersymmetry and J_μ generates the diffeomorphism symmetry. The last statement can be refined once one notices that, since $\{e_1, e_2, e_3, \epsilon_n\}$ defines a local basis of $\mathcal{V}$, it is possible to split $\mu = \lambda\epsilon_n + \iota_\xi e$, with $\lambda \in \mathcal{C}^\infty(\Sigma)[1]$ and $\xi \in \mathfrak{X}[1](\Sigma)$. Then ξ and λ can be interpreted respectively as the gauge parameters associated to the tangential and transversal diffeomorphisms with respect to Σ. The constraint J_μ splits into

$$P_\xi = \int_\Sigma \frac{1}{2}\iota_\xi(e^2)F_\omega + \frac{1}{3!}\iota_\xi e\bar{\psi}\gamma^3 d_\omega\psi \quad \text{and} \quad H_\lambda = \int_\Sigma \lambda\epsilon_n\left(eF_\omega + \frac{1}{3!}\bar{\psi}\gamma^3 d_\omega\psi\right).$$

[17]In view of the BFV description of the theory in the following chapter, we shift the degree of the Lagrange multiplier by one, as they will later represent the ghosts of the theory.

Remark 4.8. Notice that L_c can be rewritten in a nicer form, in particular one finds[18]

$$L_c = \int_\Sigma ced_\omega e + \frac{1}{3!}e\bar{\psi}\gamma^3[c,\psi].$$

To further simplify the computations, we introduce a reference connection ω_0 and use Cartan magic formula to define, for any field ϕ, the Lie derivative along ξ with respect to ω_0 as

$$\mathrm{L}_\xi^{\omega_0}\phi := [\iota_\xi, d_{\omega_0}]\phi = \iota_\xi d_{\omega_0}\phi - d_{\omega_0}\iota_\xi\phi.$$

Then, to make the dependence of P_ξ on the newly defined Lie derivative apparent and to make the Hamiltonian vector field of P_ξ well defined, we take into consideration the following redefinition:[19] $P_\xi \to P_\xi + L_{\iota_\xi(\omega-\omega_0)} + M_{\iota_\xi\psi}$, yielding

$$P_\xi = \int_\Sigma \frac{1}{2}\iota_\xi e^2 F_\omega + \iota_\xi(\omega-\omega_0)ed_\omega e - \frac{1}{3!}e\bar{\psi}\gamma^3\mathrm{L}_\xi^{\omega_0}\psi.$$

Lastly, it will be convenient to rewrite M_χ in the following form, obtained by integrating by parts

$$M_\chi = \int_\Sigma \frac{1}{3!}e(d_\omega\bar{\chi}\gamma^3\psi + \bar{\chi}\gamma^3 d_\omega\psi)$$

Theorem 4.2. *Let g^∂ be non-degenerate on Σ. Then the functions P_ξ, H_λ, L_c and M_χ form a first-class set of constraints, defining a coisotropic submanifold as their zero-locus. In particular*

$$\begin{aligned}
\{L_c, L_c\} &= -\frac{1}{2}L_{[c,c]} & \{L_c, M_\chi\} &= M_{[c,\chi]}\\
\{L_c, P_\xi\} &= L_{\mathrm{L}_\xi^{\omega_0}c} & \{P_\xi, M_\chi\} &= -M_{\mathrm{L}_\xi^{\omega_0}\chi}\\
\{P_\xi, P_\xi\} &= \frac{1}{2}P_{[\xi.\xi]} - \frac{1}{2}L_{\iota_\xi\iota_\xi F_{\omega_0}} & \{H_\lambda, H_\lambda\} &= 0\\
\{L_c, H_\lambda\} &= -P_\zeta + L_{\iota_\zeta(\omega-\omega_0)} + M_{\iota_\zeta\psi} - H_{\zeta^n} & \{M_\chi, H_\lambda\} &= 0\\
\{P_\xi, H_\lambda\} &= P_\vartheta - L_{\iota_\vartheta(\omega-\omega_0)} - M_{\iota_\vartheta\psi} + H_{\vartheta^{(n)}}\\
\{M_\chi, M_\chi\} &= \frac{1}{2}P_\varphi - \frac{1}{2}L_{\iota_\varphi(\omega-\omega_0)} - \frac{1}{2}M_{\iota_\varphi\psi} + \frac{1}{2}H_{\varphi^{(n)}}
\end{aligned}\tag{4.13}$$

where, setting $\{x^i\}$ local coordinates on Σ, one has $\zeta = e_a^i[c,\lambda\epsilon_n]^a\partial_i$, $\vartheta = e_a^i(\mathrm{L}_\xi^{\omega_0}(\lambda\epsilon_n))^a\partial_i$, $\varphi = e_a^i\bar{\chi}\gamma^a\chi\partial_i$, while $\zeta^{(n)} = [c,\lambda\epsilon_n]^n$, $\vartheta^{(n)} = (\mathrm{L}_\xi^{\omega_0}(\lambda\epsilon_n))^n$, $\varphi^{(n)} = \bar{\chi}\gamma^n\chi$. Lastly, we have the following

$$\frac{1}{3!}\bar{\chi}\gamma^3 d_\omega\chi = e\alpha^\partial(\chi, d_\omega\chi) + \epsilon_n\beta^\partial(\chi, d_\omega\chi)$$
$$\epsilon_n\mathbb{M}_\omega = e\alpha^\partial(\epsilon_n\mathbb{M}_\omega) + \epsilon_n\beta^\partial(\epsilon_n\mathbb{M}_\omega),$$

where $\mathbb{M}_\omega$ is the component of the Hamiltonian vector field M_χ along ω.

[18]Here we used the fact that

$$-\frac{1}{2}ce\bar{\psi}\gamma\psi = \frac{i}{2\cdot 3!}e\left(\bar{\psi}\gamma^3[c,\psi] + [c,\psi]\gamma^3\psi\right) = \frac{i}{3!}\bar{\psi}\gamma^3[c,\psi].$$

[19]Redefining the constraint set as a $\mathcal{C}^\infty(F_{SG}^{1,4})$-linear combination of the original constraints does not change the zero-locus of such constraints, which ultimately is what we will be interested in.

Proof. We begin by computing the Hamiltonian vector fields of the constraints, defined by $\iota_{\mathbb{X}_f}\varpi^\partial_{SG} = \delta f$.

We see

$$\iota_{\mathbb{X}}\varpi^\partial_{SG} = \int_\Sigma e\mathbb{X}_e\delta\omega + \left(e\mathbb{X}_\omega + \frac{1}{3!}\mathbb{X}_{\bar\psi}\gamma^3\psi\right)\delta e + \frac{1}{3}\left(\frac{1}{2}\mathbb{X}_e\bar\psi\gamma^3 + e\mathbb{X}_{\bar\psi}\gamma^3\right)\delta\psi. \tag{4.14}$$

$$\begin{aligned}
\delta L_c &= \int_\Sigma [c,e]e\delta\omega + ed_\omega c\delta e + \frac{1}{3!}\bar\psi\gamma^3[c,\psi]\delta e + \frac{1}{3!}e\left(\delta\bar\psi\gamma^3[c,\psi] + \bar\psi\gamma^3[c,\delta\psi]\right)\\
&= \int_\Sigma [c,e]e\delta\omega + \left(ed_\omega c + \frac{1}{3!}\bar\psi\gamma^3[c,\psi]\right)\delta e + \frac{1}{3!}e\left(\delta\bar\psi\gamma^3[c,\psi] + [c,\bar\psi]\gamma^3\delta\psi - \bar\psi[c,\gamma^3]\delta\psi\right)\\
&= \int_\Sigma [c,e]e\delta\omega + \left(ed_\omega c + \frac{1}{3!}\bar\psi\gamma^3[c,\psi]\right)\delta e + \frac{1}{3}e\left([c,\bar\psi]\gamma^3\delta\psi - \frac{1}{2}\bar\psi[c,\gamma^3]_V\delta\psi\right)\\
&= \int_\Sigma [c,e]e\delta\omega + \left(ed_\omega c + \frac{1}{3!}\bar\psi\gamma^3[c,\psi]\right)\delta e + \frac{1}{3}\left(\frac{1}{2}[c,e]\bar\psi\gamma^3 + e[c,\bar\psi]\gamma^3\right)\delta\psi
\end{aligned}$$

Remark 4.9. In the second step we used the fact that an element in $\wedge^2 V$ acts on γ both in the spinor representation and in the vector one, and they cancel each other out. Explicitly, using the fact that $\wedge^2 V \simeq \mathfrak{so}(3,1) \simeq \mathfrak{spin}(3,1)$, a basis for it is given by $\{-\frac{1}{4}\gamma_{[a}\gamma_{b]}\}$ or equivalently by $\{v_a \wedge v_b\}$. Now, since $\gamma = \gamma^a v_a$ has values in $\mathcal{C}(V)\otimes V$, when acting with c we have $[c,\gamma] = [c,\gamma]_S + [c,\gamma]_V = 0$, in fact[20]

$$[c,\gamma]_S := \frac{1}{4}[\![[\gamma,[\gamma,c]],\gamma]\!] = -\frac{1}{4}(c^{ab}\gamma_a\gamma_b\gamma + \gamma c^{ab}\gamma_a\gamma_b) = -c^{ab}\gamma_a v_b = -[c,\gamma]_V.$$

Now, since $\bar\chi\gamma^N\psi$ has no spinor indices (for any two arbitrary spinors χ and ψ), $[c,\bar\chi\gamma^N\psi] = (-1)^{|\chi|}\bar\chi[c,\gamma^N]_V\psi$, but at the same time the Leibniz rule for $[c,\cdot]$ holds, we find

$$[c,\bar\chi\gamma^N\psi] = (-1)^{|\chi|}\bar\chi[c,\gamma^N]_V\psi = [c,\chi]\gamma^N\psi - (-1)^{|\chi|}\bar\chi\gamma^N[c,\psi]. \tag{4.15}$$

$$\begin{aligned}
\delta P_\xi &= \int_\Sigma -\mathrm{L}_\xi^{\omega_0}e\delta\omega - \left(\iota_\xi F_{\omega_0} + \mathrm{L}_\xi^{\omega_0}(\omega-\omega_0) + \frac{1}{3!}\bar\psi\gamma^3\mathrm{L}_\xi^{\omega_0}\psi\right)\delta e - \frac{1}{3!}e\left(\delta\bar\psi\gamma^3\mathrm{L}_\xi^{\omega_0}\psi + \bar\psi\gamma^3\mathrm{L}_\xi^{\omega_0}\delta\psi\right)\\
&= \int_\Sigma -\mathrm{L}_\xi^{\omega_0}e\delta\omega - \left(\iota_\xi F_{\omega_0} + \mathrm{L}_\xi^{\omega_0}(\omega-\omega_0) + \frac{1}{3!}\bar\psi\gamma^3\mathrm{L}_\xi^{\omega_0}\psi\right)\delta e - \frac{1}{3}\left(e\mathrm{L}_\xi^{\omega_0}\bar\psi\gamma^3 + \frac{1}{2}\mathrm{L}_\xi^{\omega_0}e\bar\psi\gamma^3\right)\delta\psi;
\end{aligned}$$

$$\begin{aligned}
\delta H_\lambda &= \int_\Sigma d_\omega(\lambda\epsilon_n e)\delta\omega + \lambda\epsilon_n F_\omega\delta e + \frac{1}{3!}\lambda\epsilon_n(\delta\bar\psi\gamma^3 d_\omega\psi - \bar\psi\gamma^3[\delta\omega,\psi] + \bar\psi\gamma^3 d_\omega\delta\psi)\\
&= \int_\Sigma \left(d_\omega(\lambda\epsilon_n e) - \frac{1}{2}\lambda\epsilon_n\bar\psi\gamma\psi\right)\delta\omega + \lambda\epsilon_n F_\omega\delta e + \frac{1}{3}\left(\lambda\epsilon_n d_\omega\bar\psi\gamma^3 + \frac{1}{2}d_\omega(\lambda\epsilon_n)\bar\psi\gamma^3\right)\delta\psi;
\end{aligned}$$

$$\begin{aligned}
\delta M_\chi &= \int_\Sigma \frac{1}{3!}\delta e(d_\omega\bar\chi\gamma^3\psi + \bar\chi\gamma^3 d_\omega\psi) + \frac{1}{3!}e([\delta\omega,\bar\chi]\gamma^3\psi - \bar\chi\gamma^3[\delta\omega,\psi]) + \frac{1}{3}\delta\bar\psi\left(e\gamma^3 d_\omega\chi - \frac{1}{2}d_\omega e\gamma^3\chi\right)\\
&= \int_\Sigma \frac{1}{3!}(d_\omega\bar\chi\gamma^3\psi + \bar\chi\gamma^3 d_\omega\psi)\delta e - ie\bar\chi\gamma\psi\delta\omega + \frac{1}{3}\delta\bar\psi\left(e\gamma^3 d_\omega\chi - \frac{1}{2}d_\omega e\gamma^3\chi\right)
\end{aligned}$$

[20]We indicate the graded commutator by double square brackets $[\![A,B]\!] = AB - (-1)^{|A||B|}BA$.

This allows to extract the following vector fields[21]

$$
\begin{aligned}
&\mathbb{L}_e = [c, e] && \mathbb{L}_\omega = d_\omega c + \mathbb{V}_L && \mathbb{L}_\psi = [c, \psi]\\
&\mathbb{P}_e = -\mathrm{L}_\xi^{\omega_0} e && \mathbb{P}_\omega = -\iota_\xi F_{\omega_0} - \mathrm{L}_\xi^{\omega_0}(\omega - \omega_0) + \mathbb{V}_P && \mathbb{P}_\psi = -\mathrm{L}_\xi^{\omega_0}\psi\\
&\mathbb{H}_e = d_\omega(\lambda\epsilon_n) + \lambda\sigma && e\mathbb{H}_\omega = \lambda\epsilon_n F_\omega - \frac{1}{3!}\mathbb{H}_{\bar\psi}\gamma^3\psi && e\gamma^3\mathbb{H}_\psi = \lambda\epsilon_n\gamma^3 d_\omega\psi + \frac{1}{2}\lambda\sigma\gamma^3\psi\\
&\mathbb{M}_e = -\bar\chi\gamma\psi && e\mathbb{M}_\omega = \frac{1}{3!}(d_\omega\bar\chi\gamma^3\psi + \bar\chi\gamma^3 d_\omega\psi) - \frac{1}{3!}\bar\psi\gamma^3\mathbb{M}_\psi && e\gamma^3\mathbb{M}_\psi = e\gamma^3 d_\omega\chi - \frac{1}{2}d_\omega e\gamma^3\chi
\end{aligned}
\tag{4.16}
$$

Remark 4.10. Notice that the components along ω of the Hamiltonian vector fields are defined up to a term $\mathbb{V} \in \mathrm{Ker}(W_e^{\partial(1,2)})$, which is fixed by requiring that $\mathbb{X}_\omega$ preserves the structural constraint 4.12. In most of the future calculations we will only need $e\mathbb{X}_\omega$, but one can in any case prove [CCS21a] that elements in $\Omega_\partial^{(2,3)}$ are in the image of $W_e^{\partial(1,2)}$, hence such $\mathbb{X}_\omega$ always exists.

The discussion is analogous in the computation of $\mathbb{H}_\psi$ and $\mathbb{M}_\psi$, indeed by considering $e\gamma^3\mathbb{X}_\psi$, one sees

$$e\gamma^3\mathbb{X}_\psi = e^a\gamma^{bcd}\mathbb{X}_\psi\epsilon_{abcd}\mathrm{Vol}_V \overset{(A.44)}{=} i\gamma^5[e,\gamma]\mathbb{X}_\psi.$$

Now, since $\Omega_\partial^{(1,0)}(\Pi\mathbb{S}_M)$ and $\Omega_\partial^{(2,0)}(\Pi\mathbb{S}_M)$ have the same local dimension, showing $e\gamma^3$ is injective proves that it is also an isomorphism, but that amounts to show that

$$[e,\gamma]\mathbb{X}_\psi = 0 \quad \Rightarrow \quad \mathbb{X}_\psi = 0,$$

or in other words that $\gamma_{[i}\mathbb{X}_{\psi,j]} = 0$ implies $\mathbb{X}_{\psi,j} = 0$, which is immediately verified by solving the system of three equations.[22][23] One can then define $\mathbb{M}_\psi^e$ such that

$$e\gamma^3\mathbb{M}_\psi^e := -\frac{1}{2}d_\omega e\gamma^3\chi \overset{A.15}{=} -\frac{1}{2}\left(d_\omega e - \frac{1}{2}\bar\psi\gamma\psi\right)\gamma^3\chi. \tag{4.17}$$

The rest of the proof, which amounts to showing that the constraints form a first class set, is found in B.2.1. □

4.2 Towards a BFV description

Having proved that the constraints form a first-class set, under the assumption of a regular boundary, Theorem 1.1 tells us that there must exist a BFV structure on $\mathbf{F}_\Sigma$. Indeed, one can consider the bundle

$$\mathcal{F}_{SG}^\partial \to \Omega_{\partial,n.d.}^{(1,1)} \times \Omega_\partial^1(S_{\Sigma,m})$$

with local trivialization on an open $\mathcal{U}_\Sigma \subset \Omega_{\partial,n.d.}^{(1,1)} \times \Omega_\partial^1(S_{\Sigma,m})$

$$\mathcal{U}_\Sigma \times \mathcal{A}_{\Sigma,\mathrm{red}} \times T^*(\underset{(c,k^\dagger)}{\Omega_\partial^{(0,2)}[1]} \oplus \underset{(\xi,\zeta^\dagger)}{\mathfrak{X}[1](\Sigma)} \oplus \underset{(\lambda,\lambda^\dagger)}{\mathcal{C}^\infty[1](\Sigma)} \oplus \underset{(\chi,\theta_\dagger)}{\Gamma[1](\Pi\mathbb{S}_M)})$$

[21] Strictly speaking, in the computation of $\mathbb{M}_\psi$ one would have a term proportional to $\mathbb{M}_e\bar\psi\gamma^3\delta\psi = -\bar\psi\gamma\chi\bar\psi\gamma^3\delta\psi$, but this one vanishes because of A.15.

[22] We defined $\gamma_i := \gamma_a e_i^a$, which is still an invertible matrix.

[23] This computation also explicitly shows that the kernel of the pre-symplectic form (4.10) does not contain any term of the kind $\mathbb{X}_\psi\frac{\delta}{\delta\psi}$.

where $e \in \Omega^{(1,1)}_{\partial,n.d.}$, $\omega \in \mathcal{A}_{\Sigma,\text{red}}$ and $\Omega^1_\partial(S_{\Sigma,m})$, while the antighosts of c, ξ and χ are denoted respectively by $k^\daleth \in \Omega^{(3,2)}_\partial[-1]$, $\zeta^\dagger \in \Omega^{(1,0)}_\partial[-1] \otimes \Omega^{(3,4)}_\partial$ and $\theta_\daleth \in \Omega^{(3,4)}[-1](\Pi\mathbb{S}_M)$. In particular, generalizing [CCS21a], one defines $\mathcal{A}_{\Sigma,\text{red}}$ as the space of connections modeled over $\Omega^{(1,2)}_\partial$ satisfying a modified version of structural constraint (4.11), called the **BFV structural constraint**

$$\epsilon_n \left(d_\omega e - \frac{1}{2}\bar{\psi}\gamma\psi \right) + \left(\mathrm{L}^{\omega_0}_\xi (\epsilon_n)^i - [c, \epsilon_n]^i \chi \right) k^\dagger_i + \cdots = e\sigma,$$

where $\cdots$ regroups possible extra terms to ensure the constraint is invariant under the action of the cohomological vector field Q^∂_{SG}.

The canonical symplectic form is defined as

$$\varpi^\Sigma_{SG} = \int_\Sigma e\delta e\delta\omega + \frac{1}{3!}\bar{\psi}\gamma^3\delta\psi\delta e + \frac{1}{3}e\delta\bar{\psi}\gamma^3\delta\psi + \delta c\delta k^\daleth + \iota_{\delta\xi}\delta\zeta^\daleth + \delta\lambda\delta\lambda^\daleth + i\delta\bar{\chi}\delta\theta_\daleth. \tag{4.18}$$

Now, following the discussion in 1.2.1, one can, as a first attempt, define the tentative BFV action from the structure of the constraints as

$$\begin{aligned} \mathcal{S}^\Sigma_{SG} :=& L_c + M_\chi + P_\xi + H_\lambda \\ &+ \int_\Sigma \left(\frac{1}{2}[c, c] + \frac{1}{2}\iota_\xi\iota_\xi F_{\omega_0} - \mathrm{L}^{\omega_0}_\xi c \right) k^\daleth + i(\mathrm{L}^{\omega_0}_\xi \bar{\chi} - [c, \bar{\chi}])\theta_\daleth \\ &- \frac{1}{2}\iota_{[\xi,\xi]}\zeta^\daleth + \left([c, \lambda\epsilon_n]^n - \mathrm{L}^{\omega_0}_\xi (\lambda\epsilon_n)^n + \frac{1}{2}\bar{\chi}\gamma^n\chi \right) \lambda^\daleth \\ &+ \left([c, \lambda\epsilon_n]^j - \mathrm{L}^{\omega_0}_\xi (\lambda\epsilon_n)^j + \frac{1}{2}\bar{\chi}\gamma^j\chi \right) (\zeta^\dagger_j - (\omega - \omega_0)_j k^\daleth - i\bar{\psi}_j\theta_\daleth) \end{aligned} \tag{4.19}$$

At this point, in order to obtain a BFV structure, one needs to show that $\{\mathcal{S}^\Sigma_{SG}, \mathcal{S}^\Sigma_{SG}\}_{BFV} = 0$. However, doing so is a computationally hard task, which, when carried out fully[24] does not yield the desired result, hence $\mathcal{S}^\Sigma_{SG}$ needs to be complemented with terms of higher order in the antighosts.

Introducing new variables

We know from diagram (A.32) that $W^{\partial,(2,1)}_e$ is surjective, therefore it is possible to rewrite $k^\daleth = e\tilde{k}$, for (more than) a $\tilde{k} \in \Omega^{(2,1)}_\partial$. Considering the field redefiniton with $\tilde{k}$ as the new field, we immediately notice that the new symplectic form contains a term $e\delta\tilde{k}\delta c$ leading to a degeneracy, in particular

$$\mathrm{Ker}(\varpi^\Sigma_{SG}) = \left\{ \mathbb{X}_{\tilde{k}} \frac{\delta}{\delta\tilde{k}},\ \mathbb{X}_{\tilde{k}} \in \Omega^{(2,1)}_\partial \ \middle|\ e\mathbb{X}_{\tilde{k}} = 0 \right\}.$$

To obtain a well defined symplectic form one needs to consider $[\tilde{k}] \in \Omega^{(2,1)}_\partial / \mathrm{Ker}(W^{\partial,(2,1)}_e)$. However, we can cleverly fix a representative thanks to the following.

Theorem 4.3. *For all $\tilde{k} \in \Omega^{(2,1)}_\partial$ there exist a unique decomposition*

$$\tilde{k} = \check{k} + r$$

with $\check{k}, r \in \Omega^{(2,1)}_\partial$ such that

$$er = 0, \qquad \epsilon_n\check{k} = e\check{a}, \tag{4.20}$$

[24] We spare the reader of the cumbersome details of this computation.

for some $\check{a} \in \Omega_\partial^{(1,1)}$. *Furthermore, the field* $\check{k}$ *in the decomposition above only depends on the equivalence class* $[\tilde{k}] \in \Omega_{\partial,red}^{(2,1)}$.

Remark 4.11. Notice that, as an immediate consequence of the first statement we obtain $[\tilde{k}] = [\check{k}]$,

Proof. The decomposition is a direct consequence of lemma A.8. Now consider $\tilde{k}_1, \tilde{k}_2 \in [\tilde{k}]$, then by definition $\tilde{k}_1 - \tilde{k}_2 = r' \in \mathrm{Ker}(W_e^{\partial(1,2)})$. By A.8, $\tilde{k}_1 = \check{k}_1 + r_1$ and $\tilde{k}_2 = \check{k}_2 + r_2$ such that $\epsilon_n \check{k}_1, \epsilon_n \check{k}_2 \in \mathrm{Im}(W_e^{\partial,(1,1)})$ and $er_1 = er_2 = 0$, hence

$$\begin{cases} \check{k}_1 - \check{k}_2 = r_2 - r_1 - r' \in \mathrm{Ker}(W_e^{\partial(1,2)}) \\ \epsilon_n(\check{k}_1 - \check{k}_2) \in \mathrm{Im}(W_e^{\partial,(1,1)}), \end{cases}$$

which implies, by lemma A.5, that $\check{k}_1 = \check{k}_2$. □

At this point, we recall from remark 4.8 that we altered the original constraint set (to an equivalent one) by redefining P_ξ. In particular, in order to obtain nicer Hamiltonian vector fields, we had

$$P_\xi \mapsto P_\xi - L_{\iota_\xi(\omega-\omega_0)} - M_{\iota_\xi \psi}.$$

Following [CCS21a] it is convenient to change variables in order to get rid of the redundancies. We introduce

$$c' = c + \iota_\xi(\omega - \omega_0) \qquad \chi' = \chi + \iota_\xi \psi \qquad \zeta_\bullet^{\lrcorner'} = \zeta_\bullet^{\lrcorner} - (\omega - \omega_0)_\bullet k^{\lrcorner} - i\bar{\psi}_\bullet \theta_{\lrcorner}. \tag{4.21}$$

Lastly, we can define the new variable $y^{\lrcorner} \in \Omega_\partial^{(3,3)}[-1]$ such that $e_i y^{\lrcorner} = \zeta_i^{\lrcorner'}$ and $\epsilon_n y^{\lrcorner} = \lambda^{\lrcorner}$, which, combined with the field redefinition $k^{\lrcorner} = e\check{k}$, yields (omitting the $'$ apex)

$$\begin{aligned} \mathcal{S}_{SG}^{\Sigma} = \int_\Sigma & (\iota_\xi e + \lambda \epsilon_n)\left(eF_\omega + \frac{1}{3!}\bar{\psi}\gamma^3 d_\omega \psi\right) + c\left(ed_\omega e - \frac{1}{2}\bar{\psi}\gamma\psi\right) \\ & + \frac{1}{3}\bar{\chi}\left(e\gamma^3 d_\omega\psi - \frac{1}{2}d_\omega e\gamma^3\psi\right) + \left(\frac{1}{2}[c,c] - \frac{1}{2}\iota_\xi\iota_\xi F_\omega - \mathrm{L}_\xi^\omega c\right)e\check{k} \\ & + \check{k}\frac{1}{3!}(\bar{\chi} - \iota_\xi\bar{\psi})\gamma^3 d_\omega\chi - \check{a}\frac{1}{3!}(\bar{\chi} - \iota_\xi\bar{\psi})\gamma^3 d_\omega\psi - \frac{1}{2}\iota_{[\xi,\xi]}ey^{\lrcorner} - i\iota_\xi\big(\mathrm{L}_\xi^\omega\bar{\psi} - [c,\psi]\big)\theta_{\lrcorner} \\ & + i\big(\mathrm{L}_\xi^\omega\bar{\chi} - [c,\bar{\chi}]\big)\theta_{\lrcorner} + \left([c,\lambda\epsilon_n] - \mathrm{L}_\xi^\omega(\lambda\epsilon_n) + \frac{1}{2}\bar{\chi}'\gamma\chi - \iota_\xi(\bar{\chi}\gamma\psi)\right)y^{\lrcorner} \end{aligned} \tag{4.22}$$

$$\begin{aligned} \varpi_{SG}^{\Sigma} = \int_\Sigma & e\delta e\delta\omega + \frac{1}{3!}\bar{\psi}\gamma^3\delta\psi\delta e + \frac{1}{3}e\delta\bar{\psi}\gamma^3\delta\psi + \delta c'\delta(e\check{k}) + \delta\omega\delta\big(\iota_\xi(e\check{k})\big) \\ & + i\delta\bar{\chi}\delta\theta_{\lrcorner} + i\delta\bar{\psi}\delta(\iota_\xi\theta_{\lrcorner}) - \delta\big(\iota_{\delta\xi}(e)y^{\lrcorner}\big) - \delta\lambda\epsilon_n\delta y^{\lrcorner}. \end{aligned} \tag{4.23}$$

This is a good starting point but, as remarked before, it is not yet the full BFV action. In order to obtain it, one way would be to extract the Hamiltonian vector field Q_{SG}^{∂} of $\mathcal{S}_{SG}^{\Sigma}$ and compute $(Q_{SG})^2$, which would allow us to algorithmically produce the extra terms in the action. Such method, while theoretically feasible, provides many challenges. The other way to obtain a BFV action is to induce it from the BV structure in the bulk. In the following chapter, we choose the latter option.

Chapter 5

The BV-BFV description of $N = 1$, $D = 4$ Supergravity

5.1 The BV $N = 1$, $D = 4$ Supergravity Action

A BV description of on-shell $\mathcal{N} = 1$, $D = 4$ supergravity has been provided in [Bau+90], where it was shown that the BV action is of rank 2 (i.e. quadratic in the anti-fields). However, to the best of our knowledge, no off-shell BV description of it has been obtained.

We start here by applying the simplest procedure from section 1.2, defining the space of BV fields as

$$\mathcal{F}_{SG} = T^*[-1]\big(\Omega^{(1,1)}_{\text{n.d.}} \otimes \mathcal{A}_M \otimes \Omega^1(M, \Pi\mathbb{S}_M) \otimes \Omega^{(0,2)}[1] \times \mathfrak{X}[1](M) \otimes \Gamma[1](M, \Pi\mathbb{S}_M)\big),$$

where

- $e \in \Omega^{(1,1)}_{\text{n.d.}}$, $\omega \in \mathcal{A}_M$ and $\psi \in \Omega^1(M, \Pi\mathbb{S}_M)$ are the classical fields;
- $c \in \Omega^{(0,2)}[1] = \Gamma[1](M, \wedge^2\mathcal{V}) \simeq \Gamma[1](M, \mathfrak{so}(1,3))$, $\xi \in \mathfrak{X}[1](M)$ and $\chi \in \Gamma[1](M, \Pi\mathbb{S}_M)$ are the ghost fields,[1] seen as odd generators respectively to the internal Lorentz symmetry, the diffeomorphism symmetry and the local supersymmetry;
- $e^\dashv \in \Omega^{(3,3)}[-1]$, $\omega^\dashv \in \Omega^{(3,2)}[-1]$ and $\psi^\dashv \in \Omega^{(3,4)}[-1](M, \Pi\mathbb{S}_M)$ are the field momenta, while $c^\dashv \in \Omega^{(4,2)}[-2]$, $\xi^\dashv \in \Omega^1(M)[-2] \otimes \Omega^{(4,4)}$ and $\chi^\dashv \in \Omega^{(4,4)}[-2](M, \Pi\mathbb{S}_M)$ are the ghost momenta.

The -1–symplectic forms reads

$$\varpi_{SG} = \int_M \delta e \delta e^\dashv + \delta\omega\delta\omega^\dashv + i\delta\bar{\psi}\delta\psi^\dashv + \delta c\delta c^\dashv + \iota_{\delta\xi}\delta\xi^\dashv + i\delta\bar{\chi}\delta\chi^\dashv. \tag{5.1}$$

Our first attempt of finding a suitable BV action requires finding the vector field Q_0 describing

[1] Note that all the ghosts have ghost number 1, yet χ, unlike c and ξ, has even Grassmann parity.

the symmetries of the theory. We define[2]

$$Q_0 e = L_\xi^\omega e - [c, e] + \bar{\chi}\gamma\psi \qquad Q_0\omega = \iota_\xi F_\omega - d_\omega c + \delta_\chi \omega$$
$$Q_0\psi = L_\xi^\omega \psi - [c, \psi] - d_\omega \chi \qquad Q_0\xi = \frac{1}{2}[\xi, \xi] + \frac{1}{2}\varphi$$
$$Q_0 c = \frac{1}{2}(\iota_\xi \iota_\xi F_\omega - [c.c]) + \iota_\xi \delta_\chi \omega \qquad Q_0\chi = L_\xi^\omega \chi - [c, \chi] - \frac{1}{2}\iota_\varphi \psi,$$

where $e\delta_\chi\omega = -\frac{1}{3!}\bar{\chi}\gamma^3 d_\omega \psi$ and $\varphi^\mu = \bar{\chi}\gamma^\mu\chi$. In particular, for the fields on which it is defined, one can notice that $Q_0 = Q_{PC} + \delta_\chi$, having borrowed Q_{PC} from [CS19b]. Since we know $Q_{PC}^2 = 0$, we obtain

$$Q_0^2 = [Q_{PC}, \delta_\chi] + \delta_\chi^2.$$

The classical action $\mathcal{S}_0$ is then complemented with a contribution s_1 linear in the anti–fields, obtaining

$$\begin{aligned}\mathcal{S}_1 = \mathcal{S}_0 + s_1 = &\int_M \frac{e^2}{2}F_\omega + \frac{1}{3!}e\bar{\psi}\gamma^3 d_\omega \psi \\ &+ \int_M -(L_\xi^\omega e - [c, e] + \bar{\chi}\gamma\psi)e^\dashv + (\iota_\xi F_\omega - d_\omega c + \delta_\chi\omega)\omega^\dashv \\ &- i(L_\xi^\omega \bar{\psi} - [c, \bar{\psi}] - d_\omega \bar{\chi})\psi^\dashv + \left(\frac{1}{2}\iota_\xi\iota_\xi F_\omega - \frac{1}{2}[c, c] + \iota_\xi \delta_\chi \omega\right) c^\dashv \\ &+ \frac{1}{2}(\iota_{[\xi,\xi]} + \iota_\varphi)\xi^\dashv - i\left(L_\xi^\omega \bar{\chi} - [c, \bar{\chi}] - \frac{1}{2}\iota_\varphi \bar{\psi}\right)\chi^\dashv.\end{aligned}$$

In principle, to check the classical master equation $\{\mathcal{S}_1, \mathcal{S}_1\}_{BV} = 0$ it is sufficient to prove $Q_0^2 = 0$ on the fields and ghosts. Proceeding by stages, we first obtain

$$\begin{aligned}\delta_\chi^2 e &= -\frac{1}{2}\mathrm{L}_\varphi^\omega e + \frac{1}{2}\iota_\varphi\left(d_\omega e - \frac{1}{2}\bar{\psi}\gamma\psi\right) \\ \delta_\chi^2 \psi &= -\frac{1}{2}\mathrm{L}_\varphi^\omega \psi + \frac{1}{2}\iota_\varphi d_\omega \psi - \left(\bar{\chi}\kappa(< e, \underline{\gamma} d_\omega \psi >) + \frac{1}{8}\bar{\chi}\iota_{\hat{\gamma}}\iota_{\hat{\gamma}}(\underline{\gamma} d_\omega \psi)\right)\chi \\ e\delta_\chi^2 \omega &= -\frac{1}{2}e\iota_\varphi F_\omega + \frac{1}{2}\iota_\varphi\left(eF_\omega + \frac{1}{3!}\bar{\psi}\gamma^3 d_\omega\psi\right) - \frac{1}{2\cdot 3!}\bar{\psi}\iota_\varphi(\gamma^3 d_\omega \psi) \\ &\quad - \frac{1}{3!}\bar{\psi}\gamma^3\chi\left(\bar{\chi}\kappa(< e, \underline{\gamma} d_\omega \psi >) + \frac{1}{8}\bar{\chi}\iota_{\hat{\gamma}}\iota_{\hat{\gamma}}(\underline{\gamma} d_\omega \psi)\right)\end{aligned}$$
$$\delta_\chi^2 c = \frac{1}{2}\iota_\varphi \delta_\chi \omega + \iota_\xi \delta_\chi^2 \omega \qquad \delta_\chi^2 \chi = -\frac{1}{2}\mathrm{L}_\varphi^\omega \chi \qquad \delta_\chi^2 \xi = 0,$$

where $\widehat{\gamma} := \gamma^\mu \partial_\mu = e_a^\mu \gamma^a \partial_\mu$, $\underline{\gamma} := [e, \gamma] = \gamma_\mu dx^\mu$ and the map $< e, - >$ is defined via the inverse vielbein as

$$\begin{aligned}< e, - >: \Omega^{(i,j)} &\longrightarrow \Omega^{(i-1,j+1)} \\ \sigma &\longmapsto v_a \eta^{ab} e_b^\mu \iota_{\partial_\mu}\sigma.\end{aligned}$$

The interested reader can find the full computation of δ_χ^2 in B.3.1.

[2]One could obtain the correct SUSY transformations by inspecting the boundary structure and phase space Hamiltonian of supergravity.

Remark 5.1. Notice that, as expected, the square of the supersymmetry transformation is proportional to the diffeomorphisms[3] with respect to the generator $\varphi := \bar{\chi}\hat{\gamma}\chi$, plus a term which is proportional to the equations of motion.

The full expression of Q_0^2, whose full computation can be found in B.3.2, is given by

$$
\begin{aligned}
Q_0^2 e &= \frac{1}{2}\iota_\varphi \left(d_\omega e - \frac{1}{2}\bar{\psi}\gamma\psi \right) \\
Q_0^2 \psi &= \frac{1}{2}\iota_\varphi d_\omega \psi - \left(\bar{\chi}\kappa(< e, \underline{\gamma} d_\omega \psi >) + \frac{1}{8}\bar{\chi}\iota_{\hat{\gamma}}\iota_{\hat{\gamma}}(\underline{\gamma} d_\omega \psi) \right) \chi \\
e Q_0^2 \omega &= \frac{1}{2}\iota_\varphi \left(eF_\omega + \frac{1}{3!}\bar{\psi}\gamma^3 d_\omega \psi \right) + \frac{1}{2\cdot 3!}\bar{\psi}\gamma^3 \iota_\varphi d_\omega \psi \\
&\quad - \frac{1}{3!}\bar{\psi}\gamma^3\chi \left(\bar{\chi}\kappa(< e, \underline{\gamma} d_\omega \psi >) + \frac{1}{8}\bar{\chi}\iota_{\hat{\gamma}}\iota_{\hat{\gamma}}(\underline{\gamma} d_\omega \psi) \right) \\
Q_0^2 c &= \frac{1}{2}\iota_\varphi \delta_\chi \omega + \iota_\xi Q_0^2 \omega \qquad\qquad Q_0^2 \chi = 0 \qquad\qquad Q_0^2 \xi = 0,
\end{aligned}
$$

This tells us that the BV description of $\mathcal{N} = 1$, $D = 4$ SuGra is at least of second rank, hence we need to correct the action.

Lastly, for the sake of completeness, we also provide the expression of Q_0 for the anti–fields. It is obtained by computing $\delta_{\text{fields}}\mathcal{S}_1$. In particular, as we saw in section 1.3.1 , $Q_0\Phi^\dashv$ will be proportional to the equations of motion for the respective fields, and for $c^\dashv$ and $\omega^\dashv$ we will have $Q_0 = Q_{PC} + \delta_\chi$. We compute δ_χ as the Hamiltonian vector field of s_1, namely such that $\iota_{\delta_\chi}\varpi_{BV} = \delta s_1$, using

$$
\begin{aligned}
\iota_{\mathbb{X}}\varpi_{BV} = \int_M & \mathbb{X}_e \delta e^\dashv + \delta e(\mathbb{X}_{e^\dashv} + \mathbb{X}_\omega \check{\omega} + e\mathbb{X}_c \check{c}) + e\mathbb{X}_\omega \delta\check{\omega} + \delta\omega(e\mathbb{X}_\omega + \check{\omega}\mathbb{X}_e) \\
& + i\mathbb{X}_{\bar{\psi}}\delta\psi^\dashv + i\delta\bar{\psi}\mathbb{X}_{\psi^\dashv} + \frac{e^2}{2}\mathbb{X}_c \delta\check{c} + \delta c\left(\frac{e^2}{2}\mathbb{X}_{\check{c}} + e\check{c}\mathbb{X}_e \right) \\
& + \iota_{\mathbb{X}_\xi}\delta\xi^\dashv + \iota_{\delta\xi}\mathbb{X}_{\xi^\dashv} + i\mathbb{X}_{\bar{\chi}}\delta\chi^\dashv + i\delta\bar{\chi}\mathbb{X}_{\chi^\dashv},
\end{aligned}
$$

Furthermore, one can also split $s_1 = s_1^{PC} + s_1^\chi$, where s_1^{PC} is the part coming from the free

[3] This is in line with the fact that supersymmetry squares to the translations, which in their local version are realized by the diffeomorphisms.

gravity BV theory. We are then left with

$$\begin{aligned}
Q_0 e^{\lrcorner} &= eF_\omega + \frac{1}{3!}\bar{\psi}\gamma^3 d_\omega\psi + \mathrm{L}_\xi^\omega e^{\lrcorner} - [c, e^{\lrcorner}] - \frac{i}{2\cdot 3!}\iota_\varphi(e^3\bar{\psi})\chi_0^{\lrcorner} + \frac{1}{2}\iota_\varphi[v_c, e_b^\mu \eta^{bc}\xi_\mu^{\lrcorner}] \\
&\quad - \frac{1}{2\cdot 3!}\bar{\chi}\gamma^3 d_\omega\psi + \frac{1}{3!}\iota_\xi(\bar{\chi}\gamma^3 d_\omega\psi\check{c}) - \check{\omega}\delta_\chi\omega - e\check{c}\iota_\xi\delta_\chi\omega \\
eQ_0\check{\omega} &= e\left(d_\omega e - \frac{1}{2}\bar{\psi}\gamma\psi\right) - \iota_\xi[e^{\lrcorner}, e] - d_\omega(\iota_\xi\omega^{\lrcorner}) - e[c,\check{\omega}] + \frac{1}{2}d_\omega\iota_\xi\iota_\xi c^{\lrcorner} - \check{\omega}\mathrm{L}_\xi^\omega e \\
&\quad - \frac{1}{2}\bar{\chi}\gamma\psi\check{\omega} - \frac{1}{2\cdot 3!}\bar{\chi}[\check{\omega},\gamma^3]_V\psi - \frac{1}{2}\bar{\chi}\gamma\psi\left(\check{c}\iota_\xi e + \frac{1}{2}e\iota_\xi\check{c}\right) + \frac{1}{2\cdot 3!}\bar{\chi}[\check{c}\iota_\xi e + \frac{1}{2}e\iota_\xi\check{c}, \gamma^3]\psi \\
&\quad + \frac{i}{2}\iota_\xi\left(e\bar{\psi}_0^{\lrcorner}\underline{\gamma}\gamma\psi - \frac{1}{3!}\psi_0^{\lrcorner}\underline{\gamma}[e,\gamma^3]\psi\right) + \frac{i}{2}e\bar{\psi}_0^{\lrcorner}\underline{\gamma}\gamma\chi - \frac{i}{2\cdot 3!}\bar{\psi}_0^{\lrcorner}\underline{\gamma}[e,\gamma^3]\chi + \frac{i}{8}\iota_\xi\left(e^2\bar{\chi}_0^{\lrcorner}\underline{\gamma}^2\chi\right) \\
Q_0\psi^{\lrcorner} &= -\frac{i}{3}\left(e\gamma^3 d_\omega\psi - \frac{1}{2}d_\omega e\gamma^3\psi\right) - \frac{i}{3!}\gamma^3 d_\omega(\check{\omega}\chi) + \mathrm{L}_\xi^\omega\psi^{\lrcorner} - [c,\psi^{\lrcorner}] + i\gamma\chi e^{\lrcorner} \\
&\quad - \frac{i}{3!}d_\omega\left(\check{c}\iota_\xi e\gamma^3\chi + \frac{1}{2}e\iota_\xi\check{c}\gamma^3\chi\right) - \frac{1}{2}\iota_\varphi\chi^{\lrcorner} \\
\frac{e^2}{2}Q_0\check{c} &= -d_\omega\omega^{\lrcorner} - [e,e^{\lrcorner}] + \frac{i}{8}e^2\bar{\chi}_0^{\lrcorner}\underline{\gamma}^2\chi + \frac{i}{2}e\bar{\psi}_0^{\lrcorner}\underline{\gamma}\gamma\psi - \frac{i}{2\cdot 3!}\bar{\psi}_0^{\lrcorner}\underline{\gamma}[e,\gamma^3]\psi - \frac{1}{2}\check{c}\mathrm{L}_\xi^\omega e^2 - \check{c}e\bar{\chi}\gamma\psi, \\
Q_0\xi_\bullet^{\lrcorner} &= -e_\bullet^{\lrcorner} d_\omega e - d_\omega e e_\bullet^{\lrcorner} - \omega_\bullet^{\lrcorner} F_\omega - (\iota_\xi c^{\lrcorner})_\bullet F_\omega + \iota_{[\bullet,\xi]}\xi^{\lrcorner} - id\omega\bar{\psi}(\psi^{\lrcorner})_\bullet \\
&\quad + i\bar{\psi}_\bullet d_\omega\psi^{\lrcorner} - i(d_\omega\bar{\chi})_\bullet\chi^{\lrcorner} + \frac{1}{3!}\check{c}e_\bullet\bar{\chi}\gamma^3 d_\omega\psi + \frac{1}{2\cdot 3!}e\check{c}_\bullet\bar{\chi}\gamma^3 d_\omega\psi \\
Q_0\chi^{\lrcorner} &= \frac{i}{3!}\gamma^3 d_\omega\psi + i\gamma\psi e^{\lrcorner} - d_\omega\psi^{\lrcorner} - \iota_{\hat{\gamma}}\xi^{\lrcorner}\chi,
\end{aligned} \tag{5.2}$$

having used A.4.4 to redefine

$$\chi^{\lrcorner} = \frac{e^4}{4!}\chi_0^{\lrcorner}. \tag{5.3}$$

5.1.1 The second rank BV action

Before continuing, for computational purposes, it is convenient to redefine some of the fields. In particular, using A.4.2 and looking at the diagram A.31, we notice that one can uniquely define $\check{c} \in \Omega^{(2,0)}[-1]$ and $\check{\omega} \in \Omega^{(2,1)}[-1]$ such that

$$c^{\lrcorner} = \frac{e^2}{2}\check{c} \qquad \text{and} \qquad \omega^{\lrcorner} = e\check{\omega}. \tag{5.4}$$

With this redefinition, we then see

$$\begin{aligned}
\frac{e^2}{2}Q_0^2 c =& \frac{i}{8}\bar{\chi}\iota_\varphi(\mathrm{EoM}_\psi) - \frac{1}{8\cdot 3!}\iota_\varphi\big((\mathrm{EoM}_\omega)\bar{\chi}\gamma^3\psi\big) - \frac{1}{2}\iota_\xi e\iota_\varphi(\mathrm{EoM}_e) \\
& - \frac{1}{2\cdot 3!}\iota_\xi e\bar{\psi}\gamma^3\iota_\varphi d_\omega\psi + \iota_\xi\left(\frac{e}{4}\iota_\varphi(\mathrm{EoM}_e)\right) + \frac{i}{8}\iota_\xi\iota_\varphi(\bar{\psi}\mathrm{EoM}_\psi) \\
& - \frac{e}{2}\iota_\xi\left(\frac{1}{3!}\bar{\psi}\gamma^3\chi\left(\bar{\chi}\kappa(< e, \underline{\gamma}d_\omega\psi >) + \frac{1}{8}\bar{\chi}\iota_{\hat{\gamma}}\iota_{\hat{\gamma}}(\underline{\gamma}d_\omega\psi)\right)\right) \\
& + \frac{1}{2}\iota_\xi e\frac{1}{3!}\bar{\psi}\gamma^3\chi\left(\bar{\chi}\kappa(< e, \underline{\gamma}d_\omega\psi >) + \frac{1}{8}\bar{\chi}\iota_{\hat{\gamma}}\iota_{\hat{\gamma}}(\underline{\gamma}d_\omega\psi)\right)
\end{aligned}$$

There are still some terms that are not immediately recognizable as proportional to the equations of motion. In order to achieve that, one needs lemmata A.12, A.13 and A.5. In particular, setting $\underline{\gamma} := [e, \gamma] = \gamma_\mu dx^\mu$, thanks to A.12 we can redefine $\psi^\dashv$ as

$$\psi^\dashv := \frac{1}{3!} e\gamma^3 \underline{\gamma} \psi_0^\dashv,$$

while from A.13 and A.5 we have the following maps

$$\alpha\colon \Omega^{(3,1)}(\Pi\mathbb{S}_M) \to \Omega^{(1,0)}(\Pi\mathbb{S}_M) \qquad \beta\colon \Omega^{(3,1)}(\Pi\mathbb{S}_M) \to \ker(\gamma^3_{(3,1)})$$
$$\kappa\colon \Omega^{(2,1)}(\Pi\mathbb{S}_M) \to \Omega^{(1,0)}(\Pi\mathbb{S}_M) \qquad \varkappa\colon \Omega^{(2,1)}(\Pi\mathbb{S}_M) \to \ker(\underline{\gamma}\gamma^3_{(2,1)})$$

such that for all $\theta \in \Omega^{(3,1)}$ and $\omega \in \Omega^{(2,1)}$ one has

$$\theta = ie\underline{\gamma}\alpha(\theta) + \beta(\theta), \qquad \omega = e\kappa(\omega) + \varkappa(\omega).$$

Lastly, one can use the fact that $v_a v_b v_c v_d = \epsilon_{abcd}\mathrm{Vol}_V$ and (A.44) to show that the equation of motion for the gravitino reduces to

$$\frac{i}{3}\left(e\gamma^3 d_\omega \psi - \frac{1}{2} d_\omega e \gamma^3 \psi\right) = -\frac{1}{3}\gamma^5 \left(\underline{\gamma} d_\omega \psi - \frac{1}{2}[d_\omega e, \gamma]\psi\right)\mathrm{Vol}_V = 0.$$

In the end, from the terms of the kind $\int \Phi^\dashv_\alpha Q_0^2 \Phi^\alpha$ inside $(\mathcal{S}_1, \mathcal{S}_1)$, we can use (1.6) to obtain the coefficients of the rank–2 action, obtaining $\mathcal{S}_2 = \mathcal{S}_0 + s_1 + s_2$, with

$$\begin{aligned} s_2 = \int_M &\frac{1}{2}\left(\check{\omega} - \frac{1}{2}e\iota_\xi\check{c} - \check{c}\iota_\xi e\right)\iota_\varphi e^\dashv + \frac{1}{4}\left(\frac{1}{2}\bar{\psi}_0^\dashv\underline{\gamma} + \alpha(\check{\omega}\bar{\psi})\underline{\gamma} - \frac{i}{2}\iota_\xi\check{c}\bar{\psi} - \alpha(\check{c}\iota_\xi e\bar{\psi})\underline{\gamma} - \frac{i}{2}\check{c}\bar{\chi}\right)\iota_\varphi\psi^\dashv \\ &+ \frac{i}{4\cdot 3!}\left(\frac{1}{2}\alpha(\check{\omega}\bar{\psi})\underline{\gamma} - \frac{i}{2}\iota_\xi\check{c}\bar{\psi} - \alpha(\check{c}\iota_\xi e\bar{\psi})\underline{\gamma} - \frac{i}{2}\check{c}\bar{\chi}\right)\gamma^3\iota_\varphi(\check{\omega}\psi) \\ &- \frac{i}{2\cdot 3!}\left(\frac{1}{2}\bar{\psi}_0^\dashv\underline{\gamma} + \frac{1}{2}\alpha(\check{\omega}\bar{\psi})\underline{\gamma} - \frac{i}{2}\iota_\xi\check{c}\bar{\psi} - \alpha(\check{c}\iota_\xi e\bar{\psi})\underline{\gamma}\right)\gamma^3\chi < e, \bar{\chi}[\check{\omega}, \gamma]\psi > \\ &+ \frac{1}{2\cdot 3!}\left(\frac{1}{4}\bar{\psi}_0^\dashv\underline{\gamma} - \frac{i}{2}\iota_\xi\check{c}\bar{\psi} - \alpha(\check{c}\iota_\xi e\bar{\psi})\underline{\gamma}\right)\gamma^3\chi < e, \bar{\chi}\underline{\gamma}^2\psi_0^\dashv > \\ &- \frac{1}{32}\left(i\bar{\psi}^\dashv\chi + \frac{1}{3!}(\check{\omega} - e\iota_\xi\check{c} - 2\check{c}\iota_\xi e)\bar{\psi}\gamma^3\chi\right)\bar{\chi}\iota_{\hat{\gamma}}\iota_{\hat{\gamma}}([\check{\omega}, \gamma]\psi) \\ &- \frac{i}{32}\left(i\bar{\psi}^\dashv\chi + \frac{1}{3!}(e\iota_\xi\check{c} + 2\check{c}\iota_\xi e)\bar{\psi}\gamma^3\chi\right)\bar{\chi}\iota_{\hat{\gamma}}\iota_{\hat{\gamma}}(\underline{\gamma}^2\psi_0^\dashv), \end{aligned} \tag{5.5}$$

Now, letting $\mathfrak{q}$ be the Hamiltonian vector field of s_2, we see $Q = Q_0 + \mathfrak{q}$, and, after a long but straightforward computation, we have

$$
\begin{aligned}
\mathbb{q}_e &= \frac{1}{2}\iota_\varphi \check{\omega} - \frac{1}{2}\iota_\varphi \check{c}\iota_\xi e - \frac{1}{4}\iota_\varphi (e\iota_\xi \check{c}) \\
e\mathbb{q}_\omega &= \frac{1}{2}\iota_\varphi e^\dashv + \frac{i}{4\cdot 3!}\iota_\varphi(\bar{\psi}_0^\dashv \underline{\gamma})\gamma^3\psi + \frac{i}{4\cdot 3!}\bar{\psi}\gamma^3\iota_\varphi\left(\underline{\gamma}\alpha(\check{\omega}\psi)\right) - \frac{1}{8\cdot 3!}\iota_\varphi \check{c}\bar{\chi}\gamma^3\psi - \frac{1}{8\cdot 3!}\iota_\xi \check{c}\bar{\psi}\gamma^3\iota_\varphi\psi \\
&\quad - \frac{i}{4\cdot 3!}\bar{\psi}\gamma^3\iota_\varphi\left(\underline{\gamma}\alpha(\check{c}\iota_\xi e\psi)\right) + \frac{1}{2\cdot 3!}\bar{\psi}\gamma^3\chi\kappa\left[< e, \bar{\chi}\left(-\frac{i}{2}\underline{\gamma}^2\psi_0^\dashv - [\check{\omega},\gamma]\psi - \frac{1}{2}\underline{\gamma}\iota_\xi\check{c}\psi - \iota_\xi\underline{\gamma}\check{c}\psi\right) >\right] \\
&\quad + \frac{1}{16\cdot 3!}\bar{\psi}\gamma^3\chi\bar{\chi}\iota_{\hat{\gamma}}\iota_{\hat{\gamma}}\left(-\frac{i}{2}\underline{\gamma}^2\psi_0^\dashv - [\check{\omega},\gamma]\psi - \frac{1}{2}\underline{\gamma}\iota_\xi\check{c}\psi - \iota_\xi\underline{\gamma}\check{c}\psi\right) \\
\mathbb{q}_\psi &= \frac{i}{4}\iota_\varphi(\underline{\gamma}\psi_0^\dashv) - \frac{i}{4}\iota_\varphi\left(\underline{\gamma}\alpha(\check{\omega}\psi)\right) - \frac{i}{4}\iota_\varphi\left(\underline{\gamma}\alpha(\check{c}\iota_\xi e\psi)\right) + \frac{1}{8}\iota_\varphi\check{c}\chi - \frac{1}{8}\iota_\varphi(\iota_\xi\check{c}\psi) \\
&\quad + \frac{i}{4}\chi\kappa\left(< e, \bar{\chi}\underline{\gamma}^2\psi_0^\dashv + i\bar{\chi}[\check{\omega} - \frac{i}{2}\iota_\xi\check{c}e + \iota_\xi e\check{c}] >\right) + \frac{1}{16}\chi\bar{\chi}\iota_{\hat{\gamma}}\iota_{\hat{\gamma}}(\underline{\gamma}^2\psi_0^\dashv + i[\check{\omega} - \frac{i}{2}\iota_\xi\check{c}e + \iota_\xi e\check{c}]) \\
\frac{e^2}{2}\mathbb{q}_c &= -\frac{i}{8}\bar{\chi}\iota_\varphi\psi^\dashv - \frac{i}{8\cdot 3!}\iota_\varphi(\check{\omega}\bar{\chi}\gamma^3\psi) - \frac{1}{2}\iota_\xi e\iota_\varphi e^\dashv + \frac{1}{4}\iota_\xi(e\iota_\varphi e^\dashv) - \frac{i}{4\cdot 3!}\iota_\varphi(\bar{\psi}_0^\dashv\underline{\gamma})\gamma^3\iota_\xi e\psi \\
&\quad + \frac{i}{4\cdot 3!}\iota_\varphi\left(\alpha(\check{\omega}\bar{\psi})\underline{\gamma}\right)\gamma^3\iota_\xi e\psi - \frac{i}{8}\iota_\xi(\bar{\psi}\iota_\varphi\psi^\dashv) - \frac{1}{8\cdot 3!}\iota_\xi(\check{\omega}\bar{\psi}\gamma^3\iota_\varphi\psi) \\
&\quad + \frac{1}{4\cdot 3!}\iota_\xi\left(\bar{\psi}\gamma^3\chi < e, \bar{\chi}([\check{\omega},\gamma]\psi + i\underline{\gamma}^2\psi_0^\dashv) >\right) - \frac{1}{2\cdot 3!}\iota_\xi e\bar{\psi}\gamma^3\chi\kappa\left(< e, \bar{\chi}([\check{\omega},\gamma]\psi + i\underline{\gamma}^2\psi_0^\dashv) >\right) \\
&\quad + \frac{1}{32\cdot 3!}\iota_\xi e\bar{\psi}\gamma^3\chi\bar{\chi}\iota_{\hat{\gamma}}\iota_{\hat{\gamma}}([\check{\omega},\gamma]\psi + i\underline{\gamma}^2\psi_0^\dashv) - \frac{1}{32\cdot 3!}e\iota_\xi\left(\bar{\psi}\gamma^3\chi\bar{\chi}\iota_{\hat{\gamma}}\iota_{\hat{\gamma}}([\check{\omega},\gamma]\psi + i\underline{\gamma}^2\psi_0^\dashv)\right),
\end{aligned}
\tag{5.6}
$$

while one can immediately see $\mathbb{q}_\chi = 0$ and $\mathbb{q}_\xi = 0$.

Unfortunately, it turns out that (5.5) is not yet the full rank–2 action. Indeed, one needs to require the cohomological vector field Q along c to contain terms proportional to $\iota_\xi Q\omega$. This is not the case here as s_2 is missing terms quadratic in the antighost $\check{c}$.

Remark 5.2. As stated above, one can use equation (1.6)

$$Q_0^2\Phi^\alpha - (-1)^{\beta(\alpha+1)}\frac{\delta L_M}{\delta\Phi^\alpha}M^{\alpha\beta} = 0$$

to define $M^{\alpha\beta}(\Phi)$, which are exactly the coefficients appearing in the quadratic part of the action, where the equation of motion $\frac{\delta L_M}{\delta\Phi^\alpha}$ is replaced by the corresponding antifield $\Phi_\alpha^\dashv$. However, since there is no equation of motion for the ghosts, and in particular no equation of motion for c, the terms quadratic in $\check{c}$ have to be found by hand by checking $Q^2 = 0$, or equivalently by imposing the consistency equations (1.7),(1.8).

As it turns out, defining $e\mathbb{l}(\check{c},\xi,\varphi,\psi)$ as the terms inside $e\mathbb{q}_\omega$ that contain $\check{c}$,[4] we have the following theorem.

Theorem 5.1. *The collection $(\mathcal{F}_{SG}, \varpi_{SG}, Q, \mathcal{S})$ defines a BV structure, where*

$$\mathcal{S} = \mathcal{S}_2 + \int_M \frac{1}{2}c^\dashv\iota_\xi\mathbb{l}(\check{c},\xi,\varphi,\psi),$$

[4] For the full expression, see (B.13)

and $\mathbb{l}(\check{c}, \xi, \varphi, \psi)$ *implicitly defined by*

$$\begin{aligned} e\mathbb{l}(\check{c}, \xi, \chi, \psi) := &- \frac{1}{8 \cdot 3!}\iota_\xi \check{c}\bar{\psi}\gamma^3\iota_\varphi\psi - \frac{i}{4 \cdot 3!}\bar{\psi}\gamma^3\iota_\varphi\left(\underline{\gamma}\alpha(\check{c}\iota_\xi e\psi)\right) \\ &- \frac{1}{2 \cdot 3!}\bar{\psi}\gamma^3\chi\kappa\left[< e, \bar{\chi}\left(\frac{1}{2}\underline{\gamma}\iota_\xi\check{c}\psi + \iota_\xi\underline{\gamma}\check{c}\psi\right) >\right] \\ &- \frac{1}{16 \cdot 3!}\bar{\psi}\gamma^3\chi\bar{\chi}\iota_{\hat{\gamma}}\iota_{\hat{\gamma}}\left(\frac{1}{2}\underline{\gamma}\iota_\xi\check{c}\psi + \iota_\xi\underline{\gamma}\check{c}\psi\right). \end{aligned}$$

Remark 5.3. We can significantly simplify the notation in the definition of the action by defining the combination of fields $k^\dashv := \omega^\dashv - \iota_\xi c^\dashv$ and $k^\dashv = e\check{k}$. With this, the the extra term $\frac{1}{2}c^\dashv\iota_\xi\mathbb{l}(\check{c}, \xi, \varphi, \psi)$ is automatically contained in the following

$$\begin{aligned} s_2 = \int_M &\frac{1}{2}e^\dashv\iota_\varphi\check{k} + \frac{1}{4}\left(\frac{1}{2}\bar{\psi}_0^\dashv + \alpha(\check{k}\bar{\psi})\underline{\gamma} - \frac{1}{2}\check{c}\bar{\chi}\right)\iota_\varphi\psi^\dashv \\ &- \frac{i}{4 \cdot 3!}\left(\bar{\psi}_0^\dashv\underline{\gamma} + \alpha(\check{k}\bar{\psi})\underline{\gamma}\right)\gamma^3\chi < e, \bar{\chi}[\check{k}, \gamma]\psi > \\ &+ \frac{i}{8 \cdot 3!}\left(\alpha(\check{k}\bar{\psi})\underline{\gamma} - i\check{c}\bar{\chi}\right)\gamma^3\iota_\varphi(\check{k}\psi) + \frac{1}{8 \cdot 3!}\bar{\psi}_0^\dashv\underline{\gamma}\gamma^3\chi < e, \bar{\chi}\underline{\gamma}^2\psi_0^\dashv > \\ &- \frac{1}{32}\left(i\bar{\psi}^\dashv\chi + \frac{1}{3!}\check{k}\bar{\psi}\gamma^3\chi\right)\bar{\chi}\iota_{\hat{\gamma}}\iota_{\hat{\gamma}}([\check{k}, \gamma]\psi) + \frac{1}{32}\bar{\psi}^\dashv\chi\bar{\chi}\iota_{\hat{\gamma}}\iota_{\hat{\gamma}}(\underline{\gamma}^2\psi_0^\dashv). \end{aligned} \tag{5.7}$$

Proof. We leave the proof for section B.3.3. □

5.2 The BV pushforward of $N = 1$, $D = 4$ Supergravity

Having found a BV action for $N = 1, D = 4$ Supergravity raises the question on whether one can induce a well–defined BFV structure on the boundary, following the construction of 1.2. However, as remarked in section 2.4.1, already in the case of Palatini–Cartan gravity – of which supergravity is the supersymmetric extension – the boundary pre–symplectic form induced by the BV action is singular, hence not suitable to obtain a BFV structure. By repeating the same computations of 2.4.1 in the SUGRA setting, one immediately sees that the singularity of the induced pre–symplectic form is not affected by the introduction of the gravitino, which allows us to employ the same strategy employed in PC gravity, with the appropriate distinctions.

In 2.5.1 we saw how the authors in [CCS21b] obtained a BV-BFV extendible theory of PC gravity by employing the 1–dimensional AKSZ construction, considering the PC BFV theory as the target. Such strategy is obviously not suitable for the problem at hand, as we do not yet have a full BFV theory for SUGRA, but rather we want to induce one from the BV data in the bulk.

The solution is then offered by the BV pushforward, which was computed for the case of PC gravity in [CC25b] and reviewed in 2.5.2. There we saw how one can ”integrate out" the components of the spin connection ω responsible for the singularity of the induced boundary symplectic form. The result of the construction is a reduced theory, on which the fields are constrained. Originally, such constraints were found by looking at the 1–D AKSZ PC theory, but in the case of SUGRA we need to obtain them in an independent way, which is provided by generalising proposition 2.2, starting from a generalization of constraint 4.20 found in the study of the boundary structure of SUGRA. Thanks to this, we can define the following.

Definition 5.1. Letting $M = I \times \Sigma$, the reduced $\mathcal{N} = 1, D = 4$ Supergravity theory is given by $\mathfrak{F}^r_{SG} := (\mathcal{F}^r_{SG}, \varpi^r_{SG}, Q^r_{SG}, \mathcal{S}^r_{SG})$, where

- $\mathcal{F}_{SG}^r$ is given by the the subspace of $\mathcal{F}_{SG}(I \times \Sigma)$ satisfying the following constraints

$$\epsilon_n \left(\tilde{\check{\omega}} - \iota_{\tilde{\xi}} \tilde{e}\tilde{\check{c}} - \tilde{e}_n \tilde{\check{c}} \tilde{\xi}^n - \tfrac{e}{2}(\iota_{\tilde{\xi}} \tilde{\check{c}} - \tilde{\check{c}}_n \tilde{\xi}^n) \right) = \tilde{e}\tilde{\check{a}} \tag{5.8}$$

$$Q_{SG} \left[\epsilon_n \left(\tilde{\check{\omega}} - \iota_{\tilde{\xi}} \tilde{e}\tilde{\check{c}} - \tilde{e}_n \tilde{\check{c}} \tilde{\xi}^n - \tfrac{e}{2}(\iota_{\tilde{\xi}} \tilde{\check{c}} - \tilde{\check{c}}_n \tilde{\xi}^n) \right) - \tilde{e}\tilde{\check{a}} \right] = 0, \tag{5.9}$$

  for some $\tilde{\check{a}} \in \mathcal{C}^\infty(I) \otimes \Omega_\partial^{(1,1)}[-1]$;

- $\varpi_{SG}^r := \varpi_{SG}(I \times \Sigma)|_{\mathcal{F}_{SG}^r}$;
- $\mathcal{S}_{SG}^r := \mathcal{S}_{SG}|_{\mathcal{F}_{SG}^r}$;
- $Q_{SG}^r = Q_{SG}$.

Remark 5.4. Carrying out some computations, (5.9) can be expressed as

$$\epsilon_n \left(d_{\tilde{\omega}} \tilde{e} - \frac{1}{2} \bar{\tilde{\psi}} \gamma \tilde{\psi} \right) - \epsilon_n \underline{\tilde{\tau}}^{\dashv} d\tilde{\xi}^n + \iota_{\hat{X}} \tilde{k}^{\dashv} + \mathfrak{Q} = \tilde{e}\sigma, \tag{5.10}$$

where terms $\mathfrak{Q}$ are left implicitly defined by applying the terms in Q_{SG} depending on the $\tilde{\psi}$ and $\tilde{\chi}$ to $\epsilon_n \check{k}$. Furthermore, we can also define $x^{\dashv} \in \mathcal{C}^\infty(I) \otimes \Omega_\partial^{(1,1)}$ and $y^{\dashv} \in \mathcal{C}^\infty(I) \otimes \Omega_\partial^{(1,2)}$ such that

$$\mathfrak{Q} = \tilde{e}x^{\dashv} + \epsilon_n[\tilde{e}, y^{\dashv}]. \tag{5.11}$$

Lastly, we have

$$\tilde{X} := \mathrm{L}_{\tilde{\xi}}^{\tilde{\omega}}(\epsilon_n) - d_{\tilde{\omega}_n}(\epsilon_n)\tilde{\xi}^n - [\tilde{c}, \epsilon_n], \qquad \hat{X} = \tilde{e}_a^i \tilde{X}^a \partial_i.$$

Remark 5.5. As of now, we do not know if such reduced theory is a genuine BV theory, since we did not check explicitly that $(\mathcal{S}_{SG}^r, \mathcal{S}_{SG}^r) = 0$ holds. However, showing that one can recover the reduced theory as the BV pushfoward of the supergravity hedgehog will automatically assure that the CME holds.

The first goal is to find a symplectomorphism between $\mathfrak{F}_{SG} = (\mathcal{F}_{SG}(I \times \Sigma), \varpi_{SG}(I \times \Sigma))$ and $\mathfrak{F}_{SG}^H = \left(\mathcal{F}_{SG}(I \times \Sigma), \varpi_{SG}^r + \int_{I \times \Sigma} \delta\tilde{v}\delta\underline{\tilde{v}}^{\dashv} \right)$. To do so, we follow [CC25b], splitting the symplectomorphism in two steps.

Lemma 5.1. *There exists a symplectomorphism* $\phi_1 \colon \mathcal{F}_{SG}(I \times \Sigma) \to \mathcal{F}_{SG}(I \times \Sigma)$ *such that*

$$\phi_1^* \left(\varpi_{SG}^r + \int_{I \times \Sigma} \delta\tilde{v}\delta\underline{\tilde{v}}^{\dashv} + \delta\hat{\omega}\delta\underline{\tilde{v}}^{\dashv} \right) = \varpi_{SG}(I \times \Sigma).$$

Explicitly, we have that all the fields are preserved by the symplectomorphism[5] *except*

$$\phi_1^*(\tilde{e}^{\dashv}) = \tilde{e}^{\dashv} + \tilde{v}\tilde{\check{k}} \qquad \phi_1^*(\underline{\tilde{e}}_n^{\dashv}) = \underline{\tilde{e}}_n^{\dashv} - \tilde{v}\underline{\tilde{\check{k}}}_n - \iota_{\underline{z}}\tilde{v}\tilde{\check{k}}$$

$$\phi_1^*(\tilde{c}) = \tilde{c} - \iota_{\tilde{\xi}}\tilde{v} + \iota_z \tilde{v}\tilde{\xi}^n \qquad \phi_1^*(\underline{\tilde{\xi}}^{\dashv}) = \underline{\tilde{\xi}}^{\dashv} - \tilde{v}\underline{\tilde{c}}_n^{\dashv} \qquad \phi_1^*(\underline{\tilde{\xi}}_n^{\dashv}) = \underline{\tilde{\xi}}_n^{\dashv} + \iota_z \tilde{v}\underline{\tilde{c}}_n^{\dashv} \qquad \phi_1^*(\underline{\tilde{\omega}}_n) = \underline{\tilde{\omega}}_n + \iota_{\underline{z}}\tilde{v}$$

Proof. The proof can be copied, mutatis mutandis, from [CC25b], since $\varpi_{SG} = \varpi_{PC} + \int_M i\delta\bar{\boldsymbol{\psi}}\delta\boldsymbol{\psi}^{\dashv} + i\delta\bar{\boldsymbol{\chi}}\delta\boldsymbol{\chi}^{\dashv}$, and only the field inside ϖ_{PC} transform under ϕ_1. ☐

[5] In particular we also have $\phi_1^*(\tilde{v}) = \tilde{v}$, $\phi_1^*(\hat{\omega}) = \hat{\omega}$, $\phi_1^*(\underline{\tilde{\tau}}^{\dashv}) = \underline{\tilde{\tau}}^{\dashv}$ and $\phi_1^*(\underline{\tilde{\mu}}^{\dashv}) = \underline{\tilde{\mu}}^{\dashv}$.

Lemma 5.2. *There exists a symplectomorphism $\phi_2\colon \mathcal{F}_{SG}(I\times\Sigma)\to\mathcal{F}_{SG}(I\times\Sigma)$ such that*

$$\phi_2^*\left(\varpi^r_{SG}+\int_{I\times\Sigma}\delta\tilde{v}\delta\underline{\tilde{v}}^{\lrcorner}\right)=\varpi^r_{SG}+\int_{I\times\Sigma}\delta\tilde{v}\delta\underline{\tilde{v}}^{\lrcorner}+\delta\hat{\omega}\delta\underline{\tilde{v}}^{\lrcorner}.$$

In particular, defining $\alpha\in\mathcal{C}^\infty(I)\otimes\Omega_\partial^{(1,1)}$ and $\beta\in\mathcal{C}^\infty(I)\otimes\Omega_\partial^{(1,2)}$ such that

$$d\tilde{\xi}^n\tilde{\mu}^{\lrcorner}=\tilde{e}\alpha+\epsilon_n[e,\tilde{\beta}], \tag{5.12}$$

then $\epsilon_n d\tilde{\xi}^n\tilde{\mu}^{\lrcorner}=\epsilon_n\tilde{e}\alpha=:\tilde{e}\nu$, where $\nu=\epsilon_n\alpha\in\mathcal{C}^\infty(I)\otimes\Omega_\partial^{(1,2)}$.

The action of ϕ_2 is given by

$$\begin{aligned}
&\phi_2^*(\tilde{e}^{\lrcorner})=\tilde{e}^{\lrcorner}+\nu\tilde{\check{k}} && \phi_2^*(\tilde{v})=\tilde{v}+y^{\lrcorner}\\
&\phi_2^*(\tilde{\omega}_n)=\tilde{\omega}_n+\iota_z\nu+\iota_{\tilde{X}}\tilde{\mu}^{\lrcorner} && \phi_2^*(\hat{\omega})=\hat{\omega}+\nu\\
&\phi_2^*(\tilde{c})=\tilde{c}+\iota_{\tilde{X}}\tilde{\mu}^{\lrcorner}\tilde{\xi}^n+\iota_{\tilde{\xi}}\nu+\iota_z\nu\tilde{\xi}^n && \phi_2^*(\tilde{c}^{\lrcorner}_n)=\tilde{c}^{\lrcorner}_n+[\epsilon_n,\tilde{\check{k}}\tilde{\mu}^{\lrcorner}]\\
&\phi_2^*(\tilde{\omega}^{\lrcorner})=\tilde{\omega}^{\lrcorner}+[\epsilon_n,\tilde{\xi}^n\tilde{\check{k}}\tilde{\mu}^{\lrcorner}] && \phi_2^*(\tilde{\omega}_n)=\tilde{\omega}_n+[\epsilon_n,\iota_\xi(\tilde{\check{k}}\tilde{\mu}^{\lrcorner})]\\
&\phi_2^*(\tilde{\phi}^{\lrcorner}_n)=\tilde{\phi}^{\lrcorner}_n-i\epsilon_n\gamma\tilde{\psi}\tilde{\mu}^{\lrcorner},
\end{aligned}$$

$$\begin{aligned}
\phi_2^*(\underline{\tilde{e}}^{\lrcorner}_n)&=\underline{\tilde{e}}^{\lrcorner}_n+d_{\tilde{\omega}}(\epsilon_n\underline{\tilde{\mu}}^{\lrcorner})+\sigma\underline{\tilde{\mu}}^{\lrcorner}+\iota_{\tilde{X}}(\tilde{\check{k}}\underline{\tilde{\mu}}^{\lrcorner})-\nu\underline{\tilde{\check{k}}}_n-\iota_{\underline{z}}\nu\tilde{\check{k}}-x^{\lrcorner}\underline{\tilde{\mu}}^{\lrcorner}+[\epsilon_n y^{\lrcorner},\underline{\tilde{y}}^{\lrcorner}]\\
\phi_2^*(\tilde{\xi}^{\lrcorner}_n)&=\tilde{\xi}^{\lrcorner}_n+\iota_{\tilde{X}}\tilde{c}^{\lrcorner}_n\tilde{\mu}^{\lrcorner}+d(\epsilon_n\tau^{\lrcorner}\tilde{\mu}^{\lrcorner})+\iota_z\tilde{c}^{\lrcorner}_n\nu+\iota_z\nu[\epsilon_n,\tilde{\check{k}}\tilde{\mu}^{\lrcorner}]+(d_{\tilde{\omega}_n}\epsilon_n)\tilde{\check{k}}\tilde{\mu}^{\lrcorner}+\iota_{\tilde{X}}\tilde{\mu}^{\lrcorner}[\epsilon_n,\tilde{\check{k}}\tilde{\mu}^{\lrcorner}]\\
\phi_2^*(\tilde{\xi}^{\lrcorner})&=\tilde{\xi}^{\lrcorner}+\tilde{c}^{\lrcorner}_n\nu+d_{\hat{\omega}}\epsilon_n(\tilde{\check{k}}\tilde{\mu}^{\lrcorner})+\nu[\epsilon_n,\tilde{\check{k}}\tilde{\mu}^{\lrcorner}].
\end{aligned}$$

Proof. Once again, we can refer to [CC25b] and the proof therein. In particular, we notice that the ϕ_2 defined above is exactly the same as ϕ_2^{PC} of section 2.5.2, except a few cases. Indeed, defining $\phi_2=\phi_2^{PC}+\phi_3$, we have

$$\phi_3^*(\tilde{v})=y^{\lrcorner}\qquad \phi_3^*(\tilde{\psi}^{\lrcorner}_n)=-i\epsilon_n\gamma\tilde{\psi}\tilde{\mu}^{\lrcorner}\qquad \phi_3^*(\tilde{e}^{\lrcorner}_n)=x^{\lrcorner}\tilde{\mu}^{\lrcorner}+[\epsilon_n y^{\lrcorner},\tilde{\mu}^{\lrcorner}]$$

where we recall $\mathfrak{Q}=\tilde{e}x^{\lrcorner}+\epsilon_n[\tilde{e},y^{\lrcorner}]$. Then, by [CC25b], we have

$$\phi_2^*\left(\varpi^r_{PC}+\int_{I\times\Sigma}\delta\tilde{v}\delta\underline{\tilde{v}}^{\lrcorner}\right)=\varpi^r_{PC}+\int_{I\times\Sigma}\delta\tilde{v}\delta\underline{\tilde{v}}^{\lrcorner}+\delta\hat{\omega}\delta\underline{\tilde{v}}^{\lrcorner}+\delta\tilde{\bar{\psi}}\delta(\epsilon_n\gamma\psi\tilde{\mu}^{\lrcorner})+\delta\mathfrak{Q}\delta\tilde{\mu}^{\lrcorner},$$

where we have taken into account that the structural constraint that was used in [CC25b] has been changed to take the gravitino interaction into consideration.

The term $\epsilon_n\delta\tilde{\bar{\psi}}\gamma\tilde{\psi}\tilde{\mu}^{\lrcorner}$ is balanced by the term $\phi_3^*\left(\int_{I\times\Sigma}\delta\tilde{\psi}\delta\underline{\tilde{\psi}}_{n,\lrcorner}\right)=\phi_3^*(\varphi^r_{SG}-\varpi^r_{PC})$ while

$$\phi_3^*\left(\varpi^r_{PC}+\int_{I\times\Sigma}\delta v\delta\underline{\tilde{v}}^{\lrcorner}\right)=\int_{I\times\Sigma}\delta\tilde{e}\delta(x^{\lrcorner}\tilde{\mu}^{\lrcorner}+[\epsilon_n y^{\lrcorner},\tilde{\mu}^{\lrcorner}])+\delta y^{\lrcorner}\delta(\epsilon_n[\tilde{e},\tilde{\mu}^{\lrcorner}])=\int_{I\times\Sigma}-\delta\mathfrak{Q}\delta\tilde{\mu}^{\lrcorner},$$

balancing exactly the remaining term and showing the lemma.[6] □

Proposition 5.1. *Let $\mathfrak{F}^H_{SG}$ be the BV theory given by*

$$(\mathcal{F}^H_{SG},\varpi^H_{SG},\mathcal{S}^H_{SG}),$$

[6]We remark we have used the property that $\tilde{e}\tilde{\mu}^{\lrcorner}=0$.

where

$$\varpi^H_{SG} = \varpi^r_{SG} + \int_{I\times\Sigma} \delta\tilde{v}\delta\underline{\tilde{v}}^{\lrcorner}$$

and

$$\mathcal{S}^H_{SG} = S^r_{SG} + \int_{I\times\Sigma} \frac{1}{2}\underline{\tilde{e}}_n \tilde{e}[\tilde{v}-y^{\lrcorner}, \tilde{v}-y^{\lrcorner}] + g(\tilde{v}^{\lrcorner}), \tag{5.13}$$

with

$$g(\tilde{v}^{\lrcorner}) = f(v^{\lrcorner}) + (\delta_\chi(\tilde{v}+\nu) + \mathbb{q}(\tilde{v}+\nu))\,\underline{\tilde{v}}^{\lrcorner}, \tag{5.14}$$

having defined δ_χ as the supersymmetry transformation and $\mathbb{q}$ as the Hamiltonian vector field of the rank–2 part of the BV action defined in (5.7) *and $f(\tilde{v}^{\lrcorner})$ as in* (2.51)*.*

Then, letting $\Phi := \phi_1 \circ \phi_2$, we have

$$\Phi^*(\mathcal{S}^H_{SG}) = \mathcal{S}_{SG}.$$

The proposition will follow immediately from the following lemmata.

Lemma 5.3. *The symplectomorphism ϕ_2 is such that*

$$\phi_2^*\left(\mathcal{S}^r_{SG} + \int_{I\times\Sigma}\frac{1}{2}\underline{\tilde{e}}_n\tilde{e}[\tilde{v}-y^{\lrcorner},\tilde{v}-y^{\lrcorner}] + g(\tilde{v}^{\lrcorner})\right) = \mathcal{S}^r_{SG} + \int_{I\times\Sigma}\frac{1}{2}\underline{\tilde{e}}_n\tilde{e}[\tilde{v},\tilde{v}] + h(\tilde{v}^{\lrcorner}),$$

where $h(\tilde{v}^{\lrcorner}) = g(\tilde{v}^{\lrcorner}) + (\iota_{\tilde{\xi}}F_{\hat{\omega}} + F_{\tilde{\omega}_n}\tilde{\xi}^n + d_{\hat{\omega}}\tilde{c} + \delta_\chi\hat{\omega} + \mathbb{q}\hat{\omega})\tilde{v}^{\lrcorner}$.[7]

Lemma 5.4. *The symplectomorphism ϕ_1 is such that*

$$\phi_1^*\left(\mathcal{S}^r_{SG} + \int_{I\times\Sigma}\frac{1}{2}\underline{\tilde{e}}_n\tilde{e}[\tilde{v},\tilde{v}] + h(\tilde{v}^{\lrcorner})\right) = \mathcal{S}_{SG}$$

Proof. The proofs of the above lemmatas are found in B.3.4 □

Remark 5.6. So far, we have showed that the BV theory $\mathfrak{F}^H_{SG}$ is BV–equivalent to the full BV theory. In particular, this implies that the BV action is equivalent to the reduced BV action with the addition of the terms depending on $\tilde{v}^{\lrcorner}$ and $\tilde{v}$. Ultimately, we are interested in integrating out the field $\tilde{v}$, which is responsible for the singularity of the pre–symplectic form induced on the boundary by the full BV action, therefore obtaining a decoupling of S_r from the dynamics of $\tilde{v}$ is fundamental.

Furthermore, we can copy the content of remark 2.18 to see that we have a fiber bundle $\mathcal{F}^H_{SG} \to \mathcal{F}^r_{SG}$ whose fiber is locally given by $\mathcal{F}_f := T^*[1](\ker W^{(1,2)}_{\tilde{e}_0})$, for a reference non–degenerate tetrad e_0. We can furthermore obtain a BV bundle

$$\begin{array}{ccc} \mathcal{F}^H_{SG} & \xrightarrow{\quad\Xi\quad} & \tilde{\mathcal{F}}^H_{SG} \\ & {\scriptstyle\pi^H}\searrow \quad \swarrow{\scriptstyle\tilde{\pi}^H} & \\ & \mathcal{F}^r_{SG} & \end{array}$$

where $\tilde{\mathcal{F}}^H_{SG} := \mathcal{F}^r_{SG} \times F_f$ is a product of -1–symplectic manifolds. The symplectomorphism Ξ is found by noticing that at any spacetime point $x \in M$, we can find a unique orthogonal transformation Λ_x[8] such that $\tilde{e}_0 = \Lambda_x\tilde{e}$, hence providing an isomorphism

$$\ker W^{(1,2)}_{\tilde{e}_0} = \{\tilde{v} \in \Omega^{(1,2)}_\partial \mid \forall x \in M,\ \Lambda_x\tilde{e}_x\tilde{v}_x = 0\} \simeq \ker W^{(1,2)}_{\tilde{e}}.$$

[7]Notice that $h(\tilde{v}^{\lrcorner}) = Q_{SG}(\hat{\omega}+\tilde{v})\tilde{v}^{\lrcorner}$

[8]This is the content of remark A.11

This extends as a symplectomorphism $\mathcal{F}_f = T^*[1] \ker W_{\tilde{e}_0}^{(1,2)} \simeq T^*[1] \ker W_{\tilde{e}}^{(1,2)}$. Therefore, considering an open $\mathcal{U} \subset \mathcal{F}_{SG}^r$, with the local trivialization of $\mathcal{F}_{SG}^H \to \mathcal{F}_{SG}^r$ is given by

$$\mathcal{F}_{SG}^H \Big|_{\mathcal{U}} \simeq \mathcal{U} \times T^*[1] \ker W_{\tilde{e}}^{(1,2)},$$

we see that the above symplectomorphism induces $\Xi : \mathcal{F}_{SG}^H \to \tilde{\mathcal{F}}_{SG}^H$.

Theorem 5.2.

$$\tilde{\mathcal{P}}_{\mathcal{L}} : \mathrm{Dens}^{\frac{1}{2}}(\mathcal{F}_{SG}) \to \mathrm{Dens}^{\frac{1}{2}}(\mathcal{F}_{SG}^r)$$

defined by the composition

$$\mathrm{Dens}^{\frac{1}{2}}(\mathcal{F}_{SG}) \xrightarrow{\Phi^*} \mathrm{Dens}^{\frac{1}{2}}(\mathcal{F}_{SG}^H) \xrightarrow{\mathcal{P}_{\mathcal{L}}} \mathrm{Dens}^{\frac{1}{2}}(\mathcal{F}_{SG}^r),$$

wehere $\Phi = \phi_2 \circ \phi_1$ as defined in proposition 5.1 and $\mathcal{P}_{\mathcal{L}}$ is the BV pushforward of the BV bundle $\mathcal{F}_{SG}^H \to \mathcal{F}_{SG}^r$ along the Lagrangian submanifold $\mathcal{L}_f := \{(\tilde{v}, \tilde{v}^{\dashv}) \in \mathcal{F}_f \mid \tilde{v}^{\dashv} = 0\}$.

Then

$$\mu_r^{\frac{1}{2}} \exp\left(\frac{i}{\hbar} \mathcal{S}_{SG}^r\right) = \tilde{P}_{\mathcal{L}} \mu^{\frac{1}{2}} \exp\left(\frac{i}{\hbar} \mathcal{S}_{SG}\right)$$

Proof. The proof amounts to showing that $\mu_r^{\frac{1}{2}} \exp\left(\frac{i}{\hbar} \mathcal{S}_{SG}^r\right) = P_{\mathcal{L}} \mu^{\frac{1}{2}} \exp\left(\frac{i}{\hbar} \mathcal{S}_{SG}^H\right)$. This is a consequence of the fact that on $\mathcal{L}_f$ we have

$$\mathcal{S}_{SG}^H \Big|_{\mathcal{L}_f} = \mathcal{S}_{SG}^r + \int_{I \times \Sigma} \frac{1}{2} \tilde{e}_n \tilde{e}[\tilde{v} - y^{\dashv}, \tilde{v} - y^{\dashv}].$$

and that the quadratic form $\frac{1}{2} \tilde{e}_n \tilde{e}[-,-] : \Omega_\partial^{(1,2)} \times \Omega_\partial^{(1,2)} \to \mathcal{C}^\infty(M)$ is non–degenerate [CC25b], hence producing a well-defined Gaussian integral

$$\int_{\mathcal{L}_f} \mu_f^{\frac{1}{2}} \exp\left(\frac{i}{\hbar} \int_{I \times \Sigma} \frac{1}{2} \tilde{e}_n \tilde{e}[\tilde{v} - y^{\dashv}, \tilde{v} - y^{\dashv}]\right),$$

which contributes to $P_{\mathcal{L}} \mu^{\frac{1}{2}} \exp\left(\frac{i}{\hbar} \mathcal{S}_{SG}^H\right)$ just with a constant factor. □

5.2.1 The induced $N = 1, D = 4$ Supergravity BFV action

So far, we worked on a cylinder to obtain the reduced BV action of $N = 1, D = 4$ supergravity. Such theory is a suitable candidate to obtain a BFV structure on the boundary, since the induced boundary symplectic form is now non–degenerate, hence solving the original issue outlined at the beginning of the previous section.

Traditionally, one would need to compute the boundary potential 1–form α_{SG}^Σ arising as a boundary term from

$$\delta S_{SG}^r = \iota_{Q_{SG}^r} \varpi_{SG}^r + (\pi_{SG}^{r,\Sigma})^* (\vartheta_{SG}^{r,\Sigma}),$$

where $\pi_{SG}^{r,\Sigma} : \mathcal{F}_{SG}^r \to \mathcal{F}_{SG}^{r,\Sigma}$ is the surjective submersion to the space of boundary fields. In particular, we notice that, from the variation of S_{SG}^H, we have

$$\delta S_{SG}^H = \iota_{Q_{SG}^H} \varpi_{SG}^H + (\pi_{SG}^\Sigma)^* \left(\vartheta_{SG}^{r,\Sigma} + \int_\Sigma \tilde{\xi}^n \tilde{v}^{\dashv} \delta \tilde{v}\right).$$

The induced action on the boundary is obtained as the boundary term given by the (failure of) the CME in the bulk

$$\frac{1}{2}\iota_{Q_{SG}}\iota_{Q_{SG}}\varpi_{SG}^{M} = \mathcal{S}_{SG}^{r,\Sigma} + \mathcal{S}_{f}^{\Sigma},$$

where $\mathcal{S}_{SG}^{\Sigma}$ is exactly the boundary BFV action we are seeking, given as $\mathcal{S}_{SG}^{\Sigma} = \frac{1}{2}\iota_{Q_{SG}^{r}}\iota_{Q_{SG}^{r}}\varpi_{SG}^{r}$, while $\mathcal{S}_{f}^{\Sigma}$ can be seen as the Hamiltonian generating the gauge transformations of v, defining a coisotropic submanifold.[9] In particular, one sees

$$\mathcal{S}_{f}^{\Sigma} = \int_{\Sigma}\left(\mathrm{L}_{\tilde{\xi}}^{\hat{\omega}}\tilde{v} + \iota_{z}\tilde{v}d\tilde{\xi}^{n} - [\tilde{c},\tilde{v}] + \delta_{\chi}v + \mathfrak{q}\tilde{v}\right)\tilde{v}^{\dashv}\tilde{\xi}^{n} + \frac{1}{2}\epsilon_{n}\tilde{\xi}^{n}\tilde{e}[\tilde{v},\tilde{v}] \tag{5.15}$$

This is not significantly relevant at the moment, as we are are interested in $\mathcal{S}_{SG}^{r,\Sigma}$. In particular, we know from 4.2 that a BFV structure $\mathfrak{F}^{\Sigma} := (\mathcal{F}_{SG}^{\Sigma}, \mathcal{S}_{SG}^{\Sigma}, \varpi_{SG}^{\Sigma})$ must exist such that $H^{0}_{Q_{SG}^{\Sigma}}(\mathcal{F}_{SG}^{\Sigma}) \simeq \mathcal{C}^{\infty}(\underline{C}_{SG}^{\Sigma})$. The goal is then to find a symplectomorphsim

$$(\mathcal{F}_{SG}^{\Sigma}, \varpi_{SG}^{\Sigma}) \xrightarrow{\Phi_r} (\mathcal{F}_{SG}^{r,\Sigma}, \varpi_{SG}^{r,\Sigma})$$

and then define $\mathcal{S}_{SG}^{\Sigma} := \Phi_r^{*}\left(\mathcal{S}_{SG}^{\Sigma,r}\right)$, as we know that it satisfies the CME by construction.

Symplectomorphism via 1–D AKSZ

In this subsection, we employ the methods from the 1–D AKSZ construction described in 1.4.1 to obtain the following -1–symplectic supermanifold

$$\mathcal{F}_{SG}^{AKSZ} := \mathrm{Map}(T[1]I, \mathcal{F}_{SG}^{\Sigma})$$
$$\varpi_{SG}^{AKSZ} := \mathfrak{T}_{I}^{(2)}(\varpi_{SG}^{\Sigma}).$$

and look for a symplectomorphism $\Phi_r\colon (\mathcal{F}_{SG}^{r}, \varpi_{SG}^{r}) \to (\mathcal{F}_{SG}^{AKSZ}, \varpi_{SG}^{AKSZ})$.

For starters we define, as in chapter 3.3, the AKSZ fields

$$\begin{aligned}
&\mathfrak{e} = e + f^{\dashv} && \mathfrak{w} = \omega + u^{\dashv}\\
&\mathfrak{p} = \psi + \varsigma^{\dashv} && \bar{\mathfrak{p}} = \bar{\psi} + \bar{\varsigma}^{\dashv}\\
&\mathfrak{x} = \chi + \epsilon && \bar{\mathfrak{x}} = \bar{\chi} + \bar{\epsilon}\\
&\mathfrak{c} = c + w && \mathfrak{z} = \xi + z\\
&\mathfrak{l} = \lambda + \mu && \mathfrak{c}^{\dashv} = k^{\dashv} + c^{\dashv}\\
&\mathfrak{x}^{\dashv} = \theta^{\dashv} + \chi^{\dashv} && \bar{\mathfrak{x}}^{\dashv} = \bar{\theta}^{\dashv} + \bar{\chi}^{\dashv}\\
&\mathfrak{y}^{\dashv} = e^{\dashv} + y^{\dashv},
\end{aligned} \tag{5.16}$$

[9] Indeed by construction one has $\{\mathcal{S}_{f}^{\Sigma}, \mathcal{S}_{f}^{\Sigma}\} = 0$, where $\{-,-\}$ is the Poisson bracket induced by $\varpi_{f}^{\Sigma} = \int_{\Sigma}\delta(\tilde{\xi}^{n}\tilde{v}^{\dashv})\delta\tilde{v}$.

having used the splitting $\mathcal{F}_{SG}^{AKSZ} := \Omega^\bullet(I) \times \mathcal{F}_{SG}^{\Sigma}$. Recalling (1.10), we have

$$\begin{aligned}
&e \in \mathcal{C}^\infty(I) \otimes \Omega_\partial^{(1,1)} && f^{\lrcorner} \in \Omega^1[-1](I) \otimes \Omega_\partial^{(1,1)}\\
&\omega \in \mathcal{C}^\infty(I) \otimes \Omega_\partial^{(1,2)} && u^{\lrcorner} \in \Omega^1[-1](I) \otimes \Omega_\partial^{(1,2)}\\
&\psi \in \mathcal{C}^\infty(I) \otimes \Omega_\partial^{(1,0)}(\Pi\mathbb{S}_M) && \varsigma^{\lrcorner} \in \Omega^1[-1](I) \otimes \Omega_\partial^{(1,0)}(\Pi\mathbb{S}_M)\\
&\chi \in \mathcal{C}^\infty(I) \otimes \Omega_\partial^{(0,0)}[1](\Pi\mathbb{S}_M) && \epsilon \in \Omega^1[-1](I) \otimes \Omega_\partial^{(0,0)}[1](\Pi\mathbb{S}_M)\\
&\xi \in \mathcal{C}^\infty(I) \otimes \mathfrak{X}[1](\Sigma) && z \in \Omega^1[-1](I) \otimes \mathfrak{X}[1](\Sigma)\\
&c \in \mathcal{C}^\infty(I) \otimes \Omega_\partial^{(0,2)}[1] && w \in \Omega^1[-1](I) \otimes \Omega_\partial^{(0,2)}[1]\\
&\lambda \in \mathcal{C}^\infty(I) \otimes \mathcal{C}^\infty[1](\Sigma) && \mu \in \Omega^1[-1](I) \otimes \mathcal{C}^\infty[1](\Sigma)\\
&k^{\lrcorner} \in \mathcal{C}^\infty(I) \otimes \Omega_\partial^{(3,2)}[-1] && c^{\lrcorner} \in \Omega^1[-1](I) \otimes \Omega_\partial^{(3,2)}[-1]\\
&\theta^{\lrcorner} \in \mathcal{C}^\infty(I) \otimes \Omega_\partial^{(3,4)}[-1](\Pi\mathbb{S}_M) && \chi^{\lrcorner} \in \Omega^1[-1](I) \otimes \Omega_\partial^{(3,4)}[-1](\Pi\mathbb{S}_M)\\
&\eta^{\lrcorner} \in \mathcal{C}^\infty(I) \otimes \Omega_\partial^{(3,3)}[-1] && e^{\lrcorner} \in \Omega^1[-1](I) \otimes \Omega_\partial^{(3,3)}[-1]
\end{aligned}$$

where we have omitted the $\sim$ symbol for boundary fields, as in this section there is not ambiguity of meaning.[10]

The AKSZ symplectic form is then given by

$$\begin{aligned}
\varpi_{SG}^{AKSZ} = \varpi_{PC}^{AKSZ} + \int_{I\times\Sigma} \frac{1}{3!}&\left(\underline{\bar\varsigma}^{\lrcorner}\gamma^3\delta\psi\delta e + \bar\psi\gamma^3\delta\underline{\varsigma}^{\lrcorner}\delta e + \bar\psi\gamma^3\delta\psi\delta\underline{f}^{\lrcorner} + \underline{f}^{\lrcorner}\delta\bar\psi\gamma^3\delta\psi\right)\\
&+ \frac{1}{3}e\delta\bar\psi\gamma^3\delta\underline{\varsigma}^{\lrcorner} + i\delta\underline{\bar\epsilon}\delta\theta^{\lrcorner} + i\delta\bar\chi\delta\underline{\chi}^{\lrcorner} + i\delta\underline{\bar\varsigma}(\iota_{\delta\xi}\theta^{\lrcorner} + \iota_\xi\delta\theta^{\lrcorner})\\
&+ i\delta\bar\psi\left(\iota_{\delta\underline{z}}\theta^{\lrcorner} + \iota_{\underline{z}}\delta\theta^{\lrcorner} + \iota_{\delta\xi}\underline{\chi}^{\lrcorner} + \iota_\xi\delta\underline{\chi}^{\lrcorner}\right)
\end{aligned}$$

Proposition 5.2. *There exists a symplectomorphism $\Phi_r\colon \mathcal{F}_{SG}^r \to \mathcal{F}_{SG}^{AKSZ}$ such that*

$$\begin{aligned}
&\Phi_r^*(e) = e + \lambda\mu^{-1}f^{\lrcorner} && \Phi_r^*(\underline{e}_n) = \mu\epsilon_n + \iota_{\underline{z}}e + \lambda\epsilon_n^i\underline{f}_i^{\lrcorner}\\
&\Phi_r^*(\omega) = \omega - \lambda\mu^{-1}u^{\lrcorner} && \Phi_r^*(\underline{\omega}_n) = \underline{w} - \iota_\xi\underline{u}^{\lrcorner} - \lambda\epsilon_n^i\underline{u}_i\\
&\Phi_r^*(\psi) = \psi + \lambda\mu^{-1}\varsigma^{\lrcorner} && \Phi_r^*(\underline{\psi}_n) = \underline{\epsilon} - \iota_\xi\underline{\varsigma}_d ag + \lambda\mu^{-1}\iota_{\underline{z}}\varsigma^{\lrcorner}\\
&\Phi_r^*(\psi^{\lrcorner}) = \theta^{\lrcorner} && \Phi_r^*(\underline{\psi}_n^{\lrcorner}) = \iota_{\underline{z}}\theta^{\lrcorner} + \iota_\xi\underline{\chi}_{\lrcorner} - \frac{e}{3}\gamma^3\underline{\varsigma}^{\lrcorner} - \frac{1}{3}\underline{f}^{\lrcorner}\gamma^3\psi + \frac{1}{3}\lambda\mu^{-1}\underline{f}\gamma^3\varsigma^{\lrcorner}\\
&\Phi_r^*(\chi) = \chi + \lambda\mu^{-1}\iota_\xi\varsigma^{\lrcorner} && \Phi_r^*(\underline{\chi}_n^{\lrcorner}) = \underline{\chi}_{\lrcorner}\\
&\Phi_r^*(c) = c - \lambda\mu^{-1}\iota_\xi(u^{\lrcorner}) && \Phi_r^*(\underline{c}_n^{\lrcorner}) = \underline{c}^{\lrcorner}\\
&\Phi_r^*(\omega^{\lrcorner}) = k^{\lrcorner} && \Phi_r^*(\underline{\omega}_n^{\lrcorner}) = e\underline{f}^{\lrcorner} + \iota_{\underline{z}}k^{\lrcorner} + \iota_\xi\underline{c}^{\lrcorner}\\
&\Phi_r^*(\xi^i) = \xi^i - \lambda\mu^{-1}z^i && \Phi_r^*(\underline{\xi}^{\lrcorner}) = e\underline{y}^{\lrcorner} + \underline{f}^{\lrcorner}e^{\lrcorner} - \underline{u}^{\lrcorner}k^{\lrcorner} + \underline{c}^{\lrcorner}\lambda\mu^{-1}u^{\lrcorner} + i\underline{\bar\varsigma}^{\lrcorner}\theta^{\lrcorner} - i\lambda\mu^{-1}\bar\varsigma^{\lrcorner}\underline{\chi}^{\lrcorner}\\
&\Phi_r^*(e^{\lrcorner}) = e^{\lrcorner} - \lambda\mu^{-1}y^{\lrcorner} && \Phi_r^*(\xi^n) = \lambda\mu^{-1}
\end{aligned}$$

$$\begin{aligned}
\Phi_r^*(\underline{e}_n^{\lrcorner}) =& e\underline{u}^{\lrcorner} + \iota_{\underline{z}}e^{\lrcorner} - \lambda\epsilon_n^i y_i^{\lrcorner} + \lambda\mu^{-1}f^{\lrcorner}\underline{u}^{\lrcorner} - \frac{1}{3!}\underline{\bar\varsigma}^{\lrcorner}\gamma^3\psi - \frac{1}{3!}\lambda\mu^{-1}\underline{\bar\varsigma}^{\lrcorner}\gamma^3\varsigma^{\lrcorner}\\
\Phi_r^*(\underline{\xi}_n^{\lrcorner}) =& e_n\underline{y}^{\lrcorner} + ef^{\lrcorner}\underline{u}\lrcorner + f^{\lrcorner}\iota_{\underline{z}}e^{\lrcorner} + u^{\lrcorner}\iota_{\underline{z}}k^{\lrcorner} + c^{\lrcorner}\lambda\epsilon_n^i\underline{u}_i^{\lrcorner} - \frac{1}{3!}\underline{f}^{\lrcorner}\bar\psi\gamma^3\underline{\varsigma}^{\lrcorner}\\
&- \frac{1}{3!}e\underline{\bar\varsigma}^{\lrcorner}\gamma^3\underline{\varsigma}^{\lrcorner} + i\iota_{\underline{z}}\bar\varsigma^{\lrcorner}\theta^{\lrcorner} - i\lambda\mu^{-1}\iota_{\underline{z}}\bar\varsigma^{\lrcorner}\chi^{\lrcorner}
\end{aligned}$$

[10]Note that, starting from **??**, we will reintroduce the $\sim$ notation, and the "untilded" fields will refer to the bulk.

Proof. We leave this long calculation for section B.3.5. □

With the above isomorphism, we can define the boundary BFV action as $\mathcal{S}^{\Sigma}_{SG} := (\Phi_r)^* \mathcal{S}^{\Sigma,r}_{SG}$ and

$$S^{AKSZ}_{SG} := \mathfrak{T}^{(0)}_I(\mathcal{S}^{\Sigma}_{SG}) + \iota_{d_I}\mathfrak{T}^{(1)}_I(\alpha^{\Sigma}_{SG}).$$

By definition one has

$$\begin{aligned}\mathcal{S}^{\Sigma}_{SG} &= \frac{1}{2}\iota_{Q^{AKSZ}_{SG}}\iota_{Q^{AKSZ}_{SG}}\varpi^{AKSZ}_{SG}\\ \mathcal{S}^{\Sigma,r}_{SG} &= \frac{1}{2}\iota_{Q^{r}_{SG}}\iota_{Q^{r}_{SG}}\varpi^{r}_{SG},\end{aligned}$$

but since $\varpi^{AKSZ}_{SG} = \Phi^*_r(\varpi^r_{SG})$ and $\mathcal{S}^{\Sigma}_{SG} = \Phi^*_r\left(\mathcal{S}^{\Sigma,r}_{SG}\right)$, then one must have $Q^{AKSZ}_{SG} = (\Phi_r)_* Q^r_{SG}$.

At the same time

$$\begin{aligned}\delta S^{AKSZ}_{SG} &= \iota_{Q^{AKSZ}_{SG}}\varpi^{AKSZ}_{SG} + \pi^*\vartheta_{\Sigma}\\ &= \Phi^*_r\left(\iota_{Q^r_{SG}}\varpi^r_{SG}\right) + \pi^*\vartheta_{\Sigma}\\ &= \delta S^r_{SG} + \pi^*(\vartheta_{\Sigma} - \vartheta^r_{SG}).\end{aligned}$$

Taking the variation of the above expression gives $\delta\vartheta_{\Sigma} - \delta\Phi^*_r(\vartheta^r_{SG}) = 0$, implying

$$\Phi^*_r(\varpi^{r,\Sigma}_{SG}) = \varpi^{\Sigma}_{SG},$$

which is exactly the symplectomorphism we were seeking on the space of BFV boundary fields.

Remark 5.7. We notice that, in section 4.2 we defined the BFV space of fields $\mathcal{F}^{\Sigma}_{SG}$ as the space of boundary fields subject to the structural constraint

$$\epsilon_n\left(d_\omega e - \frac{1}{2}\bar{\psi}\gamma\psi\right) + \cdots = e\sigma,$$

where the terms $(\cdots)$ were left undefined to account for contribution that would render the structural constraint invariant with respect to the cohomological vector field Q^{Σ}_{SG}, which was yet to be obtained in 4.2. In principle, when defining $\mathcal{F}^{AKSZ}_{SG}$, the structural constraints splits into a tangent constraint to Σ and a part containing the transversal component along the interval I. In order to overcome the ambiguity in the definition of $\mathcal{F}^{SG}_{AKSZ}$, we simply define the structural constraints to be

$$\Phi^*_r(\epsilon_n\check{k} - e\check{a}) = 0 \tag{5.17}$$

$$\Phi^*_r\left[\epsilon_n\left(d_\omega e - \frac{1}{2}\bar{\psi}\gamma\psi - \tau^{\lrcorner}d\xi^n\right) + \iota_X k^{\lrcorner} + \mathfrak{Q} - e\sigma\right] = 0. \tag{5.18}$$

In particular, we see that (5.17) is automatically satisfied by construction. Indeed, (5.17) is equivalent to $\mathfrak{W}^{\lrcorner} \in \mathrm{Im}(W^{\partial(1,1)}_e)$, therefore, recalling the definition (2.36) of $\mathfrak{W}^{\lrcorner}$, we have

$$\begin{aligned}&\Phi^*_r\left(\omega^{\lrcorner}_n - \iota_z\omega^{\lrcorner} - \iota_\xi c^{\lrcorner}_n + \iota_z c^{\lrcorner}_n\xi^n\right) =\\ &= ef^{\lrcorner} + \iota_z k^{\lrcorner} + \iota_\xi c^{\lrcorner} - z^i k^{\lrcorner}_i - \iota_\xi c^{\lrcorner} - z^i\lambda\mu^{-1}c^{\lrcorner} + z^i\lambda\mu^{-1}c^{\lrcorner}\\ &= ef^{\lrcorner}.\end{aligned}$$

This tells us that the only relevant constraint in the space of AKSZ fields is given by (5.18), which in turn splits into the structural constraint of $\mathcal{F}^{\Sigma}_{SG}$, simply given as

$$\epsilon_n \left(d_\omega e - \frac{1}{2}\bar{\psi}\gamma\psi \right) + \iota_X k^{\lrcorner} + \mathfrak{Q}' = e\sigma \tag{5.19}$$

and the part proportional to $\lambda\mu^{-1}$, which is given by

$$\epsilon_n \left([u^{\lrcorner}, e] + d_\omega f^{\lrcorner} - \bar{\psi}\gamma\varsigma^{\lrcorner} \right) + \left(\mathrm{L}^{\omega}_{z}\epsilon_n + [\iota_\xi u^{\lrcorner} - w, \epsilon_n] \right)^i k^{\lrcorner}_i + \iota_X c^{\lrcorner} + (X^i f^{\lrcorner}_i)^j k^{\lrcorner}_j + \mathfrak{U} =^{\lrcorner} \sigma + eB,$$

having defined $\Phi^*_r(\mathfrak{Q}) = \mathfrak{Q}' + \lambda\mu^{-1}\mathfrak{U}$. The above two equations then define respectively the tangential and transversal AKSZ constraints.

Appendix A

Spinors and spin coframes on manifolds: a review with technical results

Spinors are fundamental in the description of supersymmetry theories and, specifically, of supergravities. Historically, mathematicians and physicists have adopted different notations for the same objects, used in different context. In particular, the definition of supergravity theories requires considering different kinds of spinors depending on the spacetime dimensions, which is often a source of confusion to the uninitiated reader.

This review is an attempt to provide a self-contained account on the fundamental tools used in the study of spinors, trying to reconcile the rigorous mathematical definitions with the less precise physics terminology. The secondary scope of these notes is to be a repository of results which are used by the author in forthcoming papers on supergravity, removing the necessity to provide a series of heavy technical proofs, which are conveniently regrouped here.

The work is organized as follows: the first part of section A.1 provides all the necessary definitions and a classification of real and complex Clifford algebras, followed by the definition and main results on spin groups (and their Lie algebras). Furthermore, a systematic classification of the representations of Clifford algebras is presented, with a particular interest in the Lorentzian case, in which we provide a constructive method to obtain the so–called gamma representation, commonly used in physics. The last part of this section is devoted to the definition of Majorana spinors, a central object in the theories of supergravity, showing the direct correlation between the existence of a real structure and the so–called charge conjugation matrix.

In section A.2, we employ the ideas developed in the previous chapter within the context of differential geometry, providing a global description of spin structures and, specifically, spin coframes, a concept which is particularly useful in supergravity. Indeed one can show that the notion of spin coframes is equivalent to that of spin structure (and, in particular, requires the same topological assumptions to exist), with the advantage of providing a framework which allows to define spinor fields without the necessity of fixing a metric, which is ultimately considered as a dynamical object in the context of physics.

Lastly, section A.3 contains some very well known identities, as well as some lesser known ones. A full description on how to obtain Fierz rearrangements in $D = 4$ is presented, with a particularly useful example in the mostly plus Lorentzian signature. Most of these results are rephrased in the index–free notation provided by the spin coframe formalism. Finally, the last part presents a series of technical lemmata in dimension 4, which, as previously anticipated, acts as a repository of results useful in future works of the author.

A.1 Clifford algebras and spin groups

Some of the introductory content in the following section has appeared in [CCF22] and has been reported to have a complete discussion with a consistent notation. The remaining section on Clifford algebras mainly follows ,[KS87a],[**fatibene2018**], [LM90], [RS17] and [Fig].

The parts regarding Majorana spinors and Fierz identities follow [Fig], [CDF91a; CDF91b], [FV12], [KT83], [Sch79] and [Dab88]. References [Van85; Har90] are also recommended.

A.1.1 Clifford algebras

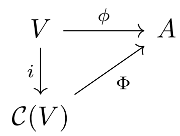
Let V be a real vector space of dimension D with an inner product of signature (r, s). Let η_{ab} be the matrix $\mathrm{diag}(-1, \cdots, -1, 1, \cdots, 1)$ with r plus 1 and s minus 1, giving the inner product on V with respect to an orthonormal basis $\{v_a\}$, $a = 1, \cdots, D$.

We define the Clifford algebra on V by means of its universal property. In particular

Definition A.1 (Clifford map)**.** A Clifford map is given by the pair (A, ϕ) where A is an associative algebra with unity and ϕ is a linear map $\phi\colon V \to A$ such that $\forall u, v \in V$

$$\phi(u)\phi(u) = -\eta(u, u)\mathbb{1}_A \tag{A.1}$$

Definition A.2 (Clifford algebra)**.** The Clifford algebra $\mathcal{C}(V)$ is an associative algebra with unit together with a Clifford map $i\colon V \to \mathcal{C}(V)$ such that any Clifford map factors through a unique algebra homomorphism from $C(V)$. In other words, given any Clifford map (A, ϕ) there is a unique algebra homomorphism $\Phi\colon \mathcal{C}(V) \to A$ such that $\phi = \Phi \circ i$



Proposition A.1. *The Clifford algebra of V is unique up to isomorphisms.*

We give a model for such an algebra. Consider the tensor algebra $T(V) := \mathbb{R} \oplus V \oplus V^{\otimes 2} \oplus \cdots$ and quotient it out by the two-sided ideal $I(V)$ generated by $v \otimes v + \eta(v, v)\mathbb{1}$, i.e.

$$\mathcal{C}(V) := \frac{T(V)}{I(V)}.$$

Indeed one can set i to be the composition of the canonical projection $\rho : T(V) \to \mathcal{C}(V)$ with the inclusion $V \hookrightarrow T(V)$. Every linear map $\phi : V \to A$ extends uniquely to an algebra homomorphism $\tilde{\Phi} : T(V) \to A$, which identically vanishes on $I(V)$ by (A.1). This implies that $\tilde{\Phi}$ uniquely descends to a homomorphism $\Phi : \mathcal{C}(V) \to A$, satisfying

$$\Phi \circ i = \phi.$$

Notice that $T(V)$ is a $\mathbb{Z}$-graded algebra. The ideal $I(V)$ is spanned by elements that are not necessarily homogeneous, therefore the $\mathbb{Z}$–grading is lost in the Clifford algebra. However, the generators of $I(V)$ are even, therefore $\mathcal{C}(V)$ will be $\mathbb{Z}_2$-graded. In particular, it splits into

$$\mathcal{C}(V) = \mathcal{C}_0(V) \oplus \mathcal{C}_1(V).$$

Another important property, for any two vectors $v, w \in V$, is the following

$$\begin{aligned}(v+w)^2 = v^2 + vw + wv + w^2 &= -\eta(v,v)\mathbb{1} - \eta(w,w)\mathbb{1} + \{v,w\} \\ &= -\eta(v+w, v+w)\mathbb{1} = -\eta(v,v)\mathbb{1} - \eta(w,w)\mathbb{1} - 2\eta(v,w)\mathbb{1} \\ \Rightarrow \quad \{v,w\} &:= vw + wv = -2\eta(v,w)\mathbb{1}.\end{aligned}$$

Now, considering an orthonormal basis $\{v_a\}$ of V, setting the first s elements $\{v_A\}$ such that $\eta(v_A, v_A) = -1$ and the second r elements $\{v_i\}$ such that $\eta(v_i, v_i) = 1$, we obtain $\{v_a, v_b\} = -2\eta_{ab}\mathbb{1}$. This means that when $a \neq b$, $v_a v_b = -v_b v_a$ and that $v_a v_a = \pm\mathbb{1}$.

At this point, since every element in the tensor algebra $T(V)$ is a finite linear combination of the product of finite elements in the basis of V, to obtain elements in $\mathcal{C}(V)$ we simply apply the constraint $\{v_a, v_b\} = -2\eta_{ab}\mathbb{1}$. Indeed, since the elements of V are multiplicative generators of $T(V)$, they must also generate $\mathcal{C}(V)$, hence a basis of Clifford algebra is given in the form

$$\mathbb{1} \quad v_a \quad \underset{a<b}{v_{ab} := v_a v_b} \quad \underset{a<b<c}{v_{abc} := v_a v_b v_c} \quad \cdots \quad v_* := v_1 \cdots v_D \tag{A.2}$$

The $\mathbb{Z}_2$-grading is now clearer, as we can interpret even (odd) elements of $\mathcal{C}(V)$ to be finite linear combinations of products of an even (odd) number of elements of the basis V. In particular, the even part $\mathcal{C}_0(V)$ is a sub-algebra of $\mathcal{C}(V)$, while the odd part $\mathcal{C}_1(V)$ is not (it does not contain the unity). They are both 2^{d-1}-dimensional, making $\mathcal{C}(V)$ 2^d-dimensional.

Proposition A.2. *There exists a canonical isomorphism between the Clifford algebra and the exterior algebra of V*

$$\sigma\colon \mathcal{C}(V) \to \wedge^\bullet V \tag{A.3}$$

Proof. For any $u = u^a v_a \in V$, consider the dual vector $\underline{u} = \eta_{ab} u^b \nu^a$, where $\{\nu^a\}$ is a basis of covectors such that $\nu^a(v_b) = \delta^a_b$. Let θ be the mapping

$$\theta\colon V \to \mathrm{End}(\wedge^\bullet V) \qquad \text{s.t.} \qquad \theta(u)(\alpha) = u \wedge \alpha + \iota_{\underline{u}}\alpha,$$

where $\iota_{\underline{u}}\alpha$ is the contraction with the covector of u for all $\alpha \in \wedge^\bullet V$. Then one finds

$$\begin{aligned}\theta(u)^2\alpha &= u \wedge u \wedge \alpha + u \wedge \iota_{\underline{u}}\alpha + \iota_{\underline{u}}(u \wedge \alpha) + \iota_{\underline{u}}\iota_{\underline{u}}\alpha \\ &= u \wedge \iota_{\underline{u}}\alpha + \iota_{\underline{u}} u \wedge \alpha - u \wedge \iota_{\underline{u}}\alpha = \iota_{\underline{u}} u \alpha \\ &= \eta(u,u)\alpha.\end{aligned}$$

This implies, by the universal property, the existence of an algebra morphism

$$\hat{\theta} : \mathcal{C}(V) \to \mathrm{End}(\wedge^\bullet V),$$

which, composed with with the identity element in $\mathrm{End}(\wedge^\bullet V)$, yields

$$\sigma : \mathcal{C}(V) \to \wedge^\bullet V.$$

It is immediate to check that an element $u_1 \cdots u_k \in \mathcal{C}(V)$ is sent to $u_1 \wedge \cdots \wedge u_k \in \wedge^\bullet V$, hence one obtains that a basis of $\mathcal{C}(V)$ is sent to a basis of $\wedge^\bullet V$, proving that σ defines an isomorphism.

Remark A.1. The highest grade basis element v_* is also known as volume element, in analogy with its image under σ, defining the volume form on V.

□

A.1.2 Classification of Clifford Algebras

We start by classifying real Clifford algebras. In this section, we denote by $\mathcal{C}(r,s)$ the Clifford algebra over the D-dimensional real vector space V endowed with a non-degenerate metric of signature (r,s). We will also denote by $\mathbb{K}(N)$ the $N \times N$ matrices over the field $\mathbb{K}$, while, in view of the future definition of gamma matrices, in this section we will denote the generators of the Clifford algebra by Γ_a. We first consider the low-dimensional Clifford algebras, which will provide the fundamental building blocks to obtain the higher dimensional ones.

Lemma A.1.

$$\text{(i) } \mathcal{C}(1,0) \simeq \mathbb{C}, \quad \text{(ii) } \mathcal{C}(0,1) \simeq \mathbb{R} \oplus \mathbb{R}, \quad \text{(iii) } \mathcal{C}(1,1) \simeq \mathbb{R}(2),$$
$$\text{(iv) } \mathcal{C}(0,2) \simeq \mathbb{R}(2), \quad \text{(v) } \mathcal{C}(2,0) \simeq \mathbb{H}.$$

Proof. In order to prove the above statements, we pick a representation of the Clifford algebra in terms of matrices,

(i) there is only one element $\{v_1\}$ in the basis of V, such that $\Gamma_1^2 = -\mathbb{1}$, defining a complex structure on $T(V)$, hence $\mathcal{C}(1,0) = \mathbb{C}$;

(ii) analogously, we find $\Gamma_1^2 = \mathbb{1}$, hence $\mathcal{C}(0,1) = \mathbb{R} \oplus \mathbb{R}$;

(iii) following the physics notation and setting $\{v_0, v_1\}$ as basis of V such that $\Gamma_0^2 = \mathbb{1}$ and $\Gamma_1^2 = -\mathbb{1}$, we can choose the following anticommuting matrices

$$\Gamma_0 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \Gamma_1 = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{A.4}$$

They are 2×2 real matrices, hence they generate $\mathcal{C}(1,1) = \mathbb{R}(2)$. The even part is generated by $\mathbb{1}$ and $\Gamma_* = \Gamma_0\Gamma_1$, given by

$$\Gamma_* = -\sigma_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

hence obtaining $\mathcal{C}_0(1,1)$ as the diagonal 2×2 real matrices;

(iv) in the case of $\mathcal{C}(0,2)$, we pick anticommuting matrices Γ_1 and Γ_2 squaring to $\mathbb{1}$, which are explicitly realized by

$$\Gamma_1 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \Gamma_2 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

as before we obtain $\mathcal{C}(0,2) = \mathbb{R}(2)$, and the volume element is given by

$$\Gamma_* = -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

which defines a complex structure as it squares to $-\mathbb{1}$. Hence the even subalgebra, being generated by $\mathbb{1}$ and Γ_*, is $\mathcal{C}_0(0,2) = \mathbb{C}$;

(v) for $\mathcal{C}(2,0)$ we need two anticommuting matrices Γ_1 and Γ_2 squaring to $-\mathbb{1}$, which are explicitly realized by

$$\Gamma_1 = i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad \text{and} \qquad \Gamma_2 = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

both squaring to $-\mathbb{1}$, hence defining two anticommuting complex structures. The Clifford algebra then has to coincide with the algebra of quaternions $\mathbb{H}$, explicitly realized by the identification $\{1, i, j, k\} = \{\mathbb{1}, \Gamma_1, \Gamma_2, \Gamma_*\}$, where

$$\Gamma_* = -i\sigma_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},$$

The even subalgebra is generated by $\mathbb{1}$ and Γ_*, which again defines a complex structure, hence obtaining $\mathcal{C}_0(0,2) = \mathbb{C}$.

□

The following lemma allows to recover higher dimensional Clifford algebras from the lower dimensional ones

Lemma A.2. *The following statements are true*

1. $\mathcal{C}(d,0) \otimes \mathcal{C}(0,2) \simeq \mathcal{C}(0,d+2)$
2. $\mathcal{C}(0,d) \otimes \mathcal{C}(2,0) \simeq \mathcal{C}(d+2,0)$
3. $\mathcal{C}(r,s) \otimes \mathcal{C}(1,1) \simeq \mathcal{C}(r+1,s+1)$

Proof. For (i), consider $\{v_i\}$, $i = 1, \cdots, d$ and $\{v_\alpha\}$, $\alpha = d+1, d+2$ respectively generating $\mathcal{C}(d,0)$ and $\mathcal{C}(0,2)$. Then there are relations

$$v_i \cdot v_j = -2\delta_{ij}\mathbb{1} \qquad \text{and} \qquad v_\alpha \cdot v_\beta = 2\delta_{\alpha\beta}.$$

We can define new elements $\{v_a\}$, $a = 1, \cdots, d+2$ as

$$v_a := \begin{cases} v_i \otimes v_{d+1} \cdot v_{d+2} & a \leq d \\ \mathbb{1} \otimes v_\alpha & a > d \end{cases}$$

A quick computation gives

$$v_a \cdot v_b = 2\delta_{ab}\mathbb{1},$$

hence proving the v_a's generate $\mathcal{C}(0,d+2)$.

The case of (ii) is analogous. For (iii) consider $\{v_1, \cdots, v_r, v_{r+1}, \cdots, v_{r+s}\}$ as a basis of $\mathbb{R}^{r,s}$, generating $\mathcal{C}(r,s)$, and $\{v'_1, v'_2\}$ as generating $\mathcal{C}(1,1)$. Then we define a new set of vectors $\{v_a\}$, $a = 1, \cdots, d+2$ such that

$$v_a = \begin{cases} v_a \otimes v'_1 \cdot v'_2, & 1 \leq a \leq r \\ \mathbb{1} \otimes v'_1 & a = r+1 \\ v_{a-1} \otimes v'_1 \cdot v'_2 & r+1 \leq a \leq d+1 \\ \mathbb{1} \otimes v'_2 & a = d+2 \end{cases}$$

A quick computation shows that the newly defined $v'_a s$ generate $\mathcal{C}(r+1,s+1)$. □

As a result, one can show that structure of the (r,s) real Clifford algebra has periodicity 8 in $r - s$. The following prposition allows us to classify the even Clifford subalgebras.

Proposition A.3. *The even Clifford subalgebra is related to the full one in the following way*

$$\mathcal{C}_0(r+1,s) \simeq \mathcal{C}(s,r) \qquad \text{and} \qquad \mathcal{C}_0(r,s+1) \simeq \mathcal{C}(r,s), \tag{A.5}$$

furthermore,

$$\mathcal{C}_0(r,s) \simeq \mathcal{C}_0(s,r). \tag{A.6}$$

Taking into account the periodicity of the structure of Clifford algebras, we obtain the following classification

$r-s$ mod 8	$\mathcal{C}(r,s)$	N
0,6	$\mathbb{R}(2^{\frac{N}{2}})$	D
2,4	$\mathbb{H}(2^{\frac{N}{2}})$	$D-2$
1,5	$\mathbb{C}(2^{\frac{N}{2}})$	$D-1$
3	$\mathbb{H}(2^{\frac{N}{2}})\oplus\mathbb{H}(2^{\frac{N}{2}})$	$D-3$
7	$\mathbb{R}(2^{\frac{N}{2}})\oplus\mathbb{R}(2^{\frac{N}{2}})$	$D-1$

$r-s$ mod 8	$\mathcal{C}_0(r,s)$	N
1,7	$\mathbb{R}(2^{\frac{N}{2}})$	$D-1$
3,5	$\mathbb{H}(2^{\frac{N}{2}})$	$D-3$
2,6	$\mathbb{C}(2^{\frac{N}{2}})$	$D-2$
4	$\mathbb{H}(2^{\frac{N}{2}})\oplus\mathbb{H}(2^{\frac{N}{2}})$	$D-4$
0	$\mathbb{R}(2^{\frac{N}{2}})\oplus\mathbb{R}(2^{\frac{N}{2}})$	$D-2$

Table A.1: Clifford algebras and even Clifford subalgebras in various dimensions

The situation is significantly simplified when one takes into consideration the complexification of the Clifford algebras. Consider $V_{\mathbb{C}} = V\otimes_{\mathbb{R}}\mathbb{C}$ and define the mapping

$$\hat{\imath}\colon V\otimes_{\mathbb{R}}\mathbb{C}\to\mathcal{C}(V)\otimes_{\mathbb{R}}\mathbb{C}\colon u\otimes z\mapsto i(u)\otimes z,$$

then $\hat{\imath}(u\otimes z)^2 = i(u)^2\otimes z^2 = -\eta(u,u)\mathbb{1}\otimes z^2 = -\eta(u\otimes z,u\otimes z)\mathbb{1}$, proving that

$$\mathcal{C}(V)_{\mathbb{C}} = \mathcal{C}(V)\otimes_{\mathbb{R}}\mathbb{C} = \mathcal{C}(V_{\mathbb{C}}). \tag{A.7}$$

Now, since on $V_{\mathbb{C}}$ it is always possible to diagonalize η to a Euclidean metric, denoting by $\mathcal{C}(D)$ the complex Clifford algebra over $\mathbb{C}^D$, one obtains

$$\mathcal{C}(D)\simeq\mathcal{C}(D,0)_{\mathbb{C}}\simeq\mathcal{C}(D-1,1)_{\mathbb{C}}\simeq\cdots\simeq\mathcal{C}(0,D)_{\mathbb{C}}. \tag{A.8}$$

Notice also that the above statement, together with proposition A.3, implies that

$$\mathcal{C}_0(D)\simeq\mathcal{C}(D-1). \tag{A.9}$$

Proposition A.4.

$$\mathcal{C}(n+2)\simeq\mathcal{C}(n)\otimes\mathbb{C}(2),\qquad\mathcal{C}(2k)\simeq\mathbb{C}(2^k),\qquad\mathcal{C}(2k+1)\simeq\mathbb{C}(2^k)\oplus\mathbb{C}(2^k). \tag{A.10}$$

Proof. Using lemma A.2 and eq. (A.8), we see that

$$\mathcal{C}(n+2)\simeq(\mathcal{C}(n,0)\otimes_{\mathbb{R}}\mathbb{C})\otimes_{\mathbb{C}}(\mathcal{C}(0.2)\otimes_{\mathbb{R}}\mathbb{C})\simeq\mathcal{C}(n)\otimes_{\mathbb{C}}\mathcal{C}(2).$$

By lemma A.1 and (A.7) we obtain

$$\mathcal{C}(1)\simeq\mathbb{C}\oplus\mathbb{C}\qquad\text{and}\qquad\mathcal{C}(2)\simeq\mathbb{C}(2),$$

thanks to which, by iteration of the above result, we obtain

$$\mathcal{C}(2k)\simeq\bigotimes^{k}\mathbb{C}(2)\simeq\mathrm{End}(\bigotimes^{k}\mathbb{C}^2)\simeq\mathbb{C}(2^k)$$

and

$$\mathcal{C}(2k+1)\simeq\bigotimes^{k}\mathbb{C}(2)\oplus\bigotimes^{k}\mathbb{C}(2)\simeq\mathbb{C}(2^k)\oplus\mathbb{C}(2^k).$$

□

Therefore, proposition (A.9), allows to obtain the following classification

D mod 2	$\mathcal{C}(D)$	N
0	$\mathbb{C}(2^{\frac{N}{2}})$	D
1	$\mathbb{C}(2^{\frac{N}{2}}) \oplus \mathbb{C}(2^{\frac{N}{2}})$	$D-1$

D mod 2	$\mathcal{C}_0(D)$	N
0	$\mathbb{C}(2^{\frac{N}{2}}) \oplus \mathbb{C}(2^{\frac{N}{2}})$	$D-2$
1	$\mathbb{C}(2^{\frac{N}{2}})$	$D-1$

Table A.2: Complex Clifford algebra and even subalgebra in various dimensions

A.1.3 Pin and Spin groups

Definition A.3 (grading map)**.** Consider the Clifford map $i\colon V \to \mathcal{C}(V)$. By abuse of notation, this map sends v to v inside $\mathcal{C}(V)$. Defining $\alpha := -i\colon v \to \mathcal{C}(V) : v \mapsto -v$, it has the property that $\alpha(v)\alpha(v) = -\eta(v,v)\mathbb{1}$. We can extend it to the whole $\mathcal{C}(V)$ as $\alpha\colon \mathcal{C}(V) \to \mathcal{C}(V)$ by restricting it to the identity on even elements, to minus the identity on odd elements. This map is called grading (or parity) since it essentially defines the $\mathbb{Z}_2$-grading on $\mathcal{C}(V)$.

Clearly we have that $\alpha \circ \alpha = \mathbb{1}$, therefore α is invertible and equal to its inverse.

Definition A.4 (transpose)**.** Let $S = u_1u_2\cdots u_k \in \mathcal{C}(V)$. We define the transpose of S to be

$$^t(S) = {}^t(u_1u_2\cdots u_k) := u_k\cdots u_2u_1 =: \overline{S}$$

It is well defined since the generators of the Clifford ideal are invariant under the transposition.

Furthermore, the transpose preserves the grading, namely $^t(\alpha(S)) = \alpha({}^t(S))$.

It is a well known fact that not all elements in $\mathcal{C}(V)$ are invertible. Let us define the multiplicative subgroup $\mathcal{C}^*(V) \subset \mathcal{C}(V)$ of invertible elements. Clearly every subgroup of $\mathcal{C}(V)$ is contained in $\mathcal{C}^*(V)$.

Definition A.5 (Clifford group)**.** The Clifford group is defined to be the Lie subgroup of $\mathcal{C}^*(V)$, given by

$$\Gamma(V) := \{S \in \mathcal{C}^*(V) \mid \forall u \in V, \alpha(S)uS^{-1} \in V\}.$$

The map $l : \Gamma(V) \to \mathrm{Aut}(V)$ defined by $\alpha(S)(u) = \alpha(S)uS^{-1}$ is by definition a representation of $\Gamma(V)$, called twisted adjoint representation.

Lemma A.3. *The twisted adjoint representation is such that*

1. $l(\alpha(S)) = l(S)$ *for all* $S \in \Gamma(V)$*;*
2. *for any vector* $v \in V$ *such that* $\eta(v,v) = \pm 1$*, the map* $l(v)$ *is a reflection about the plane orthogonal to the unit vector* v*;*
3. $\ker(l) \simeq \mathbb{R}^*$*.*

Proof. We prove each point separately:

1. $l(S)(u) = -\alpha(l(S)(u) = -\alpha(\alpha(S)uS^{-1}) = Su\alpha(S)^{-1} = l(\alpha(S)(u))$.
2. Recalling that $vv = -\eta(v,v)\mathbb{1} = -|v|^2\mathbb{1}$, we have $v^{-1} = -\frac{v}{|v|^2}$. For all $w \in V$, denote $w^{\|} := \frac{\eta(v,w)}{\eta(v,v)}v$ to be the component of w parallel to $v \in V$. The perpendicular component is defined as $w^{\perp} := w - w^{\|}$. Then

$$\begin{aligned}\alpha(v)wv^{-1} &= -vwv^{-1} = |v|^{-2}vwv = |v|^{-2}\left(uw^{\perp}v + vw^{\|}v\right)\\ &= |v|^{-2}(-vvw^{\perp} - \eta(v,w^{\perp})v - |v|^2w^{\|})\\ &= w^{\perp} - w^{\|} = l(v)w.\end{aligned}$$

3. Setting $\{v_a\}$ as the usual orthonormal basis of V, let $S \in \ker(l)$, then for all $u \in V$, $\alpha(S)uS^{-1} = u$, implying $\alpha(S)u = uS$. Splitting $S = S_0 + S_1$ into even and odd part, we obtain
$$uS_0 = S_0 u \qquad uS_1 = -S_1 u.$$
Without loss of generality, we can set $S_0 = p_0 + v_1 p_1$, where p_0 and p_1 are respectively even and odd polynomials in $v_2, \cdots, v_D$. Then, using the above equation with $u = v_1$, we see
$$v_1 p_0 + v_1^2 p_1 = p_0 v_1 + v_1 p_1 v_1 = p_0 v_1 - v_1^2 p_1,$$
hence $v_1^2 p_1 = 0$, implying $p_1 = 0$. As a consequence S_0 does not contain v_1, but this procedure can be iterated for all basis elements v_a, hence one must have $S_0 = \lambda \mathbb{1}$ for some $\lambda \in \mathbb{R}^*$. The same argument can be repeated for S_1, hence showing $\ker l = \mathbb{1} \cdot \mathbb{R}^*$.

□

Theorem A.1. *The following is a short exact sequence*
$$1 \to \mathbb{R}^* \to \Gamma(V) \to O(V) \to 1 \tag{A.11}$$

Proof. By point 3 of the previous lemma, $\ker(l) = \mathbb{R}^*$, hence we just need to show l is surjective onto $O(V)$. Notice
$$\begin{aligned}\eta\big(l(S)u, l(S)w\big) &= -\frac{1}{2}\big(l(S)ul(S)w + l(S)wl(S)u\big)\\ &= -\frac{1}{2}\big(l(S)ul(\alpha(S))w + l(S)wl(\alpha(S))u\big)\\ &= -\frac{1}{2}\alpha(S)(uw + wu)\alpha(S^{-1})\\ &= \eta(u, w),\end{aligned}$$
hence proving that $l : \Gamma(V) \to O(V)$ and l is a homomorphism.

Now, by Cartan-Dieudonne theorem, for all $R \in O(V)$, $R = R_1 \cdots R_k$ for $k \leq D = \dim(V)$ and R_i are reflections. By point 2 we know there exist unit vectors $u_i \in V$ such that $R_i = l(u_i)$ and therefore $R = l(u_1) \cdots l(u_k) = l(u_1 \cdots u_k)$, hence showing that l is surjective.

□

One can define the further subgroup $S(V) \subset \mathcal{C}^*(V) \subset \mathcal{C}(V)$ of invertible elements S whose inverse is proportional to their transpose, namely such that $S\overline{S} \propto \mathbb{1}$.

Definition A.6 (Pin and Spin groups)**.** We define the Pin group $\mathrm{Pin}(V)$ to be the subgroup of $S(V)$ generated by unit vectors (i.e. such that $v^2 = \eta(v, v) = \pm 1$), while the Spin group $\mathrm{Spin}(V)$ is defined to be the intersection of $\mathrm{Pin}(V)$ with the even Clifford subalgebra $\mathcal{C}(V)$. In other words
$$\mathrm{Pin}(V) := \{u_1 \cdots u_k \mid u_i^2 = \pm 1\} \tag{A.12}$$
$$\mathrm{Spin}(V) := \{u_1 \cdots u_k \mid \ k \text{ even and } u_i^2 = \pm 1\} = \mathrm{Pin}(V) \cap \mathcal{C}_0(V). \tag{A.13}$$

Elements in $\mathrm{Spin}(V)$ are products of an even number of unit vectors, $S = u_1 u_2 \cdots u_{2k}$. In this case it is easy to find the inverse of S, as
$$S^{-1} = \frac{\pm 1}{|u_1|^2 \cdots |u_{2k}|^2} u_{2k} \cdots u_2 u_1$$

As an immediate consequence of the above theorem, we have the following

Corollary A.1. *The restriction of l to the Pin and Spin groups defines the following short exact sequences*

$$\begin{aligned} 1 \to \mathbb{Z}_2 \to \mathrm{Pin}(V) \to O(V) \to 1, \\ 1 \to \mathbb{Z}_2 \to \mathrm{Spin}(V) \to SO(V) \to 1. \end{aligned} \tag{A.14}$$

Lie Algebra of Spin group

Proposition A.5. *Let V be a D–dimensional real vector space. Lie(Spin(V)) is a Lie subalgebra of $\mathcal{C}(V)$, given by*

$$\mathrm{Lie}(\mathrm{Spin}(V)) = \wedge^2 V$$

Proof. This can be seen by noticing that the double cover $l\colon \mathrm{Spin}(V) \to SO(V)$ reduces to an isomorphism of Lie algebras (locally their tangent space at the identity is the same)

$$\begin{aligned} \dot{l}\colon \mathfrak{spin}(V) &\to \mathfrak{so}(V) \\ a &\longmapsto \dot{l}(i) = [a, \cdot], \end{aligned}$$

where, for all $u \in V$, the $[a, u] \in SO(V)$ is given by

$$[a, u] := \frac{\partial}{\partial t}\Big|_{t=0} (e^{-ta} u e^{ta}).$$

Now, knowing that $\{v_a \wedge v_b\}$ is a basis for $\mathfrak{so}(V)$, we compute a basis for $\mathfrak{spin}(V)$.
Define $v_{ab} := \frac{1}{4}[v_a, v_b]$, then, for all $u = u^c v_c \in V$

$$\begin{aligned} \dot{l}(v_{ab})u &= \frac{1}{4}[[v_a, v_b], u] = \frac{1}{2}[v_a v_b, u] \\ &= \frac{1}{2}\left(v_a v_b u - u v_a v_b\right) = \frac{1}{2}\left(v_a v_b u - u v_a v_b + v_a u v_b - v_a u v_b\right) \\ &= \eta(u, v_a) v_b - \eta(v_b, u) v_a = u^c(\delta^d_b \eta_{ac} - \delta^d_a \eta_{bc}) v_d, \end{aligned}$$

hence

$$\dot{l}(v_{ab})^d_c = \delta^d_b \eta_{ac} - \delta^d_a \eta_{bc} = -(M_{ab})^d_c$$

where M_{ab} are the generators of the Lorentz group $SO(V)$ in the fundamental representation. This implies that $-\frac{1}{4}[v_a, v_b]$ defines a basis for $\mathfrak{spin}(V)$.

□

A.1.4 Representations

Given the definition of the Pin and Spin groups seen respectively as subgroups of $\mathcal{C}(V)$ and $\mathcal{C}_0(V)$, classifying irreducible representations of $\mathcal{C}(V)$ and $\mathcal{C}_0(V)$ will automatically produce a classification of irreps of the Pin and Spin groups, which are called respectively pinor and spinor representations.

Looking at table A.1.2, we can already classify the irreducible pinor representations, as $\mathbb{H}(N)$ and $\mathbb{R}(N)$ have a unique irreducible representation given respectively by $\mathbb{H}^N$ and $\mathbb{R}^N$, whereas $\mathbb{C}(N)$ has two, one isormorphic to $\mathbb{C}^N$ and the complex conjugate one. Therefore the number of irreducible pinor representations is given by

$$p_{r,s} = \begin{cases} 2 \text{ if } r - s = 1, 3 \bmod 4, \\ 1 \text{ if } r - s = 2, 4 \bmod 4. \end{cases}$$

Table A.1.2 also tells us whether the representation is real, complex or quaternionic.

For the spinor representations, since Spin is a subspace of the even Clifford algebra, we only need to look at table A.1.2, which implies that the number of irreducible inequivalent spinor representations is

$$s_{r,s} = \begin{cases} 2 \text{ if } r-s = 2,4 \bmod 4, \\ 1 \text{ if } r-s = 1,3 \bmod 4. \end{cases}$$

Notice that in the even dimensional case $D = 2k$ there are two inequivalent irreducible spinor representations (known as Weyl representations), which correspond to the Weyl spinors and can be understood by looking at the volume element $v_* = v_1 \cdots v_D$. Being v_* the product of an even number of generators, it anticommutes with them, i.e. $\{v_*, v_a\} = 0$ for all $a = 1, \cdots, D$, but it commutes with all the elements in the even Clifford subalgebra, hence with all the group elements in $\mathrm{Spin}(V)$, which implies that it must act as a scalar in the spinor representations. This means that the inequivalent Weyl representations can be labelled by the eigenvalues of v_*. Furthermore, a straightforward computation gives, for even $D = r + s = 2k$

$$v_*^2 = (-1)^{\frac{r-s}{2}} \mathbb{1}. \tag{A.15}$$

One can classify the inequivalent spinor representation as follows

- $r - s = 0 \bmod 8$. There are two inequivalent real spinor representations, of real dimension $2^{\frac{D-2}{2}}$, labeled by the eigenvalue of v_* being 1 or -1;
- $r - s = 1, 7 \bmod 8$. There is a unique spinor representation, which is real and of real dimension $2^{\frac{D-1}{2}}$;
- $r - s = 2, 6 \bmod 8$. There are two inequivalent complex spinor representations, of complex dimension $2^{\frac{D-2}{2}}$, labeled by the eigenvalue of v_* being i or $-i$;
- $r - s = 3, 5 \bmod 8$. There is a unique spinor representation, which is quaternionic and of quaternionic dimension $2^{\frac{D-3}{2}}$;
- $r - s = 4 \bmod 8$. There are two inequivalent quaternionic spinor representations, of quaternionic dimension $2^{\frac{D-4}{2}}$, labeled by the eigenvalue of v_* being 1 or -1;

Complex representations and the Lorentzian signature case

As it is significantly easier to deal with complex Clifford algebras, we turn our attention to complex representations of $\mathcal{C}(D)$.

We recall

$$\mathcal{C}(2k) \simeq \mathbb{C}(2^k) \qquad \text{and} \qquad \mathcal{C}(2k+1) \simeq \mathbb{C}(2^k) \oplus \mathbb{C}(2^k), \tag{A.16}$$

which implies there are faithful representations

$$\Gamma_{(2k)} : \mathcal{C}(2k) \to \mathrm{End}(\mathbb{C}^{2^k}) \tag{A.17}$$

$$\Gamma_{(2k+1)} : \mathcal{C}(2k+1) \to \mathrm{End}(\mathbb{C}^{2^k}) \oplus \mathrm{End}(\mathbb{C}^{2^k}), \tag{A.18}$$

where Γ_{2k} is irreducible and Γ_{2k+1} splits into two irreducible representations. These are precisely the pinor representations of the complex Clifford algebra.

Remark A.2. The above irreducible representations are unique up to conjugacy with unitary matrices. From now on, we drop the subscript and denote such representations just by Γ.

Proposition A.6. *Let $s = 1$ (i.e. only one time–like direction), $d := r$ and $k := \lfloor \frac{d+1}{2} \rfloor$. Furthermore, let $\{v_0, v_1 \cdots, v_d\}$ be a basis of $V_{\mathbb{C}}$ such that in the Clifford algebra $v_0^2 = \mathbb{1}$ and $v_i v_j + v_j v_i = -2\delta_{ij}\mathbb{1}$ for all $i, j = 1, \cdots, d$. Then there exists a choice of complex representation Γ of $\mathcal{C}(d+1)$ on $\mathbb{C}^{2^k}$ (called Gamma representation) such that*

(i) $\Gamma_0 := \Gamma(v_0)$ *is hermitian;*

(ii) $\Gamma_i := \Gamma(v_i)$ *is anti–hermitian for all* $i = 1, \cdots, d$*;*

(iii) Γ_0 *defines a hermitian form*[1] *for all* $\psi_1, \psi_2 \in \mathbb{C}^{2^k}$ *as*

$$\langle \psi_1, \psi_2 \rangle := \psi_1^\dagger \Gamma_0 \psi_2, \tag{A.19}$$

where $\psi^\dagger := (\psi^*)^t$ denotes the canonical hermitian conjugate in $\mathbb{C}^{2^k}$. Such pairing is called Dirac pairing and, upon defining the Dirac conjugate as $\bar{\psi} := \psi^\dagger \Gamma_0$, can be redefined as

$$\langle \psi_1, \psi_2 \rangle = \bar{\psi}_1 \psi_2.$$

Remark A.3. In the physics context, ψ in $\mathbb{C}^{2^k}$ is called a Dirac spinor, although strictly speaking it is a pinor, since $\mathbb{C}^{2^k}$ is the complex pinor representation as seen in (A.17) and (A.18). Furthermore, following Dirac's nomenclature, the matrices Γ_a in $\mathbb{C}(2^k)$ are called gamma matrices.

Remark A.4. The Dirac conjugate definition extends to any operator $A \in$End$(\mathbb{C}^{2^k})$ as

$$\bar{A} := \Gamma_0^{-1} A^\dagger \Gamma_0.$$

As it turns out, from the above proposition it follows

$$\Gamma_a^\dagger := \Gamma_0^{-1} \Gamma_a \Gamma_0 \qquad \forall a = 0, \cdots, d, \tag{A.20}$$

and noticing that $\Gamma_0^{-1} = \Gamma_0$, it is easy to see that the gamma matrices are invariant under Dirac conjugation, i.e. $\bar{\Gamma}_a = \Gamma_a$.

Furthermore, it is possible to prove that the spin group representation on $\mathbb{C}^{2^k}$ induced by the gamma representation is unitary. Indeed, recalling that $\{\frac{1}{4} v_{ab}\}$ defines a basis of the Lie algebra $\mathfrak{spin}(1, d)$, and having defined $\Gamma_{ab} := \Gamma(v_{ab}) = \frac{1}{2}\Gamma(v_a v_b - v_b v_a) = \frac{1}{2}(\Gamma_a \Gamma_b - \Gamma_b \Gamma_a)$, one sees

$$\Gamma_0^{-1} (\Gamma_a \Gamma_b)^\dagger \Gamma_0 = \Gamma_0^{-1} \Gamma_b^\dagger \Gamma_a^\dagger \Gamma_0 = \Gamma_0^{-1} \Gamma_0^{-1} \Gamma_b \Gamma_0 \Gamma_0^{-1} \Gamma_a \Gamma_0 = \Gamma_b \Gamma_a,$$

therefore $\Gamma_0^{-1}(\Gamma_{ab})^\dagger \Gamma_0 = \Gamma_{ba} = -\Gamma_{ab}$, hence implying (expanding the exponential)

$$\Gamma_0^{-1} \exp\left(\frac{1}{4}\omega^{ab}\Gamma_{ab}\right)^\dagger \Gamma_0 = \exp\left(-\frac{1}{4}\omega^{ab}\Gamma_{ab}\right) = \exp\left(\frac{1}{4}\omega^{ab}\Gamma_{ab}\right)^{-1}. \tag{A.21}$$

Proof. We start by the case of $D = 3 + 1$ and prove the proposition by induction[2]. Consider the Pauli matrices σ_a defined by

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

[1] A hermitian form on $V_{\mathbb{C}}$ is given by an $\mathbb{R}-$bilinear form $\langle -, - \rangle : V \times V \to \mathbb{C}$ such that for all $v_1, v_2 \in V$ and $\lambda \in \mathbb{C}$

- $\langle v_1, \lambda v_2 \rangle = \lambda \langle v_1, v_2 \rangle$;
- $\langle v_1, v_2 \rangle^* = \langle v_2, v_1 \rangle$, where $(-)^*$ denotes complex conjugation

[2] The cases where $D = 1$ and $D = 2$ have appeared in previous examples, while the case for $D = 3$ can be derived from the $D = 2$ using the same induction method

and set $\bar{\sigma}_0 = \sigma_0$ and $\bar{\sigma}_i = -\sigma_i$ for $i = 1, 2, 3$. Then a choice of gamma matrices is given by

$$\Gamma_a = \begin{pmatrix} 0 & \sigma_a \\ \bar{\sigma}_a & 0 \end{pmatrix}.$$

One can easily check that this choice satisfies the Clifford condition, while Γ_0 is hermitian and Γ_i antihermitian for $i = 1, 2, 3$.

The fact that $\langle -, - \rangle$ is a hermitian form is an immediate consequence of Γ_0 being hermitian.

Now, assuming there exists a gamma representation for $D = 2k$, given by matrices Γ_a, one can define a gamma matrices Γ'_a for $a = 0, \cdots, D$ (i.e. a gamma representation for $D + 1$) as follows

$$\Gamma'_a := \begin{cases} \Gamma_a & \text{for } a \leq d \\ \Gamma'_{d+1} = \alpha\Gamma_* = \alpha\Gamma_0\Gamma_1 \cdots \Gamma_d \end{cases}$$

where

$$\alpha = \begin{cases} 1 & \text{for } k \text{ even} \\ i & \text{for } k \text{ odd} \end{cases}$$

In a similar way, starting from gamma matrices for $D = 2k$, one obtains gamma matrices Γ''_a representing the complex $D + 2$ Clifford algebra as

$$\Gamma''_{a\leq d} := \begin{pmatrix} 0 & \Gamma_a \\ \Gamma_a & 0 \end{pmatrix}, \qquad \Gamma''_{d+1} := \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, \qquad \Gamma''_{d+2} = \begin{pmatrix} i\mathbb{1} & 0 \\ 0 & i\mathbb{1} \end{pmatrix}.$$

□

A.1.5 Charge conjugation and Majorana spinors in the Lorentzian signature

Before giving the definition of Majorana spinors, we first notice that the sets $\{\pm\Gamma^*_a\}$ define two new (equivalent) representations of the complex Clifford algebra $\mathcal{C}(D)$, therefore there must exist a unitary matrix B such that

$$\Gamma_a = \eta B^{-1}\Gamma^*_a B, \tag{A.22}$$

with $\eta = \pm 1$. By separately taking the complex conjugate and inverting the equation above we find

$$\Gamma^*_a = \eta B^*\Gamma_a(B^*)^{-1} = \eta B\Gamma_a B^{-1},$$

implying $\Gamma_a = B^{-1}B^*\Gamma_a(B^*)^{-1}B$, which yelds

$$B^* = \epsilon B^{-1}, \qquad \epsilon = \pm 1.$$

Notice that since B is unitary, then $B^\dagger = B^{-1} = \epsilon B^*$, implying $B^t = \epsilon B$. In general, ϵ depends on η and can be found using a method due to Scherk [Sch79; KT83]. Upon defining the charge conjugation matrix as

$$C := B^t\Gamma_0, \tag{A.23}$$

from remark A.4 one can see $\Gamma^\dagger_a = \Gamma_0\Gamma_a\Gamma_0$, but at the same time $\Gamma^\dagger_a = (\Gamma^t_a)^* = \eta(B^{-1})\Gamma^t_a B^t$, hence finding

$$\Gamma^t_a C = \eta C\Gamma_a, \qquad CC^\dagger = \mathbb{1} \qquad \text{and} \qquad C^t = \epsilon\eta C. \tag{A.24}$$

Now, first considering $D = 2k$, it is clear that the set $\{\Gamma_A\} := \{\mathbb{1}, \Gamma_a, \Gamma_{ab}, \cdots, \Gamma_0\Gamma_1 \cdots \Gamma_d\}$, generates the whole algebra of $2^k \times 2^k$ complex matrices, as it is the image of (A.2) under the gamma representation. Clearly, for all A, $C\Gamma_A$ are still generators of the whole algebra and either symmetric or antisymmetric, depending on η as can be seen from (A.24). The problem of counting how many of these matrices are antisymmetric is addressed in [Sch79], and it depends on η, ϵ and D. However, we know that on $\mathbb{C}^{2^k}$ there are $\frac{2^k}{2}(2^k-1)$ independent antisymmetric matrices. Eventually, one finds

$$\epsilon = \cos\left(\frac{\pi}{4}(d-1)\right) - \eta \sin\left(\frac{\pi}{4}(d-1)\right).$$

In the even-dimensional case one can choose either signs for $\eta = \pm 1$, while in the case where $D = 2k+1$, one needs to require that Γ_{d+1} transforms correctly under B (i.e. as in (A.22)), which fixes η as

$$\eta = (-1)^k.$$

We can now start discussing about Majorana spinors.

Definition A.7. A Majorana representation is a particular real representation (of $\mathcal{C}(V)$). It is possible to understand what types of Clifford algebras allow for such real representations by looking at table A.1.2, but, in the context described above, we regard a Majorana representation as a complex representation endowed with a real structure[3].

The following theorem allows us to relate the real structure to the charge conjugation matrix.

Theorem A.2. *Let D be such that $\epsilon = 1$ as defined above, then*

$$\phi : \mathbb{C}^{2^k} \to \mathbb{C}^{2^k} : \psi \mapsto B\psi^*$$

defines a real structure.

Remark A.5. In this particular case, one can use the charge conjugation matrix to define a $\mathrm{Spin}(d,1)$–invariant complex bilinear form $C : \mathbb{C}^{2^k} \times \mathbb{C}^{2^k} \to$ as

$$C(\psi_1, \psi_2) := \psi_1^t \cdot C \cdot \psi_2, \qquad \forall \psi_1, \psi_2 \in \mathbb{C}^{2^k}.$$

Furthermore, there exist a choice of gamma matrices for which C is real.

Proof. Clearly ϕ is connjugate linear, while

$$\phi^2\psi = BB^*\psi = \epsilon\psi = \psi.$$

Lastly, $\mathrm{Spin}(d,1)$–invariance amounts to checking that C is $\mathrm{Spin}(d,1)$–invariant, namely

$$(\Gamma_a\Gamma_b)^t C = \Gamma_b^t \Gamma_a^t C = \eta \Gamma_b^t C \Gamma_a = C\Gamma_b\Gamma_a,$$

implying $(\Gamma_{ab})^t C = -C\Gamma_{ab}$, hence satisfying

$$\exp\left(\frac{1}{4}\omega^{ab}\Gamma_{ab}\right)^t C \exp\left(\frac{1}{4}\omega^{ab}\Gamma_{ab}\right) = C.$$

□

[3]Given a complex linear representation of a Lie group $\rho : G \to \mathrm{End}(W)$ on a complex vector space W, a real or quaternionic structure is a real linear map $\varphi : W \to W$ such that

- $\varphi^2 = id_W$
- $\varphi(\lambda v) = \lambda^* v$, i.e. φ is conjugate linear;
- φ is invariant under ρ, i.e. it commutes with the image of all elements of G under ρ, $[\varphi, \rho(g)] = 0$.

Remark A.6. When $\epsilon = -1$, the above proof still holds, but in this case $\phi^2 = -id$, hence defining a quaternionic structure.

Definition A.8. Assume $\epsilon = 1$ for some k, then a (s)pinor $\psi \in \mathbb{C}^{2^k}$ satisfying the reality condition

$$\phi(\psi) = B\psi^* = \psi, \tag{A.25}$$

is said to be Majorana when $\eta = -1$ and pseudo-Majorana when $\eta = 1$.

Remark A.7. It is customary to rephrase the above condition in terms of the charge conjugation matrix, noticing that $B = \epsilon C\Gamma_0$, one obtains that, under the above assumptions, ψ is Majorana when

$$C\Gamma_0\psi^* = \psi.$$

The following table contains information on the allowed values of ϵ and η in various dimensions.

	$\eta = 1$	$\eta = -1$
$\epsilon = 1$	$D = 1, 2, 8 \text{ mod } 8$	$D = 2, 3, 4 \text{ mod } 8$
$\epsilon = -1$	$D = 4, 5, 6 \text{ mod } 8$	$D = 6, 7, 8 \text{ mod } 8$

A.2 Spin coframe formalism, i.e. defining spinor fields on manifolds

In the previous section, we saw the algebraic construction and classification of Clifford algebras and spinors. This section is dedicated to investigating the local structure of such objects in the context of differential geometry, with the goal of providing a framework that allows to treat the definition of supergravity in the same formulation found in [CS19b; CCS21a]. The main part follows [LM90], [Fat+98] and [NF22] for the spin frame definition. For a detailed review of the "bosonic" coframe formalism, reference [Tec19a] is recommended.

A.2.1 Basic notions on principal bundles

In the following, we assume M to be a pseudo-riemannian manifold of dimension D.

Definition A.9. Let G be a Lie group. A principal G-bundle $\pi\colon P \to M$ is a fiber bundle such that

- There exists a smooth right action $R\colon P \times G \to P$ which is free, i.e. such that $R(p, e) := p\dot{e} = p$ for all $p \in P$, letting $e \in G$ be the identity;
- $\pi\colon P \to M$ is diffeomorphic as a bundle to $P \to P/G$.

Remark A.8. Notice that, since R is free, any orbit $\mathcal{O}_p := \{q \in P \mid \exists g \in G \text{ s.t. } q = p \cdot g\} = [p] \in P/G$ is isomorphic to G. Then points $x \in M$ is in one-to-one correspondence with orbits $[p] \in P/G$, and each fiber is isomorphic to the group G, as $\pi^{-1}(x) = [p] = \mathcal{O}_p \simeq G$.

Definition A.10. Given a G-principal bundle P, a trivialization of P is a collection (U_α, ϕ_α), with α is an element of an index set I, such that

- $\mathcal{U} := \{U_\alpha\}_{\alpha\in I}$ is an open cover of M,

- $\phi_\alpha \colon \pi^{-1}(U_\alpha) \to U_\alpha \times G$ are diffeomorphisms
- letting $U_{\alpha\beta} := U_\alpha \cap U_\beta$, transition functions $\phi_{\alpha\beta} \colon U_{\alpha\beta} \times G \to U_{\alpha\beta} \times G$ are given by (smooth) functions $g_{\alpha\beta} \colon U_{\alpha\beta} \to G$ via left action as $\phi_{\alpha\beta} \colon (x, h) \mapsto (x, g_{\alpha\beta} \cdot h)$ and must respect the cocycle identity
$$g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = e, \qquad \text{for all } x \in U_{\alpha\beta\gamma}. \tag{A.26}$$

Remark A.9. In general, a principal bundle can be recovered by pasting together its local data, i.e. pasting the local products $\{U_\alpha \times G\}$ via the transition functions $g_{\alpha\beta}$. Indeed, it is possible to show that any principal $G-$bundle P is equivalent to a pair $(\mathcal{U}, \{g_{\alpha\beta}\})$, where $\mathcal{U}$ is an open cover and $\{g_{\alpha\beta} \colon U_{\alpha\beta} \to G\}$ are functions satisfying the cocycle condition.[4]

Definition A.11. Two G-principal bundles P and P' over M are equivalent if there exists a homeomorphism $H : P \to P'$ such that the following diagram commutes

$$\begin{array}{ccc} P & \xrightarrow{\;H\;} & P' \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi'} & \\ & M & \end{array}$$

and such that H is equivariant, i.e. $H(p \cdot g) = H(p) \cdot g$ for all $g \in G$ and $p \in P$.

It is interesting to understand such definition at the level of local trivialization, which will allow us to describe the set of inequivalent principal G–bundles over M. First of all, let P and P' be defined respectively by $(\mathcal{U}, \{g_{\alpha\beta}\})$ and $(\mathcal{U}, \{g'_{\alpha\beta}\})$ as in the above remark. They induce trivializations (ϕ_α) and (ϕ'_α), which allow to define

$$H_\alpha := \phi'_\alpha \circ H \circ \phi_\alpha^{-1} \colon U_\alpha \times G \to U_\alpha \times G.$$

Now, since $\pi' \circ H = \pi$, we must have that $H_\alpha(x, g) = (x, h_\alpha(x, g))$ for some $h_\alpha \colon U_\alpha \times G \to G$. Now, using equivariance, we obtain

$$H_\alpha(x, g \cdot f) = H_\alpha(x, g) \cdot f \qquad \Rightarrow \qquad h_\alpha(x, g \cdot f) = h_\alpha(x, g) \cdot f,$$

which implies that $H_\alpha(x, g) = (x, h_\alpha(x, e) \cdot g) = (x, g_\alpha(x) \cdot g)$, having defined $g_\alpha(x) := h_\alpha(x, e)$. The relation between the transition functions can be understood by noticing that, by definition of equivalence, the following diagram must commute

$$\begin{array}{ccccc} & & \overset{\phi_{\beta\alpha}}{\frown\!\!\longrightarrow} & & \\ U_{\alpha\beta} \times G & \xrightarrow{\;\phi_\alpha^{-1}\;} & \pi^{-1}(U_{\alpha\beta}) & \xrightarrow{\;\phi_\beta\;} & U_{\alpha\beta} \times G \\ \downarrow{\scriptstyle H_\alpha} & & \downarrow{\scriptstyle H} & & \downarrow{\scriptstyle H_\beta} \\ U_{\alpha\beta} \times G & \xrightarrow[\;\phi_\alpha'^{-1}\;]{} & (\pi')^{-1}(U_{\alpha\beta}) & \xrightarrow[\;\phi'_\beta\;]{} & U_{\alpha\beta} \times G \\ & & \underset{\phi'_{\beta\alpha}}{\smile\!\!\longrightarrow} & & \end{array}$$

therefore $\phi'_{\beta\alpha} = H_\beta \circ \phi_{\beta\alpha} \circ H_\alpha^{-1}$, implying

$$g'_{\alpha\beta} = g_\alpha^{-1} \cdot g_{\alpha\beta} \cdot g_\beta.$$

[4]The cocycle condition is equivalent to the Cech coboundary condition, and $g_{\alpha\beta}$ is nothing but a Cech 1–cocycle with coefficients in G (to be precise, with coefficients in the sheaf of germs of smooth maps to G).

Hence $(\mathcal{U}, \{g_{\alpha\beta}\})$ and $(\mathcal{U}, \{g'_{\alpha\beta}\})$ define equivalent bundles P and P' iff there exists a family $g_\alpha : U_\alpha \to G$ of smooth functions such that $g'_{\alpha\beta} = g_\alpha^{-1} \cdot g_{\alpha\beta} \cdot g_\beta$. Upon inspection, one realizes that this is nothing but a Cech-coboundary condition (in the multiplicative sense), and therefore $g'_{\alpha\beta}$ and $g_{\alpha\beta}$ only "differ by an exact term", where g_α acts as a "Cech 0-cochain". Therefore one can see an equivalence class of principal G–bundles as an element of $H^1(\mathcal{U}; G)$.

Letting $(\mathcal{U}_i)$ be a family of open covers such that for $i > j$ $\mathcal{U}_i \subset \mathcal{U}_j$, then one can define $H^1(M; G)$ as the direct limit (in the categorical sense)

$$H^1(M; G) = \mathbf{lim}_i H^1(\mathcal{U}_i; G).$$

Notice that this set is strictly speaking not a group, but contains an identity given by the trivial principal bundle $M \times G$. If G is abelian, then $H^1(M; G)$ is just the first Cech cohomology group with coefficients in G.

Definition A.12. The frame bundle $LM \to M$ is a principal $\mathrm{GL}(D, \mathbb{R})$–bundle defined by

$$LM = \bigcup_x L_x M, \qquad L_x M := \{e_a = (e_0, \cdots, e_d) \mid (e_a) \text{ is a basis of } T_x M\}.$$

with trivialisation given by $\phi_\alpha \colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathrm{GL}(D, \mathbb{R}) \colon (x, e_a) \mapsto (x, e_a^\mu)$, having set $\mu = 0, \cdots, d$

Remark A.10. One can see that the transition functions between two charts with local coordinates $\{x\}$ and $\{x'\}$ act via left action as $e'^{\mu}_{a} = \frac{\partial x'^\mu}{\partial x^\nu} e^\nu_a$.

Assuming that M is orientable, and having chosen a Lorentzian metric g on it, one can define the orthonormal frame bundle as the subbundle of the frame bundle containing orthonormal frames, i.e.

$$SO(M, g) := \{e_a \in LM \mid g(e_a, e_b) = \eta_{ab}\},$$

where η_{ab} is the Minkowski metric.

Remark A.11. Notice that $SO(M, g)$ is a principal $\mathrm{SO}(d, 1)$–bundle. Furthermore, for a given metric g, there exist more than one ON basis, as for any e_a ON and for any $\Lambda \in \mathrm{SO}(d, 1)$, also $e'_b = e_a \Lambda^a_b$ satisfies $g(e'_a, e'_b) = \eta_{ab}$. However, the viceversa is not true, indeed for each ON basis e_a there is a unique metric g with respect to which it is orthonormal.

As it turns out, it is particularly useful to consider the dual notion of an ON frame, namely an ON coframe, the dual basis e^b with respect to a given frame e_a, i.e. such that

$$e^b(e_a) = \delta^b_a.$$

This motivates the following definition

Definition A.13. Given a principal $\mathrm{SO}(d, 1)$–bundle P_{SO}, a veilbein map is a principal bundle morphism $\tilde{e} \colon P_{SO} \to LM$ satisfying verticality and equivariance, i.e. such that the following two diagrams commute

$$\begin{array}{ccc} P_{SO} & \xrightarrow{\tilde{e}} & LM \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & M & \end{array} \qquad\qquad \begin{array}{ccc} P_{SO} & \xrightarrow{\tilde{e}} & LM \\ \downarrow{\scriptstyle\cdot\Lambda} & & \downarrow{\scriptstyle\cdot\Lambda} \\ P_{SO} & \xrightarrow{\tilde{e}} & LM \end{array}$$

where Λ is an element of $\mathrm{SO}(d, 1)$ possibly seen as an element of $\mathrm{GL}(d + 1, \mathbb{R})$.

Choosing a local section $s_\alpha : U_\alpha \to P_{SO}$, on the overlap of two patches $U_{\alpha\beta}$ transition functions $\Lambda_{\alpha\beta}\colon U_{\alpha\beta} \to \mathrm{SO}(d,1)$ act via right action as

$$s_\beta = s_\alpha \cdot \Lambda_{\alpha\beta}.$$

As it turns out, it is sufficient to know $\tilde{e}$ on s_α to know it on the whole $\pi^{-1}(U_\alpha)$, indeed $\tilde{e}(s_\alpha(x)) = (x, {}_\alpha e_a(x))$, where $e_a(x)$ is a frame defining a basis of T_xM, then thanks to equivariance, for all $p \in \pi^{-1}(U_\alpha)$, there exists a $\Lambda \in \mathrm{SO}(d,1)$ such that $p = s_\alpha \cdot \Lambda$, hence $\tilde{e}(p) = \tilde{e}(s_\alpha \cdot \Lambda) = \tilde{e}(s_\alpha) \cdot \Lambda$.

It is then clear that a vielbein map uniquely defines a family of frames differing by orthonormal transformations on overlaps of the patches. It is also easy to see that the viceversa is true, and as a consequence, keeping in mind remark A.11, a vielbein map uniquely defines a metric g (with respect to which e_a is ON) on M via the dual frame, i.e.

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}.$$

It becomes even clearer when one takes the image of P_{SO} under $\tilde{e}$, finding

$$\tilde{e}(P_{SO}) = \{(x, e_a) \mid (e_a) \text{ is a basis of } T_xM, \text{ and } g(e_a, e_b) = \eta_{ab}\} \simeq SO(M,g).$$

This observation motivates the following definition:

Definition A.14. Let (V, η) be a real $D-$dimensional vector space endowed with the Minkowski metric, and let $\rho : \mathrm{SO}(d,1) \to V$ be the fundamental representation of the Lorentz group on V, then the Minkwoski bundle $\mathcal{V}$ is the associated bundle[5]

$$\mathcal{V} = P_{SO} \times_\rho V,$$

With this definition, it is clear that the vielbein map is in 1-to-1 correspondence with linear isomorphisms between TM and $\mathcal{V}$, as they are given by coframes (called vielbein field) dual to the ones defined by the vielbein map. In particular, choosing a local basis $\{v_a\}$ of $\mathcal{V}$ and local coordinates x on M, one has

$$e\colon TM \xrightarrow{\sim} \mathcal{V}, \qquad e = e^a_\mu dx^\mu v_a \quad \text{s.t.} \quad e^a_\mu e^\mu_b = \delta^a_b.$$

A.2.2 Spin structures and the equivalence with spin (co)frames

Definition A.15. Let P_s be a $\mathrm{Spin}(d,1)$–principal bundle over (M,g), a spin structure is a pair (P_s, Σ) such that $\Sigma\colon P_s \to SO(M,g)$ is an equivariant principal morphism, i.e. such that the following diagrams commute

$$\begin{array}{ccc} P_s & \xrightarrow{\quad\Lambda\quad} & SO(M,g) \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & M & \end{array} \qquad\qquad \begin{array}{ccc} P_s & \xrightarrow{\Lambda} & SO(M,g) \\ \downarrow{\scriptstyle \cdot S} & & \downarrow{\scriptstyle \cdot l(S)} \\ P_s & \xrightarrow{\Lambda} & SO(M,g) \end{array}$$

where $S \in \mathrm{Spin}(d,1)$ and $l : \mathrm{Spin}(d,1) \to \mathrm{SO}(d,1)$ is the double covering defined in the previous chapter.

Remark A.12. In general it is not true that every orientable pseudoriemannian manifold admits a spin structure, but, as we will see, there are topological requirements that need to be assumed for it to be true.

[5]Here $P_{SO} \times_\rho V$ is defined to be the quotient $P_{SO} \times V/\sim$, where $(p,v) \sim (q,w)$ if there exists a $\Lambda \in \mathrm{SO}(d,1)$ such that $q = p \cdot \Lambda$ and $w = \rho(\Lambda)^{-1} \cdot v$

We notice that the notion of spin structure is similar to the one of equivalence of principal bundles, so it might be useful to rephrase the problem of understanding when a spin structure exists in terms of equivalence of bundles.

We saw earlier that $H^1(M; G)$ is the set of inequivalent principal G–bundles. Borrowing some results from the theory of Cech cohomology, one can prove that if

$$1 \to K \xrightarrow{i} G \xrightarrow{j} G' \to 1$$

is a short exact sequence of topological groups, then there is an exact sequence at the level of cohomology, given by

$$1 \to H^0(M; K) \xrightarrow{i_*} H^0(M; G) \xrightarrow{j_*} H^0(M; G') \xrightarrow{\partial_*} H^1(M; K) \xrightarrow{i_*} H^1(M; G) \xrightarrow{j_*} H^1(M; G'),$$

where ∂ is the Cech coboundary operator and $H^0(M; G)$ is the global sections of G seen as 0-cocycles.[6] It is also possible to prove, if K is abelian, that the sequence can be extended to

$$\cdots \to H^1(M; K) \to H^1(M; G) \to H^1(M; G') \to H^2(M; K).$$

Therefore, considering the short exact sequence (A.14) $0 \to \mathbb{Z}_2 \to \mathrm{Spin}(d,1) \xrightarrow{l} \mathrm{SO}(d,1) \to 0$, we can define the second Stiefel-Whitney class as the induced map w_2 in the exact sequence

$$w_2 : H^1(M; \mathrm{SO}(d,1)) \to H^2(M; \mathbb{Z}_2)$$
$$H^1(M; \mathrm{Spin}(d,1)) \xrightarrow{l_*} (M; \mathrm{SO}(d,1)) \xrightarrow{w_2} H^2(M; \mathbb{Z}_2)$$

Theorem A.3. *(M, G) admits a spin structure if and only if $w_2([SO(M,g)]) = 0$.*

Proof. When considering the orthonormal frame bundle $SO(M,g)$, and in particular its equivalence class $[SO(M,g)] \in H^1(M; \mathrm{SO}(d,1))$, we see that the second Stiefel-Whitney class of $[SO(M,g)]$ vanishes if and only if $[SO(M,g)] \in \mathrm{Im}(l_*)$, which tells us that the orthonormal bundle is induced by a Spin bundle, and in particular l_* defines a spin structure.

To see it more explicitly, let $(\mathcal{U}, \{g_{\alpha\beta}\})$ be a cocycle representing $SO(M,g)$, with $\mathcal{U}$ defined such that each non empty $U_{\alpha\beta}$ is simply connected. We can lift the $g_{\alpha\beta}$ to functions $\{\tilde{g}_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{Spin}(d,1)\}$ and define $K_{\alpha\beta\gamma} := \tilde{g}_{\beta\gamma} \cdot (\tilde{g}_{\alpha\gamma})^{-1} \cdot \tilde{g}_{\alpha\beta}$[7] on $U_{\alpha\beta\gamma}$. Clearly $l(K_{\alpha\beta\gamma}) = 1$ as this is exaclty the cocycle identity for $SO(M,g)$, implying $K_{\alpha\beta\gamma} \in \mathbb{Z}_2$. Furthemore, it is easy to notice that $(\partial K)_{\alpha\beta\gamma\delta} = 1$, hence it defines a cocycle, which represents the second Stiefel-Whitney class. In particular $w_2 = 0$ translates to

$$[K] = \{K_{\alpha\beta\gamma} \cdot (\partial\lambda)_{\alpha\beta\gamma} \mid \lambda_{\alpha\beta} : U_{\alpha\beta} \to \mathbb{Z}_2\} = 1$$

It is clear that $[K] = 1$ iff $K = \partial\lambda$. Defining $\tilde{g}'_{\alpha\beta} := \lambda^{-1}_{\alpha\beta} \cdot g_{\alpha\beta}$, it is easy to show that $\tilde{g}'_{\alpha\beta}$ is a cocycle (i.e. satisfy the cocycle identity), hence it is possible to reconstruct a Spin-bundle from $(\mathcal{U}, \{\tilde{g}'_{\alpha\beta}\})$, showing there are no obstructions for the existence of a spin structure.

Conversely, if one assumes that a spin structure exists, then it is immediate to see $[K] = 1$ because the lifted transition functions automatically satisfy the cocycle identity.

We are only left with showing that $[K]$ is independent of the choice of trivialization and on the choice of the lift. We start by showing the independence on the choice of lift of $\{g_{\alpha\beta}\}$. Let

[6] Indeed the cocycle identity is exactly the requirement that the local sections can be glued to a global one on the overlap of the patches U_α.

[7] Notice that this is almost a coboundary as $\tilde{g}_{\beta\gamma}(\tilde{g}_{\alpha\gamma})^{-1}\tilde{g}_{\alpha\beta} = (\partial\tilde{g})_{\alpha\beta\gamma}$, however it is not because $\tilde{g}_{\alpha\beta}$ does not take values in $\mathbb{Z}_2$, namely $\tilde{g}_{\alpha\beta}$ is not a $\mathbb{Z}_2$-cochain.

$\kappa_{\alpha\beta}$ be some 1–cochain inducing $\tilde{g}"_{\alpha\beta} = \tilde{g}_{\alpha\beta} \cdot \kappa_{\alpha\beta}$. This defines a new lift $K"$ which is in the same equivalence class as K, as $K" = K \cdot \partial\kappa$.

Now we can choose different local sections $\{g'_{\alpha\beta}\}$ for the original SO–bundle. We have previously seen that $g'_{\alpha\beta} = g_\alpha^{-1} \cdot g_{\alpha\beta} \cdot g_\beta$, which, after choosing a lift $\tilde{g}_\alpha$, gives $K'_{\alpha\beta\gamma} = \tilde{g}_\beta^{-1} \cdot K_{\alpha\beta\gamma} \cdot \tilde{g}_\beta$. Now, since $K'_{\alpha\beta\gamma} \in \mathbb{Z}_2$, then $\tilde{g}_\beta$ and its inverse are both either 1 or -1, so $K'_{\alpha\beta\gamma} = K_{\alpha\beta\gamma}$.

□

As one can notice, so far we needed to fix a metric in order to define a spin strucutre. Now, we introduce an equivalent approach that does not rely on such assumption, and therefore is more suitable to work with theories where the metric is a dynamical field.

Definition A.16. Given a principal Spin$(d,1)$–bundle P_s, a spinbein map is a principal bundle morphism $\hat{e} : P_S \to LM$ satisfying verticality and equivariance, i.e. the following diagrams commute

$$\begin{array}{ccc} P_s & \xrightarrow{\tilde{e}} & LM \\ & \searrow^{\pi} \quad {}^{\pi}\swarrow & \\ & M & \end{array} \qquad \begin{array}{ccc} P_s & \xrightarrow{\tilde{e}} & LM \\ \downarrow{\cdot\Lambda} & & \downarrow{\cdot\Lambda} \\ P_{SO} & \xrightarrow{\tilde{e}} & LM \end{array}$$

As before, the spinbein defines a moving frame on sections $\hat{s}_\alpha : U_\alpha \to P_s$ as $\hat{e}(\hat{s}_\alpha(x)) = (x, {}_\alpha(e_a))$, where ${}_\alpha e_a = {}_\alpha e_a^\mu \partial_\mu$. On intersections the frames change by right action of an orthogonal transformation seen as the image under l of a Spin transformation $S_{\alpha\beta}$ defining the transition functions, i.e.

$$\hat{e}(\hat{s}_\beta) = \hat{e}(\hat{s}_\alpha) \cdot S_{\alpha\beta} \quad \Rightarrow \quad {}_\beta(e_a) = {}_\alpha(e_b) l_a^b(S_{\alpha\beta}).$$

Remark A.13. Also in this case, by dualizing the frames, one can induce unquely a metric on M as $g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}$. Exactly as before, the image of P_s under $\hat{e}$ turns out to be the orthogonal bundle $SO(M,g)$. Furthermore, lifting l to a bundle map $\hat{l} \colon P_s \to P_{SO}$, it is clear that a trivialization on P_s induces one on P_{SO} via $\hat{l}$, and for each family of sections $\hat{s}_\alpha$ we obtain sections $s_\alpha := \hat{l} \circ \hat{s}$. Equivalently, one obtains that the following diagram commutes

$$\begin{array}{ccccc} P_s & & \xrightarrow{\hat{e}} & & LM \\ & \searrow^{\hat{l}} & & \nearrow^{e} & \\ \hat{p} & & P_{SO} & & \pi \\ & & \downarrow{p} & & \\ & \searrow & M & \swarrow & \end{array} \tag{A.27}$$

Notice that also a vielbein map e equivalent to the one introduced in the previous chapter is introduced.

The reason for the last statement is clear when one defines the associated bundle

$$\hat{\mathcal{V}} := P_s \times_{\hat{\rho}} V,$$

where $\hat{\rho}$ is the vector (i.e. spin 1) representation of Spin$(d,1)$ on V. Notice however how every integer spin representation $\hat{\lambda}$ of Spin$(d,1)$ is the same as a representation of SO$(d,1)$, as it factors through the double cover $\hat{\lambda} = \lambda \circ l$. In particular, this tells us that $\hat{\mathcal{V}} \simeq \mathcal{V}$ and that a spin coframe

$$e \colon TM \xrightarrow{\sim} \hat{\mathcal{V}}$$

produces the same dynamics as the vielbein field. The advantage of using spin bundles is that of being able to define associated vector bundles with respect to half-integer spin representations, i.e. spinor bundles.

Theorem A.4. *[NF22] A spinbein map $\hat{e}$ on M exists if and only if a spin structure exists on (M, g) for some metric g.*

Proof. Given a spinbein map $\hat{e}\colon P_s \to LM$, it induces a spin structure just by restricting the target to the image of $\hat{e}$, i.e.

$$\Sigma\colon P_s \xrightarrow{\hat{e}} \hat{e}(P_s) = SO(M, g),$$

where in this case g is the metric induced by the coframe defined by $\hat{e}$. Conversely, if $\Sigma : P_s \to SO(M, g)$ is a spin structure, one can induce a spinbein map $\hat{e} := \hat{\iota} \circ \Sigma$, where $\hat{\iota} : P_{SO} \to LM$ is the inclusion in the frame bundle. □

Having proved this, it is clear that using spin coframes is allowed exactly when spin structures exist, and viceversa, hence we can regard it as an equivalent description.

Finally, we have all the ingredients to define spinor bundles.

Definition A.17. Let $V_{\mathbb{C}}$ be the complexification of the D-dimensional real vector space V. By the discussion in the previous chapter, we know that, depending on the parity of D, there exist faithful representations of the Clifford algebra $\mathcal{C}(V_{\mathbb{C}}) = \mathcal{C}(V)_{\mathbb{C}}$. In particular, we are interested in the gamma representation Γ of proposition A.6, which allows to define the Dirac spinor bundle as

$$\mathbb{S}_D := P_S \times_\Gamma \mathbb{C}^{2^{\frac{D}{2}}}$$

Sections of $\mathbb{S}_D$ are called Dirac spinor fields. Furthermore, when the dimension allows it, one can also define the subbundle of Majorana spinors as

$$\mathbb{S}_M := \bigcup_{x \in M} \{(x, \psi) \in \mathbb{S}_{D,x} \mid C\Gamma_0 \psi^* = \psi\}.$$

A.2.3 Lemmata about spin coframes

As anticipated in chapter 2, throughout the thesis we will be relying heavily on the properties of coframes. Here we provide a quick recap of some results appearing in [Can24] and some other original ones, both in the bulk and on the boundary.[8]

We start by defining the following spaces

$$\Omega^{(i,j)} := \Omega^i(M, \wedge^j \mathcal{V}) \qquad \Omega_\partial^{(i,j)} := \Omega^i(\Sigma, \wedge^j \mathcal{V}|_\Sigma) \tag{A.28}$$

where $\mathcal{V}$ is identified with $\hat{\mathcal{V}}$ and e is a spin coframe, and maps

$$W_k^{(i,j)} \colon \Omega^{(i,j)} \longrightarrow \Omega^{(i+k,j+k)} \colon \alpha \longmapsto e^k \wedge \alpha, \tag{A.29}$$

$$W_k^{\partial(i,j)} \colon \Omega_\partial^{(i,j)} \longrightarrow \Omega_\partial^{(i+k,j+k)} \colon \alpha \longmapsto e^k \wedge \alpha, \tag{A.30}$$

where

$$e^k := \underbrace{e \wedge \cdots \wedge e}_{\text{k times}}.$$

Such maps have been studied in previous papers (notably in [CS19c; CCS21a] and [Can24]. The following diagram [Can24] indicates the properties of $W_1^{(i,j)}$ and $W_1^{\partial,(i,j)}$, in particular a hooked

[8]There will be no distinction between the boundary and the bulk fields, as their definition will be clear from the context.

arrow indicates injectivity while a two–headed arrow indicates surjectivity. In the bulk we have

$$
\begin{array}{ccccccccc}
\Omega^{(0,0)} & & \Omega^{(0,1)} & & \Omega^{(0,2)} & & \Omega^{(0,3)} & & \Omega^{(0,4)} \\
 & \hookrightarrow & & \hookrightarrow & & \hookrightarrow & & \twoheadrightarrow & \\
\Omega^{(1,0)} & & \Omega^{(1,1)} & & \Omega^{(1,2)} & & \Omega^{(1,3)} & & \Omega^{(1,4)} \\
 & \hookrightarrow & & \hookrightarrow & & \hookrightarrow\!\!\!\!\twoheadrightarrow & & \twoheadrightarrow & \\
\Omega^{(2,0)} & & \Omega^{(2,1)} & & \Omega^{(2,2)} & & \Omega^{(2,3)} & & \Omega^{(2,4)} \\
 & \hookrightarrow & & \hookrightarrow\!\!\!\!\twoheadrightarrow & & \twoheadrightarrow & & \twoheadrightarrow & \\
\Omega^{(3,0)} & & \Omega^{(3,1)} & & \Omega^{(3,2)} & & \Omega^{(3,3)} & & \Omega^{(3,4)} \\
 & \hookrightarrow\!\!\!\!\twoheadrightarrow & & \twoheadrightarrow & & \twoheadrightarrow & & \twoheadrightarrow & \\
\Omega^{(4,0)} & & \Omega^{(4,1)} & & \Omega^{(4,2)} & & \Omega^{(4,3)} & & \Omega^{(4,4)}
\end{array}
\tag{A.31}
$$

whereas on the boundary one obtains

$$
\begin{array}{ccccccccc}
\Omega_\partial^{(0,0)} & & \Omega_\partial^{(0,1)} & & \Omega_\partial^{(0,2)} & & \Omega_\partial^{(0,3)} & & \Omega_\partial^{(0,4)} \\
 & \hookrightarrow & & \hookrightarrow & & \hookrightarrow & & \twoheadrightarrow & \\
\Omega_\partial^{(1,0)} & & \Omega_\partial^{(1,1)} & & \Omega_\partial^{(1,2)} & & \Omega_\partial^{(1,3)} & & \Omega_\partial^{(1,4)} \\
 & \hookrightarrow & & \hookrightarrow & & \twoheadrightarrow & & \twoheadrightarrow & \\
\Omega_\partial^{(2,0)} & & \Omega_\partial^{(2,1)} & & \Omega_\partial^{(2,2)} & & \Omega_\partial^{(2,3)} & & \Omega_\partial^{(2,4)} \\
 & \hookrightarrow & & \twoheadrightarrow & & \twoheadrightarrow & & \twoheadrightarrow & \\
\Omega_\partial^{(3,0)} & & \Omega_\partial^{(3,1)} & & \Omega_\partial^{(3,2)} & & \Omega_\partial^{(3,3)} & & \Omega_\partial^{(3,4)}
\end{array}
\tag{A.32}
$$

Lemma A.4. *[CS19c; CCF22; CF25] The following maps are isomorphisms:*

1. $W_2^{(0,2)} \colon \Omega^{(0,2)} \to \Omega^{(2,4)}$,
2. $W_2^{(2,0)} \colon \Omega^{(2,0)} \to \Omega^{(4,2)}$,
3. $W_2^{(1,1)} \colon \Omega^{(2,0)} \to \Omega^{(3,3)}$,
4. $W_4^{(0,0)} \colon \Omega^{(0,0)} \to \Omega^{(4,4)}$,
5. $\varrho^{(0,1)} \colon \Omega^{(0,1)} \to \Omega^{(1,0)}$.
6. $\varrho^{(3,4)} \colon \Omega^{(3,4)} \to \Omega^{(4,3)}$.

Lemma A.5 ([CCS21a])**.** *Let $\alpha \in \Omega_\partial^{2,1}$. Then*

$$
\alpha = 0 \qquad \Longleftrightarrow \qquad \begin{cases} e\alpha = 0 \\ \epsilon_n \alpha \in \operatorname{Im} W_1^{\partial,(1,1)} \end{cases} . \tag{A.33}
$$

Lemma A.6 ([CCS21a])**.** *Let $\beta \in \Omega_\partial^{2,2}$. If g^∂ is nondegenerate, there exist a unique $v \in \operatorname{Ker} W_1^{\partial,(1,2)}$ and a unique $\rho \in \Omega_\partial^{1,1}$ such that*

$$
\beta = e\rho + \epsilon_n [e, v].
$$

Lemma A.7. *Let $a \in \Omega_\partial^{(1,2)}$. Then*

$$a = 0 \iff \begin{cases} ea = 0 \\ \epsilon_n a \in \mathrm{Im}(W_e^{\partial(0,2)}) \end{cases} .$$

Proof. Let $I \subset \mathbb{R}$ be an interval with $\{x^n\}$ coordinate on it and let $\tilde{M} := \Sigma \times I$. Then $E := e + \epsilon_n dx^n \in \Gamma(\tilde{M}, \mathcal{V})$ defines a non-degenerate vielbein on M. Let $A := a + b dx^n \in \Omega^1(\tilde{M}, \wedge^2\mathcal{V})$, with $b \in \Omega_\partial^{(0,2)}$.

Then the above system is equivalent to the the equation $E \wedge A = ea + (eb - \epsilon_n a)dx^n = 0$. By diagram (A.31), $E \wedge \cdot$ is injective, hence $E \wedge A = 0$ iff $A = 0$, implying $a = 0$. □

Lemma A.8. *For all $\tilde{k} \in \Omega_\partial^{(2,1)}$ there exists a unique decomposition $\tilde{k} = \check{k} + r$ such that*

$$er = 0, \qquad \epsilon_n \check{k} \in Im(W_e^{\partial,(1,1)}).$$

Proof. From A.6, we know there exists a unique decomposition

$$\epsilon_n \tilde{k} = e\check{a} + \epsilon_n [e, \check{b}],$$

with $\check{b} \in \mathrm{Ker}(W_e^{\partial,(1,2)})$. Define $r := [e, \check{b}]$ and $\check{k} := \tilde{k} - r$. This implies

$$\epsilon_n \check{k} = e\check{a} \in \mathrm{Im}(W_e^{\partial,(1,1)}) \qquad er = e[e, \check{b}] = [e, e\check{b}] - [e, e]\check{b} = 0.$$

For uniqueness, assume there exist $\tilde{k} = \check{k}_1 + r_1 = \check{k}_2 + r_2$ such that $r_1, r_2 \in \mathrm{Ker}(W_e^{\partial(1,2)})$ and $\epsilon_n \check{k}_1, \epsilon_n \check{k}_2 \in \mathrm{Im}(W_e^{\partial,(1,1)})$. Then we obtain the following system

$$\begin{cases} \epsilon_n(\check{k}_1 - \check{k}_2) = \epsilon_n(r_2 - r_1) \in \mathrm{Im}(W_e^{\partial,(1,1)}) \\ e(r_2 - r_1) = 0. \end{cases}$$

By lemma A.7, we have that $r_2 = r_1$, implying $\check{k}_1 - \check{k}_2 = r_2 - r_1 = 0$. □

Lemma A.9. *Let $\Theta \in \Omega_\partial^{(1,3)}$. Then there exist unique $\alpha \in \Omega_\partial^{(0,2)}$ and $\beta \in \mathrm{Ker}(W_e^{\partial(1,2)})$ such that*

$$\Theta = e\alpha + \epsilon_n \beta.$$

Proof. Consider the map

$$\rho\colon \mathrm{Ker}(W_e^{\partial(1,2)}) \to \Omega_\partial^{(1,3)}$$
$$\beta \mapsto \epsilon_n \beta.$$

Then assume $\exists \beta \in \mathrm{Ker}(W_e^{\partial(1,2)})$ such that $\rho(\beta) = \epsilon_n \beta = 0$. Lemma A.7 implies that $\beta = 0$, hence ρ is injective.

We can then deduce that dim(Imρ)=dim(Ker($W_e^{\partial(1,2)}$))=6. In the same way, since $W_e^{\partial(0,2)}$ is injective, dim(Im($W_e^{\partial(0,2)}$))=6=dim($\Omega_\partial^{(0,2)}$). Hence dim(Im($W_e^{\partial(0,2)}$))+ dim(Imρ)=12=dim($\Omega_\partial^{(1,3)}$).

Now we just need to prove that $\mathrm{Im}(W_e^{\partial(0,2)}) \cap \mathrm{Im}\rho = \{0\}$.

Assume $\exists 0 \neq \beta \in \mathrm{Ker}(W_e^{\partial(1,2)})$ such that for some $\alpha \in \Omega_\partial^{(0,2)}$

$$\epsilon_n \beta = e\alpha.$$

Then, by lemma A.7, setting $a = v$, we automatically obtain $\beta = 0$, contradicting the hypothesis. Hence $\mathrm{Im}(W_e^{\partial(0,2)}) \cap \mathrm{Im}\rho = \{0\}$, implying $\Omega_\partial^{(1,3)} \simeq \mathrm{Im}W_e^{\partial(0,2)} \oplus \mathrm{Im}\rho$. Uniqueness follow from the injectivity of ρ and $W_e^{(0,2)}$. □

Definition A.18. The previous lemma allows to define maps

$$\alpha_\partial \colon \Omega_\partial^{(1,3)} \to \Omega_\partial^{(0,2)} \qquad \beta_\partial \colon \Omega_\partial^{(1,3)} \to \Omega_\partial^{(1,2)}$$
$$\Theta \mapsto \alpha_\partial(\Theta) \qquad \Theta \mapsto \beta_\partial(\Theta);$$

such that $\Theta = e\alpha_\partial(\Theta) + \epsilon_n \beta_\partial(\Theta)$.

Lemma A.10. *The map* $\frac{1}{3!}\epsilon_n e^3 \colon \Omega_\partial^{(0,0)} \to \Omega_\partial^{(3,4)}$ *is an isomorphism.*

Proof. It is immediate to see that $\Omega_\partial^{(3,0)} \simeq \Omega_\partial^{(3,4)}$ as $\Gamma(\Sigma, \wedge^4\mathcal{V}) \simeq \mathcal{C}^\infty(\Sigma)$ upon choice of an orientation. The same is true for $\Omega_\partial^{(3,0)} \simeq \Omega_\partial^{(0,0)}$, hence we have $\Omega_\partial^{(0,0)} \simeq \Omega_\partial^{(3,4)}$. Therefore showing that the above maps are isomorphisms is equivalent to showing that they are nowhere vanishing, which is obvious from their defininition. □

Remark A.14. If one does the computation directly, letting $\tilde{e}$ be the tetrad in the bulk and denoting the transversal (to the boundary) index by n, it's possible to see that $\mathrm{Vol}_M = \frac{1}{4!}\tilde{e}^4 = \frac{1}{3!}\tilde{e}^3\tilde{e}_n dx^n$. Restricting to the boundary, we have $\tilde{e}_n = \upsilon\epsilon_n + \iota_\zeta e$, where $\upsilon \in \mathcal{C}^\infty(\Sigma)$ and $\zeta \in \mathfrak{X}(\Sigma)$. In particular, υ is a nowhere vanishing function, hence, upon restriction to the boundary, one finds

$$\mathrm{Vol}_\Sigma = \frac{1}{3!}\left(\upsilon\epsilon_n e^3 + \iota_\zeta(e)e^3\right) = \frac{1}{3!}\upsilon\epsilon_n e^3 \quad \Rightarrow \quad \frac{1}{3!}\epsilon_n e^3 = \frac{1}{\upsilon}\mathrm{Vol}_\Sigma,$$

Corollary A.2. *The map*

$$\frac{1}{3!}e^3\gamma : \Omega_\partial^{(0,0)}(\Pi\mathbb{S}_M) \to \Omega_\partial^{(3,4)}(\Pi\mathbb{S}_M)$$

is an isomorphism.

Proof. By direct inspection, $\frac{1}{3!}e^3\gamma = \frac{1}{3!}\epsilon_n e^3\gamma^n$. Since γ^n is invertible[9], A.10 implies the desired result. □

A.3 Tools and identities

A.3.1 Basic Identities on gamma matrices

Let $a = 0, \cdots, d$. Setting $\Gamma_{a_1\cdots a_n} := \Gamma_{[a_1}\Gamma_{a_2}\cdots\Gamma_{a_n]}$, we present a list of well known identities[10] adjusted to the mostly plus signature:

$$\Gamma^a\Gamma_a = -D; \tag{A.34}$$
$$\Gamma^a\Gamma^b\Gamma_a = (D-2)\Gamma^b; \tag{A.35}$$
$$\Gamma^a\Gamma^b\Gamma^c\Gamma_a = (4-D)\Gamma^b\Gamma^c + 4\eta^{bc}\mathbb{1}; \tag{A.36}$$
$$\Gamma^a\Gamma^b\Gamma^c\Gamma^d\Gamma_a = (D-6)\Gamma^b\Gamma^c\Gamma^d - 4\eta^{cd}\Gamma^b - 4\eta^{bc}\Gamma^d + 4\eta^{bd}\Gamma^c; \tag{A.37}$$
$$\Gamma^{a_1\cdots a_r} = \frac{1}{2}\left(\Gamma^{a_1}\Gamma^{a_2\cdots a_r} - (-1)^r\Gamma^{a_2\cdots a_r}\Gamma^{a_1}\right); \tag{A.38}$$
$$\Gamma^a\Gamma^{a_1\cdots a_r}\Gamma_a = (-1)^{r+1}(D-2r)\Gamma^{a_1\cdots a_r}; \tag{A.39}$$
$$\Gamma^{a_1\cdots a_r b_1\cdots b_s}\Gamma_{b_1\cdots b_s} = (-1)^r\frac{(D-r)!}{(D-r-s)!}\Gamma^{a_1\cdots a_r} \tag{A.40}$$

[9] $\{\gamma_n, \gamma_n\} = 2(\gamma_n)^2 = -2g_{(nn)} \neq 0$ implies $(\gamma_n)^{-1} = -\frac{1}{g_{nn}}\gamma_n$

[10] which the reader can easily check by permuting the gamma matrices using their defining equations

In $D = 4$, where we denote gamma matrices with $\{\gamma_a\}$, having set $\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3$, the following identities hold:

$$\gamma^a\gamma^b\gamma^c\gamma^d\gamma_a = 2\gamma^d\gamma^c\gamma^b; \tag{A.41}$$
$$\gamma^a\gamma^b\gamma^c = -\eta^{ab}\gamma^c - \eta^{bc}\gamma^a + \eta^{ac}\gamma^b + i\epsilon^{dabc}\gamma_d\gamma^5; \tag{A.42}$$
$$\gamma^5\gamma^{cd} = -\frac{i}{2}\epsilon^{abcd}\gamma_{ab}; \tag{A.43}$$
$$\gamma^a\gamma^5 = i\epsilon^{abcd}\gamma_{bcd}. \tag{A.44}$$

Considering $\{v_a\}$ basis for V, we set $\Gamma := \Gamma^a v_a$[11] and define the bracket $[\cdot,\cdot]$ to encompass the action of $\mathfrak{spin}(d,1) \simeq \wedge^2 V$ on $\wedge^j V$, i.e extend by linearity and graded Leibniz (on the first and second entries) the following

$$\begin{aligned}[v_a,\cdot]\colon\ \wedge^k V &\longrightarrow \wedge^{k-1}V\\ \alpha = \frac{1}{k!}\alpha^{a_1\cdots a_k}v_{a_1}\cdots v_{a_k} &\longmapsto \frac{1}{(k-1)!}\eta_{aa_1}\alpha^{a_1\cdots a_k}v_{a_2}\cdots v_{a_k}\end{aligned}$$

we obtain

$$[v_a, \Gamma^N] = N[v_a,\Gamma]\Gamma^{N-1} + N(N-1)v_a\Gamma^{N-2}, \quad N \geq 2; \tag{A.45}$$
$$= (-1)^{N-1}(N\Gamma^{N-1}\Gamma_a + N(N-1)\Gamma^{N-2}v_a). \tag{A.46}$$

Now we are interested in the cases when the expression containing spinors is real (whether it is because it contains Majorana-type spinors or because we are dealing with real quantities defined via Dirac spinors). In particular, in most of the relevant computations, denoting complex conjugation by $(\cdot)^*$, one considers $iA - iA^*$, where A is any expression containing spinors. Here we list some recurring expressions:

$$[v_a,\Gamma]\Gamma^N - (-1)^N\Gamma^N[v_a,\Gamma] = -2Nv_a\Gamma^{N-1}; \tag{A.47}$$
$$[\Gamma,\Theta]\Gamma^2 - \Gamma^2[\Gamma,\Theta] = 4N\Gamma\Theta \quad \forall \tag{A.48}$$
$$\bar{\chi}\gamma^3[\alpha,\psi] = 3\bar{\chi}\gamma\psi + (-1)^{|\alpha|}\frac{1}{2}\bar{\chi}[\alpha,\gamma^3]_v\psi, \tag{A.49}$$

for all $\alpha \in \wedge^2 V, \theta \in \wedge^{D-3}V$, $\Theta \in \wedge^N V$ and even Majorana spinors χ and ψ, having defined

$$[\alpha,\psi] := \frac{1}{4}[\gamma,[\gamma,\alpha]]_V\psi = -\frac{1}{4}\alpha^{ab}\gamma_{ab}\psi,$$

having considered the split $[\alpha,\gamma] = [\alpha,\gamma]_{\mathcal{C}} + [\alpha,\gamma]_V = 0$, since an element in $\wedge^2 V \simeq \mathfrak{spin}(d,1)$ acts both via the Gamma representation and on V via the fundamental representation.

Another important identity derived from the ones above, is the following

$$\bar{\chi}\gamma[\alpha,\psi] = 3\bar{\chi}\gamma\psi\alpha + \frac{1}{2}\bar{\chi}[\alpha,\gamma^3]_V\psi, \tag{A.50}$$

which is true for all $\chi,\psi \in \mathbb{S}_M$ and $\alpha \in \wedge^2 V$.

[11] From now on we will omit the $\wedge$ symbol and automatically assume that for all $B \in V$, $B^N = B\wedge\cdots\wedge B \in \wedge^N V$.

A.3.2 Identities on Majorana spinors

Majorana flip relations

Let $D = 2k, 2k+1$. Given any two Majorana spinors ψ and χ (for which, we recall, $\epsilon = 1, \eta = -1$) of arbitrary parity, denoting $\Gamma = \Gamma^a v_a \in \mathrm{End}(\mathbb{C}^{2^k}) \otimes V$, we have the following

$$\bar{\chi}\psi = -(-1)^{|\chi||\psi|}\bar{\psi}\chi; \tag{A.51}$$

$$\bar{\chi}\Gamma\psi = (-1)^{|\psi|+|\chi|+|\psi||\chi|}\bar{\psi}\Gamma\chi; \tag{A.52}$$

$$\bar{\chi}\Gamma^2\psi = (-1)^{|\psi||\chi|}\bar{\psi}\Gamma^2\chi; \tag{A.53}$$

$$\bar{\chi}\Gamma^3\psi = -(-1)^{|\psi|+|\chi|+|\psi||\chi|}\bar{\psi}\Gamma^3\chi; \tag{A.54}$$

In general, one finds

$$\bar{\chi}\Gamma^N\psi = -t_N(-1)^{N(|\psi|+|\chi|)+|\psi||\chi|}\bar{\psi}\Gamma^N\chi, \tag{A.55}$$

where t_N is defined from $(C\Gamma^N)^t = -t_N C\Gamma^N$ and is such that $t_{N+4} = t_N$.[12] The first 4 parameters read

$$t_0 = 1, \qquad t_1 = -1, \qquad t_2 = -1, \qquad t_3 = 1,$$

while the general formula is

$$t_N = (-1)^{\lfloor \frac{N+1}{2} \rfloor}. \tag{A.56}$$

Fierz identities

As stated in (A.2), one can find a basis of the Clifford algebra using products of elements of the basis of V. In the context of the gamma representation, the Clifford product is mapped into matrix multiplication, hence a basis is given in terms of products of gamma matrices as $\{\Gamma^{[A]}\} := \{\mathbb{1}, \Gamma^a, \Gamma^{ab}, \cdots, \Gamma^{a_0\cdots a_d}\}$, where $[A]$ represents the number of factors in the basis element, also known as the rank of the basis element. We define $\{\Gamma_{[A]}\} :== \{\mathbb{1}, \Gamma_a, \Gamma_{ba}, \cdots, \Gamma_{a_d\cdots a_0}\}$ with lower indices in the opposite order, as it helps with signs arising in the computations.

Starting by the even dimensional case where $D = 2k$, we aim at using the generators $\{\Gamma^{[A]}\}$ to obtain any matrix on $\mathbb{C}(2^k)$. Indeed, on $\mathbb{C}(2^k)$, one has the obvious pairing introduced by the trace operator, i.e. $\forall M, N \in \mathbb{C}(2^k)$, $(M, N) := \mathrm{Tr}(MN^\dagger)$.

It can be shown that, for even dimensions $D = 2k$, one has the following property

$$\mathrm{Tr}(\Gamma^{[A]}\Gamma_{[B]}) = (-1)^{[A]}2^k\delta^{[A]}_{[B]}, \tag{A.57}$$

where for a generic index set $[A] = a_1 \cdot a_r$, $\delta^{[A]}_{[B]} := \delta^{a_1\cdots a_r}_{b_1\cdots b_r} := \delta^{[a_1}_{b_1}\cdots\delta^{a_r]}_{b_r}$. The above relation allows to expand any matrix $M \in \mathbb{C}(2^k)$ as a linear combination of products of gamma matrices, i.e.

$$M = \sum_A m_{[A]}\Gamma^{[A]} \qquad \text{with} \qquad m_{[A]} = \frac{(-1)^{[A]}}{2^m}\mathrm{Tr}(M\Gamma_{[A]}).$$

Denoting by $\alpha = 1, \cdots, 2^k$ the spinor indices, following [FV12], one can consider $\delta^\beta_\alpha\delta^\delta_\gamma$ as a matrix with entries labelled by indices β and γ, while α and δ are just dummy inert indices. Applying the above formula we obtain

$$\delta^\beta_\alpha\delta^\delta_\gamma = \sum_A (m_{[A]})^\delta_\alpha(\Gamma^{[A]})^\beta_\gamma, \qquad (m_{[A]})^\delta_\alpha = \frac{(-1)^{[A]}}{2^k}\delta^\beta_\alpha\delta^\delta_\gamma(\Gamma_{[A]})^\gamma_\beta = \frac{(-1)^{[A]}}{2^k}(\Gamma_{[A]})^\delta_\alpha,$$

[12] A closer inspection reveals $t_0 = t_3 = -\epsilon\eta = 1$, $t_1 = t_2 = -\epsilon$,

obtaining

$$\delta^\beta_\alpha \delta^\delta_\gamma = \sum_A \frac{(-1)^{[A]}}{2^k} (\Gamma_{[A]})^\delta_\alpha (\Gamma^{[A]})^\beta_\gamma$$

We are interested in the decomposition of $\gamma^a\gamma_a$ in even dimensions. We obtain

$$\begin{aligned}(\Gamma^a)^\rho_\alpha (\Gamma_a)^\delta_\sigma &= (\Gamma^a)^\rho_\beta (\Gamma_a)^\gamma_\sigma \delta^\beta_\alpha \delta^\delta_\gamma \\ &= (\Gamma^a)^\rho_\beta (\Gamma_a)^\gamma_\sigma \sum_A \frac{(-1)^{[A]}}{2^k} (\Gamma_{[A]})^\delta_\alpha (\Gamma^{[A]})^\beta_\gamma \\ &= \sum_A \frac{(-1)^{[A]}}{2^k} (\Gamma_{[A]})^\delta_\alpha (\Gamma^a \Gamma^{[A]} \Gamma_a)^\rho_\sigma .\end{aligned}$$

From (A.39), we see $\Gamma^a\Gamma^{[A]}\Gamma_a = (-1)^{[A]+1}(D-2[A])\Gamma^{[A]}$, hence obtaining

$$(\Gamma^a)^\rho_\alpha (\Gamma_a)^\delta_\sigma = \sum_A \frac{(2[A]-D)}{2^k} (\Gamma^{[A]})^\rho_\sigma (\Gamma_{[A]})^\delta_\alpha \tag{A.58}$$

Now we consider the case $D = 4$. We can use the charge conjugation matrix to lower the indices of the gamma matrices[13] and obtain $(\gamma^a)^\cdot_{\alpha\beta} := C_{\alpha\delta}(\gamma^a)^\delta_\beta$. Furthermore, we symmetrize the part in $(\beta\rho\delta)$ obtaining

$$\begin{aligned}(\gamma^a)^\cdot_{\alpha(\beta}(\gamma^a)^\cdot_{\rho\delta)} &= \frac{1}{4}\left[-4C_{\alpha(\delta}C_{\rho\beta)} - 2(\gamma^a)^\cdot_{\alpha(\delta}(\gamma^a)^\cdot_{\rho\beta)} - 0 + 2(\gamma^{abc})^\cdot_{\alpha(\delta}(\gamma_{abc})^\cdot_{\rho\beta)} + 4(\gamma^5)^\cdot_{\alpha(\delta}(\gamma^5)^\cdot_{\rho\beta)}\right] \\ &= -\frac{1}{2}(\gamma^a)^\cdot_{\alpha(\delta}(\gamma^a)^\cdot_{\rho\beta)} = 0,\end{aligned}$$

Having used that $C_{(\alpha\beta)} = 0$, $(\gamma_{abc})^\cdot_{(\rho\beta)} = 0$ and that $(\gamma^5)^\cdot_{(\rho\beta)} = 0$, as a consequence of the fact that

$$(\gamma^{a_1\cdots a_r})^\cdot_{\alpha\beta} = -t_r(\gamma^{a_1\cdots a_r})^\cdot_{\beta\alpha}. \tag{A.59}$$

Then one finds

$$(\gamma^a)^\cdot_{\alpha(\beta}(\gamma_a)^\cdot_{\rho\delta)} = 0. \tag{A.60}$$

Contracting with 4 Majorana spinors λ_i's ($i = 1, \cdots, 4$) of arbitary parity, we obtain

$$\bar\lambda_1\gamma^3\lambda_2\bar\lambda_3\gamma\lambda_4 = (-1)^{|\lambda_2||\lambda_3|}\bar\lambda_1\gamma\lambda_3\bar\lambda_2\gamma^3\lambda_4 + (-1)^{|\lambda_4|(|\lambda_2|+|\lambda_3|+1)+|\lambda_3|}\bar\lambda_1\gamma\lambda_4\bar\lambda_2\gamma^3\lambda_3; \tag{A.61}$$

$$\bar\lambda_1\gamma^3\lambda_2\bar\lambda_3\gamma\lambda_4 = -(-1)^{|\lambda_2||\lambda_3|}\bar\lambda_1\gamma^3\lambda_3\bar\lambda_2\gamma\lambda_4 - (-1)^{|\lambda_4|(|\lambda_2|+|\lambda_3|+1)+|\lambda_3|}\bar\lambda_1\gamma^3\lambda_4\bar\lambda_2\gamma\lambda_3. \tag{A.62}$$

A.3.3 Lemmata and other facts about spinor fields

The following section regroups a series of results which are cited throghout the thesis. They involve some identities and lemmata about spinor in the spin coframe formalism in $D = 4$, both in the bulk and on the boundary.

Lemma A.11. *The map*

$$\begin{aligned}\Theta^{(1,0)}\colon \Omega^{(1,0)}(\Pi\mathbb{S}_D) &\longrightarrow \Omega^{(2,4)}(\Pi\mathbb{S}_D) \\ \psi &\longmapsto \frac{1}{3!}e\gamma^3\psi\end{aligned}$$

is injective.

[13]It is also useful to adopt this formalism when dealing with scalar quantities defined in terms of spinors. For example, we have $\bar\chi_\alpha = \chi^\beta C_{\alpha\beta}$ and $C_{\beta\alpha} := \delta_{\epsilon\alpha}C^\epsilon_\beta$.

Proof. Using $v_a v_b v_c v_d = \epsilon_{abcd}\mathrm{Vol}_V$

$$\begin{aligned}\frac{1}{3!}e\gamma^3\psi &= \frac{1}{3!}e^a\gamma^{bcd}\psi v_a v_b v_c v_d\\ &= \frac{1}{3!}\epsilon_{abcd}e^a\gamma^{bcd}\psi\mathrm{Vol}_V\\ &\overset{(A.44)}{=} \frac{1}{3!}i\gamma^5 e^a\gamma_a\psi\mathrm{Vol}_V = 0 \qquad \Leftrightarrow \qquad [e,\gamma]\psi = 0.\end{aligned}$$

Now $[e,\gamma]\psi = 0$ if and only if $\gamma_{[\mu}\psi_{\nu]} = 0$, which is uniquely solved by $\psi = 0$, hence proving $\Theta^{(1,0)}$ is injective. □

Lemma A.12. *The map*

$$\begin{aligned}\Theta_\gamma^{(1,0)}\colon \Omega^{(1,0)}(\Pi\mathbb{S}_D) &\longrightarrow \Omega^{(3,4)}(\Pi\mathbb{S}_D)\\ \psi &\longmapsto \frac{1}{3!}e\gamma^3\underline{\gamma}\psi\end{aligned}$$

is an isomorphism, where $\underline{\gamma} := [e,\gamma] = \gamma_\mu dx^\mu$

Proof. First of all, from the previous proof we know $e\gamma^3 = i\gamma^5[e,\gamma] = i\gamma^5\underline{\gamma}\mathrm{Vol}_V$. Then

$$e\gamma^3\underline{\gamma}\psi = i\gamma^5\underline{\gamma}^2\psi\mathrm{Vol}_V = 0 \quad \Leftrightarrow \quad \underline{\gamma}^2\psi = 0 \quad \Leftrightarrow \quad \gamma_{[\mu}\gamma_\nu\psi_{\rho]} = 0.$$

The latter is a system of 4 equations whose solution (due to invertibility of the gamma matrices) is uniquely given by $\psi_\rho = 0$. This shows that $\Theta_\gamma^{(1,0)}$ is injective, but since $\dim\Omega^{(1,0)} = \dim\Omega^{(3,4)}$ and $\Theta_\gamma^{(1,0)}$ is linear, by the rank theorem $\dim\mathrm{Im}(\Theta_\gamma^{(1,0)}) = \dim\Omega^{(3,4)}$ hence it is also surjective. □

Remark A.15. By the same reasoning (or just by taking the Dirac conjugate of the above expression), one finds that also the map

$$\psi \longmapsto \frac{1}{3!}e\underline{\gamma}\gamma^3\psi$$

is an isomorphism.

Lemma A.13. *For all* $\theta \in \Omega^{(3,1)}(\Pi\mathbb{S}_M)$ *there exist unique* $\alpha \in \Omega^{(1,0)}(\Pi\mathbb{S}_M)$ *and* $\beta \in \Omega^{(3,1)}(\Pi\mathbb{S}_M)$ *such that*

$$\theta = ie\underline{\gamma}\alpha + \beta \qquad \text{and} \qquad \gamma^3\beta = 0. \tag{A.63}$$

Proof. We start by considering the map $(e\underline{\gamma})_{(1,0)}\colon \Omega^{(1,0)} \to \Omega^{(3,1)}\colon \alpha \mapsto e\underline{\gamma}\alpha$. We see that $e\underline{\gamma}\alpha = 0$ implies $\underline{\gamma}\alpha + 0$ due to injectivity of $W_e^{(1,0)}$[Can24], while

$$\underline{\gamma}\alpha = 0 \quad \Leftrightarrow \quad \gamma_{[\mu}\alpha_{\nu]} = 0 \quad \Leftrightarrow \quad \alpha_\nu = 0,$$

hence implying that $(e\underline{\gamma})_{(1,0)}$ is injective.

Now, defining $(\gamma^3)_{(3,1)}\colon \Omega^{(3,1)} \to \Omega^{(3,4)}\colon \beta \mapsto \gamma^3\beta$, we notice that $\gamma^3\beta = \mathrm{Vol}_V[\gamma,\beta]_V$, hence $\ker((\gamma^3)_{(3,1)}) = \{\beta \in \Omega^{(3,1)} \mid [\gamma,\beta] = 0\}$. We have

$$[\gamma,\beta] = \gamma_a\beta^a_{\mu\nu\rho} = 0,$$

which is a system of 4 independent equations, implying $\dim(\ker((\gamma^3)_{(3,1)})) = \dim(\Omega^{(3,1)}) - 4 = 12$. Now, since $(e\underline{\gamma})_{(1,0)}$ is injective, it is immediate to see that

$$\dim(\Omega^{(3,1)}) = 16 = \dim(\mathrm{Im}((e\underline{\gamma})_{(1,0)})) + \dim(\ker((\gamma^3)_{(3,1)})) = \dim(\Omega^{(1,0)}) + \dim(\ker((\gamma^3)_{(3,1)})).$$

The claim is then proved once we show that $\mathrm{Im}((e\underline{\gamma})_{(1,0)}) \cap \ker((\gamma^3)_{(3,1)}) = \{0\}$. This is immediate since, by lemma A.12, for all $\alpha \in \Omega^{(1,0)}$,

$$\gamma^3 e\underline{\gamma}\alpha = 0 \quad \Leftrightarrow \quad \alpha = 0.$$

□

Lemma A.14. *Let $n \in \mathbb{N}$ and $\underline{\gamma}\gamma^n_{(i,j)}$ be the map*

$$\underline{\gamma}\gamma^n_{(i,j)} \colon \Omega^{(i,j)} \to \Omega^{(i,j+n)} \colon \beta \mapsto \underline{\gamma}\gamma^n\beta.$$

Then, for all $\theta \in \Omega^{(2,1)}(\Pi\mathbb{S}_M)$ there exist unique $\alpha \in \Omega^{(1,0)}(\Pi\mathbb{S}_M)$ and $\beta \in \ker \underline{\gamma}\gamma^3_{(2,1)}$ such that

$$\theta = e\alpha + \beta.$$

Proof. We see $\dim \mathrm{Im}(W_1^{(1,0)}) = \dim \Omega^{(1,0)} = 4$ as $W_1^{(1,0)}$ is injective. On the other hand, from A.12, we see $\Omega^{(3,4)} = e\underline{\gamma}\gamma^3\Omega^{(1,0)}$, implying in particular that $\underline{\gamma}\gamma^3_{(2,1)}$ is surjective, hence $\dim\ker(\underline{\gamma}\gamma^3_{(2,1)}) = \dim\Omega^{(2,1)} - \dim\Omega^{(3,4)} = 20$. Now, since $\dim\Omega^{(2,1)} = \dim\mathrm{Im}(W_1^{(1,0)}) + \dim\ker(\underline{\gamma}\gamma^3_{(2,1)})$, we just need to prove that $\mathrm{Im}(W_1^{(1,0)}) \cap \ker(\underline{\gamma}\gamma^3_{(2,1)}) = \{0\}$. Choosing $\alpha \in \Omega^{(1,0)}$, we see

$$e\alpha \in \ker(\underline{\gamma}\gamma^3_{(2,1)}) \quad \Leftrightarrow \quad e\underline{\gamma}\gamma^3\alpha = 0 \quad \Leftrightarrow \quad \alpha = 0.$$

For uniqueness, assume there exist $\alpha_1, \alpha_2 \in \Omega^{(1,0)}$ and $\beta_1, \beta_2 \in \ker(\underline{\gamma}\gamma^3_{(2,1)})$ such that $\theta = e\alpha_1 + \beta_1 = e\alpha_2 + \beta_2$, then

$$e(\alpha_1 - \alpha_2) = \beta_2 - \beta_1 \in \ker(\underline{\gamma}\gamma^3_{(2,1)}),$$

which implies $\alpha_1 - \alpha_2 = 0$, and $\beta_2 - \beta_1 = 0$. □

Lemma A.15. *For all $\lambda, \psi, \chi \in \mathbb{S}_M$ such that $|\chi| = 0$ and $|\psi| = 1$, the following identities hold*

$$\bar\lambda\gamma^3\chi\bar\chi\gamma\psi = 0, \qquad \bar\chi\gamma\chi\bar\lambda\gamma^3\psi = 0 \qquad \text{and} \qquad \bar\lambda\gamma\chi\bar\chi\gamma^3\psi = 0.$$

Proof. The proof of the above identity rely on subsequent applications of Fierz identities (A.61) and (A.62) and Majorana flip relations. In particular $\bar\lambda\gamma^3\chi\bar\chi\gamma\psi \overset{(A.61)}{=} \bar\lambda\gamma\chi\bar\chi\gamma^3\psi + (-1)^{|\psi|}\bar\lambda\gamma\psi\bar\chi\gamma^3\chi = \bar\lambda\gamma\chi\bar\chi\gamma^3\psi$, having used (A.54). At the same time

$$\begin{aligned}
\bar\lambda\gamma\chi\bar\chi\gamma^3\psi &\overset{(A.54)}{=} (-1)^{|\lambda|(|\psi|+1)}\bar\chi\gamma^3\psi\bar\lambda\gamma\chi \\
&\overset{(A.61)}{=} (-1)^{|\lambda|(|\psi|+1)}\left((-1)^{|\lambda||\psi|}\bar\chi\gamma\psi\bar\psi\gamma^3\chi + (-1)^{|\lambda|}\bar\chi\gamma\chi\bar\lambda\gamma^3\psi\right) \\
&\overset{(A.52)(A.54)}{=} -(-1)^{|\psi|}\bar\lambda\gamma\chi\bar\chi\gamma^3\psi + (-1)^{|\lambda||\psi|}\bar\chi\gamma\chi\bar\lambda\gamma^3\psi,
\end{aligned}$$

hence showing that when $|\psi| = 1$, $\bar\chi\gamma\chi\bar\lambda\gamma^3\psi = 0$. Now at the same time we have

$$\bar\lambda\gamma^3\chi\bar\chi\gamma\psi \overset{(A.62)}{=} -\bar\lambda\gamma^3\chi - (-1)^{|\psi|}\bar\lambda\gamma^3\psi\bar\chi\gamma\chi,$$

hence $\bar\lambda\gamma^3\chi\bar\chi\gamma\psi = -\frac{1}{2}(-1)^{|\psi|}\bar\lambda\gamma^3\psi\bar\chi\gamma\chi = 0$. Lastly, we saw that $\bar\lambda\gamma\chi\bar\chi\gamma^3\psi = \bar\lambda\gamma^3\chi\bar\chi\gamma\psi = 0$. □

A.4 Proofs of section A.3

We now list the proofs of equations in A.3

- (A.45) and (A.46). We prove this by induction, first showing it holds for $N = 2, 3$ and then proving the inductive step, having set $\Gamma_a := \eta_{ab}\Gamma^b = [v_a, \Gamma]$.

$$\begin{aligned}[v_a, \Gamma^2] &= \Gamma_a\Gamma - \Gamma\Gamma_a = \Gamma_a\Gamma - \Gamma^b v_b \Gamma^c \eta_{ac} = \Gamma_a\Gamma + \Gamma^c\eta_{ac}\Gamma^b v_b + 2\eta^{bc}\eta_{ac}v_b = 2\Gamma_a\Gamma + 2v_a \\ &= 2\Gamma^c\Gamma^b\eta_{ac}v_b + 2v_a = -2\Gamma\Gamma_a - 4\eta^{cb}\eta_{ac}v_b + 2v_a = -2\Gamma\Gamma_a - 2v_a,\end{aligned}$$

$$\begin{aligned}[v_a, \Gamma^3] &= [v_a, \Gamma^2]\Gamma + \Gamma^2\Gamma_a = -2\Gamma^c\Gamma^d\Gamma^b\eta_{ad}v_c v_b - 2v_a\Gamma + \Gamma^2\Gamma_a \\ &= 2\Gamma^c\Gamma^b\Gamma^d\eta_{ad}v_c v_b + 4\eta^{db}\eta_{ad}\Gamma^c v_c v_b + 2\Gamma v_a + \Gamma^2\Gamma_a = 3\Gamma^2\Gamma_a + 6\Gamma v_a \\ &= \Gamma_a\Gamma^2 - \Gamma[v_a, \Gamma^2] = \Gamma_a\Gamma^2 - 2\Gamma\Gamma_a\Gamma - 2\Gamma v_a \\ &= \Gamma_a\Gamma^2 + 2\Gamma^b\Gamma^c\Gamma^d\eta_{ad}v_b v_c + 4\eta^{cd}\eta_{ad}\Gamma^b v_b v_c + 2v_a\Gamma \\ &= 3\Gamma_a\Gamma^2 + 6v_a\Gamma.\end{aligned}$$

  Now assume (A.45) and (A.46) hold for $N - 1$, then

$$\begin{aligned}[v_a, \Gamma^N] &= \Gamma_a\Gamma^{N-1} - \Gamma[v_a, \Gamma^{N-1}] = \Gamma_a\Gamma^{N-1} - (N-1)\Gamma\Gamma_a\Gamma^{N-2} - (N-1)(N-2)\Gamma v_a\Gamma^{N-3} \\ &= N\Gamma_a\Gamma^{N-1} + 2(N-1)v_a\Gamma^{N-2} + (N-1)(N-2)v_a\Gamma\Gamma^{N-2} \\ &= N\Gamma_a\Gamma^{N-1} + N(N-1)v_a\Gamma^{N-2} \\ &= [v_a, \Gamma^{N-1}]\Gamma + (-1)^{N-1}\Gamma^{N-1}\Gamma_a \\ &= (-1)^{N-2}[(N-1)\Gamma^{N-2}\Gamma_a(N-1)(N-2)\Gamma^{N-3}v_a]\Gamma + (-1)^{N-1}\Gamma^{N-1}\Gamma_a \\ &= (-1)^{N-1}(N\Gamma^{N-1}\Gamma_a + N(N-1)\Gamma^{N-2}v_a);\end{aligned}$$

- (A.47) follows immediately by subtracting (A.45) from (A.46) applied to Γ^{N+1}:
- (A.48). Consider $\Theta = \frac{1}{N!}\Theta^{a_1\cdots a_n}v_{a_1}\cdots v_{a_N}$, then

$$\begin{aligned}[\Gamma, \Theta]\Gamma^2 &= \frac{(-1)^{|\Theta|-N}}{(N-1)!}\Theta^{a_1a_2\cdots a_N}v_{a_2}\cdots v_{a_N}\eta_{a_1a}\Gamma^a\Gamma^b\Gamma^c v_b v_c \\ &= -\frac{(-1)^{|\Theta|-N}}{(N-1)!}\Theta^{a_1a_2\cdots a_N}v_c v_b v_{a_2}\cdots v_{a_N}\eta_{a_1a}(-4\eta^{ab}\Gamma^c + \Gamma^b\Gamma^c\Gamma^a) \\ &= \Gamma^2[\Gamma, \Theta] + 4N\Gamma\Theta;\end{aligned}$$

- (A.49) Consider $\alpha \in \wedge^2 V$ with parity $|\alpha|$, then for any Dirac spinor (of any parity) χ and ψ we have $\bar{\chi}\gamma^3[\alpha, \psi] = \frac{1}{4}\bar{\chi}\gamma^3\gamma^a\gamma^b[v_a, [v_b, \alpha]]\psi$, so

$$\begin{aligned}\gamma^3\gamma^a\gamma^b[v_a, [v_b, \alpha]] &= -[v_a, \gamma^3[v_b, \alpha]]\gamma^a\gamma^b + (3\gamma^2\gamma_a + 6\gamma v_a)[v_b, \alpha]\gamma^a\gamma^b \\ &= -[v_a, (3\gamma^2\gamma_b + 6\gamma v_b)\gamma^a\gamma^b\alpha] - 6\gamma^2[\gamma, \alpha]_V \\ &= -6\gamma^2[\gamma, \alpha]_V + [v_a, 12\gamma v_b\eta^{ab}\alpha] \\ &= -6\gamma^2[\gamma, \alpha]_V + 36\gamma\alpha + (-1)^{|\alpha|}12\gamma v_b\eta^{ab}\alpha^{cd}\eta_{ca}v_d \\ &= -12\gamma\alpha + (1)^{|\alpha|}6\gamma^2[\alpha, \gamma]_V.\end{aligned}$$

Now, since $[\alpha,\gamma^3]_V = 3\gamma^2[\alpha,\gamma]_V - (1)^{|\alpha|}12\gamma\alpha$, one has that

$$\gamma^3\gamma^a\gamma^b[v_a,[v_b,\alpha]] = (1)^{|\alpha|}2[\alpha,\gamma^3]_V + 12\gamma\alpha,$$

hence

$$\bar{\chi}\gamma^3[\alpha,\psi] = 3\bar{\chi}\gamma\psi + (-1)^{|\alpha|}\frac{1}{2}\bar{\chi}[\alpha,\gamma^3]_v\psi.$$

- (A.51). We use the fact that $C^t = -C$, hence

$$\bar{\chi}\psi = \chi^\alpha C_{\alpha\beta}\psi^\beta = (-1)^{|\chi||\psi|}\psi^\beta C_{\alpha\beta}\chi^\alpha = -(-1)^{|\chi||\psi|}\psi^\beta C_{\beta\alpha}\chi^\alpha = -(-1)^{|\chi||\psi|}\bar{\psi}\chi;$$

- (A.52). We denote by $(\Gamma^a)^{\cdot}_{\alpha\beta} := C_{\alpha\delta}(\Gamma^a)^\delta_\beta$ and by $(\Gamma^a)_{\alpha\beta} = \delta_{\alpha\epsilon}(\Gamma^a)^\epsilon_\beta$. Then, using $C\Gamma^a = -(\Gamma^a)^t C$, we have

$$\begin{aligned}(\Gamma^a)^{\cdot}_{\alpha\delta} &= C_{\alpha\beta}(\Gamma^a)^\beta_\delta = \delta_{\alpha\epsilon}C^\epsilon_\beta(\Gamma^a)^\beta_\delta = -\delta_{\alpha\epsilon}(\Gamma^{a,t})^\epsilon_\beta C^\beta_\delta = -(\Gamma^{a,t})_{\alpha\beta}C^\beta_\delta \\ &= -(\Gamma^a)_{\beta\alpha}C^\beta_\delta = -(\Gamma^a)^\epsilon_\alpha\delta_{\epsilon\beta}C^\beta_\delta = -C_{\epsilon\delta}(\Gamma^a)^\epsilon_\alpha = C_{\delta\epsilon}(\Gamma^a)^\epsilon_\alpha \\ &= (\Gamma^a)^{\cdot}_{\delta\alpha},\end{aligned}$$

hence finding

$$C_{\alpha\beta}(\Gamma^a)^\beta_\delta = -C_{\beta\delta}(\Gamma^a)^\beta_\alpha = C_{\delta\beta}(\Gamma^a)^\beta_\alpha. \tag{A.64}$$

Now we have

$$\begin{aligned}\bar{\chi}\Gamma\psi &= (-1)^{|\psi|}\bar{\chi}^\alpha C_{\alpha\beta}(\Gamma^a)^\beta_\delta\psi^\delta v_a = (-1)^{|\psi|}\bar{\chi}^\alpha(\Gamma^a)^{\cdot}_{\alpha\beta}\psi^\beta v_a \\ &= (-1)^{|\psi|+|\psi||\chi|}\psi^\beta(\Gamma^a)^{\cdot}_{\beta\alpha}\chi^\alpha v_a = (-1)^{|\chi|+|\psi|+|\psi||\chi|}\bar{\psi}\Gamma\chi;\end{aligned}$$

- (A.53). Recall $\Gamma^{ab} := \Gamma^{[a}\Gamma^{b]} = \frac{1}{2}[\Gamma^a,\Gamma^b]$. Now

$$\begin{aligned}(\Gamma^a\Gamma^b)^{\cdot}_{\alpha\beta} &= (\Gamma^a)^{\cdot}_{\alpha\delta}(\Gamma^b)^\delta_\beta = (\Gamma^a)^{\cdot}_{\delta\alpha}(\Gamma^b)^\delta_\beta = C_{\delta\epsilon}(\Gamma^a)^\epsilon_\alpha(\Gamma^b)^\delta_\beta \\ &= -C_{\epsilon\delta}(\Gamma^a)^\epsilon_\alpha(\Gamma^b)^\delta_\beta = -(\Gamma^b)^{\cdot}_{\epsilon\beta}(\Gamma^a)^\epsilon_\alpha = -(\Gamma^b)^{\cdot}_{\beta\epsilon}(\Gamma^a)^\epsilon_\alpha \\ &= -(\Gamma^b\Gamma^a)^{\cdot}_{\beta\alpha},\end{aligned}$$

implying $(\Gamma^{ab})^{\cdot}_{\alpha\beta} = -(\Gamma^{ba})^{\cdot}_{\beta\alpha} = (\Gamma^{ab})^{\cdot}_{\beta\alpha}$, finding

$$\bar{\chi}\Gamma^2\psi = \bar{\chi}^\alpha(\Gamma^{ab})^{\cdot}_{\alpha\beta}\psi^\beta v_a v_b = (-1)^{|\psi||\chi|}\bar{\psi}^\beta(\Gamma^{ab})^{\cdot}_{\beta\alpha}\chi^\alpha v_a v_b = (-1)^{|\psi||\chi|}\bar{\psi}\Gamma^2\chi;$$

- (A.54). Again $\Gamma^{abc} = \Gamma^{[a}\Gamma^b\Gamma^{c]}$, and

$$\begin{aligned}(\Gamma^a\Gamma^b\Gamma^c)^{\cdot}_{\alpha\beta} &= (\Gamma^a\Gamma^b)^{\cdot}_{\alpha\delta}(\Gamma^c)^\delta_\beta = -(\Gamma^b\Gamma^a)^{\cdot}_{\delta\alpha}(\Gamma^c)^\delta_\beta = -C_{\delta\epsilon}(\Gamma^b\Gamma^a)^\epsilon_\alpha(\Gamma^c)^\delta_\beta \\ &= C_{\epsilon\delta}(\Gamma^b\Gamma^a)^\epsilon_\alpha(\Gamma^c)^\delta_\beta = (\Gamma^c)^{\cdot}_{\epsilon\beta}(\Gamma^b\Gamma^a)^\epsilon_\alpha = (\Gamma^c)^{\cdot}_{\beta\epsilon}(\Gamma^b\Gamma^a)^\epsilon_\alpha \\ &= (\Gamma^c\Gamma^b\Gamma^a)^{\cdot}_{\beta\alpha},\end{aligned}$$

implying $(\Gamma^{abc})^{\cdot}_{\alpha\beta} = -(\Gamma^{abc})^{\cdot}_{\beta\alpha}$, which in turn gives

$$\begin{aligned}\bar{\chi}\Gamma^3\psi &= (-1)^{|\psi|}\chi^\alpha(\Gamma^{abc})^{\cdot}_{\alpha\beta}\psi^\beta v_a v_b v_c = -(-1)^{|\psi|+|\psi||\chi|}\psi^\beta(\Gamma^{abc})^{\cdot}_{\beta\alpha}\chi^\alpha v_a v_b v_c \\ &= -(-1)^{|\psi|+|\chi|+|\psi||\chi|}\bar{\psi}\Gamma^3\chi;\end{aligned}$$

- (A.55). In order to prove the general formula, we first have to prove

$$(\Gamma^{a_1\cdots a_r})^{\cdot}_{\alpha\beta} = -t_r(\Gamma^{a_1\cdots a_r})^{\cdot}_{\beta\alpha}.$$

In particular, we want to show that $t_r = (-1)^{\lfloor \frac{r+1}{2} \rfloor}$. We know this is true for $r = 0, 1, 2, 3$ as we showed explicitly the values of t_r in these cases. Now we prove the inductive step. Consider

$$\begin{aligned}
(\Gamma^{a_1\cdots a_r}\Gamma^{a_{r+1}})^{\cdot}_{\alpha\beta} &= (\Gamma^{a_1\cdots a_r})^{\cdot}_{\alpha\delta}(\Gamma^{a_{r+1}})^{\delta}_{\beta} = -(-1)^{\lfloor \frac{r+1}{2} \rfloor}(\Gamma^{a_1\cdots a_r})^{\cdot}_{\delta\alpha}(\Gamma^{a_{r+1}})^{\delta}_{\beta} \\
&= (-1)^{\lfloor \frac{r+1}{2} \rfloor} C_{\epsilon\delta}(\Gamma^{a_1\cdots a_r})^{\epsilon}_{\alpha}(\Gamma^{a_{r+1}})^{\delta}_{\beta} = (-1)^{\lfloor \frac{r+1}{2} \rfloor}(\Gamma^{a_{r+1}})^{\cdot}_{\beta\epsilon}(\Gamma^{a_1\cdots a_r})^{\epsilon}_{\alpha} \\
&= (-1)^{\lfloor \frac{r+1}{2} \rfloor}(\Gamma^{a_{r+1}}\Gamma^{a_1\cdots a_r})^{\cdot}_{\beta\alpha},
\end{aligned}$$

implying

$$\begin{aligned}
(\Gamma^{a_1\cdots a_{r+1}})^{\cdot}_{\alpha\beta} &= (-1)^{\lfloor \frac{r+1}{2} \rfloor}(\Gamma^{a_{r+1}a_1\cdots a_r})^{\cdot}_{\beta\alpha} = (-1)^{\lfloor \frac{r+1}{2} \rfloor + r}(\Gamma^{a_1\cdots a_{r+1}})^{\cdot}_{\beta\alpha} \\
&= -(-1)^{\lfloor \frac{r+1}{2} \rfloor + r + 1}(\Gamma^{a_1\cdots a_{r+1}})^{\cdot}_{\beta\alpha} = -(-1)^{\lfloor \frac{r+2}{2} \rfloor}(\Gamma^{a_1\cdots a_{r+1}})^{\cdot}_{\beta\alpha},
\end{aligned}$$

showing that $t_{r+1} = (-1)^{\lfloor \frac{r+2}{2} \rfloor}$ as expected.[14] With this formula, we can now easily show

$$\begin{aligned}
\bar{\chi}\Gamma^N\psi &= (-1)^{N|\psi|}\chi^{\alpha}((\Gamma^{a_1\cdots a_N})^{\cdot}_{\alpha\beta}\psi^{\beta}v_{a_1}\cdots v_{a_N} \\
&= -t_N(-1)^{N|\psi|+|\psi||\chi|}\psi^{\beta}(\Gamma^{a_1\cdots a_N})^{\cdot}_{\beta\alpha}\chi^{\alpha}v_{a_1}\cdots v_{a_N} \\
&= -t_N(-1)^{N(|\psi|+|\chi|)+|\psi||\chi|}\bar{\psi}\Gamma^N\chi
\end{aligned}$$

- (A.61) and (A.62). We consider four Majorana spinors λ_i of arbitrary parity. First we see that

$$\begin{aligned}
\bar{\lambda}_1\gamma^3\lambda_2\bar{\lambda}_3\gamma\lambda_4 &= -(-1)^{|\lambda_2|+|\lambda_3|}\bar{\lambda}_1\gamma^{bcd}\lambda_2\bar{\lambda}_3\gamma^a\lambda_4 v_a v_b v_c v_d \\
&= -4!(-1)^{|\lambda_2|+|\lambda_3|}\bar{\lambda}_1\gamma^{bcd}\lambda_2\bar{\lambda}_3\gamma^a\lambda_4\epsilon_{abcd}v_0v_1v_2v_3 \\
&\overset{(A.44)}{=} -4!i(-1)^{|\lambda_2|+|\lambda_3|}\bar{\lambda}_1\gamma^5\gamma_a\lambda_2\bar{\lambda}_3\gamma^a\lambda_4 v_0v_1v_2v_3 \\
&= 4!i(-1)^{|\lambda_2|+|\lambda_3|}\bar{\lambda}_1\gamma_a\gamma^5\lambda_2\bar{\lambda}_3\gamma^a\lambda_4 v_0v_1v_2v_3,
\end{aligned}$$

having used $\{\gamma^5, \gamma^a\} = 0.$. Redefining $\lambda_2' := \gamma^5\lambda_2$ and $\bar{\lambda}_1' := \bar{\lambda}_1\gamma^5$ and setting $v^4 = v_0v_1v_2v_3$, we obtain

$$\begin{aligned}
\bar{\lambda}_1\gamma^3\lambda_2\bar{\lambda}_3\gamma\lambda_4 &= -4!i(-1)^{|\lambda_2|+|\lambda_3|}\bar{\lambda}_1'\gamma_a\lambda_2\bar{\lambda}_3\gamma^a\lambda_4 v^4 && \text{(A.65)} \\
&= 4!i(-1)^{|\lambda_2|+|\lambda_3|}\bar{\lambda}_1\gamma_a\lambda_2'\bar{\lambda}_3\gamma^a\lambda_4 v^4 && \text{(A.66)}
\end{aligned}$$

We now apply (A.60) to the expressions containing $\gamma^a\gamma_a$. Explicitly

$$\begin{aligned}
3\lambda_1'^{\alpha}\lambda_2^{\beta}\lambda_3^{\rho}\lambda_4^{\delta}(\gamma^a)^{\cdot}_{\alpha(\beta}(\gamma_a)^{\cdot}_{\rho\delta)} &= \lambda_1'^{\alpha}\lambda_2^{\beta}\lambda_3^{\rho}\lambda_4^{\delta}((\gamma^a)^{\cdot}_{\alpha\beta}(\gamma_a)^{\cdot}_{\rho\delta} + (\gamma^a)^{\cdot}_{\alpha\rho}(\gamma_a)^{\cdot}_{\beta\delta} + (\gamma^a)^{\cdot}_{\alpha\delta}(\gamma_a)^{\cdot}_{\beta\rho}) \\
&= \bar{\lambda}_1'\gamma^a\lambda_2\bar{\lambda}_3\gamma_a\lambda_4 + (-1)^{|\lambda_2||\lambda_3|}\bar{\lambda}_1'\gamma^a\lambda_3\bar{\lambda}_2\gamma_a\lambda_4 \\
&\quad + (-1)^{|\lambda_4|(|\lambda_2|+|\lambda_3|)}\bar{\lambda}_1'\gamma^a\lambda_4\bar{\lambda}_2\gamma_a\lambda_3 \\
&= 0,
\end{aligned}$$

[14] Here we used the fact that $(-1)^{\lfloor \frac{n}{2} \rfloor + n} = (-1)^{\lfloor \frac{n+1}{2} \rfloor}$, as one can easily check by separating the cases for $n = 2k, 2k+1$.

substituing in (A.65) gives

$$\begin{aligned}\bar{\lambda}_1\gamma^3\lambda_2\bar{\lambda}_3\gamma\lambda_4 = {} & -i\cdot 4!(-1)^{|\lambda_2|+|\lambda_3|}[(-1)^{|\lambda_2|+|\lambda_3|}\bar{\lambda}_1\gamma^5\gamma_a\lambda_3\bar{\lambda}_2\gamma^a\lambda_4 \\ & + (-1)^{|\lambda_4|(|\lambda_2|+|\lambda_3|)}\bar{\lambda}_1\gamma^5\gamma_a\lambda_4\bar{\lambda}_2\gamma^a\lambda_3]v^4 \\ \overset{(A.44)}{=} {} & -(-1)^{|\lambda_2||\lambda_3|}\bar{\lambda}_1\gamma^3\lambda_3\bar{\lambda}_2\gamma\lambda_4 - (-1)^{|\lambda_3|+|\lambda_4|(|\lambda_2|+|\lambda_3|+1)}\bar{\lambda}_1\gamma^3\lambda_4\bar{\lambda}_2\gamma\lambda_3.\end{aligned}$$

(A.61) is recovered in the same way applying (A.60) to (A.66).

Appendix B

Lengthy computations

B.1 Proofs of chapter 3

B.1.1 Theorem 3.2

Proof of theorem 3.2. We first compute the Hamiltonian vector fields of the constraints. To make the notation lighter, we get rid of the apex ^{PCD}, as its use is implied in the following computations.

$$\begin{aligned}
\delta L_c &= \int_\Sigma [c,e]e\delta\omega + \left(ed_\omega c + \frac{i}{4}e^2\left(\overline{\psi}\gamma[c,\psi] - [c,\overline{\psi}]\gamma\psi\right)\right)\delta e \\
&\quad + \frac{i}{2\cdot 3!}e^3\left[\delta\bar{\psi}\gamma[c,\psi] - \bar{\psi}\gamma[c,\delta\psi] + [c,\delta\bar{\psi}]\gamma\psi + [c,\bar{\psi}]\gamma\delta\psi\right] \\
&\overset{\blacktriangledown}{=} \int_\Sigma [c,e]e\delta\omega + \left(ed_\omega c + \frac{i}{4}e^2\left(\overline{\psi}\gamma[c,\psi] - [c,\overline{\psi}]\gamma\psi\right)\right)\delta e \\
&\quad + \frac{ie^3}{3!}\left[[c,\bar{\psi}]\gamma\delta\psi + \delta\bar{\psi}\gamma[c,\psi]\right] - \frac{ie^3}{2\cdot 3!}\left[\delta\bar{\psi}[c,\gamma]\psi + \bar{\psi}[c,\gamma]\delta\psi\right] \\
&= \int_\Sigma [c,e]e\delta\omega + \left(ed_\omega c + \frac{i}{4}e^2\left([c,\overline{\psi}]\gamma\psi + \overline{\psi}\gamma[c,\psi]\right)\right)\delta e \\
&\quad + \frac{i}{3!}e^3\left[\delta\overline{\psi}\left(\frac{1}{2}[c,\gamma]\psi + \gamma[c,\psi]\right) + \left([c,\overline{\psi}]\gamma - \frac{1}{2}\bar{\psi}[c,\gamma]\right)\delta\psi\right],
\end{aligned}$$

where in the last passage we used that

$$\bar{\psi}\gamma[c,\delta\psi] = \bar{\psi}[c,\gamma]\delta\psi - [c,\bar{\psi}]\gamma\delta\psi \tag{B.1}$$

$$[c,\delta\bar{\psi}]\gamma\psi = \delta\bar{\psi}\gamma[c,\psi] - \delta\bar{\psi}[c,\gamma]\psi \tag{B.2}$$

which can easily be proved using the following identity

$$j_\gamma j_\gamma c\gamma = -\gamma j_\gamma j_\gamma c - 4 j_\gamma c = -\gamma j_\gamma j_\gamma c + 4[c,\gamma]. \tag{$\blacktriangledown$}$$

We also get

$$\begin{aligned}
\delta P_\xi = \int_\Sigma & -e\delta e \left(\mathrm{L}_\xi^{\omega_0}(\omega - \omega_0) + \iota_\xi F_{\omega_0} - \frac{i}{4} e \left(\overline{\psi}\gamma \mathrm{L}_\xi^{\omega_0}(\psi) - \mathrm{L}_\xi^{\omega_0}(\overline{\psi})\gamma\psi \right) \right) \\
& - \mathrm{L}_\xi^{\omega_0}(e) e\delta\omega + i\delta\overline{\psi} \left(-\frac{e^3}{2\cdot 3!}\gamma \mathrm{L}_\xi^{\omega_0}(\psi) \right) + \frac{ie^3}{2\cdot 3!}\overline{\psi}\gamma \mathrm{L}_\xi^{\omega_0}(\delta\psi) \\
& - \frac{ie^3}{2\cdot 3!}\mathrm{L}_\xi^{\omega_0}(\delta\overline{\psi})\gamma\psi - \frac{i}{2\cdot 3!} e^3 \mathrm{L}_\xi^{\omega_0}(\overline{\psi})\gamma\delta\psi \\
= \int_\Sigma & -e\delta e \left(\mathrm{L}_\xi^{\omega_0}(\omega - \omega_0) + \iota_\xi F_{\omega_0} - \frac{i}{4} e \left(\overline{\psi}\gamma \mathrm{L}_\xi^{\omega_0}(\psi) - \mathrm{L}_\xi^{\omega_0}(\overline{\psi})\gamma\psi \right) \right) \\
& - \mathrm{L}_\xi^{\omega_0}(e) e\delta\omega - i\delta\overline{\psi} \left(\frac{e^3}{3!}\gamma \mathrm{L}_\xi^{\omega_0}(\psi) - \frac{1}{2\cdot 3!}\mathrm{L}_\xi^{\omega_0}(e^3)\gamma\psi \right) \\
& - i \left(\frac{e^3}{3!}\mathrm{L}_\xi^{\omega_0}(\overline{\psi})\gamma + \frac{1}{2\cdot 3!}\mathrm{L}_\xi^{\omega_0}(e^3)\overline{\psi}\gamma \right) \delta\psi,
\end{aligned}$$

$$\begin{aligned}
\delta H_\lambda = \int_\Sigma & \lambda\epsilon_n \left[F_\omega + \frac{\Lambda}{2} e^2 + i\frac{e}{2} \left(\overline{\psi}\gamma d_\omega \psi - d_\omega\overline{\psi}\gamma\psi \right) \right] \delta e + d_\omega(\lambda\epsilon_n e)\delta\omega \\
& + \frac{i}{4}\lambda\epsilon_n e^2 \Big[\delta\overline{\psi}\gamma d_\omega\psi - \overline{\psi}\gamma d_\omega\delta\psi + d_\omega\delta\overline{\psi}\gamma\psi + d_\omega\overline{\psi}\gamma\delta\psi \\
& + \overline{\psi}\gamma[\delta\omega,\psi] - [\delta\omega,\overline{\psi}]\gamma\psi \Big] \\
= \int_\Sigma & \lambda\epsilon_n \left[F_\omega + \frac{\Lambda}{2} e^2 + i\frac{e}{2} \left(\overline{\psi}\gamma d_\omega \psi - d_\omega\overline{\psi}\gamma\psi \right) \right] \delta e + d_\omega(\lambda\epsilon_n e)\delta\omega \\
& + i\delta\overline{\psi} \left[\lambda\epsilon_n \frac{e^2}{4}\gamma d_\omega\psi - d_\omega \left(\lambda\epsilon_n \frac{e^2}{4}\gamma\psi \right) \right] \\
& + i \left[\lambda\epsilon_n \frac{e^2}{4} d_\omega\overline{\psi}\gamma + d_\omega \left(\lambda\epsilon_n \frac{e^2}{4}\overline{\psi}\gamma \right) \right] \delta\psi \\
& + \frac{i}{16}\lambda\overline{\psi} \left(j_\gamma j_\gamma(\epsilon_n e^2)\gamma - \gamma j_\gamma j_\gamma(\epsilon_n e^2) \right) \psi\delta\omega.
\end{aligned}$$

We are then left with

$$\begin{aligned}
\mathbb{L}_e^{PCD} &= [c,e] & \mathbb{L}_\psi^{PCD} &= [c,\psi] \\
\mathbb{L}_\omega^{PCD} &= d_\omega c + \mathbb{V}_L & \mathbb{L}_{\overline{\psi}}^{PCD} &= [c,\overline{\psi}] \\
\mathbb{P}_e^{PCD} &= -\mathrm{L}_\xi^{\omega_0} e & \mathbb{P}_\psi^{PCD} &= -\mathrm{L}_\xi^{\omega_0}(\psi) \\
\mathbb{P}_\omega^{PCD} &= -\mathrm{L}_\xi^{\omega_0}(\omega-\omega_0) - \iota_\xi F_{\omega_0} + \mathbb{V}_P & \mathbb{P}_{\overline{\psi}}^{PCD} &= -\mathrm{L}_\xi^{\omega_0}(\overline{\psi}).
\end{aligned}$$

$$\begin{aligned}
\mathbb{H}_e^{PCD} &= d_\omega(\lambda\epsilon_n) + \lambda\sigma + \frac{i}{4}\lambda\overline{\psi}\left(j_\gamma\epsilon_n j_\gamma e\gamma - \gamma j_\gamma \epsilon_n j_\gamma e \right)\psi \\
e\mathbb{H}_\omega^{PCD} &= \lambda\epsilon_n \left(F_\omega + \frac{\lambda}{2}e^2 \right) - i\frac{\lambda\epsilon_n}{4} e(\overline{\psi}\gamma d_\omega\psi - d_\omega\overline{\psi}\gamma\psi) \\
\frac{e^3}{3!}\gamma\mathbb{H}_\psi^{PCD} &= \frac{\lambda\epsilon_n}{2}e^2\gamma d_\omega\psi - \frac{\lambda\epsilon_n}{4} e d_\omega e\gamma\psi + \frac{i}{64}\lambda e \left[\overline{\psi}\left(j_\gamma j_\gamma(\epsilon_n e^2)\gamma - \gamma j_\gamma j_\gamma(\epsilon_n e^2) \right)\psi \right]\gamma\psi \\
\frac{e^3}{3!}\mathbb{H}_{\overline{\psi}}^{PCD}\gamma &= \frac{\lambda\epsilon_n}{2}e^2 d_\omega\overline{\psi}\gamma + \frac{\lambda\epsilon_n}{4} e d_\omega e\overline{\psi}\gamma - \frac{i}{64}\lambda e\overline{\psi}\gamma \left[\overline{\psi}\left(j_\gamma j_\gamma(\epsilon_n e^2)\gamma - \gamma j_\gamma j_\gamma(\epsilon_n e^2) \right)\psi \right]
\end{aligned}$$

The Poisson brackets of the constraints are:

$$\begin{aligned}
\{L_c, L_c\} &= \int_\Sigma (\cdots) - \frac{i}{4}e^2\left(-\frac{1}{4}\overline{\psi}j_\gamma j_\gamma c\gamma\psi + \frac{1}{4}\overline{\psi}\gamma j_\gamma j_\gamma c\psi\right)[c,e] \\
&\quad + \frac{i}{3!}e^3[c,\overline{\psi}]\gamma[c,\psi] \\
&= \int_\Sigma (\cdots) + \frac{i}{8\cdot 3!}\overline{\psi}(j_\gamma j_\gamma c\gamma - \gamma j_\gamma j_\gamma c)\psi[c,e^3] \\
&\quad + \frac{i}{16\cdot 3!}e^3\overline{\psi}j_\gamma j_\gamma c\gamma j_\gamma j_\gamma c\psi \\
&\overset{\blacktriangledown}{=} \int_\Sigma (\cdots) - \frac{i}{2\cdot 3!}\left([c,\overline{\psi}]\gamma\psi - \overline{\psi}\gamma[c,\psi]\right)[c,e^3] \\
&\quad + \frac{i}{32}\frac{e^3}{3!}\overline{\psi}\left(-\gamma j_\gamma j_\gamma c j_\gamma j_\gamma c + 4[c,\gamma]j_\gamma j_\gamma c - j_\gamma j_\gamma c j_\gamma j_\gamma c\gamma + 4j_\gamma j_\gamma c[c,\gamma]\right)\psi \\
&= \int_\Sigma (\cdots) + \frac{i}{2\cdot 3!}e^3\left(\overline{\psi}\gamma[c,[c,\psi]] - [c,[c,\overline{\psi}]]\gamma\psi\right) \\
&= \int_\Sigma -\frac{1}{2}[c,c]ed_\omega e + \frac{i}{4\cdot 3!}e^3\left([[c,c],\overline{\psi}]\gamma\psi - \overline{\psi}\gamma[[c,c],\psi]\right) \\
&= -\frac{1}{2}L_{[c,c]},
\end{aligned}$$

where in last few steps we used the graded Jacobi identity to prove

$$[c,[c,\psi] = -\frac{1}{2}[[c,c],\psi]$$

and the fact that

$$\gamma j_\gamma j_\gamma c j_\gamma j_\gamma c = j_\gamma j_\gamma c j_\gamma j_\gamma c\gamma + 4j_\gamma j_\gamma c j_\gamma c + 4j_\gamma c j_\gamma j_\gamma c.$$

$$\begin{aligned}
\{L_c, P_\xi\} &= \int_\Sigma (\cdots) - \frac{i}{2\cdot 3!}e^3\left([c,\overline{\psi}]\gamma \mathrm{L}_\xi^{\omega_0}\psi - \overline{\psi}\gamma \mathrm{L}_\xi^{\omega_0}([c,\psi]) + \mathrm{L}_\xi^{\omega_0}([c,\overline{\psi}])\gamma\psi + \mathrm{L}_\xi^{\omega_0}\overline{\psi}\gamma[c,\psi]\right) \\
&\quad - \frac{i}{2\cdot 3!}[c,e^3]\left(\overline{\psi}\gamma \mathrm{L}_\xi^{\omega_0}\psi - \mathrm{L}_\xi^{\omega_0}\overline{\psi}\gamma\psi\right) \\
&= \int_\Sigma (\cdots) - \frac{i}{2\cdot 3!}e^3\big([c,\overline{\psi}]\gamma \mathrm{L}_\xi^{\omega_0}\psi + \mathrm{L}_\xi^{\omega_0}\overline{\psi}\gamma[c,\psi] - \overline{\psi}\gamma[\mathrm{L}_\xi^{\omega_0}c,\psi] + \overline{\psi}\gamma[c,\mathrm{L}_\xi^{\omega_0}\psi] \\
&\quad - [\mathrm{L}_\xi^{\omega_0}c,\overline{\psi}]\gamma\psi - [c,\mathrm{L}_\xi^{\omega_0}\overline{\psi}]\gamma\psi\big) - \frac{i}{2\cdot 3!}[c,e^3](\overline{\psi}\gamma \mathrm{L}_\xi^{\omega_0}\psi - \mathrm{L}_\xi^{\omega_0}\overline{\psi}\gamma\psi) \\
&= \int_\Sigma (\cdots) - \frac{i}{2\cdot 3!}\big([c,\overline{\psi}]\gamma \mathrm{L}_\xi^{\omega_0}\psi + \mathrm{L}_\xi^{\omega_0}\overline{\psi}\gamma[c,\psi] - \overline{\psi}\gamma[\mathrm{L}_\xi^{\omega_0}c,\psi] + [\mathrm{L}_\xi^{\omega_0}c,\overline{\psi}]\gamma\psi \\
&\quad - [c,\overline{\psi}]\gamma \mathrm{L}_\xi^{\omega_0}\psi - \overline{\psi}[c,\gamma]\mathrm{L}_\xi^{\omega_0}\psi - \mathrm{L}_\xi^{\omega_0}\overline{\psi}[c,\gamma]\psi - \mathrm{L}_\xi^{\omega_0}\overline{\psi}\gamma[c,\psi] \\
&\quad + \overline{\psi}[c,\gamma]\mathrm{L}_\xi^{\omega_0}\psi + \mathrm{L}_\xi^{\omega_0}\overline{\psi}[c,\gamma]\psi\big) \\
&= \int_\Sigma \mathrm{L}_\xi^{\omega_0}ced_\omega e - \frac{i}{2\cdot 3!}e^3\big([\mathrm{L}_\xi^{\omega_0}c,\overline{\psi}]\gamma\psi - \overline{\psi}\gamma[\mathrm{L}_\xi^{\omega_0}c,\psi]\big) \\
&= L_{\mathrm{L}_\xi^{\omega_0}c},
\end{aligned}$$

where in the second to last passage we used that

$$\begin{aligned}
\overline{\psi}\gamma[c,\mathrm{L}_\xi^{\omega_0}\psi] &= -[c,\overline{\psi}]\gamma \mathrm{L}_\xi^{\omega_0}\psi - \overline{\psi}[c,\gamma]\mathrm{L}_\xi^{\omega_0}\psi, \\
[c,\mathrm{L}_\xi^{\omega_0}\overline{\psi}]\gamma\psi &= \mathrm{L}_\xi^{\omega_0}\overline{\psi}[c,\gamma]\psi + \mathrm{L}_\xi^{\omega_0}\overline{\psi}\gamma[c,\psi].
\end{aligned}$$

$$\begin{aligned}\{L_c, H_\lambda\} = \mathbb{L}_c(H_\lambda) = \int_\Sigma (\cdots) + \lambda\epsilon_n \Big\{ &\frac{i}{4}[c, e^2]\big(\overline{\psi}\gamma d_\omega \psi - d_\omega \overline{\psi}\gamma\psi\big) + \frac{i}{4} e^2 \big([c, \overline{\psi}]\gamma d_\omega \psi \\ &- \overline{\psi}\gamma d_\omega [c, \psi] + d_\omega([c, \overline{\psi}])\gamma\psi + d_\omega \overline{\psi}\gamma[c, \psi] \\ &+ \overline{\psi}\gamma[d_\omega c, \psi] - [d_\omega c, \overline{\psi}]\gamma\psi\big)\Big\} \\ \overset{\triangledown}{=} \int_\Sigma (\cdots) - [c, \lambda\epsilon_n]\frac{i}{4} e^2 \big(\overline{\psi}\gamma d_\omega\psi - d_\omega\overline{\psi}\gamma\psi\big) - i\frac{\lambda\epsilon_n}{4} e^2 \Big\{ &- \overline{\psi}[c, \gamma] d_\omega\psi - d_\omega\overline{\psi}[c, \gamma]\psi \\ &+ \overline{\psi}[c, \gamma d_\omega\psi - \overline{\psi}\gamma[c, d_\omega\psi] + [d_\omega c, \overline{\psi}]\gamma\psi - [c, d_\omega\overline{\psi}]\gamma\psi + [c, d_\omega\overline{\psi}]\gamma\psi \\ &+ d_\omega\overline{\psi}[c, \gamma]\psi + \overline{\psi}\gamma[d_\omega c, \psi] - [d_\omega c, \overline{\psi}]\gamma\psi - \overline{\psi}\gamma[d_\omega c, \psi] + \overline{\psi}\gamma[c, d_\omega\psi]\Big\} \\ = \int_\Sigma -[c, \lambda\epsilon_n]\big(eF_\omega + \frac{\Lambda}{2}e^2 + \frac{i}{4}e^2(\overline{\psi}\gamma d_\omega\psi - d_\omega\overline{\psi}\gamma\psi)\big) \\ = -P_{[c,\lambda\epsilon_n]^{(a)}} - H_{[c,\lambda\epsilon_n]^{(a)}} + L_{[c,\lambda\epsilon_n]^{(a)}(\omega-\omega_0)_{(a)}}\end{aligned}$$

having used the following identities, which can be easily found

$$\begin{aligned} d_\omega\overline{\psi}\gamma[c, \psi] &= [c, d_\omega\overline{\psi}]\gamma\psi + d_\omega\overline{\psi}[c, \gamma]\psi, \\ [c, \overline{\psi}]\gamma d_\omega\psi &= \overline{\psi}[c, \gamma] d_\omega\psi - \overline{\psi}\gamma[c, d_\omega\psi]. \end{aligned} \tag{$\triangledown$}$$

$$\begin{aligned}\{P_\xi, P_\xi\} = \int_\Sigma (\cdots) + \frac{i}{2\cdot 3!}\mathrm{L}_\xi^{\omega_0}(e^3)\big(\overline{\psi}\gamma\mathrm{L}_\xi^{\omega_0}\psi - \mathrm{L}_\xi^{\omega_0}\overline{\psi}\gamma\psi\big) - \frac{i}{2\cdot 3!}e^3\big\{ &- \mathrm{L}_\xi^{\omega_0}\overline{\psi}\gamma\mathrm{L}_\xi^{\omega_0}\psi \\ &+ \overline{\psi}\gamma\mathrm{L}_\xi^{\omega_0}\mathrm{L}_\xi^{\omega_0}\psi - \mathrm{L}_\xi^{\omega_0}\mathrm{L}_\xi^{\omega_0}\overline{\psi}\gamma\psi - \mathrm{L}_\xi^{\omega_0}\overline{\psi}\gamma\mathrm{L}_\xi^{\omega_0}\psi\big\} \\ = \int_\Sigma (\cdots) - \frac{i}{2\cdot 3!}e^3\big\{&\mathrm{L}_\xi^{\omega_0}\overline{\psi}\gamma\psi + \overline{\psi}\gamma\mathrm{L}_\xi^{\omega_0}\mathrm{L}_\xi^{\omega_0}\psi - \mathrm{L}_\xi^{\omega_0}\mathrm{L}_\xi^{\omega_0}\overline{\psi}\gamma\psi + \mathrm{L}_\xi^{\omega_0}\overline{\psi}\gamma\mathrm{L}_\xi^{\omega_0}\psi \\ &- \mathrm{L}_\xi^{\omega_0}\overline{\psi}\gamma\mathrm{L}_\xi^{\omega_0}\psi + \overline{\psi}\gamma\mathrm{L}_\xi^{\omega_0}\mathrm{L}_\xi^{\omega_0}\psi - \mathrm{L}_\xi^{\omega_0}\mathrm{L}_\xi^{\omega_0}\overline{\psi}\gamma\psi - \mathrm{L}_\xi^{\omega_0}\overline{\psi}\gamma\mathrm{L}_\xi^{\omega_0}\psi\big\} \\ = \int_\Sigma (\cdots) - \frac{i}{3!}e^3\big(&\overline{\psi}\gamma\mathrm{L}_\xi^{\omega_0}\mathrm{L}_\xi^{\omega_0}\psi - \mathrm{L}_\xi^{\omega_0}\mathrm{L}_\xi^{\omega_0}\overline{\psi}\gamma\psi\big) \\ \overset{\clubsuit}{=} \int_\Sigma (\cdots) - \frac{i}{2\cdot 3!}e^3\big(&\overline{\psi}\gamma\mathrm{L}_{[\xi,\xi]}^{\omega_0}\psi - \mathrm{L}_{[\xi,\xi]}^{\omega_0}\overline{\psi}\gamma\psi\big) \\ &+ \frac{i}{2\cdot 3!}e^3\big([\iota_\xi\iota_\xi F_{\omega_0}, \overline{\psi}]\gamma\psi - \overline{\psi}\gamma[\iota_\xi\iota_\xi F_{\omega_0}, \psi]\big) \\ = \frac{1}{2}P_{[\xi,\xi]} - \frac{1}{2}L\iota_\xi\iota_\xi F_{\omega_0};\end{aligned}$$

$$\{P_\xi, H_\lambda\} = \int_\Sigma (\cdots) + \lambda\epsilon_n\Big\{ -\frac{i}{4}\mathrm{L}_\xi^{\omega_0}(e^2)(\overline{\psi}\gamma d_\omega\psi - d_\omega\overline{\psi}\gamma\psi + \frac{i}{4}e^2\big[-\mathrm{L}_\xi^{\omega_0}\overline{\psi}\gamma d_\omega\psi$$
$$+ \overline{\psi}\gamma d_\omega \mathrm{L}_\xi^{\omega_0}\psi - d_\omega\mathrm{L}_\xi^{\omega_0}\overline{\psi}\gamma\psi - d_\omega\overline{\psi}\gamma\mathrm{L}_\xi^{\omega_0}\psi$$
$$- \overline{\psi}\gamma[\iota_\xi F_{\omega_0} + \mathrm{L}_\xi^{\omega_0}(\omega-\omega_0), \psi] + [\iota_\xi F_{\omega_0} + \mathrm{L}_\xi^{\omega_0}(\omega-\omega_0), \overline{\psi}]\gamma\psi\big]\Big\}$$
$$= \int_\Sigma (\cdots) + i\mathrm{L}_\xi^{\omega_0}(\lambda\epsilon_n)\frac{e^2}{4}\big(\overline{\psi}\gamma d_\omega\psi - d_\omega\overline{\psi}\gamma\psi\big) + i\frac{\lambda\epsilon_n}{4}e^2\big\{\overline{\psi}\gamma\mathrm{L}_\xi^{\omega_0}d_\omega\psi$$
$$- \mathrm{L}_\xi^{\omega_0}d_\omega\overline{\psi}\gamma\psi + \overline{\psi}\gamma d_\omega\mathrm{L}_\xi^{\omega_0}\psi - d_\omega\mathrm{L}_\xi^{\omega_0}\overline{\psi}\gamma\psi$$
$$- \overline{\psi}\gamma[\iota_\xi F_{\omega_0} + \mathrm{L}_\xi^{\omega_0}(\omega-\omega_0), \psi] + [\iota_\xi F_{\omega_0} + \mathrm{L}_\xi^{\omega_0}(\omega-\omega_0), \overline{\psi}]\gamma\psi\big\}$$
$$\overset{\blacklozenge}{=} \int_\Sigma (\cdots) + i\mathrm{L}_\xi^{\omega_0}(\lambda\epsilon_n)\frac{e^2}{4}\big(\overline{\psi}\gamma d_\omega\psi - d_\omega\overline{\psi}\gamma\psi\big) + i\frac{\lambda\epsilon_n}{4}e^2\big\{\overline{\psi}\gamma[\mathrm{L}_\xi^{\omega_0}\omega,\psi] - [\mathrm{L}_\xi^{\omega_0}\omega,\overline{\psi}]\gamma\psi$$
$$- \overline{\psi}\gamma[\iota_\xi F_{\omega_0} + \mathrm{L}_\xi^{\omega_0}(\omega-\omega_0), \psi] + [\iota_\xi F_{\omega_0} + \mathrm{L}_\xi^{\omega_0}(\omega-\omega_0), \overline{\psi}]\gamma\psi\big\}$$
$$= \int_\Sigma (\cdots) + i\mathrm{L}_\xi^{\omega_0}(\lambda\epsilon_n)\frac{e^2}{4}\big(\overline{\psi}\gamma d_\omega\psi - d_\omega\overline{\psi}\gamma\psi\big) + i\frac{\lambda\epsilon_n}{4}e^2\big\{\overline{\psi}\gamma[\mathrm{L}_\xi^{\omega_0}\omega_0 - \iota_\xi F_{\omega_0},\psi]$$
$$- [\mathrm{L}_\xi^{\omega_0}\omega_0 - \iota_\xi F_{\omega_0},\overline{\psi}]\gamma\psi\big\}$$
$$= \int_\Sigma \mathrm{L}_\xi^{\omega_0}(\lambda\epsilon_n)\left(eF_\omega + \frac{\Lambda}{2}e^2 + \frac{i}{4}e^2\big(\overline{\psi}\gamma d_\omega\psi - d\omega\overline{\psi}\gamma\psi\big)\right)$$
$$= P_{\mathrm{L}_\xi^{\omega_0}(\lambda\epsilon_n)^{(a)}} + H_{\mathrm{L}_\xi^{\omega_0}(\lambda\epsilon_n)^{(a)}} - L_{\mathrm{L}_\xi^{\omega_0}(\lambda\epsilon_n)^{(a)}(\omega-\omega_0)},$$

where we used that $\mathrm{L}_\xi^{\omega_0}\omega_0 - \iota_\xi F_{\omega_0} = -d\iota_\xi\omega_0$ and the following identity:

$$\mathrm{L}_\xi^{\omega_0}d_\omega\psi = -d_\omega\mathrm{L}_\xi^{\omega_0}\psi + [\mathrm{L}_\xi^{\omega_0}\omega,\psi]. \qquad (\blacklozenge)$$

Furthermore, recalling that $d_{\omega_0}\gamma = 0$, it is quite easy to see that

$$\overline{\psi}\gamma[d\iota_\xi\omega_0,\psi] - [d\iota_\xi\omega_0,\overline{\psi}]\gamma\psi = -[d\iota_\xi\omega_0,\overline{\psi}\gamma\psi] = 0.$$

Now, before computing $\{H_\lambda, H_\lambda\}$, we first notice that the Hamiltonian vector field associated to H_λ can be rewritten as

$$e\gamma\mathbb{H}_\psi = 3\lambda\epsilon_n\gamma d_\omega\psi - \frac{3}{2}\lambda\sigma\gamma\psi + \frac{3i}{8}\lambda\beta$$
$$e\mathbb{H}_{\overline{\psi}}\gamma = 3\lambda\epsilon_n d_\omega\overline{\psi}\gamma + \frac{3}{2}\overline{\psi}\gamma\lambda\sigma - \frac{3}{8}i\lambda\overline{\beta},$$

with $\beta := \overline{\psi}\big(j_\gamma\epsilon_n j_\gamma e\gamma - \gamma j_\gamma\epsilon_n j_\gamma e\big)\gamma\psi$, hence

$$\{H_\lambda, H_\lambda\} = \int_\Sigma i\left[\frac{\lambda\epsilon_n}{2}\mathbb{H}_{\overline{\psi}}e^2\gamma d_\omega\psi - \frac{1}{4}d_\omega(\lambda\epsilon_n)e^2\mathbb{H}_{\overline{\psi}}\gamma\psi - \frac{\lambda\epsilon_n}{2}d_\omega ee\mathbb{H}_{\overline{\psi}}\gamma\psi\right]$$
$$+ i\left[\left(\frac{\lambda\epsilon_n}{2}d_\omega\overline{\psi} + \frac{1}{4}d_\omega(\lambda\epsilon_n)\overline{\psi}\right)e^2\gamma\mathbb{H}_\psi + \frac{\lambda\epsilon_n}{2}d_\omega e\overline{\psi}e\gamma\mathbb{H}_\psi\right]$$
$$= \int_\Sigma \frac{3}{4\cdot 32}d_\omega(\lambda\epsilon_n)\lambda\overline{\psi}\gamma\psi\big[\overline{\psi}\big(j_\gamma j_\gamma(\epsilon_n e^2)\gamma - \gamma j_\gamma j_\gamma(\epsilon_n e^2)\big)\psi\big]$$
$$- \frac{3}{4\cdot 32}d_\omega(\lambda\epsilon_n)\lambda\overline{\psi}\gamma\psi\big[\overline{\psi}\big(j_\gamma j_\gamma(\epsilon_n e^2)\gamma - \gamma j_\gamma j_\gamma(\epsilon_n e^2)\big)\psi\big] = 0,$$

where all the remaining terms vanish because they are either proportional to $\lambda^2 = 0$ or $\epsilon_n^2 = 0$. □

B.1.2 Theorem 3.4

Proof of 3.4. First, we notice that the contraction of the symplectic form with a vector field $\mathbb{X} \in \mathfrak{X}(\mathcal{F}_\Sigma^{PCD})$ is given by

$$\iota_{\mathbb{X}}\varpi = \int_\Sigma e\mathbb{X}_e\delta\omega + \left[e\mathbb{X}_\omega + \frac{i}{4}e^2(\overline{\psi}\gamma\mathbb{X}_\psi - \mathbb{X}_{\overline{\psi}}\gamma\psi)\right]\delta e \\ + i\delta\overline{\psi}\left(-\frac{e^2}{4}\gamma\psi\mathbb{X}_e + \frac{e^3}{3!}\gamma\mathbb{X}_\psi\right) + i\left(\frac{e^2}{4}\overline{\psi}\gamma\mathbb{X}_e + \frac{e^3}{3!}\mathbb{X}_{\overline{\psi}}\gamma\right)\delta\psi. \tag{B.3}$$

Then, we start giving the Hamiltonian vector fields of the constraints. For L_c^{PCD} and P_ξ^{PCD}, from the non–degenerate case, we have

$$\begin{aligned} \mathbb{L}_e^{PCD} &= [c,e] & \mathbb{L}_\psi^{PCD} &= [c,\psi] \\ \mathbb{L}_\omega^{PCD} &= d_\omega c & \mathbb{L}_{\overline{\psi}}^{PCD} &= [c,\overline{\psi}] \\ \mathbb{P}_e^{PCD} &= -\mathcal{L}_\xi^{\omega_0} e & \mathbb{P}_\psi^{PCD} &= -\mathcal{L}_\xi^{\omega_0}(\psi) \\ \mathbb{P}_\omega^{PCD} &= -\mathcal{L}_\xi^{\omega_0}(\omega-\omega_0) - \iota_\xi F_{\omega_0} & \mathbb{P}_{\overline{\psi}}^{PCD} &= -\mathcal{L}_\xi^{\omega_0}(\overline{\psi}). \end{aligned}$$

Whereas, for H_λ^ψ, we have

$$\begin{aligned} \mathbb{H}_e^\psi &= d_\omega(\lambda\epsilon_n) + \lambda\sigma + \frac{i}{4}\lambda\overline{\psi}\left(\iota_\gamma\iota_\gamma\epsilon_n e\gamma - \gamma\iota_\gamma\iota_\gamma\epsilon_n e\right)\psi \\ e\mathbb{H}_\omega^\psi &= \lambda\epsilon_n\left(F_\omega + \frac{\Lambda}{2}e^2\right) - i\frac{\lambda\epsilon_n}{4}e(\overline{\psi}\gamma d_\omega\psi - d_\omega\overline{\psi}\gamma\psi) \\ \frac{e^3}{3!}\gamma\mathbb{H}_\psi^\psi &= \frac{\lambda\epsilon_n}{2}e^2\gamma d_\omega\psi - \frac{\lambda\epsilon_n}{4}ed_\omega e\gamma\psi \\ &\quad + \frac{i}{64}\lambda e\left[\overline{\psi}\left(\iota_\gamma\iota_\gamma(\epsilon_n e^2)\gamma - \gamma\iota_\gamma\iota_\gamma(\epsilon_n e^2)\right)\psi\right]\gamma\psi \\ \frac{e^3}{3!}\mathbb{H}_{\overline{\psi}}^\psi\gamma &= \frac{\lambda\epsilon_n}{2}e^2 d_\omega\overline{\psi}\gamma + \frac{\lambda\epsilon_n}{4}ed_\omega e\overline{\psi}\gamma \\ &\quad - \frac{i}{64}\lambda e\overline{\psi}\gamma\left[\overline{\psi}\left(\iota_\gamma\iota_\gamma(\epsilon_n e^2)\gamma - \gamma\iota_\gamma\iota_\gamma(\epsilon_n e^2)\right)\psi\right], \end{aligned}$$

where $\sigma \in \Omega_\Sigma^{1,1}$. Lastly, the Hamiltonian vector fields of R_τ^{PCD}, are given by

$$\begin{aligned} e\mathbb{R}_e^{PCD} &= [\tau,e] \\ e\mathbb{R}_\omega^{PCD} &= \frac{\delta\tau}{\delta e}d_\omega e + d_\omega\tau \\ \mathbb{R}_\psi^{PCD} &= \mathbb{R}_{\overline{\psi}}^{PCD} = 0, \end{aligned}$$

since they coincide with the ones of the Palatini–Cartan theory of 2.4. Notice that, instead of using the function $g = g(\tau,e,\omega)$, we preferred expressing the variation of τ with respect to e by means of the functional derivative $\frac{\delta\tau}{\delta e}$. However, we have the relation

$$g(\tau,e,\omega) = \frac{\delta\tau}{\delta e}d_\omega e.$$

Now, we are ready to compute the Poisson brackets of the constraints. From the non–degenerate case, we have already knowledge of the following Poisson brackets

$$\{P_\xi^{PCD}, P_\xi^{PCD}\} = \frac{1}{2}P_{[\xi,\xi]}^{PCD} - \frac{1}{2}L_{\iota_\xi\iota_\xi F_{\omega_0}}^{PCD} \qquad \{H_\lambda^\psi, H_\lambda^\psi\} = 0$$

$$\{L_c^{PCD}, P_\xi^{PCD}\} = L_{\mathcal{L}_\xi^{\omega_0}c}^{PCD} \qquad \{L_c^{PCD}, L_c^{PCD}\} = -\frac{1}{2}L_{[c,c]}^{PCD}$$

$$\{L_c^{PCD}, H_\lambda^\psi\} = -P_{X^{(a)}}^{PCD} + L_{X^{(a)}(\omega-\omega_0)_a}^{PCD} - H_{X^{(n)}}^\psi$$

$$\{P_\xi^{PCD}, H_\lambda^\psi\} = P_{Y^{(a)}}^{PCD} - L_{Y^{(a)}(\omega-\omega_0)_a}^{PCD} + H_{Y^{(n)}}^\psi,$$

with $X = [c, \lambda\epsilon_n]$ and $Y = \mathcal{L}_\xi^{\omega_0}(\lambda\epsilon_n)$ as above. Therefore, we are left with computing the remaining constraints. First, we notice that

$$\{R_\tau^{PCD}, L_c^{PCD}\} = \{R_\tau, L_c\} = -R_{p_\mathcal{S}[c,\tau]} = -R_{p_\mathcal{S}[c,\tau]}^{PCD}.$$

Similarly, we can also compute the bracket

$$\{R_\tau^{PCD}, P_\xi^{PCD}\} = \{R_\tau, P_\xi\} = R_{p_\mathcal{S}\mathcal{L}_\xi^{\omega_0}\tau} = R_{p_\mathcal{S}\mathcal{L}_\xi^{\omega_0}\tau}^{PCD}.$$

Now, we move on to compute the brackets $\{R_\tau^{PCD}, R_\tau^{PCD}\}$ and $\{R_\tau^{PCD}, H_\lambda^{PCD}\}$. The first bracket is simply given by

$$\{R_\tau^{PCD}, R_\tau^{PCD}\} = \{R_\tau, R_\tau\} \approx F_{\tau\tau}$$

with $F_{\tau\tau}$ defined in Theorem 30 of [CCT21] and which is in general non-vanishing on the constraint submanifold. Whereas, for the second one, we obtain

$$\begin{aligned}
\{R_\tau^{PCD}, H_\lambda^\psi\} = \int_\Sigma \Big(\epsilon_n \frac{\delta\beta}{\delta e} d_\omega e + d_\omega(\epsilon_n\beta)\Big)\Big(& d_\omega(\lambda\epsilon_n) + \lambda\sigma \\
& - i\lambda(\overline{\psi}\gamma[\epsilon_n e, \psi] - [\epsilon_n e, \overline{\psi}]\gamma\psi)\Big) \\
& + W_1^{-1}[\epsilon_n\beta, e]\big(\lambda\epsilon_n(F_\omega + \frac{\Lambda}{2}e^2) \\
& - \frac{i}{4}\lambda\epsilon_n e(\overline{\psi}\gamma d_\omega\psi - d_\omega\overline{\psi}\gamma\psi)\big) \\
\approx \int_\Sigma & -i\lambda\beta d_\omega\epsilon_n(\overline{\psi}\gamma[\epsilon_n e, \psi] - [\epsilon_n e, \overline{\psi}]\gamma\psi) \\
& - \frac{i}{4}[\epsilon_n\beta, e]\lambda\epsilon_n(\overline{\psi}\gamma d_\omega\psi - d_\omega\overline{\psi}\gamma\psi) \\
& + G_{\lambda\tau},
\end{aligned}$$

where, in the last passage, we used 3.2 and the fact that $\epsilon_n^2 = 0$. Moreover, the quantity $G_{\lambda\tau}$ is defined in Theorem 30 of [CCT21]. Now, we can notice that, thanks to 3.2, we can write

$$\begin{aligned}
& \lambda\beta d_\omega\epsilon_n(\overline{\psi}\gamma[\epsilon_n e, \psi] - [\epsilon_n e, \overline{\psi}]\gamma\psi) = \\
& = \lambda\epsilon_n e d_\omega\epsilon_n(\overline{\psi}\gamma[\beta, \psi] - [\beta, \overline{\psi}]\gamma\psi) \\
& = \lambda e\beta(\overline{\psi}\gamma[\epsilon_n d_\omega\epsilon_n, \psi] - [\epsilon_n d_\omega\epsilon_n, \overline{\psi}]\gamma\psi) \\
& = 0,
\end{aligned}$$

obtaining

$$\{R_\tau^{PCD}, H_\lambda^\psi\} \approx G_{\lambda\tau} - \int_\Sigma \frac{i}{4}[\epsilon_n\beta, e]\lambda\epsilon_n(\overline{\psi}\gamma d_\omega\psi - d_\omega\overline{\psi}\gamma\psi).$$

Finally, we can write the integral as

$$\{R_\tau^{PCD}, H_\lambda^\psi\} \approx G_{\lambda\tau} - \int_\Sigma \frac{i}{4}\lambda\tau[\epsilon_n, \hat{e}](\overline{\psi}\gamma d_\omega\psi - d_\omega\overline{\psi}\gamma\psi),$$

where we implemented again 3.1 and also the relation[1]

$$\epsilon_n[\tau, e] = \tau[\epsilon_n, \hat{e}] \tag{B.4}$$

with $\hat{e}$ defined as $\hat{e} \coloneqq e - \tilde{e}$ (see 2.18). More specifically, using the definition of the independent components of τ, we have

$$\{R_\tau^{PCD}, H_\lambda^\psi\} \approx G_{\lambda\tau} + K_{\lambda\tau}^{PCD},$$

with

$$K_{\lambda\tau}^{PCD} \coloneqq - \int_\Sigma i\lambda\Big(\sum_{\mu=1}^{2} \mathcal{Y}_\mu \big(\hat{g}_n d_\omega J_\psi\big)_\mu^{3\mu} + \sum_{\mu_1\neq\mu_2=1}^{2} \mathcal{X}_{\mu_1}^{\mu_2} \big(\hat{g}_n d_\omega J_\psi\big)_{3\mu_2}^{\mu_1}\Big),$$

where $\hat{g}_n \coloneqq [\epsilon_n, \hat{e}] \in \Omega_\Sigma^{1,0}$ and $d_\omega J_\psi \coloneqq d_\omega(\overline{\psi}\gamma\psi) \in \Omega_\Sigma^{1,1}$.

This final result completes the proof. □

B.1.3 Theorem 3.7

Proof of 3.7. We start by noticing that on $M = I \times \Sigma$ one has the following splitting

$$\begin{aligned}
\boldsymbol{e} &= \tilde{e} + \underline{\tilde{e}}_n & \boldsymbol{\omega} &= \tilde{\omega} + \underline{\tilde{\omega}}_n \\
\boldsymbol{e}^\dashv &= \tilde{e}^\dashv + \underline{\tilde{e}}_n^\dashv & \boldsymbol{\omega}^\dashv &= \tilde{\omega}^\dashv + \underline{\tilde{\omega}}_n^\dashv \\
\boldsymbol{\psi} &= \tilde{\psi} & \boldsymbol{\psi}^\dashv &= \underline{\tilde{\psi}}_{\dashv,n} \\
\boldsymbol{c} &= \tilde{c} & \boldsymbol{c}^\dashv &= \underline{\tilde{c}}_n^\dashv \\
\boldsymbol{\xi} &= \tilde{\xi} + \overline{\tilde{\xi}}_n & \boldsymbol{\xi}^\dashv &= \underline{\tilde{\xi}}^\dashv + \underline{\tilde{\xi}}_n^\dashv
\end{aligned}$$

If $\tilde{\phi} \in \mathcal{C}^\infty(I) \otimes \Omega^k(\Sigma)$ denotes the field in $\mathcal{F}_{PCD}(I \times \Sigma)$, we denote by ϕ the corresponding field in the BFV space of fields $\mathcal{F}_{PCD}^\Sigma$.

Remark B.1. Since by hypothesis we have that $\{\tilde{e}_i, \epsilon_n\}$ form a basis of $\mathcal{V}$, which is equivalent to asking that $\tilde{e}$ define a non-degenerate metric on the boundary, we can decompose $\underline{\tilde{e}}_n = \iota_{\underline{z}}\tilde{e} + \underline{\mu}\epsilon_n$, where $z^i \coloneqq \tilde{e}_n^i$ and $\mu \coloneqq \tilde{e}_n^n$. Notice also that $\mu \neq 0$ necessarily, otherwise $\boldsymbol{e}$ would not define a non–degenerate metric on M, which is required by assumption.

[1] It simply comes from the definition of $\mathcal{S}$.

We define φ on the above fields by

$$
\begin{aligned}
&\varphi^*(\tilde{e}) = e + \lambda\mu^{-1} f^{\lrcorner} && \varphi^*(\underline{\tilde{e}}_n) = \mu\epsilon_n + \iota_{\underline{z}} e + \lambda\epsilon_n^i \underline{f}_i^{\lrcorner}\\
&\varphi^*(\tilde{\omega}) = \omega - \lambda\mu^{-1} u^{\lrcorner} && \varphi^*(\underline{\tilde{\omega}}_n) = \underline{w} - \iota_\xi \underline{u}^{\lrcorner} - \lambda\epsilon_n^i \underline{u}_i\\
&\varphi^*(\tilde{\psi}) = \psi + \lambda\mu^{-1}\theta^{\lrcorner} && \varphi^*(\underline{\tilde{\psi}}_{\lrcorner,n}) = \frac{e^3}{3!}\gamma\underline{\theta}^{\lrcorner} - \frac{1}{2}e^2 \underline{f}^{\lrcorner}\gamma\psi + \frac{1}{2}\lambda\mu^{-1}e^2\underline{f}\gamma\theta^{\lrcorner}\\
&\varphi^*(\tilde{c}) = c - \iota_\xi(\lambda\mu^{-1}u^{\lrcorner}) && \varphi^*(\underline{\tilde{c}}_n^{\lrcorner}) = \underline{c}^{\lrcorner}\\
&\varphi^*(\tilde{\omega}^{\lrcorner}) = k^{\lrcorner} && \varphi^*(\underline{\tilde{\omega}}_n^{\lrcorner}) = e\underline{f}^{\lrcorner} + \iota_{\underline{z}} k^{\lrcorner} + \iota_\xi \underline{c}^{\lrcorner}\\
&\varphi^*(\tilde{\xi}^i) = \xi^i - \lambda\mu^{-1} z^i && \varphi^*(\underline{\tilde{\xi}}^{\lrcorner}) = e\underline{y}^{\lrcorner} + \underline{f}^{\lrcorner} e^{\lrcorner} - \underline{u}^{\lrcorner} k^{\lrcorner} + \underline{c}^{\lrcorner}\lambda\mu^{-1}u^{\lrcorner}\\
&\varphi^*(\tilde{e}^{\lrcorner}) = e^{\lrcorner} - \lambda\mu^{-1} y_n^{\lrcorner} && \varphi^*(\tilde{\xi}^n) = \lambda\mu^{-1}
\end{aligned}
$$

$$\varphi^*(\underline{\tilde{e}}_n^{\lrcorner}) = e\underline{u}^{\lrcorner} + \iota_{\underline{z}} e^{\lrcorner} - \lambda\epsilon_n^i y_i^{\lrcorner} + \lambda\mu^{-1} f^{\lrcorner}\underline{u}^{\lrcorner} - \frac{i}{4}e^2\underline{\bar{\theta}}^{\lrcorner}\gamma\psi - \frac{i}{4}e^2\lambda\mu^{-1}\underline{\bar{\theta}}^{\lrcorner}\gamma\theta^{\lrcorner} - \frac{i}{2}\lambda\mu^{-1}ef^{\lrcorner}\underline{\bar{\theta}}^{\lrcorner}\gamma\psi + \text{c.c.}$$

$$\varphi^*(\underline{\tilde{\xi}}_n^{\lrcorner}) = (\mu\epsilon_n + \iota_z e)\underline{y}^{\lrcorner} + ef^{\lrcorner}\underline{u}^{\lrcorner} + f^{\lrcorner}\iota_{\underline{z}} e^{\lrcorner} + u^{\lrcorner}\iota_{\underline{z}} k^{\lrcorner} + c^{\lrcorner}\lambda\epsilon_n^i \underline{u}_i^{\lrcorner} - \frac{i}{2\cdot 3!}e^3\underline{\bar{\theta}}^{\lrcorner}\gamma\theta^{\lrcorner} - \frac{i}{4}e^2 f^{\lrcorner}\underline{\bar{\theta}}^{\lrcorner}\gamma\psi + \text{c.c.}$$

We remark that, after setting $\psi = 0$ and $\theta^{\lrcorner} = 0$, the φ defined above coincides with the one defined in the PC AKSZ theory in [CCS21b] appearing in 2.13. This implies that, if one splits $\varpi_{PCD}^M = \varpi_{PC}^M + \varpi_D^M$ and $\mathcal{S}_{PCD}^M = \mathcal{S}_{PC}^M + \mathcal{S}_D^M$, by theorem 2.8 one has

$$
\begin{aligned}
&\varphi^*(\varpi_{PC}^M) = \\
&= \varpi_{PC}^{AKSZ} + \int_{I\times\Sigma} - \underline{\frac{i}{2\cdot 3!}\delta(e^3)\left(\delta\underline{\theta}^{\lrcorner}\gamma\psi + \underline{\theta}^{\lrcorner}\gamma\delta\psi\right)}_{1} - \underline{\frac{i}{4}\delta(\lambda\mu^{-1})f^{\lrcorner}\delta(e^2)\underline{\theta}^{\lrcorner}\gamma\psi}_{2}\\
&\quad - \frac{i}{4}\delta(\lambda\mu^{-1})f^{\lrcorner}e^2\left(\underline{\delta\underline{\theta}^{\lrcorner}\gamma\psi}_{3} + \underline{\underline{\theta}^{\lrcorner}\gamma\delta\psi}_{4}\right) + \underline{\frac{i}{4}\lambda\mu^{-1}\delta f^{\lrcorner}\delta(e^2)\underline{\theta}^{\lrcorner}\gamma\psi}_{5}\\
&\quad + \frac{i}{4}\lambda\mu^{-1}\delta f^{\lrcorner}e^2\left(\underline{\delta\underline{\theta}^{\lrcorner}\gamma\psi}_{6} + \underline{\underline{\theta}^{\lrcorner}\gamma\delta\psi}_{7}\right) + \underline{\frac{i}{2\cdot 3!}\delta(e^3)\delta(\lambda\mu^{-1})\underline{\theta}^{\lrcorner}\gamma\theta^{\lrcorner}}_{8}\\
&\quad - \underline{\frac{i}{3!}\delta(e^3)\lambda\mu^{-1}\delta\underline{\theta}^{\lrcorner}\gamma\theta^{\lrcorner}}_{9} + \underline{\frac{i}{4}\delta(\lambda\mu^{-1})f^{\lrcorner}\delta(\lambda\mu^{-1})e^2\underline{\theta}^{\lrcorner}\gamma\theta^{\lrcorner}}_{10}\\
&\quad - \frac{i}{2}\delta(\lambda\mu^{-1})f^{\lrcorner}\left[\underline{\frac{1}{2}\lambda\mu^{-1}\delta(e^2)\underline{\theta}^{\lrcorner}\gamma\theta^{\lrcorner}}_{11} + \underline{\lambda\mu^{-1}e^2\delta\underline{\theta}^{\lrcorner}\gamma\theta^{\lrcorner}}_{12}\right] \qquad\qquad \text{(B.5)}\\
&\quad - \underline{\frac{i}{4}\lambda\mu^{-1}\delta f^{\lrcorner}\delta(\lambda\mu^{-1})e^2\underline{\theta}^{\lrcorner}\gamma\theta^{\lrcorner}}_{13} + \underline{\frac{i}{2\cdot 3!}\delta(\lambda\mu^{-1})\delta(e^3)\underline{\theta}^{\lrcorner}\gamma\theta^{\lrcorner}}_{14}\\
&\quad + \underline{\frac{i}{3!}\delta(\lambda\mu^{-1})e^3\delta\underline{\theta}^{\lrcorner}\gamma\theta^{\lrcorner}}_{15} - \frac{i}{4}\delta(\lambda\mu^{-1})\left(\underline{\delta(e^2)f^{\lrcorner}\underline{\theta}^{\lrcorner}\gamma\psi}_{16} + \underline{e^2\delta f^{\lrcorner}\underline{\theta}^{\lrcorner}\gamma\psi}_{17}\right)\\
&\quad + \frac{i}{4}\delta(\lambda\mu^{-1})\left(\underline{e^2 f^{\lrcorner}\delta\underline{\theta}^{\lrcorner}\gamma\psi}_{18} + \underline{e^2 f^{\lrcorner}\underline{\theta}^{\lrcorner}\gamma\delta\psi}_{19}\right)\\
&\quad - \frac{i}{4}\delta(e^2)\left[\underline{\delta(\lambda\mu^{-1})f^{\lrcorner}\underline{\bar{\theta}}^{\lrcorner}\gamma\psi}_{20} - \underline{(\lambda\mu^{-1})\delta f^{\lrcorner}\underline{\bar{\theta}}^{\lrcorner}\gamma\psi}_{21}\right]\\
&\quad - \frac{i}{4}\delta(e^2)\left[\underline{\lambda\mu^{-1}f^{\lrcorner}\delta\underline{\bar{\theta}}^{\lrcorner}\gamma\psi}_{22} + \underline{(\lambda\mu^{-1})f^{\lrcorner}\underline{\bar{\theta}}^{\lrcorner}\gamma\delta\psi}_{23}\right] + \text{c.c.}
\end{aligned}
$$

where "c.c." encapsulates all the complex conjugates of the above terms except the real ones (e.g.

$i\delta\theta\gamma\delta\theta)$. One also finds

$$
\begin{aligned}
\varphi^*(\varpi_D^M) &= \int_{I\times\Sigma} \frac{i}{2}\delta\varphi^*(\underline{\tilde{\bar{\psi}}}^{\lrcorner})\delta(\varphi^*\tilde{\psi}) \\
&= \int_{I\times\Sigma} \frac{i}{3!}\delta(e^3)\underline{\bar{\theta}}^{\lrcorner}\gamma\left(\underline{\delta\psi}_1 + \underline{\delta(\lambda\mu^{-1})\theta^{\lrcorner}}_2 - \underline{\lambda\mu^{-1}\delta\theta^{\lrcorner}}_3\right) + \frac{i}{3!}e^3\delta\underline{\theta}\gamma\left(\underline{\delta\psi}_4 + \underline{\delta(\lambda\mu^{-1})\theta^{\lrcorner}}_5 - \underline{\lambda\mu^{-1}\delta\theta^{\lrcorner}}_6\right) \\
&\quad - \frac{i}{4}\delta(e^2)f^{\lrcorner}\bar{\psi}\gamma\left(\underline{\delta\psi}_7 + \underline{\delta(\lambda\mu^{-1})\theta^{\lrcorner}}_8 - \underline{\lambda\mu^{-1}\delta\theta^{\lrcorner}}_9\right) - \frac{i}{4}e^2\delta f^{\lrcorner}\bar{\psi}\gamma\left(\underline{\delta\psi}_{10} + \underline{\delta(\lambda\mu^{-1})\theta^{\lrcorner}}_{11} - \underline{\lambda\mu^{-1}\delta\theta^{\lrcorner}}_{12}\right) \\
&\quad + \frac{i}{4}e^2 f^{\lrcorner}\delta\bar{\psi}\gamma\left(\underline{\delta\psi}_{13} + \underline{\delta(\lambda\mu^{-1})\theta^{\lrcorner}}_{14} - \underline{\lambda\mu^{-1}\delta\theta^{\lrcorner}}_{15}\right) - \frac{i}{4}\lambda\mu^{-1}\delta(e^2)f^{\lrcorner}\underline{\theta}^{\lrcorner}\gamma\left(\underline{\delta\psi}_{16} + \underline{\delta(\lambda\mu^{-1})\theta^{\lrcorner}}_{17}\right) \\
&\quad - \frac{i}{4}\lambda\mu^{-1}e^2\delta f^{\lrcorner}\underline{\theta}^{\lrcorner}\gamma\left(\underline{\delta\psi}_{18} + \underline{\delta(\lambda\mu^{-1})\theta^{\lrcorner}}_{19}\right) + \frac{i}{4}\lambda\mu^{-1}e^2 f^{\lrcorner}\delta\underline{\theta}^{\lrcorner}\gamma\left(\underline{\delta\psi}_{20} + \underline{\delta(\lambda\mu^{-1})\theta^{\lrcorner}}_{21}\right) \\
&\quad + \frac{i}{4}\delta(\lambda\mu^{-1})e^2 f^{\lrcorner}\underline{\theta}^{\lrcorner}\gamma\left(\underline{\delta\psi}_{22} + \underline{\delta(\lambda\mu^{-1})\theta^{\lrcorner}}_{23} - \underline{\lambda\mu^{-1}\delta\theta^{\lrcorner}}_{24}\right) + \text{c.c.}
\end{aligned}
\tag{B.6}
$$

We now proceed to show that (3.37) coincides with $\varphi^*(\varpi_{PCD}^M)$. We see

- The terms $(B.5.1) + (B.6.1) + (B.6.4)(B.6.7) + (B.6.10) + (B.6.13)$ exactly reproduce the terms inside (3.37).
- $(B.5.2) + (B.5.16) + (B.5.20) + (B.6.17) = 0$
- $(B.5.3) + (B.5.18) = 0$
- $(B.5.4) + (B.5.19) + (B.6.14) + (B.6.19) = 0$
- $(B.5.5) + (B.5.21) = 0$
- $(B.5.6) + (B.6.12) = 0$
- $(B.5.7) + (B.6.18) = 0$
- $(B.5.8) + (B.5.14) = 0$
- $(B.5.9) + (B.6.3) = 0$
- $(B.5.10) + (B.6.23) = 0$
- $(B.5.11) + (B.6.17) = 0$
- $(B.5.12) + (B.6.24) = 0$
- $(B.5.13) + (B.6.19) = 0$
- $(B.5.15) + (B.6.5) = 0$
- $(B.5.17) + (B.6.11) = 0$
- $(B.5.17) + (B.6.11) = 0$
- $(B.5.22) + (B.6.9) = 0$
- $(B.5.23) + (B.6.16) = 0$
- $(B.6.15) + (B.6.20) = 0.$

We now move to show that $\varphi^*(\mathcal{S}_{PCD}^M) = \mathcal{S}_{PCD}^{AKSZ}$. We start by computing the terms inside $\varphi^*(\mathcal{S}_{PC}^M)$ that depend ψ and $\theta^{\lrcorner}$, which appear in $\varphi^*(\tilde{e}_n^{\lrcorner})$ and $\varphi^*(\tilde{\xi}_n^{\lrcorner})$. We use the following

$$\begin{aligned}
&\varphi^*(\mathrm{L}_{\tilde{\xi}}^{\tilde{\omega}}\tilde{e} + d_{\tilde{\omega}_n}\tilde{e}\tilde{\xi}^n + \tilde{e}_n d\tilde{\xi}^n - [\tilde{c}, \tilde{e}]) = \\
&= \mathrm{L}_{\xi}^{\omega} e + \mathrm{L}_{\xi}^{\omega}(\lambda\mu^{-1} f^{\lrcorner}) - \lambda\mu^{-1}\mathrm{L}_{z}^{\omega}(e) - \lambda\mu^{-1}\mathrm{L}_{z}^{\omega}(\lambda\mu^{-1}) f^{\lrcorner} + d_{\omega}(\lambda\mu^{-1})\iota_z e \\
&+ \lambda\mu^{-1}\partial_n e + \lambda\mu^{-1}\partial_n(\lambda\mu^{-1}) f^{\lrcorner} + \lambda\mu^{-1}[w, e] - \lambda\mu^{-1}[\iota_{\xi} u^{\lrcorner}, e] \\
&+ d(\lambda\mu^{-1})(\mu\epsilon_n + \iota_z e - \lambda\mu^{-1}\iota_z f^{\lrcorner}) - [c, e] + \lambda\mu^{-1}[c, f^{\lrcorner}],
\end{aligned}$$

$$\begin{aligned}
&\varphi^*(\mathrm{L}_{\tilde{\xi}}^{\tilde{\omega}}\tilde{\psi} + d_{\tilde{\omega}_n}\tilde{\psi}\tilde{\xi}^n - [\tilde{c}, \tilde{\psi}]) = \\
&= \mathrm{L}_{\xi}^{\omega}\psi + \mathrm{L}_{\xi}^{\omega}(\lambda\mu^{-1}\theta^{\lrcorner}) - \lambda\mu^{-1}\mathrm{L}_{z}^{\omega}(\psi) - \lambda\mu^{-1}\mathrm{L}_{z}^{\omega}(\lambda\mu^{-1})\theta^{\lrcorner} \\
&+ \lambda\mu^{-1}\partial_n\psi + \lambda\mu^{-1}\partial_n(\lambda\mu^{-1})\theta^{\lrcorner} + \lambda\mu^{-1}[w, \psi] - \lambda\mu^{-1}[\iota_{\xi} u^{\lrcorner}, \psi] \\
&- [c, \psi] + \lambda\mu^{-1}[c, \theta^{\lrcorner}],
\end{aligned}$$

and that $\varphi^*([\tilde{\xi}, \tilde{\xi}]^n = \mathrm{L}_{\xi}^{\omega}(\lambda\mu^{-1}) - \lambda\mu^{-1}\mathrm{L}_{z}^{\omega}(\lambda\mu^{-1})) + \mathrm{L}_{z}^{\omega}(\lambda\mu^{-1})\partial_n(\mathrm{L}_{z}^{\omega}(\lambda\mu^{-1}))$.

We then find

$$\begin{aligned}
&\varphi^*(\mathcal{S}_{PC}^M) = \mathcal{S}_{PC}^{AKSZ} + \\
&+ \int \frac{i}{2\cdot 3!} e^3 \underline{\bar{\theta}}^{\lrcorner}\gamma\theta^{\lrcorner}(\underline{\mathrm{L}_{\xi}^{\omega}(\lambda\mu^{-1})}_1 - \underline{\lambda\mu^{-1}\mathrm{L}_{z}^{\omega}(\lambda\mu^{-1})}_2 + \underline{\lambda\mu^{-1}\partial_n(\lambda\mu^{-1})}_3) \\
&+ \frac{i}{4} e^2 f^{\lrcorner}\underline{\bar{\theta}}^{\lrcorner}\gamma\psi(\underline{\mathrm{L}_{\xi}^{\omega}(\lambda\mu^{-1})}_4 - \underline{\lambda\mu^{-1}\mathrm{L}_{z}^{\omega}(\lambda\mu^{-1})}_5 + \underline{\lambda\mu^{-1}\partial_n(\lambda\mu^{-1})}_6) \\
&+ \frac{i}{4} e^2 \underline{\bar{\theta}}^{\lrcorner}\gamma\psi\left(\underline{\mathrm{L}_{\xi}^{\omega} e}_7 + \mathrm{L}_{\xi}^{\omega}(\underline{\lambda\mu^{-1} f^{\lrcorner}}_8 - \underline{\lambda\mu^{-1}\mathrm{L}_{z}^{\omega} e}_9 - \underline{\lambda\mu^{-1}\mathrm{L}_{z}^{\omega}(\lambda\mu^{-1}) f^{\lrcorner}}_{10}\right) \\
&+ \frac{i}{4} e^2 \underline{\bar{\theta}}^{\lrcorner}\gamma\psi\left(\underline{d(\lambda\mu^{-1})\iota_z e}_{11} + \underline{\lambda\mu^{-1}\partial_n e}_{12} + \underline{\lambda\mu^{-1}\partial_n(\lambda\mu^{-1}) f^{\lrcorner}}_{13} + \underline{\lambda\mu^{-1}[w, e]}_{14} - \underline{\lambda\mu^{-1}[\iota_{\xi} u^{\lrcorner}, e]}_{15}\right) \\
&+ \frac{i}{4} e^2 \underline{\bar{\theta}}^{\lrcorner}\gamma\psi\left(d(\lambda\mu^{-1})(\underline{\mu\epsilon_n}_{16} + \underline{\iota_z e}_{17} - \underline{\lambda\mu^{-1}\iota_z f^{\lrcorner}}_{18}) - \underline{[c, e]}_{19} + \underline{\lambda\mu^{-1}[c, f^{\lrcorner}]}_{20}\right) \\
&+ \frac{i}{4}\lambda\mu^{-1} e^2 \underline{\bar{\theta}}^{\lrcorner}\gamma\theta^{\lrcorner}\left(\underline{\mathrm{L}_{\xi}^{\omega} e}_{21} + \mathrm{L}_{\xi}^{\omega}\underline{(\lambda\mu^{-1}) f^{\lrcorner}}_{22} - \underline{d(\lambda\mu^{-1})\iota_z e}_{23} + d(\lambda\mu^{-1})(\underline{\mu\epsilon_n}_{24} + \underline{\iota_z e}_{25}) - \underline{[c, e]}_{26}\right) \\
&+ \frac{i}{4}\lambda\mu^{-1} e^2 f^{\lrcorner}\underline{\bar{\theta}}^{\lrcorner}\gamma\psi\left(\underline{\mathrm{L}_{\xi}^{\omega} e}_{27} - \underline{d(\lambda\mu^{-1})\iota_z e}_{28} + d(\lambda\mu^{-1})(\underline{\mu\epsilon_n}_{29} + \underline{\iota_z e}_{30}) - \underline{[c, e]}_{31}\right) + \text{c.c.}
\end{aligned} \tag{B.7}$$

We also have

$$
\begin{aligned}
\varphi^*(\mathcal{S}_D^M) &= \varphi^* \int_{I\times\Sigma} \frac{i}{4}\tilde{e^2}\underline{\tilde{e}}_n\bar{\tilde{\psi}}\gamma d_{\tilde{\omega}}\tilde{\psi} + \frac{i}{2\cdot 3!}\tilde{e}^2\bar{\tilde{\psi}}\gamma d_{\underline{\tilde{\omega}}_n}\tilde{\psi} + i\bar{\tilde{\psi}}_{\lrcorner,n}(\mathrm{L}_{\tilde{\xi}}^{\tilde{\omega}}\tilde{\psi} + d_{\tilde{\omega}_n}\tilde{\psi}\tilde{\xi}^n - [\tilde{c},\tilde{\psi}])\\
= \int_{I\times\Sigma} & \frac{i}{4}\underline{\mu}\epsilon_n\left(\bar{\psi}\gamma(\underline{d_\omega\psi}_{1} + \underline{d(\lambda\mu^{-1}\theta^{\lrcorner}}_{2})\right)) + \frac{i}{4}\lambda\epsilon_n e^2\left(\bar{\psi}\gamma(\underline{d_\omega\underline{\theta}^{\lrcorner}}_{3} + \underline{[\underline{u}^{\lrcorner},\psi]}_{4}) + \underline{\bar{\theta}}^{\lrcorner}\gamma(\underline{d_\omega\psi}_{5} + \underline{d(\lambda\mu^{-1})\theta^{\lrcorner}}_{6})\right)\\
&+ \frac{i}{2\cdot 3!}\iota_{\underline{z}}e^3\left[\bar{\psi}\gamma(\underline{d_\omega\psi}_{7} + \underline{d_\omega(\lambda\mu^{-1}\theta^{\lrcorner})}_{8} - \underline{\lambda\mu^{-1}[u^{\lrcorner},\psi]}_{9}) + \lambda\mu^{-1}\bar{\theta}^{\lrcorner}\gamma(\underline{d_\omega\psi}_{10} + \underline{d(\lambda\mu^{-1})\theta^{\lrcorner}}_{11})\right]\\
&+ \frac{i}{4}\lambda\mu^1\iota_{\underline{z}}(e^2 f^{\lrcorner})\bar{\psi}\gamma(\underline{d_\omega\psi}_{12} + \underline{d(\lambda\mu^{-1})\theta^{\lrcorner}}_{13}) + \frac{i}{2}\lambda\epsilon_n\underline{f}^{\lrcorner}\bar{\psi}\gamma\underline{d_\omega\psi}_{14} + \underline{d(\lambda\mu^{-1})\theta^{\lrcorner}}_{15})\\
&+ \frac{i}{2\cdot 3!}e^3\bar{\psi}\gamma\left(\underline{\partial}_n(\underline{\psi}_{16} + \underline{\lambda\mu^{-1}\theta^{\lrcorner}}_{17}) + [\underline{w},\underline{\psi}_{18} + \underline{\lambda\mu^{-1}\theta^{\lrcorner}}_{19}] - [\iota_\xi\underline{u}^{\lrcorner},\underline{\psi}_{20} + \underline{\lambda\mu^{-1}\theta^{\lrcorner}}_{21}]\right)\\
&+ \frac{i}{2\cdot 3!}e^3\bar{\psi}\gamma\underline{\lambda\mu^{-1}[\iota_z\underline{u}^{\lrcorner},\psi]}_{22} + \frac{i}{2\cdot 3!}\lambda\mu^{-1}e^3\bar{\theta}\gamma(\underline{\underline{\partial}_n\psi}_{23} + \underline{\underline{\partial}_n(\lambda\mu^{-1})\theta^{\lrcorner}}_{24} + \underline{[w,\psi]}_{25} - \underline{[\iota_\xi u^{\lrcorner},\psi]}_{26})\\
&+ \frac{i}{4}\lambda\mu^{-1}e^2 f^{\lrcorner}\bar{\psi}\gamma(\underline{\underline{\partial}_n\psi}_{27} + \underline{\underline{\partial}_n(\lambda\mu^{-1})\theta^{\lrcorner}}_{28} + \underline{[w,\psi]}_{29} - \underline{[\iota_\xi u^{\lrcorner},\psi]}_{30})\\
&+ \frac{i}{3!}e^3\underline{\bar{\theta}}^{\lrcorner}\gamma\left(\underline{(\mathrm{L}_\xi^\omega\psi}_{31} + \underline{\mathrm{L}_\xi^\omega(\lambda\mu^{-1}\theta^{\lrcorner})}_{32} - \underline{\lambda\mu^{-1}\mathrm{L}_z^\omega\psi}_{33} - \underline{\lambda\mu^{-1}\mathrm{L}_z^\omega(\lambda\mu^{-1})\theta^{\lrcorner}}_{34} + \underline{\lambda\mu^{-1}\partial_n\psi}_{35}\right)\\
&+ \frac{i}{3!}e^3\underline{\bar{\theta}}^{\lrcorner}\gamma\left(\underline{\lambda\mu^{-1}\partial_n(\lambda\mu^{-1})\theta^{\lrcorner}}_{36} + \underline{\lambda\mu^{-1}[w,\psi]}_{37} - \underline{\lambda\mu^{-1}[\iota_\xi u^{\lrcorner},\psi]}_{38} - \underline{[c,\psi]}_{39} + \underline{\lambda\mu^{-1}[c,\theta^{\lrcorner}]}_{40}\right)\\
&- \frac{i}{2}e^2\underline{f}^{\lrcorner}\bar{\psi}\gamma\left(\underline{(\mathrm{L}_\xi^\omega\psi}_{41} + \underline{\mathrm{L}_\xi^\omega(\lambda\mu^{-1}\theta^{\lrcorner})}_{42} - \underline{\lambda\mu^{-1}\mathrm{L}_z^\omega\psi}_{43} - \underline{\lambda\mu^{-1}\mathrm{L}_z^\omega(\lambda\mu^{-1})\theta^{\lrcorner}}_{44} + \underline{\lambda\mu^{-1}\partial_n\psi}_{45}\right)\\
&- \frac{i}{2}e^2\underline{f}^{\lrcorner}\bar{\psi}\gamma\left(\underline{\lambda\mu^{-1}\partial_n(\lambda\mu^{-1})\theta^{\lrcorner}}_{46} + \underline{\lambda\mu^{-1}[w,\psi]}_{47} - \underline{\lambda\mu^{-1}[\iota_\xi u^{\lrcorner},\psi]}_{48} - \underline{[c,\psi]}_{49} + \underline{\lambda\mu^{-1}[c,\theta^{\lrcorner}]}_{50}\right)\\
&+ \frac{i}{2}\lambda\mu^{-1}e^2\underline{f}^{\lrcorner}\bar{\theta}\gamma\left(\underline{(\mathrm{L}_\xi^\omega\psi}_{51} + \underline{\mathrm{L}_\xi^\omega(\lambda\mu^{-1}\theta^{\lrcorner})}_{52} - \underline{[c,\psi]}_{53}\right) + \mathrm{c.c}
\end{aligned}
\tag{B.8}
$$

We confront the above with equation (3.38) and see

- $(B.8.31) + (B.8.39) + (B.7.7) + (B.7.1) = -\frac{i}{2\cdot 3!}e^3\bar{\theta}^{\lrcorner}\gamma(\mathrm{L}_\xi^\omega\psi - [c,\psi]) - \frac{i}{2\cdot 3!}e^3\bar{\psi}\gamma(\mathrm{L}_\xi^\omega\theta - [c,\theta])$
- all the other terms inside (3.38) are given by $(B.8.16) + (B.8.18) + (B.8.20) + (B.8.41) + (B.8.49)$.

The remainder in (B.7) and (B.8) must vanish. Indeed

- $(B.7.1) + (B.7.21) + (B.7.23) + (B.8.32) = 0$
- $(B.7.2) + (B.7.22) + (B.7.25) + (B.8.11) + (B.8.34) + +(B.8.52) = 0$
- $(B.7.3) + (B.8.24) + (B.8.36) = 0$
- $(B.7.4) + (B.7.8) + (B.7.27) + (B.8.22) + (B.8.42) + (B.8.51) + (B.8.43) = 0$
- $(B.7.5) + (B.7.10) + (B.7.18) + (B.7.28) + (B.7.30) + (B.8.13) + (B.8.44) = 0$
- $(B.7.6) + (B.7.13) + (B.8.28) + (B.8.36) = 0$
- $(B.7.9) + (B.7.11) + (B.7.17) + (B.8.8) + (B.8.10) + (B.8.33) = 0$
- $(B.7.12) + (B.8.17) + (B.8.35) + (B.8.23) = 0$
- $(B.7.14) + (B.8.25) + (B.8.37) + (B.8.19) = 0$

- $(B.7.15) + (B.8.21) + (B.8.38) + (B.8.19) + +(B.8.38) = 0$
- $(B.7.16) + (B.8.2) + (B.8.26) + (B.8.19) = 0$
- $(B.7.20) + (B.7.31) + (B.8.50) + (B.8.53) = 0$
- $(B.7.24) + (B.8.6) = 0$
- $(B.7.26) + (B.8.40) = 0$
- $(B.7.29) + (B.8.15) = 0$
- $(B.8.18) + (B.8.22) = 0$
- $(B.8.18) + (B.8.22) = 0$
- $(B.8.27) + (B.8.45) = 0$
- $(B.8.29) + (B.8.47) = 0$
- $(B.8.30) + (B.8.48) = 0$

So far we have proved that defining $\varphi : \mathfrak{F}_{PCD}^{AKSZ} \to \mathfrak{F}_{PCD}$ is such that $\varphi^*(\varpi_{PCD}^M) = \varpi_{PCD}^{AKSZ}$ and $\varphi^*(\mathcal{S}_{PCD}^M) = \mathcal{S}_{PCD}^{AKSZ}$. However, we also notice that the image of ϕ is given by the restricted BV PCD fields, i.e. $\phi(\mathcal{F}_{PCD}^{AKSZ}) = \mathcal{F}_{PCD}^r$. Indeed

$$\varphi^*(\mathfrak{W}^\dashv) = \varphi^* \left(\underline{\tilde{\omega}}_n^\dashv - \iota_{\tilde{\xi}} \underline{\tilde{c}}_n^\dashv - \iota_{\underline{\tilde{z}}} (\tilde{\omega}^\dashv - \tilde{c}_n^\dashv \tilde{\xi}^n) \right) = e \underline{f}^\dashv \in \operatorname{Im}(W_e^{\partial,(1,1)})$$

and in the same way

$$\begin{aligned} &\varphi^* \left(\epsilon_n \left(d_{\tilde{\omega}} \tilde{e} + \frac{i}{4} (\bar{\psi}\gamma[e^2, \psi] - [e^2, \bar{\psi}]\gamma\psi) \right) - \epsilon_n W_{\tilde{e}}^{-1}(\underline{\mathfrak{W}}) d\tilde{\xi}^n + \iota_{\hat{X}} (\tilde{\omega}_n^\dashv - \tilde{c}_n^\dashv \tilde{\xi}^n) \right) \\ &= e(\sigma + \lambda\mu^{-1} B), \end{aligned}$$

for some σ and B. This concludes the proof.

□

B.2 Proofs of chapter 4

B.2.1 Theorem 4.2

proof of 4.2. Having obtained the Hamiltonian vector fields of the constraints, we can compute their Poisson brackets. The pure gravity sector has been computed in [CCS21a], we refer to it for the details, and concentrate on the Rarita-Schwinger sector.

Remark B.2. In the following, instead of the definition via the symplectic form $\{F, G\} = \iota_{\mathbb{X}_F} \iota_{\mathbb{X}_G} \varpi$, we use the equivalent formulation $\{F, G\} = \mathbb{X}_F(G)(= \iota_{\mathbb{X}_F} \delta G = \iota_{\mathbb{X}_F} \iota_{\mathbb{X}_G} \varpi)$.

$$\begin{aligned}\{L_c, L_c\} &= \int_\Sigma -[c,c]ed_\omega e + \frac{1}{3!}[c,e]\bar{\psi}\gamma^3[c,\psi] + \frac{1}{3!}e\left([c,\bar{\psi}]\gamma^3[c,\psi] + \bar{\psi}\gamma^3[c,[c,\psi]]\right)\\ &= \int_\Sigma -[c,c]ed_\omega e + \frac{1}{3!}e\left(-\bar{\psi}[c,\gamma^3]_v[c,\psi] + \bar{\psi}[c,\gamma^3]_V[c,\psi] + 2\bar{\psi}\gamma^3[c,[c,\psi]]\right)\\ &= \int_\Sigma -[c,c]ed_\omega e + \frac{1}{3!}e\bar{\psi}\gamma^3[[c,c],\psi]\\ &= -\int_\Sigma [c,c]e\left(d_\omega e - \frac{1}{2}\bar{\psi}\gamma\psi\right) = -L_{[c,c]};\end{aligned}$$

$$\begin{aligned}\{L_c, M_\chi\} &= \frac{1}{3!}\int_\Sigma [c,e](d_\omega\bar{\chi}\gamma^3\psi + \bar{\chi}\gamma^3 d_\omega\psi) + e([d_\omega c,\bar{\chi}]\gamma^3\psi + d_\omega\bar{\chi}\gamma^3[c,\psi] - \bar{\chi}\gamma^3[d_\omega c,\psi] + \bar{\chi}\gamma^3 d_\omega[c,\psi])\\ &= \frac{1}{3!}\int_\Sigma e(d_\omega\bar{\chi}[c,\gamma^3]_V\psi - \bar{\chi}[c,\gamma^3]_V d_\omega\psi + [d_\omega c,\bar{\chi}]\gamma^3\psi - d_\omega\bar{\chi}[c,\gamma^3]_V\psi - [c,d_\omega\bar{\chi}]\gamma^3\psi - \bar{\chi}\gamma^3[c,d_\omega\psi])\\ &= \frac{1}{3!}\int_\Sigma e(d_\omega[c,\bar{\chi}]\gamma^3\psi - [c,\bar{\chi}]\gamma^3 d_\omega\psi) = M_{[c,\chi]},\end{aligned}$$

having used Leibniz rule and an expression analogous to (4.15).

$$\begin{aligned}\{L_c, P_\xi\} &= \int_\Sigma \mathrm{L}_\xi^{\omega_0} c e d_\omega e - \frac{1}{3!}[c,e](\bar{\psi}\gamma^3\mathrm{L}_\xi^{\omega_0}\psi) - \frac{1}{3!}e\left([c,\bar{\psi}]\gamma^3\mathrm{L}_\xi^{\omega_0}\psi + \bar{\psi}\gamma^3\mathrm{L}_\xi^{\omega_0}[c,\psi]\right)\\ &= \int_\Sigma \mathrm{L}_\xi^{\omega_0} c e d_\omega e - \frac{1}{3!}e\left(\bar{\psi}\gamma^3[c,\mathrm{L}_\xi^{\omega_0}\psi] + \bar{\psi}\gamma^3[\mathrm{L}_\xi^{\omega_0}c,\psi] - \bar{\psi}\gamma^3[c,\mathrm{L}_\xi^{\omega_0}\psi]\right)\\ &= \int_\Sigma \mathrm{L}_\xi^{\omega_0} c e d_\omega e - \frac{1}{3!}e\bar{\psi}\gamma^3[\mathrm{L}_\xi^{\omega_0}c,\psi] = \int_\Sigma \mathrm{L}_\xi^{\omega_0} c e\left(d_\omega e - \frac{1}{2}\bar{\psi}\gamma\psi\right)\\ &= L_{\mathrm{L}_\xi^{\omega_0}c};\end{aligned}$$

$$\begin{aligned}\{L_c, H_\lambda\} &= \int_\Sigma -[c,\lambda\epsilon_n]eF_\omega + \frac{1}{3!}\lambda\epsilon_n\left([c,\bar{\psi}]\gamma^3 d_\omega\psi - \bar{\psi}\gamma^3[d_\omega c,\psi] + \bar{\psi}\gamma^3 d_\omega[c,\psi]\right)\\ &= \int_\Sigma -[c,\lambda\epsilon_n]eF_\omega + \frac{1}{3!}\lambda\epsilon_n\left([c,\bar{\psi}]\gamma^3 d_\omega\psi - \bar{\psi}\gamma^3[c,d_\omega\psi]\right)\\ &= \int_\Sigma -[c,\lambda\epsilon_n]eF_\omega + \frac{1}{3!}\lambda\epsilon_n\bar{\psi}[c,\gamma^3]_V d_\omega\psi\\ &= \int_\Sigma -[c,\lambda\epsilon_n]\left(eF_\omega + \frac{1}{3!}\bar{\psi}\gamma^3 d_\omega\psi\right)\\ &= -P_X + L_{\iota_X(\omega-\omega_0)} + M_{\iota_X\psi} - H_{X^n},\end{aligned}$$

where, letting $\{x^i\}$ be coordinates on Σ, we have $X = e_a^i[c,\lambda\epsilon_n]^{(i)}\partial_i$ and $X^{(n)} = [c,\lambda\epsilon_n]^{(n)}$, having set $e_a^i e_j^a = \delta_j^i$.

$$\begin{aligned}\{P_\xi, M_\chi\} &= \int_\Sigma -\frac{1}{3!}\mathrm{L}_\xi^{\omega_0} e(d_\omega\bar{\chi}\gamma^3\psi + \bar{\chi}\gamma^3 d_\omega\psi) - ie\bar{\chi}\gamma^3\psi(\iota_\xi F_{\omega_0} + \mathrm{L}_\xi^{\omega_0}(\omega-\omega_0))\\ &\quad - \frac{1}{3!}e(d_\omega\bar{\chi}\gamma^3\mathrm{L}_\xi^{\omega_0}\psi - \bar{\chi}\gamma^3 d_\omega\mathrm{L}_\xi^{\omega_0}\psi)\\ &\overset{\star}{=} \int_\Sigma \frac{1}{3!}e\left(\mathrm{L}_\xi^{\omega_0}\bar{\chi}\gamma^3 d_\omega\psi - d_\omega\mathrm{L}_\xi^{\omega_0}\bar{\psi}\gamma^3\psi\right) = -M_{\mathrm{L}_\xi^{\omega_0}\chi},\end{aligned}$$

where we have used integration by parts and

$$[\mathrm{L}_\xi^{\omega_0}, d_\omega]\psi = [\iota_\xi F_{\omega_0} + \mathrm{L}_\xi^{\omega_0}(\omega-\omega_0), \psi] \qquad (\star)$$

$$\begin{aligned}
\{P_\xi, H_\lambda\} &= \int_\Sigma \mathrm{L}_\xi^{\omega_0}(\lambda\epsilon_n) e F_\omega - \frac{1}{3!}\lambda\epsilon_n \left(\mathrm{L}_\xi^{\omega_0}\bar\psi\gamma^3 d_\omega\psi - \bar\psi\gamma^3[\iota_\xi F_{\omega_0} + \mathrm{L}_\xi^{\omega_0}(\omega-\omega_0),\psi] + \bar\psi\gamma^3 d_\omega \mathrm{L}_\xi^{\omega_0}\psi\right)\\
&\overset{\star}{=} \int_\Sigma \mathrm{L}_\xi^{\omega_0}(\lambda\epsilon_n) e F_\omega + \frac{1}{3!}\mathrm{L}_\xi^{\omega_0}(\lambda\epsilon_n)\bar\psi\gamma^3 d_\omega\psi\\
&= P_Y - L_{\iota_Y(\omega-\omega_0)} - M_{\iota_Y\psi} + H_{Y^{(n)}},
\end{aligned}$$

where $Y = e_a^i \mathrm{L}_\xi^{\omega_0}(\lambda\epsilon_n)^{(i)}\partial_i$, $Y^{(n)} = \mathrm{L}_\xi^{\omega_0}(\lambda\epsilon_n)^{(n)}$.

$$\begin{aligned}
\{M_\chi, M_\chi\} &= \frac{1}{2}\int_\Sigma -\frac{1}{3!}\bar\chi\gamma\psi(d_\omega\bar\chi\gamma^3\psi + \bar\chi\gamma^3 d_\omega\psi) - e\mathbb{M}_\omega\bar\chi\gamma\psi + \frac{e}{3!}d_\omega\bar\chi\gamma d_\omega\chi\\
&\qquad - \frac{1}{2\cdot 3!}d_\omega\bar\chi d_\omega e\gamma^3\chi + \frac{1}{3!}e\bar\chi\gamma^3[F_\omega,\chi] + \frac{1}{3!}e\bar\chi\gamma^3 d_\omega\mathbb{M}_\psi^{e}\\
&\overset{A.15}{=} \frac{1}{2}\int_\Sigma -\frac{1}{3!}d_\omega e\bar\chi\gamma^3 d_\omega\chi + \frac{1}{3}\bar\chi\gamma^3[F_\omega,\chi] + \frac{1}{3!}d_\omega e d_\omega\bar\chi\gamma^3\chi + \frac{1}{3!}d_\omega e\bar\chi\gamma^3\mathbb{M}_\psi^e\\
&\overset{(A.50)}{=} \int_\Sigma \frac{1}{2}\bar\chi\gamma\chi e F_\omega\\
&= \int_\Sigma \frac{1}{2}\bar\chi\gamma\chi e F_\omega - e\alpha^\partial(\chi, d_\omega\chi)\left(d_\omega e - \frac{1}{2}\bar\psi\gamma\psi\right) - \epsilon_n\beta^\partial(\chi, d_\omega\chi)\left(d_\omega e - \frac{1}{2}\bar\psi\gamma\psi\right)\\
&= \frac{1}{2}P_\varphi - \frac{1}{2}L_{\iota_\varphi(\omega-\omega_0)} - \frac{1}{2}M_{\iota_\varphi\psi} + H_{\varphi^n}
\end{aligned}$$

where $\varphi^i := \bar\chi\gamma^a\chi e_a^i$ and $\varphi^n := \bar\chi\gamma^a\chi e_a^n$, having used that $\mathbb{M}_\psi^e \propto \chi$, hence $\bar\chi\gamma^3\mathbb{M}_\psi^e \propto \bar\chi\gamma^3\chi = 0$.

$$\begin{aligned}
\{P_\xi, P_\xi\} &= \int_\Sigma e d_\omega e\iota_{[\xi,\xi]}(\omega-\omega_0) + \frac{1}{2}\iota_{[\xi,\xi]}(e^2)F_\omega - e d_\omega e F_{\omega_0} + \frac{1}{3!}\mathrm{L}_\xi^{\omega_0}e\bar\psi\gamma^3\mathrm{L}_\xi^{\omega_0}\psi\\
&\qquad + \frac{1}{3!}e\left(\mathrm{L}_\xi^{\omega_0}\bar\psi\gamma^3\psi + \bar\psi\gamma^3\mathrm{L}_\xi^{\omega_0}\mathrm{L}_\xi^{\omega_0}\psi\right)\\
&\overset{B.9}{=} \int_\Sigma e d_\omega e\iota_{[\xi,\xi]}(\omega-\omega_0) + \frac{1}{2}\iota_{[\xi,\xi]}(e^2)F_\omega + \frac{1}{3!}e\bar\psi\gamma^3\mathrm{L}_{[\xi,\xi]}^{\omega_0}\psi\\
&\qquad - e d_\omega e F_{\omega_0} + \frac{1}{3!}e\bar\psi\gamma^3[\iota_\xi\iota_\xi F_{\omega_0},\psi]\\
&= P_{[\xi,\xi]} - L_{\iota_\xi\iota_\xi F_{\omega_0}};
\end{aligned}$$

having used

$$\mathrm{L}_\xi^{\omega_0}\mathrm{L}_\xi^{\omega_0}A = \frac{1}{2}\mathrm{L}_{[\xi,\xi]}^{\omega_0}A + \frac{1}{2}[\iota_\xi\iota_\xi F_{\omega_0}, A]. \qquad (B.9)$$

$$\begin{aligned}
&= \int_\Sigma d_\omega(\lambda\epsilon_n)\left(\lambda\epsilon_n F_\omega - \frac{1}{3!}\bar\psi\gamma^3\mathbb{H}_\psi\right) + \lambda\epsilon_n\left(d_\omega e - \frac{1}{2}\bar\psi\gamma\psi\right)\mathbb{H}_\omega\\
&\qquad + \frac{1}{3!}\lambda\epsilon_n\left(\mathbb{H}_{\bar\psi}\gamma^3 d_\omega\psi + \bar\psi\gamma^3 d_\omega\mathbb{H}_\psi\right)\\
&= \int_\Sigma \lambda e\mathbb{H}_\omega\sigma + \frac{1}{3}\lambda\epsilon_n\bar\psi\gamma^3\mathbb{H}_\psi = \int_\Sigma \lambda^2(\cdots) = 0,
\end{aligned}$$

having used the fact that $\lambda^2 = \epsilon_n^2 = 0$ and that $d_\omega(\lambda\epsilon_n)\lambda\epsilon_n = 0$;

$$\begin{aligned}
\{M_\chi, H_\lambda\} &= \mathbb{M}_\chi(H_\lambda)\\
&= \int_\Sigma -\lambda\epsilon_n\bar{\chi}\gamma\psi F_\omega + d_\omega(\lambda\epsilon_n e)\mathbb{M}_\omega + \frac{1}{3!}\lambda\epsilon_n d_\omega\bar{\chi}\gamma^3 d_\omega\psi - \frac{1}{2}\lambda\epsilon_n\bar{\psi}\gamma\psi\mathbb{M}_\omega + \frac{1}{3!}\lambda\epsilon_n\bar{\psi}\gamma^3[F_\omega,\chi]\\
&= \int_\Sigma -\lambda\epsilon_n\bar{\chi}\gamma\psi F_\omega + \lambda\epsilon_n\left(d_\omega e - \frac{1}{2}\bar{\psi}\gamma\psi\right)\mathbb{M}_\omega + \frac{1}{3!}\lambda\epsilon_n\left(\bar{\psi}\gamma^3[F_\omega,\chi] + \bar{\chi}\gamma^3[F_\omega,\psi]\right)\\
&\overset{(A.50)}{=} \int_\Sigma \lambda\epsilon_n\mathbb{M}_\omega\left(d_\omega e - \frac{1}{2}\bar{\psi}\gamma\psi\right) = \int_\Sigma \left(e\alpha^\partial(\mathbb{M}_\omega) + \epsilon_n\beta^\partial(\mathbb{M}_\omega)\right)\left(d_\omega e - \frac{1}{2}\bar{\psi}\gamma\psi\right)\\
&= L_{\alpha^\partial(\epsilon_n\mathbb{M}_\omega)}
\end{aligned}$$

having used the structural constraint and that $e\beta^\partial(\chi,\psi) = 0$.

□

B.3 Proofs of chapter 5

B.3.1 Computing δ_χ^2

To obtain the full expression of δ_χ^2, we start by the simpler case of $\delta_\chi^2 e$. We have

$$\begin{aligned}
\delta_\chi^2 e = \delta_\chi(\bar{\chi}\gamma\psi) &= -\frac{1}{2}\iota_\varphi\bar{\psi}\gamma\psi + \bar{\chi}\gamma d_\omega\psi = -\frac{1}{2}d_\omega(\bar{\chi}\gamma\chi) + \frac{1}{2}\iota_\varphi\left(-\frac{1}{2}\bar{\psi}\gamma\psi\right)\\
&= -\frac{1}{2}\mathrm{L}_\varphi^\omega e + \frac{1}{2}\iota_\varphi\left(d_\omega e - \frac{1}{2}\bar{\psi}\gamma\psi\right),
\end{aligned}$$

For the computation of $\delta_\chi^2\psi$, we have

$$\delta_\chi^2\psi = \delta_\chi(-d_\omega\chi) = -[\delta_\chi\omega,\chi] - \frac{1}{2}d_\omega\iota_\varphi\psi = -[\delta_\chi\omega,\chi] - \frac{1}{2}\mathrm{L}_\varphi^\omega\psi + \frac{1}{2}\iota_\varphi d_\omega\psi.$$

We now need the explicit form of $\delta_\chi\omega$. In order to obtain it, we rewrite $d_\omega\psi$ in the veilbein basis, i.e. define

$$d_\omega\psi := \frac{1}{2}\rho_{ab}e^a e^b = \frac{1}{4}[v_a,[v_b,e^2]]\rho^{ab},$$

hence obtaining

$$\begin{aligned}
e\delta_\chi\omega &= -\frac{1}{4\cdot 3!}\bar{\chi}[v_a,[v_b,e^2]]\gamma^3\rho^{ab} = -\frac{1}{4\cdot 3!}\bar{\chi}\left([v_a,e^2[v_b,\gamma^3]] - [v_b,e^2][v_a,\gamma^3]\right)\\
&= e\bar{\chi}\left(\frac{1}{2}\gamma_a\gamma^2 e_b - \gamma v_a e_b + \frac{1}{4}e\gamma_b\gamma_a\gamma + \frac{1}{2}ev_a\gamma^b\right)\rho^{ab}.
\end{aligned}$$

Recalling the definitions of $\hat{\gamma} := \gamma^a e_a^\mu\partial_\mu \in \mathfrak{X}(M)$, and of the map[2]

$$\begin{aligned}
< e,\cdot >\colon \Omega^{(i,j)} &\to \Omega^{(i-1,j+1)}\\
\alpha &\mapsto v_a\eta^{ad}e_d^\mu\iota_{\partial_\mu}\alpha,
\end{aligned}$$

we have

$$\delta_\chi\omega = \frac{1}{2}\bar{\chi}\iota_{\hat{\gamma}}(\gamma^2 d_\omega\psi) - \bar{\chi}\gamma < e, d_\omega\psi > + \frac{1}{4}e\bar{\chi}\iota_{\hat{\gamma}}\iota_{\hat{\gamma}}(\gamma d_\omega\psi) - \frac{1}{2}e\bar{\chi}\iota_{\hat{\gamma}} < e, d_\omega\psi > . \tag{B.10}$$

[2]Notice that, with this definition, $[e, < \hat{e}, d_\omega\psi >] = 2d_\omega\psi$.

For computational reasons, rather than $[\delta_\chi \omega, \chi]$ it is easier to compute $\frac{1}{3!} e\underline{\gamma}\gamma^3[\delta_\chi\omega, \chi]$, since the multiplication by $\frac{1}{3!}e\underline{\gamma}\gamma^3$ provides an isomorphism from lemma A.12. Furthermore, without loss of generality we can contract the expression with a generic Majorana spinor λ. Using (A.50), we find

$$\begin{aligned}\frac{1}{3!}e\bar{\lambda}\underline{\gamma}\gamma^3[\delta_\chi\omega,\chi] &= \frac{1}{2}\bar{\lambda}\underline{\gamma}\gamma\chi e\delta_\chi\omega + \frac{1}{2\cdot 3!}e\bar{\lambda}\underline{\gamma}[\delta_\chi\omega,\gamma^3]\chi \\ &= -\frac{1}{2\cdot 3!}\bar{\lambda}\underline{\gamma}\gamma\chi\bar{\chi}\gamma^3 d_\omega\psi + \frac{1}{2\cdot 3!}\bar{\lambda}\underline{\gamma}\gamma^3\chi[e,\delta_\chi\omega] \\ &\overset{(A.15)}{=} \frac{1}{2\cdot 3!}\bar{\lambda}\underline{\gamma}\gamma^3\chi[e,\delta_\chi\omega].\end{aligned}$$

Now, a rather long but straightforward computation gives

$$[e,\delta_\chi\omega] = 5\bar{\chi}\gamma d_\omega\psi + \bar{\chi}\gamma\iota_{\hat{\gamma}}(\underline{\gamma}d_\omega\psi) + \bar{\chi} < e, \underline{\gamma}d_\omega\psi > + \frac{1}{4}e\bar{\chi}\iota_{\hat{\gamma}}\iota_{\hat{\gamma}}(\underline{\gamma}d_\omega\psi).$$

We notice that in $\bar{\lambda}\underline{\gamma}\gamma^3\chi[e,\delta_\chi\omega]$ all the terms containing $\bar{\chi}\gamma(\cdot)$ vanish because of lemma A.12. Hence, eliminating the arbitrary spinor λ, we are left with

$$\begin{aligned}\frac{1}{3!}e\underline{\gamma}\gamma^3[\delta_\chi\omega,\chi] &= \frac{1}{2\cdot 3!}\underline{\gamma}\gamma^3\chi\bar{\chi} < e,\underline{\gamma}d_\omega\psi > + \frac{1}{8\cdot 3!}e\underline{\gamma}\gamma^3\chi\bar{\chi}\iota_{\hat{\gamma}}\iota_{\hat{\gamma}}(\underline{\gamma}d_\omega\psi) \\ &\overset{A.5}{=} \frac{1}{3!}e\underline{\gamma}\gamma^3\chi\left(\bar{\chi}\kappa(< e,\underline{\gamma}d_\omega\psi >) + \frac{1}{8}\bar{\chi}\iota_{\hat{\gamma}}\iota_{\hat{\gamma}}(\underline{\gamma}d_\omega\psi)\right),\end{aligned}$$

hence showing

$$[\delta_\chi\omega,\chi] = \left(\bar{\chi}\kappa(< e,\underline{\gamma}d_\omega\psi >) + \frac{1}{8}\bar{\chi}\iota_{\hat{\gamma}}\iota_{\hat{\gamma}}(\underline{\gamma}d_\omega\psi)\right)\chi,$$

and

$$\delta_\chi^2\psi = -\frac{1}{2}\mathrm{L}_\varphi^\omega\psi + \frac{1}{2}\iota_\varphi d_\omega\psi - \left(\bar{\chi}\kappa(< e,\underline{\gamma}d_\omega\psi >) + \frac{1}{8}\bar{\chi}\iota_{\hat{\gamma}}\iota_{\hat{\gamma}}(\underline{\gamma}d_\omega\psi)\right)\chi.$$

We notice that computing $e\delta_\chi^2\omega$ defines $\delta_\chi^2\omega$ uniquely due the fact that the map $e\wedge\cdot$ is an isomorphism on $\Omega^{(1,2)}$. Hence we obtain

$$\begin{aligned}e\delta_\chi^2\omega &= \delta_\chi(e\delta_\chi\omega) - \bar{\chi}\gamma\psi\delta_\chi\omega = \delta_\chi\left(-\frac{1}{3!}\bar{\chi}\gamma^3 d_\omega\psi\right) - \bar{\chi}\gamma\psi\delta_\chi\omega \\ &= \frac{1}{2\cdot 3!}\iota_\varphi(\bar{\psi})\gamma^3 d_\omega\psi + \frac{1}{3!}\bar{\chi}\gamma^3[\delta_\chi\omega,\psi] + \frac{1}{3!}\bar{\chi}\gamma^3[F_\omega,\chi] - \bar{\chi}\gamma\psi\delta_\chi\omega \\ &= \frac{1}{2}\iota_\varphi\left(\frac{1}{3!}\bar{\psi}\gamma^3 d_\omega\psi\right) - \frac{1}{2\cdot 3!}\bar{\psi}\iota_\varphi(\gamma^3 d_\omega\psi) + \frac{1}{3!}\bar{\chi}\gamma^3[\delta_\chi\omega,\psi] \\ &\quad + \frac{1}{2}F_\omega\bar{\chi}\gamma\chi - \bar{\chi}\gamma\psi\delta_\chi\omega \\ &= -\frac{1}{2}e\iota_\varphi F_\omega + \frac{1}{2}\iota_\varphi\left(eF_\omega + \frac{1}{3!}\bar{\psi}\gamma^3 d_\omega\psi\right) - \frac{1}{2\cdot 3!}\bar{\psi}\iota_\varphi(\gamma^3 d_\omega\psi) \\ &\quad + \frac{1}{3!}\bar{\chi}\gamma^3[\delta_\chi\omega,\psi] - \bar{\chi}\gamma\psi\delta_\chi\omega.\end{aligned}$$

Now we can use (A.50) to see

$$\bar{\chi}\gamma\psi\delta_\chi\omega = \frac{1}{3!}\bar{\chi}\gamma^3[\delta_\chi\omega,\psi] - \frac{1}{3!}[\delta_\chi\omega,\bar{\chi}]\gamma^3\psi,$$

hence

$$\begin{aligned} e\delta_\chi^2\omega =& -\frac{1}{2}e\iota_\varphi F_\omega + \frac{1}{2}\iota_\varphi\left(eF_\omega + \frac{1}{3!}\bar\psi\gamma^3 d_\omega\psi\right) - \frac{1}{2\cdot 3!}\bar\psi\iota_\varphi(\gamma^3 d_\omega\psi) - \frac{1}{3!}\bar\psi\gamma^3[\delta_\chi\omega,\chi] \\ =& -\frac{1}{2}e\iota_\varphi F_\omega + \frac{1}{2}\iota_\varphi\left(eF_\omega + \frac{1}{3!}\bar\psi\gamma^3 d_\omega\psi\right) - \frac{1}{2\cdot 3!}\bar\psi\iota_\varphi(\gamma^3 d_\omega\psi) \\ &- \frac{1}{3!}\bar\psi\gamma^3\chi\left(\bar\chi\kappa(< e, \underline{\gamma} d_\omega\psi >) + \frac{1}{8}\bar\chi\iota_{\hat\gamma}\iota_{\hat\gamma}(\underline{\gamma} d_\omega\psi)\right). \end{aligned}$$

B.3.2 Computing Q_0^2

The detailed computation of Q_0^2 goes as follows: we start by using $Q_0^2 = [Q_P C, \delta_\chi] + \delta_\chi^2$, obtaining

$$\begin{aligned} Q_0^2 e =& Q_{PC}(\bar\chi\gamma\psi) + \delta_\chi(\mathrm{L}_\xi^\omega e - [c,e]) - \frac{1}{2}\mathrm{L}_\varphi^\omega e + \frac{1}{2}\iota_\varphi(\mathrm{EoM}_\omega) \\ =& \mathrm{L}_\xi^\omega\bar\chi\gamma\psi - [c,\bar\chi]\gamma\psi - \bar\chi\gamma\mathrm{L}_\xi^\omega\psi + \bar\chi\gamma[c,\psi] + \frac{1}{2}\mathrm{L}_\varphi^\omega e + [\iota_\xi\omega_\chi, e] \\ &- \mathrm{L}_\xi^\omega(\bar\chi\gamma\psi) - [\iota_\xi\delta_\chi\omega,\chi] + [c,\bar\chi\gamma\psi] - \frac{1}{2}\mathrm{L}_\varphi^\omega e + \frac{1}{2}\iota_\varphi(\mathrm{EoM}_\omega) \\ =& \frac{1}{2}\iota_\varphi(\mathrm{EoM}_\omega), \end{aligned}$$

similarly, we have

$$\begin{aligned} Q_0^2\psi =& Q_0(\mathrm{L}_\xi^\omega\psi - [c,\psi] - d_\omega\chi) \\ =& \frac{1}{2}\mathrm{L}_{[\xi,\xi]}^\omega\psi + \frac{1}{2}\mathrm{L}_\varphi^\omega\psi + [\iota_\xi\iota_\xi F_\omega,\psi] - [\mathrm{L}_\xi^\omega c,\psi] + [\iota_\xi\delta_\chi\omega,\psi] \\ &- \mathrm{L}_\xi^\omega\mathrm{L}_\xi^\omega\psi + \mathrm{L}_\xi^\omega[c,\psi] + \mathrm{L}_\xi^\omega d_\omega\chi + \frac{1}{2}[[c,c],\psi] - \frac{1}{2}[\iota_\xi\iota_\xi F_\omega,\psi] \\ &- [\iota_\xi\delta_\chi\omega,\psi] + [c,\mathrm{L}_\xi^\omega\psi] - [c,[c,\psi]] - [c,d_\omega\chi] + \delta_\chi^2\psi \\ &- [\iota_\xi F_\omega,\chi] + [-\mathrm{L}_\xi^\omega c,\chi] + d_\omega\left(-\mathrm{L}_\xi^\omega\chi - [c,\chi]\right) \\ =& \frac{1}{2}\iota_\varphi d_\omega\psi - \left(\bar\chi\kappa(< e, \underline{\gamma} d_\omega\psi >) + \frac{1}{8}\bar\chi\iota_{\hat\gamma}\iota_{\hat\gamma}(\underline{\gamma} d_\omega\psi)\right)\chi, \end{aligned}$$

having noticed the following

- $\frac{1}{2}\mathrm{L}_{[\xi,\xi]}^\omega\psi + [\iota_\xi\iota_\xi F_\omega,\psi] - \mathrm{L}_\xi^\omega\mathrm{L}_\xi^\omega\psi - \frac{1}{2}[\iota_\xi\iota_\xi F_\omega,\psi] = 0$;
- $\mathrm{L}_\xi^\omega d_\omega\chi + d_\omega\mathrm{L}_\xi^\omega\chi - [\iota_\xi F_\omega,\chi] = 0$, since $[\mathrm{L}_\xi^\omega, d_\omega] = [\iota_\xi F_\omega,\cdot]$ on any field;
- $\frac{1}{2}[[c,c],\psi] - [c,[c,\psi]] = 0$ using graded Jacobi identity.

For the connection, we see

$$\begin{aligned} eQ_0(\delta_\chi\omega) =& -\frac{1}{3!}Q_0(\bar\chi\gamma^3 d_\omega\psi) - (\mathrm{L}_\xi^\omega e - [c,e])\delta_\chi\omega + e\delta_\chi^2\omega \\ =& -\frac{1}{3!}(\mathrm{L}_\xi^\omega\bar\chi - [c,\bar\chi])\gamma^3 d_\omega\psi + \frac{1}{3!}\bar\chi\gamma^3[\iota_\xi F_\omega - d_\omega c,\psi] \\ &- \frac{1}{3!}\bar\chi\gamma^3 d_\omega(\mathrm{L}_\xi^\omega\psi - [c,\psi]) - (\mathrm{L}_\xi^\omega e - [c,e])\delta_\chi\omega + e\delta_\chi^2\omega \\ =& -\frac{1}{3!}\mathrm{L}_\xi^\omega(\bar\chi\gamma^3 d_\omega\psi) + \frac{1}{3!}[c,\bar\chi\gamma^3 d_\omega\psi] - (\mathrm{L}_\xi^\omega e - [c,e])\delta_\chi\omega + e\delta_\chi^2\omega \\ =& e(\mathrm{L}_\xi^\omega\delta_\chi\omega - [c,\delta_\chi\omega]) + e\delta_\chi^2\omega, \end{aligned}$$

hence obtaining

$$\begin{aligned}
eQ_0^2\omega &= eQ_0(\iota_\xi F_\omega - d_\omega c + \delta_\chi \omega)\\
&= \frac{1}{2}e\iota_\varphi F_\omega - e\iota_\xi d_\omega \delta_\chi\omega - e[\delta_\chi\omega, c] + ed_\omega \iota_\xi \delta_\chi \omega + e\mathrm{L}_\xi^\omega \delta_\chi\omega\\
&\quad - e[c,\delta_\chi\omega] + e\delta_\chi^2\omega\\
&= \frac{1}{2}\iota_\varphi\,(\mathrm{EoM}_e) - \frac{1}{2\cdot 3!}\bar{\psi}\iota_\varphi(\gamma^3 d_\omega\psi)\\
&\quad - \frac{1}{3!}\bar{\psi}\gamma^3\chi\left(\bar{\chi}\kappa(< e, \underline{\gamma} d_\omega \psi >) + \frac{1}{8}\bar{\chi}\iota_{\hat{\gamma}}\iota_{\hat{\gamma}}(\underline{\gamma} d_\omega\psi)\right)
\end{aligned}$$

For c, χ and ξ, we can do the computations of Q_0^2 right away, obtaining

$$\begin{aligned}
Q_0^2 c &= Q_{PC}(\iota_\xi\delta_\chi\omega) + \delta_\chi\left(\frac{1}{2}\iota_\xi\iota_\xi F_\omega - \frac{1}{2}[c,c] + \iota_\xi\delta_\chi\omega\right)\\
&= \frac{1}{2}\iota_{[\xi,\xi]}\delta_\chi\omega + \iota_\xi \mathrm{L}_\xi^\omega\delta_\chi\omega - \iota_\xi[c,\delta_\chi\omega] + \frac{1}{2}\iota_\xi\iota_\varphi F_\omega - \frac{1}{2}\iota_\xi\iota_\xi d_\omega\delta_\chi\omega\\
&\quad - [\iota_\xi\delta_\chi\omega, c] + \frac{1}{2}\iota_\varphi\delta_\chi\omega + \iota_\xi\delta_\chi^2\omega\\
&= \frac{1}{2}\iota_\varphi\delta_\chi\omega + \iota_\xi Q_0^2\omega,
\end{aligned}$$

having used the fact that $\frac{1}{2}\iota_{[\xi,\xi]}\delta_\chi\omega + \iota_\xi\mathrm{L}_\xi^\omega\delta_\chi\omega - \frac{1}{2}\iota_\xi\iota_\xi d_\omega\delta_\chi\omega = 0$.

For $Q_0^2\xi$, we see

$$\begin{aligned}
Q_0^2(\xi) &= \frac{1}{2}Q_0([\xi,\xi]+\varphi) = \frac{1}{2}[\xi,\varphi] + \frac{1}{2}Q_0(\bar{\chi}\gamma^a\chi e_a^\mu)\partial_\mu\\
&= \frac{1}{2}[\xi,\varphi] + \mathrm{L}_\xi^\omega(\bar{\chi})\gamma^\mu\chi\partial_\mu - [c,\bar{\chi}]\gamma^\mu\chi\partial_\mu - \frac{1}{2}\iota_\varphi\bar{\psi}\gamma^\mu\chi\partial_\mu + \frac{1}{2}\bar{\chi}\gamma^a\chi Q_0(e_a^\mu)\partial_\mu.
\end{aligned}$$

Now, since $e_a^\mu e_\mu^b = \delta_a^b$, we have $Q_0(e_a^\mu) = -e_a^\nu e_b^\mu Q_0(e_\nu^b)$, obtaining

$$\begin{aligned}
Q_0^2(\xi) &= \frac{1}{2}[\varphi,\xi] + \mathrm{L}_\xi^\omega(\bar{\chi})\gamma^\mu\chi\partial_\mu - [c,\bar{\chi}]\gamma^\mu\chi\partial_\mu - \frac{1}{2}\iota_\varphi\bar{\psi}\gamma^\mu\chi\partial_\mu\\
&\quad - \frac{1}{2}e_b^\mu\bar{\chi}\gamma^\nu\chi\left((\mathrm{L}_\xi^\omega e)_\nu^b - [c,e_\nu]^b + \bar{\chi}\gamma^b\psi_\nu\right)\partial_\mu\\
&= -\frac{1}{2}[\xi,\varphi] + \mathrm{L}_\xi^\omega(\bar{\chi})\gamma^\mu\chi\partial_\mu - [c,\bar{\chi}]\gamma^\mu\chi\partial_\mu - \frac{1}{2}\iota_\varphi\bar{\psi}\gamma^\mu\chi\partial_\mu\\
&\quad + \frac{1}{2}\bar{\chi}[c,\gamma]_V^\mu\chi\partial_\mu - \frac{1}{2}\bar{\chi}\gamma^\mu\iota_\varphi\psi\partial_\mu + \frac{1}{2}\bar{\chi}\mathrm{L}_\xi^\omega(\gamma^\mu\partial_\mu)\chi\\
&= -\frac{1}{2}\mathrm{L}_\xi^\omega(\phi) + \frac{1}{2}\mathrm{L}_\xi^\omega(\bar{\chi}\gamma^\mu\chi\partial_\mu) = 0.
\end{aligned}$$

Notice also that this tells us that $Q_0\varphi = \mathrm{L}^{\omega}_{\xi}(\varphi) = [\xi, \varphi]$. Lastly,

$$
\begin{aligned}
Q_0^2\chi &= Q_0\left(\mathrm{L}^{\omega}_{\xi}\chi - [c,\chi] - \frac{1}{2}\iota_\varphi\psi\right)\\
&= \frac{1}{2}\mathrm{L}^{\omega}_{[\xi,\xi]}\chi + \frac{1}{2}\mathrm{L}^{\omega}_{\varphi}\chi + [\iota_\xi\iota_\xi F_\omega, \chi] - [\mathrm{L}^{\omega}_{\xi}c, \chi] + [\iota_\xi\delta_\chi\omega, \chi] - \mathrm{L}^{\omega}_{\xi}\mathrm{L}^{\omega}_{\xi}\chi\\
&\quad + \mathrm{L}^{\omega}_{\xi}[c,\chi] + \frac{1}{2}\mathrm{L}^{\omega}_{\xi}\iota_\varphi\psi - \frac{1}{2}[\iota_\xi\iota_\xi F_\omega, \chi] + \frac{1}{2}[[c,c],\chi] - [\iota_\xi\delta_\chi\omega, \chi]\\
&\quad + [c, \mathrm{L}^{\omega}_{\xi}\chi] - [c,[c,\chi]] - \frac{1}{2}[c, \iota_\varphi\psi] - \frac{1}{2}\iota_{[\xi,\varphi]}\psi + \frac{1}{2}\iota_\varphi\left(\mathrm{L}^{\omega}_{\xi}\psi - [c,\psi] - d_\omega\chi\right)\\
&= 0.
\end{aligned}
$$

B.3.3 Showing the CME

Proof of 5.1. We start by considering the variation of 5.5. The full computation is long and tedious, hence we do not provide the details. However, carefully carrying it out yields the following Hamiltonian vector fields

$$
\begin{aligned}
\mathbb{q}_e &= \frac{1}{2}\iota_\varphi\check{\omega} - \frac{1}{2}\iota_\varphi\check{c}\iota_\xi e - \frac{1}{4}\iota_\varphi(e\iota_\xi\check{c})\\
e\mathbb{q}_\omega &= \frac{1}{2}\iota_\varphi e^{\dashv} + \frac{i}{4\cdot 3!}\iota_\varphi(\bar{\psi}_0^{\dashv}\underline{\gamma})\gamma^3\psi + \frac{i}{4\cdot 3!}\bar{\psi}\gamma^3\iota_\varphi\left(\underline{\gamma}\alpha(\check{\omega}\psi)\right) - \frac{1}{8\cdot 3!}\iota_\varphi\check{c}\bar{\chi}\gamma^3\psi - \frac{1}{8\cdot 3!}\iota_\xi\check{c}\bar{\psi}\gamma^3\iota_\varphi\psi\\
&\quad - \frac{i}{4\cdot 3!}\bar{\psi}\gamma^3\iota_\varphi\left(\underline{\gamma}\alpha(\check{c}\iota_\xi e\psi)\right) + \frac{1}{2\cdot 3!}\bar{\psi}\gamma^3\chi\kappa\left[< e, \bar{\chi}\left(-\frac{i}{2}\underline{\gamma}^2\psi_0^{\dashv} - [\check{\omega},\gamma]\psi - \frac{1}{2}\underline{\gamma}\iota_\xi\check{c}\psi - \iota_\xi\underline{\gamma}\check{c}\psi\right) >\right]\\
&\quad + \frac{1}{16\cdot 3!}\bar{\psi}\gamma^3\chi\bar{\chi}\iota_{\hat{\gamma}}\iota_{\hat{\gamma}}\left(-\frac{i}{2}\underline{\gamma}^2\psi_0^{\dashv} - [\check{\omega},\gamma]\psi - \frac{1}{2}\underline{\gamma}\iota_\xi\check{c}\psi - \iota_\xi\underline{\gamma}\check{c}\psi\right)\\
\mathbb{q}_\psi &= \frac{i}{4}\iota_\varphi(\underline{\gamma}\psi_0^{\dashv}) - \frac{i}{4}\iota_\varphi\left(\underline{\gamma}\alpha(\check{\omega}\psi)\right) - \frac{i}{4}\iota_\varphi\left(\underline{\gamma}\alpha(\check{c}\iota_\xi e\psi)\right) + \frac{1}{8}\iota_\varphi\check{c}\chi - \frac{1}{8}\iota_\varphi(\iota_\xi\check{c}\psi)\\
&\quad + \frac{i}{4}\chi\kappa\left(< e, \bar{\chi}\underline{\gamma}^2\psi_0^{\dashv} + i\bar{\chi}[\check{\omega} - \frac{i}{2}\iota_\xi\check{c}e + \iota_\xi e\check{c}] >\right) + \frac{1}{16}\chi\bar{\chi}\iota_{\hat{\gamma}}\iota_{\hat{\gamma}}(\underline{\gamma}^2\psi_0^{\dashv} + i[\check{\omega} - \frac{i}{2}\iota_\xi\check{c}e + \iota_\xi e\check{c}])\\
\frac{e^2}{2}\mathbb{q}_c &= -\frac{i}{8}\bar{\chi}\iota_\varphi\psi^{\dashv} - \frac{i}{8\cdot 3!}\iota_\varphi(\check{\omega}\bar{\chi}\gamma^3\psi) - \frac{1}{2}\iota_\xi e\iota_\varphi e^{\dashv} + \frac{1}{4}\iota_\xi(e\iota_\varphi e^{\dashv}) - \frac{i}{4\cdot 3!}\iota_\varphi(\bar{\psi}_0^{\dashv}\underline{\gamma})\gamma^3\iota_\xi e\psi\\
&\quad + \frac{i}{4\cdot 3!}\iota_\varphi\left(\alpha(\check{\omega}\bar{\psi})\underline{\gamma}\right)\gamma^3\iota_\xi e\psi - \frac{i}{8}\iota_\xi(\bar{\psi}\iota_\varphi\psi^{\dashv}) - \frac{1}{8\cdot 3!}\iota_\xi(\check{\omega}\bar{\psi}\gamma^3\iota_\varphi\psi)\\
&\quad + \frac{1}{4\cdot 3!}\iota_\xi\left(\bar{\psi}\gamma^3\chi < e, \bar{\chi}([\check{\omega},\gamma]\psi + i\underline{\gamma}^2\psi_0^{\dashv}) >\right) - \frac{1}{2\cdot 3!}\iota_\xi e\bar{\psi}\gamma^3\chi\kappa\left(< e, \bar{\chi}([\check{\omega},\gamma]\psi + i\underline{\gamma}^2\psi_0^{\dashv}) >\right)\\
&\quad + \frac{1}{32\cdot 3!}\iota_\xi e\bar{\psi}\gamma^3\chi\bar{\chi}\iota_{\hat{\gamma}}\iota_{\hat{\gamma}}([\check{\omega},\gamma]\psi + i\underline{\gamma}^2\psi_0^{\dashv}) - \frac{1}{32\cdot 3!}e\iota_\xi\left(\bar{\psi}\gamma^3\chi\bar{\chi}\iota_{\hat{\gamma}}\iota_{\hat{\gamma}}([\check{\omega},\gamma]\psi + i\underline{\gamma}^2\psi_0^{\dashv})\right),
\end{aligned}
$$

while the full vector field Q is obtained by summing $Q = Q_0 + \mathbb{q}$, $Q\chi = Q_0\chi$ and $Q\xi = Q_0\xi$.

Now, to keep the discussion somewhat contained, we explicitly compute Q^2e and show it vanishes, as similar computations and arguments work for the other fields and ghosts too.

Before we begin, we remark that $e\mathbb{q}_{\check{\omega}} = -\check{\omega}\mathbb{q}_e$ and $\frac{e^2}{2}\mathbb{q}_{\check{c}} = -e\mathbb{q}_e\check{c}$. We then start by computing

$\mathfrak{q}(\varphi) = \bar{\chi}\gamma^a\chi\mathfrak{q}(e^a_\mu)\partial_\mu$, obtaining

$$\begin{aligned}\mathfrak{q}(\varphi^\mu) &= -e^\nu_a e^\mu_b \bar{\chi}\gamma^a\chi\mathfrak{q}(e^b_\nu)\\ &= -e^\mu_b\bar{\chi}\gamma^\nu\chi\left(\frac{1}{2}(\iota_\varphi\check{\omega})^b_\nu - \frac{1}{2}(\iota_\varphi\check{c})_\nu\iota_\xi e^b - \frac{1}{4}(\iota_\varphi(e^b\iota_\xi\check{c}))_\nu\right)\\ &= -e^\mu_b\left(\frac{1}{2}\iota_\varphi\iota_\varphi\check{\omega}^b - \frac{1}{2}\iota_\varphi\iota_\varphi\check{c}\iota_\xi e^b - \frac{1}{4}\iota_\varphi\iota_\varphi(e^b\iota_\xi\check{c})\right) = 0,\end{aligned}$$

since ι_φ is odd. Now we have

$$Q^2 e = Q_0^2 e + Q_0\mathfrak{q}e + \mathfrak{q}Q_0 e + \mathfrak{q}^2 e.$$

Notice that $\mathfrak{q}^2 e$ is quadratic in the anti–fields, while the other terms are at most linear, hence we proceed to show $\mathfrak{q}^2 e = 0$ separately. Notice first that from lemma A.4.3, we can equivalently compute $\frac{e^2}{2}Q^2 e$, obtaining

$$\begin{aligned}\frac{e^2}{2}\mathfrak{q}^2 e =& \frac{e^2}{2}\mathfrak{q}\left(\frac{1}{2}\iota_\varphi\check{\omega} - \frac{1}{2}\iota_\varphi\check{c}\iota_\xi e - \frac{1}{4}\iota_\varphi(e\iota_\xi\check{c})\right)\\ =& \frac{e^2}{2}\Bigg[-\frac{1}{2}\iota_\varphi\mathfrak{q}_{\check{\omega}} + \frac{1}{2}\iota_\varphi\mathfrak{q}_{\check{c}}\iota_\xi e + \frac{1}{2}\iota_\varphi\check{c}\iota_\xi\left(\frac{1}{2}\iota_\varphi\check{\omega} - \frac{1}{2}\iota_\varphi\check{c}\iota_\xi e - \frac{1}{4}\iota_\varphi(e\iota_\xi\check{c})\right)\\ &+ \frac{1}{4}\iota_\varphi(e\iota_\xi\mathfrak{q}_{\check{c}}) + \frac{1}{4}\iota_\varphi\left(\iota_\xi\check{c}\left(\frac{1}{2}\iota_\varphi\check{\omega} - \frac{1}{2}\iota_\varphi\check{c}\iota_\xi e - \frac{1}{4}\iota_\varphi(e\iota_\xi\check{c})\right)\right)\Bigg]\\ =& -\frac{e}{4}\iota_\varphi(e\mathfrak{q}_{\check{\omega}}) + \frac{1}{4}\bar{\chi}\gamma\chi e\mathfrak{q}_{\check{\omega}} - \frac{1}{4}\bar{\chi}\gamma\chi\iota_\xi\left(\frac{e^2}{2}\mathfrak{q}_{\check{c}}\right) + \frac{e}{4}\iota_\varphi\iota_\xi\left(\frac{e^2}{2}\mathfrak{q}_{\check{c}}\right)\\ &+ \frac{e^2}{2}\Bigg[\frac{1}{4}\iota_\varphi\check{c}\iota_\xi\iota_\varphi\check{\omega} - \frac{1}{4}\iota_\varphi\check{c}\iota_\xi\iota_\varphi\check{c}\iota_\xi e + \frac{1}{8}\bar{\chi}\gamma\chi\iota_\varphi\check{c}\iota_\xi\iota_\xi\check{c} - \frac{1}{8}\iota_\varphi\check{c}\iota_\varphi\iota_\xi\check{c}\iota_\xi e\\ &+ \frac{1}{4}\iota_\xi\iota_\varphi\check{c}\left(\frac{1}{2}\iota_\varphi\check{\omega} - \frac{1}{2}\iota_\varphi\check{c}\iota_\xi e - \frac{1}{4}\iota_\varphi(e\iota_\xi\check{c})\right)\Bigg].\end{aligned}$$

Making the expressions containing $\mathfrak{q}_{\check{c}}$ and $\mathfrak{q}_{\check{\omega}}$ explicit is quite a cumbersome challenge. The reader will excuse us for not providing all the steps, however, when the dust settles, we are left with

$$\begin{aligned}\frac{e^2}{2}\mathfrak{q}^2 e =& \frac{1}{16}\bar{\chi}\gamma\chi\iota_\varphi\left(\check{\omega}\iota_\xi(e\check{c}) - \check{\omega}^2 - (\iota_\xi e)^2\check{c}^2 + \frac{1}{4}\iota_\xi(e^2)\iota_\xi(\check{c}^2) + e^2(\iota_\xi\check{c})^2\right)\\ =& -\frac{1}{16}\iota_\varphi\left[\bar{\chi}\gamma\chi\left(\check{\omega}\iota_\xi(e\check{c}) - \check{\omega}^2 - (\iota_\xi e)^2\check{c}^2 + \frac{1}{4}\iota_\xi(e^2)\iota_\xi(\check{c}^2) + e^2(\iota_\xi\check{c})^2\right)\right] = 0.\end{aligned}$$

To show that $\frac{e^2}{2}\mathfrak{q}^2 e = 0$, first consider any $\Xi \in \Omega^{(4,2)}$, then the expression above is of the type $\bar{\chi}\gamma\chi\Xi \in \Omega^{(4,3)}$. Now, thanks to lemma A.4.6 and A.12, we can see that there must exist a $\tilde{\theta} \in \Omega^{(1,0)}(\mathbb{S}_M)$ such that

$$\bar{\chi}\gamma\Xi\chi = \frac{1}{3!}[e, e\bar{\chi}\gamma^3\underline{\gamma}\tilde{\theta}].$$

Similarly, using A.4.5, there must exist a $\theta \in \Omega^{(0,1)}$ such that $\tilde{\theta} = [e, \theta]$, hence finding

$$\begin{aligned}
\iota_\varphi(\bar{\chi}\gamma\chi\Xi) &= \frac{1}{3!}\iota_\varphi[e, e\bar{\chi}\gamma^3\underline{\gamma}\tilde{\theta}] \\
&= \frac{1}{3!}[\bar{\chi}\gamma\chi, e\bar{\chi}\gamma^3\underline{\gamma}\tilde{\theta}] - \frac{1}{3!}\underbrace{[e, \bar{\chi}\gamma^a\chi\bar{\chi}\gamma^3\gamma_a\tilde{\theta}]}_{0 \text{ from (A.61)}} + \frac{1}{3!}[e, e\bar{\chi}\gamma^3\underline{\gamma}[\bar{\chi}\gamma\chi, \theta]] \\
&= -\frac{1}{3!}[e, [\bar{\chi}\gamma\chi, e\bar{\chi}\gamma^3\underline{\gamma}]]\theta = -\frac{1}{3!}[e, [\bar{\chi}\gamma\chi, e\bar{\chi}\gamma^3\underline{\gamma}\theta^b]]v_b.
\end{aligned}$$

As it turns out, after some manipulation involving a mixture of Leibniz rule and Fierz identities, we have $[e, [\bar{\chi}\gamma\chi, e\bar{\chi}\gamma^3\underline{\gamma}\theta^b]] = 3[e, \bar{\chi}\gamma\chi\bar{\chi}\gamma^2\underline{\gamma}^2\theta^b]$, while, thanks to (A.50),

$$\begin{aligned}
\bar{\chi}\gamma\chi\bar{\chi}\gamma^2\underline{\gamma}^2\theta^b &= \frac{1}{3}\bar{\chi}\gamma^3[\bar{\chi}\gamma^2\underline{\gamma}^2\theta^b, \chi] = \frac{1}{3!}\bar{\chi}\gamma^3\gamma^a\gamma^c\chi\bar{\chi}\gamma_c\gamma_a\underline{\gamma}^2\theta^b \\
&\overset{(A.60)}{=} -\frac{1}{3!}\bar{\chi}\gamma^3\gamma^a\underline{\gamma}^2\theta^b\bar{\chi}\gamma_a\chi \overset{(A.61)}{=} -\frac{1}{3}\bar{\chi}\gamma^3\gamma^a\chi\bar{\chi}\gamma^a\underline{\gamma}^2\theta^b \\
&\overset{(A.54)}{=} \frac{1}{3}\bar{\chi}\gamma^3\gamma^a\chi\bar{\theta}^b\underline{\gamma}^2\gamma^a\chi \overset{(A.61)}{=} \frac{1}{3!}\bar{\chi}\gamma^3\gamma^a\underline{\gamma}^2\theta^b\bar{\chi}\gamma_a\chi \\
&\overset{(A.61)}{=} -\bar{\chi}\gamma\chi\bar{\chi}\gamma^2\underline{\gamma}^2\theta^b = 0,
\end{aligned} \tag{B.11}$$

hence showing $\frac{e^2}{2}\mathbb{q}^2 e = 0$.

We now consider the remaining terms of Q^2e, which, after some rearranging, read

$$\begin{aligned}
&\mathbb{q}\left(\mathrm{L}_\xi^\omega e - [c, e]c + \bar{\chi}\gamma\psi\right) + Q_0\left(\frac{1}{2}\iota_\varphi\check{\omega} - \frac{1}{2}\iota_\varphi\check{c}\iota_\xi e - \frac{1}{4}\iota_\varphi(\iota_\xi ce)\right) = \\
=&[\iota_\xi\mathbb{q}_\omega, e] - [\mathbb{q}_c, e] - \bar{\chi}\gamma\mathbb{q}_\psi - \frac{1}{2}\iota_\varphi(Q_0\check{\omega}) + \frac{1}{2}\iota_\varphi(Q_0\check{c}\iota_\xi e) + \frac{1}{4}\iota_\varphi(e\iota_\xi Q_0\check{c}) \\
&+ \frac{1}{2}\iota_\varphi\mathrm{L}_\xi^\omega\check{\omega} - \frac{1}{2}\iota_\varphi\mathrm{L}_\xi^\omega\check{c}\iota_\xi e - \frac{1}{4}\iota_\varphi\check{c}\iota_{[\xi,\xi]}e - \frac{1}{8}\iota_\varphi\iota_{[\xi,\xi]}\check{c}e + \frac{1}{8}\iota_{[\xi,\xi]}\check{c}\bar{\chi}\gamma\chi \\
&- \frac{1}{4}e\iota_\varphi\iota_\xi\mathrm{L}_\xi^\omega\check{c} + \frac{1}{4}\iota_\xi\mathrm{L}_\xi^\omega\check{c}\bar{\chi}\gamma\chi - \frac{1}{2}\iota_\varphi[c, \check{\omega}] - \frac{1}{8}\bar{\chi}\gamma\chi\iota_\varphi\check{c} + \frac{1}{2}\iota_\varphi\check{c}\bar{\chi}\gamma\iota_\xi\psi \\
&+ \frac{1}{4}\iota_\varphi\iota_\xi\check{c}\bar{\chi}\gamma\psi - \frac{1}{4}\iota_\xi\check{c}\bar{\chi}\gamma\iota_\varphi\psi.
\end{aligned} \tag{B.12}$$

A few remarks are in order. First of all, notice that the term $-\frac{1}{2}\iota_\varphi Q_0\check{\omega}$ contains a term (proportional to the equations of motion) that cancels out exactly the non zero part of $Q_0^2 e$. Secondly, we notice that, in order to obtain $Q^2e = 0$, we need to implement some terms in $\mathbb{q}_c$ to balance out $\iota_\xi\mathbb{q}_\omega$, in particular we are missing all the terms proportional to $\check{c}$. Explicitly, $e\mathbb{q}_\omega$ contains

$$\begin{aligned}
e\mathbb{I}(\check{c}, \xi, \chi, \psi) \coloneqq &- \frac{1}{8\cdot 3!}\iota_\xi\check{c}\bar{\psi}\gamma^3\iota_\varphi\psi - \frac{i}{4\cdot 3!}\bar{\psi}\gamma^3\iota_\varphi\left(\underline{\gamma}\alpha(\check{c}\iota_\xi e\psi)\right) \\
&- \frac{1}{2\cdot 3!}\bar{\psi}\gamma^3\chi\kappa\left[< e, \bar{\chi}\left(\frac{1}{2}\underline{\gamma}\iota_\xi\check{c}\psi + \iota_\xi\underline{\gamma}\check{c}\psi\right) >\right] \\
&- \frac{1}{16\cdot 3!}\bar{\psi}\gamma^3\chi\bar{\chi}\iota_{\hat{\gamma}}\iota_{\hat{\gamma}}\left(\frac{1}{2}\underline{\gamma}\iota_\xi\check{c}\psi + \iota_\xi\underline{\gamma}\check{c}\psi\right),
\end{aligned} \tag{B.13}$$

hence, to cancel them in the computation of $Q_0^2 e$ (and in general Q_0^2), we need to add[3] to $\mathbb{q}_c$ terms of the kind $\iota_\xi\mathbb{I}(\check{c}, \xi, \varphi, \psi)$, resulting in a correction term in s_2

$$\frac{1}{2}c^\dashv\iota_\xi\mathbb{I}(\check{c}, \xi, \varphi, \psi).$$

[3]Similarly to adding $\iota_\xi\delta_\chi\omega$ to Q_0c

Hence, in the computation of $\frac{e^2}{2}\big([\iota_\xi \mathbb{q}_\omega, e] - [\mathbb{q}_c, e]\big)$ we are left with

$$\begin{aligned}\frac{e^2}{2}\big([\iota_\xi \mathbb{q}_\omega, e] - [\mathbb{q}_c, e]\big) =& [e, \frac{e^2}{2}\mathbb{q}_c] + \frac{1}{2}[e, e\mathbb{q}_\omega \iota_\xi e] - \frac{1}{2}[e, e\iota_\xi(e\mathbb{q}_\omega)] \\ =& \frac{e}{16\cdot 3!}[\iota_\xi\left(\iota_\varphi \check{c}\bar{\chi}\gamma^3\psi\right), e] - \frac{1}{16\cdot 3!}[\iota_\xi e \iota_\varphi \check{c}\bar{\chi}\gamma^3\psi, e] \\ &- \frac{i}{8}[\bar{\chi}\iota_\varphi\psi^{\lrcorner}, e] + \frac{1}{8\cdot 3!}[\iota_\varphi(\check{\omega}\bar{\chi}\gamma^3\psi), e].\end{aligned}$$

Similarly, when computing $-\frac{1}{2}\iota_\varphi Q_0\check{\omega} + \frac{1}{2}\iota_\varphi Q_0\check{c}\iota_\xi e + \frac{1}{4}\iota_\varphi(e\iota_\xi Q_0\check{c})$, we notice that a lot of the terms in $Q_0\check{\omega}$ are canceled out by $\iota_\xi Q_0\check{c}$.[4] In particular, after noticing that

$$-\frac{e^2}{4}\iota_\varphi Q_0\check{\omega} = -\frac{e}{4}\iota_\varphi(eQ_0\check{\omega}) + \frac{1}{4}\bar{\chi}\gamma\chi e Q_0\check{\omega}$$

and

$$\frac{e^2}{2}\left(\frac{1}{2}\iota_\varphi Q_0\check{c}\iota_\xi e + \frac{1}{4}\iota_\varphi(e\iota_\xi Q_0\check{c})\right) = \frac{e}{4}\iota_\varphi\iota_\xi\left(\frac{e^2}{2}Q_0\check{c}\right) - \frac{1}{4}\bar{\chi}\gamma\chi\iota_\xi\left(\frac{e^2}{2}Q_0\check{c}\right),$$

one finds that the remaining terms are

$$\begin{aligned}&\frac{e^2}{2}\left(-\frac{1}{2}\iota_\varphi Q_0\check{\omega} + \frac{1}{2}\iota_\varphi Q_0\check{c}\iota_\xi e + \frac{1}{4}\iota_\varphi(e\iota_\xi Q_0\check{c})\right) = \\ &= -\frac{e^2}{4}\iota_\varphi \mathrm{EoM}_\omega + \underline{\frac{e}{4}\iota_\varphi d_\omega \iota_\xi \omega^{\lrcorner}}_{1} - \underline{\frac{1}{4}\bar{\chi}\gamma\chi d_\omega\iota_\xi\omega^{\lrcorner}}_{2} - \frac{e^2}{4}[c, \iota_\varphi\check{\omega}] - \underline{\frac{e}{8}\iota_\varphi(d_\omega\iota_\xi\iota_\xi c^{\lrcorner} + \iota_\xi\iota_\xi d_\omega c^{\lrcorner})}_{3} \\ &\quad + \underline{\frac{1}{8}\bar{\chi}\gamma\chi(d_\omega\iota_\xi\iota_\xi c^{\lrcorner} + \iota_\xi\iota_\xi d_\omega c^{\lrcorner})}_{4} + \underline{\frac{e}{4}\iota_\varphi(\check{\omega}\mathrm{L}_\xi^\omega e)}_{5} - \underline{\frac{1}{4}\bar{\chi}\gamma\chi\check{\omega}\mathrm{L}_\xi^\omega e}_{6} + \frac{e}{4}\iota_\varphi\left(\frac{1}{2}\bar{\chi}\gamma\psi\check{\omega} + \frac{1}{2\cdot 3!}\bar{\chi}[\check{\omega}, \gamma^3]\psi\right) \\ &\quad - \frac{1}{4}\bar{\chi}\gamma\chi\left(\frac{1}{2}\bar{\chi}\gamma\psi\check{\omega} + \frac{1}{2\cdot 3!}\bar{\chi}[\check{\omega}, \gamma^3]\psi\right) + \frac{e}{4}\iota_\varphi\left[\frac{1}{2}\bar{\chi}\gamma\psi\left(\iota_\xi e\check{c} + \frac{e}{2}\iota_\xi\check{c}\right) - \frac{1}{2\cdot 3!}\bar{\chi}[\left(\iota_\xi e\check{c} + \frac{e}{2}\iota_\xi\check{c}\right), \gamma^3]\psi\right] \\ &\quad - \frac{1}{4}\bar{\chi}\gamma\chi\left[\frac{1}{2}\bar{\chi}\gamma\psi\left(\iota_\xi e\check{c} + \frac{e}{2}\iota_\xi\check{c}\right) - \frac{1}{2\cdot 3!}\bar{\chi}[\left(\iota_\xi e\check{c} + \frac{e}{2}\iota_\xi\check{c}\right), \gamma^3]\psi\right] \\ &\quad - \frac{e}{4}\iota_\varphi\left(\frac{i}{2}e\bar{\psi}_0^{\lrcorner}\underline{\gamma}\gamma\chi - \frac{1}{2\cdot 3!}\bar{\psi}_0^{\lrcorner}\underline{\gamma}[e, \gamma^3]\chi\right) + \frac{1}{4}\bar{\chi}\gamma\chi\left(\frac{i}{2}e\bar{\psi}_0^{\lrcorner}\underline{\gamma}\gamma\chi - \frac{1}{2\cdot 3!}\bar{\psi}_0^{\lrcorner}\underline{\gamma}[e, \gamma^3]\chi\right) \\ &\quad - \underline{\frac{e}{4}\iota_\varphi\iota_\xi\left(\frac{1}{2}\mathrm{L}_\xi^\omega(e^2)\check{c}\right)}_{7} + \underline{\frac{1}{4}\bar{\chi}\gamma\chi\iota_\xi\left(\frac{1}{2}\mathrm{L}_\xi^\omega(e^2)\check{c}\right)}_{8} - \frac{e}{4}\iota_\varphi\iota_\xi(\bar{\chi}\gamma\psi e\check{c}) + \frac{1}{4}\bar{\chi}\gamma\chi\iota_\xi(\bar{\chi}\gamma\psi e\check{c}) \\ &\quad - \underline{\frac{e}{4}\iota_\varphi\iota_\xi d_\omega\omega^{\lrcorner}}_{9} + \underline{\frac{1}{4}\iota_\xi d_\omega\omega^{\lrcorner}}_{10},\end{aligned} \tag{B.14}$$

where we added terms proportional to $\iota_\xi\iota_\xi d_\omega c^{\lrcorner}$, which vanish since $d_\omega c^{\lrcorner} = 0$.

Now we notice that

- $(B.14.1) + (B.14.2) + (B.14.5) + (B.14.6) + (B.14.9) + (B.14.10) = -\frac{e^2}{2}\iota_\varphi \mathrm{L}_\xi^\omega\check{\omega}$;
- using the identity [CCS21a]

$$\frac{1}{2}\iota_{[\xi,\xi]}A = -\frac{1}{2}\iota_\xi\iota_\xi d_\omega A + \iota_\xi d_\omega\iota_\xi A - \frac{1}{2}d_\omega\iota_\xi\iota_\xi A$$

[4]All the terms in $Q_0\check{\omega}$ coming from the variation in $\mathcal{S}_1$ of $\mathrm{L}_\xi^\omega(\cdot)$ with respect of ω are exactly canceled by the ones in $\iota_\xi Q_0\check{c}$ coming from the variation of $[c,\cdot]$ with respect to c

and the fact that $L_\xi \omega c^{\lrcorner} = -d_\omega \iota_\xi c^{\lrcorner}$, then

$$(B.14.3)+(B.14.4)+(B.14.7)+(B.14.8)=$$
$$=\frac{e^2}{2}\left(\frac{1}{4}\iota_\varphi\check{c}\iota_{[\xi,\xi]}e+\frac{1}{8}e\iota_\varphi\iota_{[\xi,\xi]}\check{c}+\frac{1}{2}\iota_\xi e\iota_\varphi L_\xi^\omega\check{c}+\frac{1}{8}\bar\chi\gamma\chi\iota_{[\xi,\xi]}\check{c}+\frac{1}{4}\bar\chi\gamma\chi\iota_\xi L_\xi^\omega\check{c}+\frac{e}{4}\iota_\varphi\iota_\xi L_\xi^\omega\check{c}\right).$$

Now we can finally compute the full $Q^2 e$, taking into consideration the full expression of $\bar\chi\gamma\mathfrak{q}_\psi$, from (B.12)

$$\begin{aligned}
Q^2e =&\underline{\frac{e}{16\cdot 3!}[\iota_\xi\left(\iota_\varphi\check{c}\bar\chi\gamma^3\psi\right),e]}_1-\underline{\frac{1}{16\cdot 3!}[\iota_\xi e\iota_\varphi\check{c}\bar\chi\gamma^3\psi,e]}_2-\underline{\frac{i}{8}[\bar\chi\iota_\varphi\psi^{\lrcorner},e]}_3+\underline{\frac{1}{8\cdot 3!}[\iota_\varphi(\check\omega\bar\chi\gamma^3\psi),e]}_4\\
&-\underline{\frac{e^2}{2}\iota_\varphi L_\xi^\omega\check\omega}_5-\underline{\frac{e^2}{4}[c,\iota_\varphi\check\omega]}_6+\underline{\frac{e}{4}\iota_\varphi\left(\frac{1}{2}\bar\chi\gamma\psi\check\omega+\frac{1}{2\cdot 3!}\bar\chi[\check\omega,\gamma^3]\psi\right)}_7\\
&+\frac{e^2}{2}\left(\underline{\frac{1}{4}\iota_\varphi\check{c}\iota_{[\xi,\xi]}e}_8+\underline{\frac{1}{8}e\iota_\varphi\iota_{[\xi,\xi]}\check{c}}_9+\underline{\frac{1}{2}\iota_\xi e\iota_\varphi L_\xi^\omega\check{c}}_{10}+\underline{\frac{1}{8}\bar\chi\gamma\chi\iota_{[\xi,\xi]}\check{c}}_{11}+\underline{\frac{1}{4}\bar\chi\gamma\chi\iota_\xi L_\xi^\omega\check{c}}_{12}+\underline{\frac{e}{4}\iota_\varphi\iota_\xi L_\xi^\omega\check{c}}_{13}\right)\\
&-\underline{\frac{1}{4}\bar\chi\gamma\chi\left(\frac{1}{2}\bar\chi\gamma\psi\check\omega+\frac{1}{2\cdot 3!}\bar\chi[\check\omega,\gamma^3]\psi\right)}_{14}+\frac{e}{4}\iota_\varphi\left[\underline{\frac{1}{2}\bar\chi\gamma\psi\left(\iota_\xi e\check{c}+\frac{e}{2}\iota_\xi\check{c}\right)}_{15}-\underline{\frac{1}{2\cdot 3!}\bar\chi[\left(\iota_\xi e\check{c}+\frac{e}{2}\iota_\xi\check{c}\right)}_{16},\gamma^3]\psi\right]\\
&-\frac{1}{4}\bar\chi\gamma\chi\left[\underline{\frac{1}{2}\bar\chi\gamma\psi\left(\iota_\xi e\check{c}+\frac{e}{2}\iota_\xi\check{c}\right)}_{17}-\underline{\frac{1}{2\cdot 3!}\bar\chi[\left(\iota_\xi e\check{c}+\frac{e}{2}\iota_\xi\check{c}\right)}_{18},\gamma^3]\psi\right]\\
&-\underline{\frac{e}{4}\iota_\varphi\left(\frac{i}{2}e\bar\psi_0^{\lrcorner}\underline{\gamma}\gamma\chi-\frac{1}{2\cdot 3!}\bar\psi_0^{\lrcorner}\underline{\gamma}[e,\gamma^3]\chi\right)}_{19}+\underline{\frac{1}{4}\bar\chi\gamma\chi\left(\frac{i}{2}e\bar\psi_0^{\lrcorner}\underline{\gamma}\gamma\chi-\frac{1}{2\cdot 3!}\bar\psi_0^{\lrcorner}\underline{\gamma}[e,\gamma^3]\chi\right)}_{20}\\
&-\underline{\frac{e}{4}\iota_\varphi\iota_\xi(\bar\chi\gamma\psi e\check{c})}_{21}+\underline{\frac{1}{4}\bar\chi\gamma\chi\iota_\xi(\bar\chi\gamma\psi e\check{c})}_{22}-\frac{e^2}{2}\bar\chi\gamma\left[\underline{\frac{i}{4}\iota_\varphi(\underline{\gamma}\psi_0^{\lrcorner})}_{23}-\underline{\frac{i}{4}\iota_\varphi\left(\underline{\gamma}\alpha(\check\omega\psi)\right)}_{24}-\underline{\frac{i}{4}\iota_\varphi\left(\underline{\gamma}\alpha(\check{c}\iota_\xi e\psi)\right)}_{25}\right]\\
&-\frac{e^2}{2}\bar\chi\gamma\left[\underline{\frac{1}{8}\iota_\varphi\check{c}\chi}_{26}-\underline{\frac{1}{8}\iota_\varphi(\iota_\xi\check{c}\psi)}_{27}+\frac{i}{4}\chi\kappa\left(< e,\underline{\bar\chi\underline{\gamma}^2\psi_0^{\lrcorner}}_{28}+i\bar\chi[\underline{\check\omega}_{29}-\underline{\frac{i}{2}\iota_\xi\check{c}e}_{30}+\underline{\iota_\xi e\check{c}}_{31},\gamma]\psi>\right)\right]\\
&-\frac{e^2}{2}\bar\chi\gamma\left(\frac{1}{16}\chi\bar\chi\iota_{\hat\gamma}\iota_{\hat\gamma}(\underline{\underline{\gamma}^2\psi_0^{\lrcorner}}_{32}+i[\underline{\check\omega}_{33}-\underline{\frac{i}{2}\iota_\xi\check{c}e}_{34}+\underline{\iota_\xi e\check{c}}_{35},\gamma]\psi)\right)+\frac{e^2}{2}\left(\underline{\frac{1}{4}\iota_\varphi\iota_\xi\check{c}\bar\chi\gamma\psi}_{36}-\underline{\frac{1}{4}\iota_\xi\check{c}\bar\chi\gamma\iota_\varphi\psi}_{37}\right)\\
&+\frac{e^2}{2}\left(\underline{\frac{1}{2}\iota_\varphi L_\xi^\omega\check\omega}_{38}-\underline{\frac{1}{2}\iota_\varphi L_\xi^\omega\check{c}\iota_\xi e}_{39}-\underline{\frac{1}{4}\iota_\varphi\check{c}\iota_{[\xi,\xi]}e}_{40}-\underline{\frac{1}{8}\iota_\varphi\iota_{[\xi,\xi]}\check{c}e}_{41}+\underline{\frac{1}{8}\iota_{[\xi,\xi]}\check{c}\bar\chi\gamma\chi}_{42}\right)\\
&+\frac{e^2}{2}\left(-\underline{\frac{1}{4}e\iota_\varphi\iota_\xi L_\xi^\omega\check{c}}_{43}+\underline{\frac{1}{4}\iota_\xi L_\xi^\omega\check{c}\bar\chi\gamma\chi}_{44}-\underline{\frac{1}{2}\iota_\varphi[c,\check\omega]}_{45}-\underline{\frac{1}{8}\bar\chi\gamma\chi\iota_\varphi\check{c}}_{46}+\underline{\frac{1}{2}\iota_\varphi\check{c}\bar\chi\gamma\iota_\xi\psi}_{47}\right).
\end{aligned} \tag{B.15}$$

We can regroup the above terms to show that the total sum is zero. We immediately see

- $(B.15.5)+(B.15.38)=0$,
- $(B.15.6)+(B.15.45)=0$,
- $(B.15.8)+(B.15.40)=0$,
- $(B.15.9)+(B.15.41)=0$,
- $(B.15.10)+(B.15.39)=0$,
- $(B.15.11)+(B.15.42)=0$,
- $(B.15.12)+(B.15.44)=0$,

- $(B.15.13) + (B.15.43) = 0$,
- $(B.15.26) + (B.15.46) = 0$,

For the remaining terms there is a recurring pattern which we explicitly show just once. Consider for example $(B.15.3) + (B.15.19) + (B.15.20) + (B.15.23) + (B.15.28) + (B.15.32)$, we have, after expanding the terms

- $(B.15.3) = \frac{i}{8} e \left(\frac{1}{2} \bar{\chi}\gamma^2 \underline{\gamma} \iota_\varphi (\underline{\gamma}\psi_0^{\dashv}) + e\bar{\chi}\gamma \iota_\varphi (\underline{\gamma}\psi_0^{\dashv}) \right)$,
- $(B.15.19) + (B.15.20) = \frac{i}{16} e \iota_\varphi (\bar{\psi}_0^{\dashv} \underline{\gamma}) \underline{\gamma} \gamma^2 \chi + \frac{i}{8 \cdot 3!} e \bar{\psi}_0^{\dashv} \underline{\gamma} [\bar{\chi}\gamma\chi, \gamma^3]\chi - \frac{i}{8 \cdot 3!} \bar{\chi}\gamma\chi \bar{\psi}_0^{\dashv} \underline{\gamma} [e, \gamma^3]\chi$,
- the term $(B.15.28) + (B.15.32)$ presents an added difficulty, which can be resolved once one notices that, backwards engineering the methods used to compute $\delta_\chi^2 \psi$ in the previous section,[5] it can be rewritten as

$$\begin{aligned} (B.15.28) + (B.15.32) &= \frac{e^2}{8} \left[(W_1^{(2,3)})^{-1} \left(\frac{i}{3!} \bar{\chi}\gamma^3 \underline{\gamma}\psi_0^{\dashv} \right), \bar{\chi}\gamma\chi \right] \\ &= -\frac{e}{8} \left[\bar{\chi}\gamma\chi, \frac{i}{3!} \bar{\chi}\gamma^3 \underline{\gamma}\psi_0^{\dashv} \right] - \frac{i}{8 \cdot 3!} \bar{\chi}\gamma^3 \underline{\gamma}\psi_0^{\dashv} [e, \bar{\chi}\gamma\chi] \\ &\overset{A.15}{=} -\frac{e}{8} \left[\bar{\chi}\gamma\chi, \frac{i}{3!} \bar{\chi}\gamma^3 \underline{\gamma}\psi_0^{\dashv} \right] + \frac{i}{8 \cdot 3!} \bar{\chi} [e, \gamma^3] \underline{\gamma}\psi_0^{\dashv} \bar{\chi}\gamma\chi. \end{aligned}$$

  it is a simple matter of algebra to see $(B.15.3) + (B.15.19) + (B.15.20) + (B.15.23) + (B.15.28) + (B.15.32) = 0$.

As previously anticipated, one can analogously show the following terms vanish

- $(B.15.1) + (B.15.2) + (B.15.2) + (B.15.15) + (B.15.16) + (B.15.17) + (B.15.18) + (B.15.21) + (B.15.22) + (B.15.25) + (B.15.27) + (B.15.30) + (B.15.31) + (B.15.34) + (B.15.35) + (B.15.36) + (B.15.37) + (B.15.47) = 0$
- $(B.15.4) + (B.15.7) + (B.15.14) + (B.15.29) + (B.15.33) = 0$

Now, in order to show that $Q^2 = 0$ when computed on the other fields and ghosts, one needs to perform similar manipulations as in the case of $Q^2 e$, but we think that explicitly carrying them out, while equally (if not more) challenging, does not provide any further insight. □

B.3.4 BV pushforward computations

Proof. Proof of 5.3 Adapting the proof from [CC25b], one can easily see that, considering the

[5] In this particular case, it suffices to notice that $(B.15.28) + (B.15.32)$ is exactly equal to

$$-\frac{e^2}{4} \bar{\chi}\gamma[\alpha, \chi],$$

which is easily seen after comparing it with the expression of $[\delta_\chi \omega, \chi]$. One then just substitutes $d_\omega \psi$ in $\delta_\chi \omega$ with $i\underline{\gamma}^2 \psi_0^{\dashv}$ to find that α is such that $e\alpha = \frac{i}{3!} \bar{\chi}\gamma^3 \underline{\gamma}\psi_0^{\dashv}$. Then the above expression becomes

$$-\frac{e^2}{4} \bar{\chi}\gamma[\alpha, \chi] = -\frac{e^2}{8} [\alpha, \bar{\chi}\gamma\chi].$$

extra terms inside the supergravity structural constraint, we have

$$
\begin{aligned}
&\left(\phi_2^{PC}\right)^* \left(\mathcal{S}_{PC}^r + \int_{I\times\Sigma} \frac{1}{2}\underline{\tilde{e}_n}\tilde{e}[\tilde{v}-y^{\dashv}, \tilde{v}-y^{\dashv}] + g(\tilde{v}^{\dashv})\right) = \mathcal{S}_{PC}^r + \int_{I\times\Sigma}\frac{1}{2}\underline{\tilde{e}_n}\tilde{e}[\tilde{v},\tilde{v}] + h_{PC}(\tilde{v}^{\dashv})\\
&+\int_{I\times\Sigma} \underline{\mathrm{L}_{\tilde{\xi}}^{\hat{\omega}}\tilde{e}x^{\dashv}\tilde{\mu}^{\dashv}}_1 - \underline{\epsilon_n[\mathrm{L}_{\tilde{\xi}}^{\hat{\omega}}e, y^{\dashv}]\tilde{\mu}^{\dashv}}_2\\
&\quad - \underline{\epsilon_n[e, \mathrm{L}_{\tilde{\xi}}^{\hat{\omega}}y^{\dashv}]\tilde{\mu}^{\dashv}}_3 + \underline{\epsilon_n \mathrm{L}_{\tilde{\xi}}^{\hat{\omega}}(\tilde{\bar{\psi}})\gamma\tilde{\psi}\tilde{\mu}^{\dashv}}_4 + \underline{\frac{1}{2}\mathrm{L}_{\tilde{\xi}}^{\hat{\omega}}(\epsilon_n)^i\tilde{\mu}_i^{\dashv}e\tilde{\bar{\psi}}\gamma\tilde{\psi}}_5\\
&\quad + \underline{\frac{1}{2}(d_{\tilde{\omega}_n}\epsilon_n)^i\tilde{\mu}_i^{\dashv}\tilde{\xi}^n\tilde{\bar{\psi}}\gamma\tilde{\psi}}_6 + \underline{(d_{\tilde{\omega}_n}e)\tilde{\xi}^n x^{\dashv}\tilde{\mu}^{\dashv}}_7 + \underline{\epsilon_n[d_{\tilde{\omega}_n}e, y^{\dashv}]\tilde{\xi}^n\tilde{\mu}^{\dashv}}_8\\
&\quad + \underline{\epsilon_n[e, d_{\tilde{\omega}_n}y^{\dashv}]\tilde{\xi}^n\tilde{\mu}^{\dashv}}_9 - \underline{\epsilon_n d_{\tilde{\omega}_n}\tilde{\bar{\psi}}\gamma\tilde{\psi}\tilde{\xi}^n\tilde{\mu}^{\dashv}}_{10} + \underline{\frac{1}{2}\tilde{e}_n\nu\tilde{\bar{\psi}}\gamma\tilde{\psi}}_{11} + \underline{\frac{1}{2}\tilde{e}\iota_z\nu\tilde{\bar{\psi}}\gamma\tilde{\psi}}_{12}\\
&\quad - \underline{\tilde{e}_n x^{\dashv}d\tilde{\xi}^n\tilde{\mu}^{\dashv}}_{13} + \underline{\epsilon_n[\tilde{e}_n, y^{\dashv}]d\tilde{\xi}^n\tilde{\mu}^{\dashv}}_{14} + \underline{\iota_z y^{\dashv}d\tilde{\xi}^n\tilde{v}^{\dashv}}_{15} - \underline{\epsilon_n[\tilde{c}, \tilde{\bar{\psi}}]\gamma\tilde{\psi}\tilde{\mu}^{\dashv}}_{16}\\
&\quad + \underline{\epsilon_n[[\tilde{c},\tilde{e}], y^{\dashv}]\tilde{\mu}^{\dashv}}_{17} - \underline{[\tilde{c}, y^{\dashv}]\tilde{v}^{\dashv}}_{18} - \underline{[c,e]x^{\dashv}\tilde{\mu}^{\dashv}}_{19} - \underline{\frac{1}{2}[\tilde{c},\epsilon_n]^i\tilde{\mu}_i^{\dashv}\tilde{e}\tilde{\bar{\psi}}\gamma\tilde{\psi}}_{20} + \underline{(\delta_\chi\nu + \mathbb{q}\nu)\,\underline{\tilde{v}}^{\dashv}}_{21}
\end{aligned}
\tag{B.16}
$$

where $h_{PC}(\tilde{v}^{\dashv}) = f(v^{\dashv}) + (\iota_{\tilde{\xi}}F_{\hat{\omega}} + F_{\tilde{\omega}_n}\tilde{\xi}^n + d_{\hat{\omega}}\tilde{c})\tilde{v}^{\dashv}$. We also easily see that

$$
\begin{aligned}
&\phi_3^*\left(\mathcal{S}_{PC}^r + \int_{I\times\Sigma}\frac{1}{2}\underline{\tilde{e}_n}\tilde{e}[\tilde{v}-y^{\dashv}, \tilde{v}-y^{\dashv}] + g(\tilde{v}^{\dashv})\right) =\\
&= \int_{I\times\Sigma}\underline{(\delta_\chi y^{\dashv} + \mathbb{q}y^{\dashv})\tilde{v}^{\dashv}}_1 + \mathrm{L}_{\tilde{\xi}}^{\hat{\omega}}\tilde{e}\left(-\underline{x^{\dashv}\tilde{\mu}^{\dashv}}_2 + \underline{[\epsilon_n\tilde{\mu}^{\dashv}, y^{\dashv}]}_3\right) + d_{\tilde{\omega}\tilde{e}}\tilde{\xi}^n\left(-\underline{x^{\dashv}\tilde{\mu}^{\dashv}}_4 + \underline{[\epsilon_n\tilde{\mu}^{\dashv}, y^{\dashv}]}_5\right)\\
&\quad + \tilde{e}_n d\tilde{\xi}^n\left(-\underline{x^{\dashv}\tilde{\mu}^{\dashv}}_6 + \underline{[\epsilon_n\tilde{\mu}^{\dashv}, y^{\dashv}]}_7\right) - [\tilde{c},\tilde{e}]\left(-\underline{x^{\dashv}\tilde{\mu}^{\dashv}}_8 + \underline{[\epsilon_n\tilde{\mu}^{\dashv}, y^{\dashv}]}_9\right) + \underline{\mathrm{L}_{\tilde{\xi}}^{\hat{\omega}}y^{\dashv}\tilde{v}^{\dashv}}_{10}\\
&\quad + \underline{d_{\tilde{\omega}_n}y^{\dashv}\tilde{\xi}^n\tilde{v}^{\dashv}}_{11} - \underline{[c, y^{\dashv}]\tilde{v}^{\dashv}}_{12}.
\end{aligned}
\tag{B.17}
$$

We immediately see

- $(B.17.2) + (B.16.1) = 0.$
- $(B.17.3) + (B.16.3) = 0.$
- $(B.17.4) + (B.16.8) = 0.$
- $(B.17.5) + (B.16.9) = 0.$
- $(B.17.6) + (B.16.14) = 0.$
- $(B.17.7) + (B.16.15) = 0.$
- $(B.17.8) + (B.16.20) = 0.$
- $(B.17.9) + (B.16.19) = 0.$
- $(B.17.10) + (B.16.3) = 0.$
- $(B.17.11) + (B.16.9) = 0.$
- $(B.17.12) + (B.16.18) = 0.$

We are left with computing $\phi_2^*(s^r_\psi + s^r_2)$, where s_2 is given by (5.7) and

$$
\begin{aligned}
s^r_\psi :=& \int_{I\times M} \frac{1}{3!}\left(\underline{\tilde{e}}_n \tilde{\bar{\psi}}\gamma^3 d_{\hat{\omega}}\tilde{\psi} + \tilde{e}\underline{\tilde{\bar{\psi}}}_n\gamma^3 d_{\hat{\omega}}\tilde{\psi} + \tilde{e}\tilde{\bar{\psi}}\gamma^3 d_{\underline{\tilde{\omega}}_n}\tilde{\psi} + \tilde{e}\tilde{\bar{\psi}}\gamma^3 d_{\hat{\omega}}\underline{\tilde{\psi}}_n\right)\\
&- \tilde{\bar{\chi}}\gamma\underline{\tilde{\psi}}_n\tilde{e}^{\lrcorner} - \tilde{\bar{\chi}}\gamma\tilde{\psi}\underline{\tilde{e}}_n^{\lrcorner} - \frac{1}{3!}\tilde{\bar{\chi}}\gamma^3\left(d_{\underline{\tilde{\omega}}_n}\tilde{\psi} + d_{\hat{\omega}}\underline{\tilde{\psi}}_n\right)\tilde{\check{k}} - \frac{1}{3!}\tilde{\bar{\chi}}\gamma^3 d_{\hat{\omega}}\tilde{\psi}\underline{\tilde{\check{k}}}_n\\
&- i\left(\mathrm{L}_{\tilde{\xi}}^{\hat{\omega}}\tilde{\bar{\psi}} + d_{\tilde{\omega}_n}\tilde{\bar{\psi}}\tilde{\xi}^n + \tilde{\bar{\psi}}_n d\tilde{\xi}^n - [\tilde{c},\tilde{\bar{\psi}}] - d_{\hat{\omega}}\tilde{\bar{\chi}}\right)\underline{\tilde{\psi}^{\lrcorner}}_n\\
&- i\left(\mathrm{L}_{\tilde{\xi}}^{\hat{\omega}}\underline{\tilde{\bar{\psi}}}_n + \iota_{\underline{\partial_n}\tilde{\xi}}\tilde{\bar{\psi}} - d_{\underline{\tilde{\omega}}_n}(\tilde{\bar{\psi}}_n\tilde{\xi}_n) - [\tilde{c},\tilde{\bar{\psi}}_n] - d_{\underline{\tilde{\omega}}_n}\tilde{\bar{\chi}}\right)\tilde{\psi}^{\lrcorner}\\
&- i\left(\mathrm{L}_{\tilde{\xi}}^{\hat{\omega}}\tilde{\bar{\chi}} + d_{\tilde{\omega}_n}\tilde{\bar{\chi}}\tilde{\xi}^n - [\tilde{c},\tilde{\bar{\chi}}] - \frac{1}{2}\iota_{\tilde{\varphi}}\tilde{\bar{\psi}} - \frac{1}{2}\tilde{\bar{\psi}}_n\tilde{\varphi}^n\right)\underline{\tilde{\chi}^{\lrcorner}}_n\\
&+ \frac{1}{2}\iota_{\tilde{\varphi}}\underline{\tilde{\xi}^{\lrcorner}} + \frac{1}{2}\underline{\tilde{\xi}}^{\lrcorner}_n\tilde{\varphi}^n.
\end{aligned}
$$

Notice that $\phi_2^*(\tilde{k}^{\lrcorner}) = 0$, hence we obtain

$$
\begin{aligned}
&\phi_2^*\left(s^r_\psi + s^r_2\right) = s^r_\psi + s^r_2\\
=& \int_{I\times\Sigma}\frac{1}{3!}\left(\underline{\tilde{e}_n\tilde{\bar{\psi}}\gamma^3[\nu,\tilde{\psi}]}_1 + \underline{\tilde{e}\tilde{\bar{\psi}}_n\gamma^3[\nu,\tilde{\psi}]}_2 + \underline{\tilde{e}\tilde{\bar{\psi}}\gamma^3[\iota_z\nu}_3 + \underline{\iota_{\tilde{X}}\tilde{\mu}^{\lrcorner},\tilde{\psi}]}_4 + \underline{e\tilde{\bar{\psi}}\gamma^3[\nu,\tilde{\psi}_n]}_5\right)\\
&+ \underline{\epsilon_n \mathbb{q}_{\tilde{\bar{\psi}}}\gamma\tilde{\psi}\tilde{\mu}^{\lrcorner}}_6 + \tilde{\bar{\chi}}\gamma\tilde{\psi}\left(\underline{d_{\hat{\omega}}(\epsilon_n\tilde{\mu}^{\lrcorner})}_7 + \underline{\sigma\tilde{\mu}^{\lrcorner}}_8 + \underline{\tilde{\check{k}}_n\nu}_9 + \underline{\iota_z\nu\tilde{\check{k}}}_{10} + \underline{\iota_{\tilde{X}}(\tilde{\check{k}}\tilde{\mu}^{\lrcorner})}_{11}\right)\\
&+ \tilde{\bar{\chi}}\gamma\tilde{\psi}\left(\underline{x^{\lrcorner}\tilde{\mu}^{\lrcorner}}_{12} + \underline{\epsilon_n[\tilde{e},y^{\lrcorner}]}_{13}\right) + \underline{\tilde{\bar{\chi}}\gamma\tilde{\psi}_n\tilde{\check{k}}}_{14} - \frac{1}{3!}\tilde{\bar{\psi}}\gamma^3\left([\underline{\iota_z\nu}_{15} + \underline{\iota_{\tilde{X}}\tilde{\mu}^{\lrcorner}}_{16},\psi] + \underline{[\nu,\tilde{\psi}_n]}_{17}\right)\tilde{\check{k}}\\
&- \underline{\frac{1}{3!}\tilde{\bar{\chi}}\gamma^3[\nu,\tilde{\psi}]\tilde{\check{k}}_n}_{18} - \underline{i[\nu,\tilde{\bar{\chi}}]\tilde{\psi}_n^{\lrcorner}}_{19} + \epsilon_n\left(\underline{\mathrm{L}_{\tilde{\xi}}^{\hat{\omega}}\tilde{\bar{\psi}}}_{20} + \underline{d_{\tilde{\omega}_n}\tilde{\bar{\psi}}\tilde{\xi}^n}_{21} + \underline{\tilde{\bar{\psi}}_n d\tilde{\xi}^n}_{22} - \underline{[\tilde{c},\tilde{\bar{\psi}}]}_{23}\right)\gamma\tilde{\psi}\tilde{\mu}^{\lrcorner}\\
&- \epsilon_n\left(\underline{d_{\hat{\omega}}\tilde{\bar{\chi}}}_{24} + \underline{[\nu,\tilde{\bar{\chi}}]}_{25}\right)\gamma\tilde{\psi}\tilde{\mu}^{\lrcorner} - i[\underline{\iota_z\nu}_{26} + \underline{\iota_{\tilde{X}}\tilde{\mu}^{\lrcorner}}_{27},\tilde{\bar{\chi}}]\tilde{\psi}^{\lrcorner} + \frac{1}{2}\iota_{\tilde{\varphi}}\nu\left(\underline{\tilde{c}_n^{\lrcorner}}_{28} + \underline{[\epsilon_n,\tilde{\check{k}}\tilde{\mu}^{\lrcorner}]}_{29}\right)\\
&+ \underline{\frac{1}{2}\mathrm{L}_{\tilde{\varphi}}^{\hat{\omega}}(\epsilon_n)\check{k}\tilde{\tilde{\mu}}^{\lrcorner}}_{30} + \frac{1}{2}\tilde{\varphi}^n\left(\underline{\iota_{\tilde{X}}\tilde{c}_n^{\lrcorner}\tilde{\mu}^{\lrcorner} + d(\epsilon_n\tau^{\lrcorner}\tilde{\mu}^{\lrcorner})}_{31} + \underline{\iota_z\tilde{c}_n^{\lrcorner}\nu}_{32} + \underline{\iota_z\nu[\epsilon_n,\tilde{\check{k}}\tilde{\mu}^{\lrcorner}]}_{33}\right)\\
&+ \tilde{\varphi}^n\left(\underline{(d_{\tilde{\omega}_n}\epsilon_n)\tilde{\check{k}}\tilde{\mu}^{\lrcorner}}_{34} + \underline{\iota_{\tilde{X}}\tilde{\mu}^{\lrcorner}[\epsilon_n,\tilde{\check{k}}\tilde{\mu}^{\lrcorner}]}_{35}\right) + \frac{1}{2}\left(\underline{d_{\hat{\omega}}(\epsilon_n\tilde{\mu}^{\lrcorner})}_{36} + \underline{\sigma\tilde{\mu}^{\lrcorner}}_{37} + \underline{\tilde{\check{k}}_n\nu}_{38} + \underline{\iota_z\nu\tilde{\check{k}}}_{39}\right)\iota_{\tilde{\varphi}}\tilde{\check{k}}\\
&+ \frac{1}{2}\left(\underline{\iota_{\tilde{X}}(\tilde{\check{k}}\tilde{\mu}^{\lrcorner})}_{40} + \underline{x^{\lrcorner}\tilde{\mu}^{\lrcorner}}_{41} + \underline{\epsilon_n[\tilde{e},y^{\lrcorner}]}_{42}\right)\iota_{\tilde{\varphi}}\tilde{\check{k}} + \frac{1}{2}\left(\underline{d_{\hat{\omega}}(\epsilon_n\tilde{\mu}^{\lrcorner})}_{43} + \underline{\sigma\tilde{\mu}^{\lrcorner}}_{44} + \underline{\tilde{\check{k}}_n\nu}_{45} + \underline{\iota_z\nu\tilde{\check{k}}}_{46}\right)\tilde{\check{k}}_n\tilde{\varphi}^n\\
&+ \frac{1}{2}\left(\underline{\iota_{\tilde{X}}(\tilde{\check{k}}\tilde{\mu}^{\lrcorner})}_{47} + \underline{x^{\lrcorner}\tilde{\mu}^{\lrcorner}}_{48} + \underline{\epsilon_n[\tilde{e},y^{\lrcorner}]}_{49}\right)\tilde{\check{k}}_n\tilde{\varphi}^n + \underline{\frac{1}{2}\nu\tilde{\check{k}}\iota_{\tilde{\varphi}}\tilde{\check{k}}_n}_{50}
\end{aligned}
\tag{B.18}
$$

We can then see

- $(B.18.1) + (B.16.12) = 0$.
- $(B.18.2) + (B.18.5) + (B.18.21) = 0$.
- $(B.18.3) + (B.16.13) = 0$.
- $(B.18.4) + (B.16.6) + (B.16.7) + (B.16.21) = 0$.
- $(B.18.20) + (B.16.5) = 0$.
- $(B.18.21) + (B.16.11) = 0$.

- $(B.18.23) + (B.16.17) = 0$.

Therefore, noticing that $\mathcal{S}^r_{SG} = \mathcal{S}^r_{PC} + s^r_\psi + s^r_2$, we have

$$
\begin{aligned}
&\phi_2^* \left(\mathcal{S}^r_{SG} + \int_{I\times\Sigma} \frac{1}{2} \underline{\tilde{e}}_n \tilde{e}[\tilde{v} - y^{\lrcorner}, \tilde{v} - y^{\lrcorner}] + g(\tilde{v}^{\lrcorner}) \right) = \\
&= \mathcal{S}^r_{SG} + \int_{I\times\Sigma} \frac{1}{2} \underline{\tilde{e}}_n \tilde{e}[\tilde{v}, \tilde{v}] + f(\tilde{v}^{\lrcorner}) + \left((\delta_{\tilde{\chi}} + \mathbb{q})\tilde{v} + \underline{(\delta_{\tilde{\chi}} + \mathbb{q}) y^{\lrcorner}}_1 + \right) \tilde{v}^{\lrcorner} \\
&\quad + \underline{\epsilon_n \mathbb{q}_{\tilde{\tilde{\psi}}} \gamma \tilde{\psi} \tilde{\mu}^{\lrcorner}}_2 + \tilde{\tilde{\chi}} \gamma \tilde{\psi} \left(\underline{d_{\hat{\omega}}(\epsilon_n \tilde{\mu}^{\lrcorner})}_3 + \underline{\sigma \tilde{\mu}^{\lrcorner}}_4 + \underline{\tilde{\tilde{\check{k}}}_n \nu}_5 + \underline{\iota_z \nu \tilde{\tilde{\check{k}}}}_6 + \underline{\iota_{\tilde{X}} (\tilde{\tilde{\check{k}}} \tilde{\mu}^{\lrcorner})}_7 \right) \\
&\quad + \tilde{\tilde{\chi}} \gamma \tilde{\psi} \left(\underline{x^{\lrcorner} \tilde{\mu}^{\lrcorner}}_8 + \underline{\epsilon_n [\tilde{e}, y^{\lrcorner}]}_9 \right) + \underline{\tilde{\tilde{\chi}} \gamma \tilde{\psi}_n \tilde{\tilde{\check{k}}} \nu}_{10} - \frac{1}{3!} \tilde{\tilde{\chi}} \gamma^3 \left([\underline{\iota_z \nu}_{11} + \underline{\iota_{\tilde{X}} \tilde{\mu}^{\lrcorner}}_{12}, \psi] + \underline{[\nu, \tilde{\psi}_n]}_{13} \right) \tilde{\tilde{\check{k}}} \\
&\quad - \underline{\frac{1}{3!} \tilde{\tilde{\chi}} \gamma^3 [\nu, \tilde{\psi}] \tilde{\tilde{\check{k}}}_n}_{14} - \underline{i[\nu, \tilde{\tilde{\chi}}] \tilde{\psi}_n^{\lrcorner}}_{15} - \epsilon_n \left(\underline{d_{\hat{\omega}} \tilde{\tilde{\chi}}}_{16} + \underline{[\nu, \tilde{\tilde{\chi}}]}_{17} \right) \gamma \tilde{\psi} \tilde{\mu}^{\lrcorner} \\
&\quad - i[\underline{\iota_z \nu}_{18} + \underline{\iota_{\tilde{X}} \tilde{\mu}^{\lrcorner}}_{19}, \tilde{\tilde{\chi}}] \tilde{\psi}^{\lrcorner} + \frac{1}{2} \iota_{\tilde{\varphi}} \nu \left(\underline{\tilde{c}_n^{\lrcorner}}_{20} + \underline{[\epsilon_n, \tilde{\tilde{\check{k}}} \tilde{\mu}^{\lrcorner}]}_{21} \right) \\
&\quad + \underline{\frac{1}{2} \mathrm{L}^{\hat{\omega}}_{\tilde{\varphi}}(\epsilon_n) \check{k} \tilde{\tilde{\mu}}^{\lrcorner}}_{22} + \frac{1}{2} \tilde{\varphi}^n \left(\underline{\iota_{\tilde{X}} \tilde{c}_n^{\lrcorner} \tilde{\mu}^{\lrcorner} + d(\epsilon_n \tau^{\lrcorner} \tilde{\mu}^{\lrcorner})}_{23} + \underline{\iota_z \tilde{c}_n^{\lrcorner} \nu}_{24} + \underline{\iota_z \nu [\epsilon_n, \tilde{\tilde{\check{k}}} \tilde{\mu}^{\lrcorner}]}_{25} \right) \\
&\quad + \tilde{\varphi}^n \left(\underline{(d_{\tilde{\omega}_n} \epsilon_n) \tilde{\tilde{\check{k}}} \tilde{\mu}^{\lrcorner}}_{26} + \underline{\iota_{\tilde{X}} \tilde{\mu}^{\lrcorner} [\epsilon_n, \tilde{\tilde{\check{k}}} \tilde{\mu}^{\lrcorner}]}_{27} \right) + \frac{1}{2} \left(\underline{d_{\hat{\omega}}(\epsilon_n \tilde{\mu}^{\lrcorner})}_{28} + \underline{\sigma \tilde{\mu}^{\lrcorner}}_{29} + \underline{\tilde{\tilde{\check{k}}}_n \nu}_{30} + \underline{\iota_z \nu \tilde{\tilde{\check{k}}}}_{31} \right) \iota_{\tilde{\varphi}} \tilde{\tilde{\check{k}}} \\
&\quad + \frac{1}{2} \left(\underline{\iota_{\tilde{X}} (\tilde{\tilde{\check{k}}} \tilde{\mu}^{\lrcorner})}_{32} + \underline{x^{\lrcorner} \tilde{\mu}^{\lrcorner}}_{33} + \underline{\epsilon_n [\tilde{e}, y^{\lrcorner}]}_{34} \right) \iota_{\tilde{\varphi}} \tilde{\tilde{\check{k}}} + \frac{1}{2} \left(\underline{d_{\hat{\omega}}(\epsilon_n \tilde{\mu}^{\lrcorner})}_{35} + \underline{\sigma \tilde{\mu}^{\lrcorner}}_{36} + \underline{\tilde{\tilde{\check{k}}}_n \nu}_{37} + \underline{\iota_z \nu \tilde{\tilde{\check{k}}}}_{38} \right) \tilde{\tilde{\check{k}}}_n \tilde{\varphi}^n \\
&\quad + \frac{1}{2} \left(\underline{\iota_{\tilde{X}} (\tilde{\tilde{\check{k}}} \tilde{\mu}^{\lrcorner})}_{39} + \underline{x^{\lrcorner} \tilde{\mu}^{\lrcorner}}_{40} + \underline{\epsilon_n [\tilde{e}, y^{\lrcorner}]}_{41} \right) \tilde{\tilde{\check{k}}}_n \tilde{\varphi}^n + \underline{\frac{1}{2} \nu \tilde{\tilde{\check{k}}} \iota_{\tilde{\varphi}} \tilde{\tilde{\check{k}}}_n}_{42}.
\end{aligned}
\tag{B.19}
$$

We are therefore left with showing that all the underlined terms above exacltly correspond to $(\delta_{\tilde{\chi}} + \mathbb{q})\hat{\omega}\tilde{v}^{\lrcorner}$. We already remarked that we only know $e(\delta_{\tilde{\chi}} + \mathbb{q})$ and not the full expression. However, we notice

$$
(\delta_{\tilde{\chi}} + \mathbb{q})\hat{\omega}\tilde{v}^{\lrcorner} = \epsilon_n [(\delta_{\tilde{\chi}} + \mathbb{q})\hat{\omega}, \tilde{e}] \tilde{\mu}^{\lrcorner}.
$$

It is then enough to apply the operator $(\delta_{\tilde{\chi}} + \mathbb{q})$ to the constraint (5.10), and isolate the term we need exactly. In particular, we see that

$$
(\delta_{\tilde{\chi}} + \mathbb{q}) \left[\epsilon_n \left(d_{\hat{\omega}} \tilde{e} - \frac{1}{2} \bar{\tilde{\psi}} \gamma \tilde{\psi} + (\tilde{\tilde{\check{k}}}_n - \iota_z \tilde{\tilde{\check{k}}}) d\xi^n \right) + \iota_{\tilde{X}} \tilde{k}^{\lrcorner} + \tilde{e} x^{\lrcorner} + \epsilon_n [\tilde{e}, y^{\lrcorner}] - \tilde{e} \sigma \right] \tilde{\mu}^{\lrcorner} = 0
$$

yields the relevant term $\epsilon_n [(\delta_{\tilde{\chi}} + \mathbb{q})\hat{\omega}, \tilde{e}] \tilde{\mu}^{\lrcorner}$.[6] We apply the operator to each addend one by one,

[6] We have used that $\tau^{\lrcorner} = \tilde{\tilde{\check{k}}}_n - \iota_z \tilde{\tilde{\check{k}}} + \check{a}$ and that $\epsilon_n \check{a} = 0$.

obtaining

$$
\begin{aligned}
&\epsilon_n[(\delta_{\tilde{\chi}}+\mathbb{q})(\hat{\omega}+\nu),\tilde{e}]\tilde{\mu}^{\lrcorner} = \\
&=\epsilon_n\left(-\underline{\tilde{\bar{\chi}}\gamma d_{\hat{\omega}}\tilde{\psi}}_1 + \underline{d_{\hat{\omega}}(\iota_{\tilde{\varphi}}\tilde{\check{k}}}_2 + \underline{\tilde{\check{k}}_n\tilde{\varphi}^n)}_3 + \underline{\mathbb{q}_{\tilde{\bar{\psi}}}\gamma\tilde{\psi}}_4\right)\mu^{\lrcorner} + \underline{\frac{1}{3!}[\nu,\tilde{\bar{\chi}}]\gamma^3\tilde{\psi}_n\tilde{\check{k}}}_5 \\
&\quad + \underline{\frac{1}{3!}[\nu,\tilde{\bar{\chi}}]\gamma^3\tilde{\psi}\tilde{\check{k}}_n}_6 - \underline{i[\nu,\tilde{\bar{\chi}}]\tilde{\psi}_n^{\lrcorner}}_7 - \underline{\frac{1}{2}\nu\iota_{\tilde{\varphi}}\tilde{c}_n^{\lrcorner}}_8 + \underline{\frac{1}{3!}[\iota_z\nu,\tilde{\bar{\chi}}]\gamma^3\tilde{\psi}\tilde{\check{k}}}_9 \\
&\quad - \underline{i[\iota_z\nu,\tilde{\bar{\chi}}]\tilde{\psi}^{\lrcorner}}_{10} - \underline{\frac{1}{2}\iota_z\nu\tilde{c}_n^{\lrcorner}\tilde{\varphi}^n}_{11} - \underline{\frac{1}{2}\nu\iota_{\tilde{\varphi}}(\tilde{\check{k}}_n\tilde{\check{k}})}_{12} - \underline{\frac{1}{2}\iota_z\nu\tilde{\check{k}}(\iota_{\tilde{\varphi}}\tilde{\check{k}}+\tilde{\check{k}}\tilde{\varphi}^n}_{13} \\
&\quad + \underline{(\tilde{\check{k}}_n-\iota_z\tilde{\check{k}})d\tilde{\varphi}^n\tilde{\mu}^{\lrcorner}}_{14} + \frac{1}{2}\tilde{\mu}^{\lrcorner}\left(\underline{\mathrm{L}_{\tilde{\varphi}}^{\hat{\omega}}\epsilon_n}_{15} - \underline{d_{\tilde{\omega}_n}\epsilon_n\tilde{\varphi}^n}_{16}\right)\tilde{\check{k}} + \underline{\frac{1}{3!}\tilde{\bar{\chi}}\gamma^3[\iota_{\tilde{X}}\tilde{\mu}^{\lrcorner},\tilde{\psi}]}_{17} \\
&\quad - \underline{\frac{1}{2}\tilde{c}_n^{\lrcorner}\tilde{\varphi}^n\iota_{\tilde{X}}\tilde{\mu}^{\lrcorner}}_{18} - \underline{i[\iota_{\tilde{X}}\tilde{\mu}^{\lrcorner},\tilde{\bar{\chi}}]\tilde{\psi}^{\lrcorner}}_{19} + \underline{\tilde{\bar{\chi}}\gamma\tilde{\psi}x^{\lrcorner}\tilde{\mu}^{\lrcorner}}_{20} \\
&\quad + \underline{\tilde{\bar{\chi}}\gamma\tilde{\psi}\iota_{\tilde{X}}(\tilde{\check{k}}\tilde{\mu})}_{21} + \underline{\tilde{\mu}^{\lrcorner}(\mathbb{q}\tilde{X})\tilde{\check{k}}}_{22} - \frac{1}{2}\left(\underline{\iota_{\tilde{\varphi}}\tilde{\check{k}}}_{23} + \underline{\tilde{\check{k}}_n\tilde{\varphi}^n}_{24}\right)\iota_{\tilde{X}}(\tilde{\check{k}}\tilde{\mu}^{\lrcorner}) \\
&\quad + \frac{1}{2}\left(\underline{\iota_{\tilde{\varphi}}\tilde{\check{k}}+\tilde{\check{k}}_n\tilde{\varphi}^n}_{25}\right)x^{\lrcorner}\tilde{\mu}^{\lrcorner} + \epsilon_n\left[\underline{\tilde{\bar{\chi}}\gamma\tilde{\psi}}_{26} + \frac{1}{2}\left(\underline{\iota_{\tilde{\varphi}}\tilde{\check{k}}+\tilde{\check{k}}_n\tilde{\varphi}^n}_{27}\right), y^{\lrcorner}\right]\tilde{\mu}^{\lrcorner} \\
&\quad + \underline{\epsilon_n[\tilde{e},(\delta_\chi+\mathbb{q})y^{\lrcorner}]\tilde{\mu}^{\lrcorner}}_{28} + \underline{\tilde{\bar{\chi}}\gamma\tilde{\psi}\sigma\tilde{\mu}^{\lrcorner}}_{29} + \frac{1}{2}\left(\underline{\iota_{\tilde{\varphi}}\tilde{\check{k}}+\tilde{\check{k}}_n\tilde{\varphi}^n}_{30}\right)\sigma\tilde{\mu}^{\lrcorner} \\
&\quad - \underline{[\nu,\tilde{\bar{\chi}}]\gamma\tilde{\psi}\epsilon_n\tilde{\mu}^{\lrcorner}}_{31} - \underline{\frac{1}{2}\iota_{\tilde{\varphi}}\nu[\epsilon_n,\tilde{\check{k}}\tilde{\mu}^{\lrcorner}]}_{32} - \underline{\frac{1}{2}\iota_z\nu[\epsilon_n,\tilde{\check{k}}\tilde{\mu}^{\lrcorner}]\tilde{\varphi}^n}_{33} - \underline{\frac{1}{2}[\epsilon_n,\tilde{\check{k}}\tilde{\mu}^{\lrcorner}]\tilde{\varphi}^n\iota_{\tilde{X}}\tilde{\mu}^{\lrcorner}}_{34}
\end{aligned}
\tag{B.20}
$$

where we have used the definition of δ_χ and $\mathbb{q}$ from (5.6) and (5.2).[7] Furthermore, we have used the fact that $\hat{\omega}+\nu=\phi_2(\hat{\omega})$ to compute $\epsilon_n\mu^{\lrcorner}[\tilde{e},(\delta_\chi+\mathbb{q})\nu]=(\delta_\chi+\mathbb{q})\nu\tilde{v}^{\lrcorner}$.

We are just left with checking that all the terms in (B.20) appear in (B.19). We notice

- $(B.19.1)=(B.20.28)$;
- $(B.19.2)=(B.20.4)$;
- $(B.19.3)+(B.19.16)=(B.20.1)$ having used integration by parts;
- $(B.19.4)=(B.20.29)$;
- $(B.19.5)+(B.19.14)=(B.20.6)$;

[7]For completeness, we compute explicitly the terms $\epsilon_n(\delta_\chi\tilde{\check{k}}_n+\iota_z\delta\tilde{\check{k}})d\tilde{\xi}^n\tilde{\mu}^{\lrcorner}$. The computation works for $\mathbb{q}$ equivalently. We know from (5.2) that in the bulk

$$
\boldsymbol{e}\delta_\chi\check{\boldsymbol{k}} = -\frac{1}{2}\bar{\boldsymbol{\chi}}\gamma\boldsymbol{\psi}\check{\boldsymbol{k}} - \frac{1}{2\cdot 3!}\bar{\boldsymbol{\chi}}[\check{\boldsymbol{k}},\gamma^3]\boldsymbol{\psi} + i[\bar{\boldsymbol{\psi}}^{\lrcorner},\boldsymbol{\chi}] - \frac{1}{2}\boldsymbol{\iota_\varphi}\boldsymbol{c}^{\lrcorner}. \tag{B.21}
$$

Selecting the component along dx^n, we obtain

$$
\tilde{e}_n\delta_\chi\tilde{\check{k}} + \tilde{e}\delta_\chi\tilde{\check{k}}_n = \iota_z(\tilde{e}\delta_\chi\tilde{\check{k}}) + \mu\epsilon_n\delta_\chi\tilde{\check{k}} + \tilde{e}(\delta_\chi\tilde{\check{k}}_n - \iota_z\delta_\chi\tilde{\check{k}}),
$$

hence, using $\epsilon_n d\tilde{\xi}^n\mu^{\lrcorner}=\tilde{e}\nu$, with $\epsilon_n\nu=0$, we have

$$
\begin{aligned}
\epsilon_n(\delta_\chi\tilde{\check{k}}_n+\iota_z\delta\tilde{\check{k}})d\tilde{\xi}^n\tilde{\mu}^{\lrcorner} &= \nu\tilde{e}(\delta_\chi\tilde{\check{k}}_n-\iota_z\delta_\chi\tilde{\check{k}}) \\
&= \nu\left(\tilde{e}_n\delta_\chi\tilde{\check{k}}+\tilde{e}\delta_\chi\tilde{\check{k}}_n-\iota_z(\tilde{e}\delta_\chi\tilde{\check{k}})\right),
\end{aligned}
$$

which gives the desired result, as $\tilde{e}_n\delta_\chi\tilde{\check{k}}+\tilde{e}\delta_\chi\tilde{\check{k}}_n$ and $\tilde{e}\delta_\chi\tilde{\check{k}}$ can be found respectively as the transversal and tangential components of (B.21).

- $(B.19.6) + (B.19.11) = (B.20.9)$;
- $(B.19.7) = (B.20.21)$;
- $(B.19.9) = (B.20.26)$;
- $(B.19.10) + (B.19.13) = (B.20.5)$;
- $(B.19.12) = (B.20.17)$;
- $(B.19.15) = (B.20.7)$;
- $(B.19.17) = (B.20.31)$;
- $(B.19.18) = (B.20.10)$;
- $(B.19.19) = (B.20.19)$;
- $(B.19.20) = (B.20.8)$;
- $(B.19.21) = (B.20.32)$;
- $(B.19.22) = (B.20.15)$;
- $(B.19.23) = (B.20.18) + (B.20.14)$;
- $(B.19.24) = (B.20.11)$;
- $(B.19.25) = (B.20.33)$;
- $(B.19.26) = (B.20.16)$;
- $(B.19.27) = (B.20.34)$;
- $(B.19.28) + (B.19.35) = (B.20.2) + (B.20.3)$;
- $(B.19.29) + (B.19.36) = (B.20.30)$;
- $(B.19.30) + (B.19.42) = (B.20.12)$;
- $(B.19.31) + (B.19.38) = (B.20.13)$;
- $(B.19.32) + (B.19.39) = (B.20.23) + (B.20.24)$;
- $(B.19.33) + (B.19.40) = (B.20.25)$;
- $(B.19.34) + (B.19.41) = (B.20.27)$;
- $(B.19.37) = 0$, because $\tilde{\tilde{k}}_n^2 = 0$, since it is an odd quantity,

which concludes the proof. □

Proof of 5.4. We start by noticing that all the terms of the quadratic part s_2 of $\mathcal{S}_{SG}$ are left unchanged by ϕ_1, except for the term $\frac{1}{2}\check{\boldsymbol{k}}\iota_{\boldsymbol{\varphi}}\boldsymbol{e}^{\lrcorner}$. Therefore, letting

$$\begin{aligned}\mathcal{A}_{SG} := \int_{I\times\Sigma} &\frac{\boldsymbol{e}^2}{2}F_{\boldsymbol{\omega}} + \frac{1}{3!}\boldsymbol{e}\bar{\boldsymbol{\psi}}\gamma^3 d_{\boldsymbol{\omega}}\boldsymbol{\psi} - (\mathrm{L}_{\boldsymbol{\xi}}^{\boldsymbol{\omega}}\boldsymbol{e} - [\boldsymbol{c},\boldsymbol{e}] + \bar{\boldsymbol{\chi}}\gamma\boldsymbol{\psi})\boldsymbol{e}^{\lrcorner} \\ &+ (\iota_{\boldsymbol{\xi}}F_{\boldsymbol{\omega}} - d_{\boldsymbol{\omega}}\boldsymbol{c} + \delta_{\boldsymbol{\chi}}\boldsymbol{\omega})\boldsymbol{\omega}^{\lrcorner} - i(\mathrm{L}_{\boldsymbol{\xi}}^{\boldsymbol{\omega}}\bar{\boldsymbol{\psi}} - [\boldsymbol{c},\bar{\boldsymbol{\psi}}] - d_{\boldsymbol{\omega}}\bar{\boldsymbol{\chi}})\boldsymbol{\psi}^{\lrcorner} \\ &+ \left(\frac{1}{2}\iota_{\boldsymbol{\xi}}\iota_{\boldsymbol{\xi}}F_{\boldsymbol{\omega}} - \frac{1}{2}[\boldsymbol{c},\boldsymbol{c}] + \iota_{\boldsymbol{\xi}}\delta_{\bar{\chi}}\boldsymbol{\omega}\right)\boldsymbol{c}^{\lrcorner} + \frac{1}{2}\iota_{[\boldsymbol{\xi},\boldsymbol{\xi}]}\boldsymbol{\xi}^{\lrcorner} + \frac{1}{2}\iota_{\boldsymbol{\varphi}}\boldsymbol{\xi}^{\lrcorner} \\ &- i\left(\mathrm{L}_{\boldsymbol{\xi}}^{\boldsymbol{\omega}}\bar{\boldsymbol{\chi}} - [\boldsymbol{c},\bar{\boldsymbol{\chi}}] - \frac{1}{2}\iota_{\boldsymbol{\varphi}}\bar{\boldsymbol{\psi}}\right)\boldsymbol{\chi}^{\lrcorner} + \frac{1}{2}\left(\check{\boldsymbol{\omega}} - \iota_{\boldsymbol{\xi}}\boldsymbol{e}\check{\boldsymbol{c}} - \frac{\boldsymbol{e}}{2}\iota_{\boldsymbol{\xi}}\check{\boldsymbol{c}}\right)\iota_{\boldsymbol{\varphi}}\boldsymbol{e}^{\lrcorner},\end{aligned}$$

we just have to show that

$$\phi_1^*\left(\mathcal{A}_{SG}^r + \int_{I\times\Sigma}\underline{\tilde{e}}_n\tilde{e}^2[\tilde{v},\tilde{v}] + h(\tilde{v}^{\lrcorner})\right) = \mathcal{A}_{SG}.$$

In order to do so, we start by looking at the proof of the corresponding lemma for the pure PC theory in [CC25b], we see that, with the new definition of structural constraints, we have

$$\phi_1^*\left(\mathcal{S}_{PC}^r + \int_{I\times\Sigma}\frac{1}{2}\tilde{e}_n\tilde{e}[\tilde{v},\tilde{v}] + h(\tilde{v}^{\lrcorner})\right) = \mathcal{S}_{PC} + \int_{I\times\Sigma}\frac{1}{2}\tilde{v}\underline{\mu}\epsilon_n\tilde{\bar{\psi}}\gamma\tilde{\psi} - \tilde{v}\underline{\mu}\mathfrak{Q} + (\delta_\chi(\hat{\omega}+\tilde{v}) + \mathfrak{q}(\hat{\omega}+\tilde{v}))\,\tilde{v}^{\lrcorner}.$$

At this point, we see

$$\begin{aligned}\mathcal{A}_{SG} = \mathcal{S}_{PC} + \int_{I\times\Sigma}&\frac{1}{3!}\boldsymbol{e}\bar{\boldsymbol{\psi}}\gamma^3 d_{\boldsymbol{\omega}}\boldsymbol{\psi} - \bar{\boldsymbol{\chi}}\gamma\boldsymbol{\psi}\boldsymbol{e}^{\lrcorner} + \delta_{\boldsymbol{\chi}}\boldsymbol{\omega}\boldsymbol{\omega}^{\lrcorner} \\ &- i(\mathrm{L}_{\boldsymbol{\xi}}^{\boldsymbol{\omega}}\bar{\boldsymbol{\psi}} - [\boldsymbol{c},\bar{\boldsymbol{\psi}}] - d_{\boldsymbol{\omega}}\bar{\boldsymbol{\chi}})\boldsymbol{\psi}^{\lrcorner} + \iota_{\boldsymbol{\xi}}\delta_{\bar{\chi}}\boldsymbol{\omega}\boldsymbol{c}^{\lrcorner} \\ &+ \frac{1}{2}\iota_{\boldsymbol{\varphi}}\boldsymbol{\xi}^{\lrcorner} - i\left(\mathrm{L}_{\boldsymbol{\xi}}^{\boldsymbol{\omega}}\bar{\boldsymbol{\chi}} - [\boldsymbol{c},\bar{\boldsymbol{\chi}}] - \frac{1}{2}\iota_{\boldsymbol{\varphi}}\bar{\boldsymbol{\psi}}\right)\boldsymbol{\chi}^{\lrcorner} \\ &+ \frac{1}{2}\left(\check{\boldsymbol{\omega}} - \iota_{\boldsymbol{\xi}}\boldsymbol{e}\check{\boldsymbol{c}} - \frac{\boldsymbol{e}}{2}\iota_{\boldsymbol{\xi}}\check{\boldsymbol{c}}\right)\iota_{\boldsymbol{\varphi}}\boldsymbol{e}^{\lrcorner}\end{aligned}$$

obtaining

$$\begin{aligned}\mathcal{A}_{SG}^r = \mathcal{S}_{PC}^r + \int_{I\times M}&\frac{1}{3!}\left(\underline{\tilde{e}}_n\tilde{\bar{\psi}}\gamma^3 d_{\hat{\omega}}\tilde{\psi} + \tilde{e}\underline{\tilde{\bar{\psi}}}_n\gamma^3 d_{\hat{\omega}}\tilde{\psi} + \tilde{e}\tilde{\bar{\psi}}\gamma^3 d_{\underline{\tilde{\omega}}_n}\tilde{\psi} + \tilde{e}\tilde{\bar{\psi}}\gamma^3 d_{\hat{\omega}}\underline{\tilde{\psi}}_n\right) \\ &- \tilde{\bar{\chi}}\gamma\underline{\tilde{\psi}}_n\tilde{e}^{\lrcorner} - \tilde{\bar{\chi}}\gamma\tilde{\psi}\underline{\tilde{e}}_n^{\lrcorner} - \frac{1}{3!}\tilde{\bar{\chi}}\gamma^3\left(d_{\underline{\tilde{\omega}}_n}\tilde{\psi} + d_{\hat{\omega}}\underline{\tilde{\psi}}_n\right)\tilde{\check{k}} - \frac{1}{3!}\tilde{\bar{\chi}}\gamma^3 d_{\hat{\omega}}\tilde{\psi}\underline{\tilde{\check{k}}}_n \\ &- i\left(\mathrm{L}_{\tilde{\xi}}^{\hat{\omega}}\tilde{\bar{\psi}} + d_{\tilde{\omega}_n}\tilde{\bar{\psi}}\tilde{\xi}^n + \tilde{\bar{\psi}}_n d\tilde{\xi}^n - [\tilde{c},\tilde{\bar{\psi}}] - d_{\hat{\omega}}\tilde{\bar{\chi}}\right)\underline{\tilde{\psi}^{\lrcorner}}_n \\ &- i\left(\mathrm{L}_{\tilde{\xi}}^{\hat{\omega}}\underline{\tilde{\bar{\psi}}}_n + \iota_{\underline{\partial_n}\tilde{\xi}}\tilde{\bar{\psi}} - d_{\underline{\tilde{\omega}}_n}(\tilde{\bar{\psi}}_n\tilde{\xi}_n) - [\tilde{c},\tilde{\bar{\psi}}_n] - d_{\underline{\tilde{\omega}}_n}\tilde{\bar{\chi}}\right)\tilde{\psi}^{\lrcorner} \\ &- i\left(\mathrm{L}_{\tilde{\xi}}^{\hat{\omega}}\tilde{\bar{\chi}} + d_{\tilde{\omega}_n}\tilde{\bar{\chi}}\tilde{\xi}^n - [\tilde{c},\tilde{\bar{\chi}}] - \frac{1}{2}\iota_{\tilde{\varphi}}\tilde{\bar{\psi}} - \frac{1}{2}\tilde{\bar{\psi}}_n\tilde{\varphi}^n\right)\underline{\tilde{\chi}^{\lrcorner}}_n \\ &+ \frac{1}{2}\iota_{\tilde{\varphi}}\underline{\tilde{\xi}}^{\lrcorner} + \frac{1}{2}\underline{\tilde{\xi}}_n^{\lrcorner}\tilde{\varphi}^n + \frac{1}{2}\tilde{\check{k}}\iota_{\tilde{\varphi}}\underline{\tilde{e}}_n^{\lrcorner} + \frac{1}{2}\underline{\tilde{\check{k}}}_n\left(\iota_{\tilde{\varphi}}\tilde{e}^{\lrcorner} + \tilde{e}_n^{\lrcorner}\tilde{\varphi}^n\right),\end{aligned}$$

and

$$\begin{aligned}\mathcal{A}_{SG} - \mathcal{S}_{PC} =& \mathcal{A}^r_{SG} - \mathcal{S}^r_{PC} + \int_{I\times\Sigma} \frac{1}{3!}\left(\underline{\underline{\tilde{e}_n}\tilde{\bar{\tilde{\psi}}}\gamma^3[\tilde{v},\tilde{\psi}]}_1 + \underline{\tilde{e}\underline{\tilde{\bar{\tilde{\psi}}}}_n\gamma^3[\tilde{v},\tilde{\psi}]}_2 + \underline{\tilde{e}\tilde{\bar{\tilde{\psi}}}\gamma^3[\tilde{v},\underline{\tilde{\psi}}_n]}_3\right)\\ &+ \int_{I\times\Sigma} -\frac{1}{3!}\tilde{\bar{\tilde{\chi}}}\gamma^3\left(\underline{[\tilde{v},\underline{\tilde{\psi}}_n]\check{\tilde{k}}}_4 + \underline{[\tilde{v},\tilde{\psi}]\underline{\check{\tilde{k}}}_n}_5\right) - i\left(\underline{[\iota_{\tilde{\xi}}\tilde{v},\tilde{\bar{\tilde{\psi}}}]}_6 - \underline{[\tilde{v},\tilde{\bar{\tilde{\chi}}}]}_7\right)\underline{\tilde{\psi}^{\dashv}}_n\\ &\qquad - i\left(\underline{[\iota_{\tilde{\xi}}\tilde{v},\underline{\tilde{\bar{\tilde{\psi}}}}_n]}_8\right)\tilde{\psi}^{\dashv} - i\left(\underline{[\iota_{\tilde{\xi}}\tilde{v},\tilde{\bar{\tilde{\chi}}}]}_9\right)\underline{\tilde{\chi}^{\dashv}}_n + \left(\delta_\chi(\hat{\omega}+\tilde{v}) + \mathfrak{q}(\hat{\omega}+\tilde{v})\right)\tilde{v}^{\dashv},\end{aligned} \tag{B.22}$$

Using (A.50), we see that $(B.22.1) = \frac{1}{2}\underline{\tilde{e_n}}\tilde{v}\bar{\psi}\gamma\psi$, and that $(B.22.2) + (B.22.3)$ give

$$\frac{1}{3!}\left(\underline{\tilde{\bar{\tilde{\psi}}}}_n\gamma^3[\tilde{v},\tilde{\psi}] + \tilde{e}\tilde{\bar{\tilde{\psi}}}\gamma^3[\tilde{v},\underline{\tilde{\psi}}_n]\right) \propto \tilde{e}\tilde{v}\underline{\tilde{\bar{\tilde{\psi}}}}_n\gamma\tilde{\psi} = 0.$$

A straightforward computation gives, for equation (**??**)

$$\begin{aligned}\phi_1^*&\left(\mathcal{A}^r_{SG} - \mathcal{S}^r_{PC}\right) + \int_{I\times\Sigma}\frac{1}{2}\tilde{v}\underline{\mu}\epsilon_n\tilde{\bar{\tilde{\psi}}}\gamma\tilde{\psi} - \tilde{v}\underline{\mu}\mathfrak{Q} - \left(\delta_\chi(\hat{\omega}+\tilde{v}) + \mathfrak{q}(\hat{\omega}+\tilde{v})\right)\tilde{v}^{\dashv}\\ &= \mathcal{A}^r_{SG} - \mathcal{S}^r_{PC} + \int_{I\times\Sigma}\underline{\frac{1}{3!}\tilde{e}\tilde{\bar{\tilde{\psi}}}\gamma^3[\iota_{\underline{z}}\tilde{v},\tilde{\psi}]}_1 - \underline{\tilde{\bar{\tilde{\chi}}}\gamma\underline{\tilde{\psi}}_n\tilde{v}\check{\tilde{k}}}_2 + \underline{\tilde{\bar{\tilde{\chi}}}\gamma\tilde{\psi}\tilde{v}\underline{\check{\tilde{k}}}_n}_3 + \underline{\tilde{\bar{\tilde{\chi}}}\gamma\tilde{\psi}\iota_{\underline{z}}\tilde{v}\check{\tilde{k}}}_4\\ &\quad - \underline{\frac{1}{3!}\tilde{\bar{\tilde{\chi}}}\gamma^3[\iota_{\underline{z}}\tilde{v},\tilde{\psi}]\check{\tilde{k}}}_5 - \underline{i[\iota_{\tilde{\xi}}\tilde{v},\tilde{\bar{\tilde{\psi}}}]\underline{\tilde{\psi}^{\dashv}}_n}_6 - \underline{i[\iota_{\tilde{\xi}}\tilde{v},\underline{\tilde{\psi}}_n]\tilde{\psi}^{\dashv}}_7 + \underline{i[\iota_{\underline{z}}\tilde{v},\tilde{\bar{\tilde{\chi}}}]\tilde{\psi}^{\dashv}}_8\\ &\quad - \underline{i[\iota_{\tilde{\xi}}\tilde{v},\tilde{\bar{\tilde{\chi}}}]\underline{\tilde{\chi}^{\dashv}}_n}_9 - \underline{\frac{1}{2}\tilde{v}\iota_{\tilde{\varphi}}\underline{\tilde{c}^{\dashv}}_n}_{10} + \underline{\frac{1}{2}\iota_{\underline{z}}\tilde{v}\tilde{c}^{\dashv}_n\tilde{\varphi}^n}_{11} - \underline{\frac{1}{2}\check{\tilde{k}}\iota_{\tilde{\varphi}}(\tilde{v}\underline{\check{\tilde{k}}}_n + \iota_{\underline{z}}\tilde{v}\check{\tilde{k}})}_{12}\\ &\quad + \underline{\frac{1}{2}\underline{\check{\tilde{k}}}_n\left(\iota_{\tilde{\varphi}}(\tilde{v}\check{\tilde{k}}) - (\iota_z\tilde{v}\check{\tilde{k}})\tilde{\varphi}^n\right)}_{13} + \underline{\frac{1}{2}\tilde{v}\underline{\mu}\epsilon_n\tilde{\bar{\tilde{\psi}}}\gamma\tilde{\psi}}_{14} - \underline{\tilde{v}\underline{\mu}\mathfrak{Q}}_{15}.\end{aligned} \tag{B.23}$$

We are then only left to show that equations (B.23) and (B.22) coincide. We start by noticing the following terms match:

- $(B.23.1) + (B.23.14) \overset{A.50}{=} \frac{1}{2}\tilde{v}(\iota_{\underline{z}}\tilde{e} + \mu\epsilon_n)\tilde{\bar{\tilde{\psi}}}\gamma\tilde{\psi} = (B.22.1)$.
- $(B.22.6) = (B.23.6)$.
- $(B.22.8) = (B.23.7)$.
- $(B.22.9) = (B.23.8)$

Before we proceed, we need to find an explicit expression of $\mathfrak{Q}$. Indeed, recall that it was defined in (5.9) as the part of $Q_{SG}(\epsilon_n\check{\tilde{k}} - \tilde{e}\tilde{\check{a}}) = 0$ depending on $\tilde{\chi}$ and $\tilde{\psi}$. We start by computing $\boldsymbol{e}Q_{SG}\check{\boldsymbol{k}}$ in the bulk as in the proof of proposition **??**. It is a quick computation to see

$$\begin{aligned}\boldsymbol{e}Q_{SG}\check{\boldsymbol{k}} =& \boldsymbol{e}\left(d_{\boldsymbol{\omega}}\boldsymbol{e} - \frac{1}{2}\bar{\boldsymbol{\psi}}\gamma\boldsymbol{\psi} + \mathrm{L}^{\boldsymbol{\omega}}_{\boldsymbol{\xi}}\check{\boldsymbol{k}} - [\boldsymbol{c},\check{\boldsymbol{k}}] - \frac{1}{2}\iota_{\boldsymbol{\varphi}}(\boldsymbol{e}\check{\boldsymbol{c}})\right)\\ &- \frac{1}{4}\bar{\boldsymbol{\chi}}\gamma^2[\check{\boldsymbol{k}},\gamma]\boldsymbol{\psi} - \frac{i}{4}\bar{\boldsymbol{\psi}}^{\dashv}_0\underline{\gamma}^2\gamma^2\boldsymbol{\chi} - \frac{1}{2}\check{\boldsymbol{k}}\iota_{\boldsymbol{\varphi}}\check{\boldsymbol{k}}.\end{aligned}$$

In particular, one has

$$\mathfrak{Q} = \epsilon_n\left(Q_{SG}(\check{\tilde{k}}) - d_{\tilde{\omega}}\tilde{e} + \frac{1}{2}\tilde{\bar{\tilde{\psi}}}\gamma\tilde{\psi} - \mathrm{L}^{\tilde{\omega}}_{\tilde{\xi}}\check{\tilde{k}} + [\tilde{c},\check{\tilde{k}}]\right).$$

The only term inside $\mathfrak{Q}$ which can be easily found is given by $-\frac{1}{2}\epsilon_n\left(\bar{\tilde{\chi}}\gamma\tilde{\chi}\tilde{\check{c}}\right)$, as the terms proportional to $\tilde{e}$ can be reabsorbed in the right hand side of the structural constraint (5.9). In particular, we can define

$$\mathfrak{B} := \mathfrak{Q} + \frac{1}{2}\epsilon_n\left(\bar{\tilde{\chi}}\gamma\tilde{\chi}\tilde{\check{c}}\right).$$

Unfortunately, finding $Q_{SG}\check{\boldsymbol{k}}$ and extracting $Q_{SG}(\tilde{\check{k}})$ as the part not proportional to dx^n, is not convenient in our case, as it involves cumbersome computations. It is however more convenient to consider the part along dx^n inside $\boldsymbol{e}Q_{SG}(\check{\boldsymbol{k}})$, which is given by $\underline{\tilde{e}}_n Q_{SG}(\tilde{\check{k}}) + \tilde{e}Q_{SG}(\underline{\tilde{\check{k}}}_n)$. As in the computation of the BV pushforward we are only interested in $\underline{\mu}\tilde{v}\mathfrak{Q}$, we notice that

$$\tilde{v}\left(\underline{\tilde{e}}_n Q_{SG}(\tilde{\check{k}}) + \tilde{e}Q_{SG}(\underline{\tilde{\check{k}}}_n)\right)\tilde{v}\underline{\tilde{e}}_n Q_{SG}(\tilde{\check{k}}) = -\underline{\mu}\tilde{v}\epsilon_n Q_{SG}\tilde{\check{k}} - \iota_{\underline{z}}\tilde{v}\tilde{e}Q_{SG}(\tilde{\check{k}}),$$

having used $\tilde{e}\tilde{v} = 0$. At this point, discarding all the terms that do not depend on χ and ψ, we obtain the term $(B.23.15)$ as

$$-\underline{\mu}\tilde{v}\mathfrak{Q} = \left[\tilde{v}\left(\underline{\tilde{e}}_n Q_{SG}(\tilde{\check{k}}) + \tilde{e}Q_{SG}(\underline{\tilde{\check{k}}}_n)\right) + \iota_{\underline{z}}\tilde{v}\tilde{e}Q_{SG}(\tilde{\check{k}})\right]\Bigg|_{\psi,\chi}.$$

In particular, inside $\boldsymbol{e}Q_{SG}\check{\boldsymbol{k}}$, what we are interested in are the terms

$$-\boldsymbol{e}\frac{1}{2}\iota_{\boldsymbol{\varphi}}(\boldsymbol{e}\check{\boldsymbol{c}}) - \frac{1}{4}\bar{\boldsymbol{\chi}}\gamma^2[\check{\boldsymbol{k}},\gamma]\boldsymbol{\psi} - \frac{i}{4}\bar{\boldsymbol{\psi}}_0^{\lrcorner}\underline{\gamma}^2\gamma^2\boldsymbol{\chi} - \frac{1}{2}\check{\boldsymbol{k}}\iota_{\boldsymbol{\varphi}}\check{\boldsymbol{k}}.$$

We carry out the computation term by term:

- $-\boldsymbol{e}\frac{1}{2}\iota_{\boldsymbol{\varphi}}(\boldsymbol{e}\check{\boldsymbol{c}})$. As previously anticipated, this gives $-\frac{1}{2}\epsilon_n\left(\bar{\tilde{\chi}}\gamma\tilde{\chi}\tilde{\check{c}}\right)$. We obtain

$$\begin{aligned}\frac{1}{2}\underline{\mu}\tilde{v}\epsilon_n\left(\bar{\tilde{\chi}}\gamma\tilde{\chi}\tilde{\check{c}}\right) =& -\frac{1}{2}\tilde{v}(\underline{\tilde{e}}_n - \iota_{\underline{z}}\tilde{e})(\iota_{\varphi}\tilde{e} + \tilde{e}_n\tilde{\varphi}^n)\tilde{\check{c}}\\ =&\frac{1}{2}\tilde{v}\iota_{\varphi}(\underline{\tilde{e}}_n\tilde{e}\tilde{\check{c}}) - \frac{1}{2}\iota_{\underline{z}}\tilde{v}\left(\frac{1}{2}\iota_{\tilde{\varphi}}(e^2)\tilde{\check{c}} + \tilde{e}\tilde{e}_n\tilde{\varphi}^n\tilde{\check{c}}\right)\\ =& \frac{1}{2}\tilde{v}\iota_{\tilde{\varphi}}(\underline{\tilde{c}}_n^{\lrcorner}) - \frac{1}{2}\iota_{\underline{z}}\tilde{v}\tilde{c}_n^{\lrcorner}\tilde{\varphi}^n,\end{aligned}$$

  having used the fact that $\underline{\tilde{c}}_n^{\lrcorner} = \underline{\tilde{e}}_n\tilde{e}\tilde{\check{c}} + \frac{1}{2}\tilde{e}^2\underline{\tilde{\check{c}}}_n$ and that $\tilde{e}\tilde{v} = 0$ repeatedly. We see that the terms above exactly cancel out $(B.23.10) + (B.23.11)$.

- $-\frac{1}{4}\bar{\boldsymbol{\chi}}\gamma^2[\check{\boldsymbol{k}},\gamma]\boldsymbol{\psi}$. We have

$$\begin{aligned}&\tilde{v}\left(-\frac{1}{4}\bar{\tilde{\chi}}\gamma^2[\underline{\tilde{\check{k}}}_n\gamma]\tilde{\psi} - \frac{1}{4}\bar{\tilde{\chi}}\gamma^2[\tilde{\check{k}},\gamma]\underline{\tilde{\psi}}_n\right) - \frac{1}{4}\iota_{\underline{z}}\tilde{v}\bar{\tilde{\chi}}\gamma^2[\tilde{\check{k}},\gamma]\psi\\ &\overset{(A.50)}{=} \tilde{v}\left(\underline{\bar{\tilde{\chi}}\gamma\tilde{\psi}\underline{\tilde{\check{k}}}_n}_1 + \underline{\bar{\tilde{\chi}}\gamma\underline{\tilde{\psi}}\tilde{\check{k}}}_2\right) - \frac{1}{3!}\bar{\tilde{\chi}}\gamma^3\left(\underline{[\tilde{v},\tilde{\psi}]\underline{\tilde{\check{k}}}_n}_3 + \underline{[\tilde{v},\underline{\tilde{\psi}}_n]\tilde{\check{k}}}_4\right) - \underline{\iota_{\underline{z}}\tilde{v}\bar{\tilde{\chi}}\gamma\tilde{\psi}\tilde{\check{k}}}_5 + \underline{\frac{1}{3!}\bar{\tilde{\chi}}\gamma^3[\iota_{\underline{z}}\tilde{v},\tilde{\psi}]\tilde{\check{k}}}_6.\end{aligned} \tag{B.24}$$

  We see that $(B.24.1) + (B.24.2) + (B.24.5) + (B.24.6) + (B.23.2) + (B.23.3) + (B.23.4) + (B.23.5) = 0$, while $(B.24.3) = (B.22.5)$ and $(B.24.4) = (B.22.4)$

- $-\frac{i}{4}\bar{\boldsymbol{\psi}}_0^{\lrcorner}\underline{\gamma}^2\gamma^2\boldsymbol{\chi}$. Before carrying out the computation, we notice that $-\frac{i}{4}\bar{\boldsymbol{\psi}}_0^{\lrcorner}\underline{\gamma}^2\gamma^2\boldsymbol{\chi} = -i[\bar{\boldsymbol{\psi}}^{\lrcorner},\boldsymbol{\chi}]$, where the action on $\boldsymbol{\chi}$ is given only by the part in V of $\boldsymbol{\psi}^{\lrcorner}$. Using Leibniz and the Majorana flip relations, we then have

$$\begin{aligned}&-i\tilde{v}[\underline{\bar{\tilde{\psi}}}_n^{\lrcorner},\tilde{\chi}] - i\iota_{\underline{z}}\tilde{v}[\bar{\tilde{\psi}}^{\lrcorner},\tilde{\chi}]\\ &= \underline{-i[\tilde{v},\bar{\tilde{\chi}}]\underline{\tilde{\psi}}_n^{\lrcorner}}_1 + \underline{i[\iota_{\underline{z}}\tilde{v},\bar{\tilde{\chi}}]\tilde{\psi}^{\lrcorner}}_2.\end{aligned} \tag{B.25}$$

We see $(B.25.2) + (B.23.8) = 0$, while $(B.25.2) = (B.22.7)$, which tells us that (B.23) contains all the terms in (B.22). We are then left to show that (B.23) does not contain extra terms.

- $-\frac{1}{2}\check{\boldsymbol{k}}\iota_{\boldsymbol{\varphi}}\check{\boldsymbol{k}}$. We have simply

$$-\frac{1}{2}\tilde{v}\iota_{\tilde{\varphi}}(\underline{\tilde{\check{k}}}_n\tilde{\check{k}}) + \frac{1}{2}\iota_{\underline{z}}\tilde{v}\tilde{\check{k}}\tilde{\check{k}}_n\tilde{\varphi}^n, \tag{B.26}$$

since $\tilde{\check{k}}\iota_{\tilde{\varphi}}\tilde{\check{k}} = \frac{1}{2}\iota_{\tilde{\varphi}}(\tilde{\check{k}}^2) = 0$ because there are no 4-forms on Σ and since $\tilde{\check{k}}_n^2 = 0$ because of parity. It is easy to see that equation (B.26) exactly cancels out $(B.23.12) + (B.23.13)$.

The above computation tells us that eq. (B.23) is equal to eq. (B.22), hence showing

$$\phi_1^*\left(\mathcal{A}_{SG}^r + \int_{I\times\Sigma}\frac{1}{2}\tilde{e}_n\tilde{e}[\tilde{v},\tilde{v}] + h(\tilde{v}^{\dashv})\right) = \mathcal{A}_{SG},$$

□

B.3.5 AKSZ Symplectomorphism

We now want to show the proof of proposition 5.2. We start by recalling the form of the AKSZ symplectic form:

$$\begin{aligned}\varpi_{SG}^{AKSZ} = \varpi_{PC}^{AKSZ} + \int_{I\times\Sigma}\frac{1}{3!}\Big(&\underline{\bar{\underline{\varsigma}}^{\dashv}\gamma^3\delta\psi\delta e}_{1} + \underline{\bar{\psi}\gamma^3\delta\underline{\varsigma}^{\dashv}\delta e}_{2} + \underline{\bar{\psi}\gamma^3\delta\psi\delta\underline{f}^{\dashv}}_{3} + \underline{\underline{f}^{\dashv}\delta\bar{\psi}\gamma^3\delta\psi}_{4}\Big)\\ &+\frac{1}{3}\underline{e\delta\bar{\psi}\gamma^3\delta\underline{\varsigma}^{\dashv}}_{5} + \underline{i\delta\bar{\underline{\epsilon}}\delta\theta^{\dashv}}_{6} + \underline{i\delta\bar{\chi}\delta\underline{\chi}^{\dashv}}_{7} + i\delta\bar{\underline{\varsigma}}(\underline{\iota_{\delta\xi}\theta^{\dashv}}_{8} + \underline{\iota_{\xi}\delta\theta^{\dashv}}_{9})\\ &+ i\delta\bar{\psi}\left(\underline{\iota_{\delta\underline{z}}\theta^{\dashv}}_{10} + \underline{\iota_{\underline{z}}\delta\theta^{\dashv}}_{11} + \underline{\iota_{\delta\xi}\underline{\chi}^{\dashv}}_{12} + \underline{\iota_{\xi}\delta\underline{\chi}^{\dashv}}_{13}\right)\end{aligned} \tag{B.27}$$

The reduced BV form is given by

$$\varpi_{SG}^r = \varpi_{PC}^r + \int_{I\times\Sigma} i\delta\bar{\underline{\psi}}_n\delta\psi^{\dashv} + i\delta\bar{\psi}\delta\underline{\psi}_n^{\dashv} + i\delta\bar{\chi}\delta\underline{\chi}_n^{\dashv}$$

Notice that, as it happened in chapter 3.3, we have

$$
\begin{aligned}
&\Phi_r^*(\varpi_{PC}^r) = \varpi_{PC}^{AKSZ}+\\
&+\int_{I\times\Sigma}\delta e\left(\underline{\frac{1}{3!}\bar{\underline{\varsigma}}^{\lrcorner}\gamma^3\delta\psi}_{1}+\underline{\frac{1}{3!}\bar{\psi}\gamma^3\underline{\varsigma}^{\lrcorner}}_{2}-\underline{\frac{1}{3!}\delta(\lambda\mu^{-1})\bar{\varsigma}^{\lrcorner}\gamma^3\underline{\varsigma}^{\lrcorner}}_{3}+\underline{\frac{1}{3}\lambda\mu^{-1}\delta\bar{\varsigma}^{\lrcorner}\gamma^3\underline{\varsigma}^{\lrcorner}}_{4}\right)\\
&+\delta(\lambda\mu^{-1})f^{\lrcorner}\left(\underline{\frac{1}{3!}\bar{\underline{\varsigma}}^{\lrcorner}\gamma^3\delta\psi}_{5}+\underline{\frac{1}{3!}\bar{\psi}\gamma^3\underline{\varsigma}^{\lrcorner}}_{6}-\underline{\frac{1}{3!}\delta(\lambda\mu^{-1})\bar{\varsigma}^{\lrcorner}\gamma^3\underline{\varsigma}^{\lrcorner}}_{7}+\underline{\frac{1}{3}\lambda\mu^{-1}\delta\bar{\varsigma}^{\lrcorner}\gamma^3\underline{\varsigma}^{\lrcorner}}_{8}\right)\\
&-\lambda\mu^{-1}\delta f^{\lrcorner}\left(\underline{\frac{1}{3!}\bar{\underline{\varsigma}}^{\lrcorner}\gamma^3\delta\psi}_{9}+\underline{\frac{1}{3!}\bar{\psi}\gamma^3\underline{\varsigma}^{\lrcorner}}_{10}-\underline{\frac{1}{3!}\delta(\lambda\mu^{-1})\bar{\varsigma}^{\lrcorner}\gamma^3\underline{\varsigma}^{\lrcorner}}_{11}\right)\\
&+\underline{i\iota_{\delta\xi}\delta\bar{\underline{\varsigma}}^{\lrcorner}\theta^{\lrcorner}}_{12}+\underline{i\iota_{\delta\xi}\bar{\underline{\varsigma}}^{\lrcorner}\delta\theta^{\lrcorner}}_{13}+i\delta(\lambda\mu^{-1})\left(\underline{i\iota_z\delta\bar{\underline{\varsigma}}^{\lrcorner}\theta^{\lrcorner}}_{14}+\underline{i\iota_z\bar{\underline{\varsigma}}^{\lrcorner}\delta\theta^{\lrcorner}}_{15}\right)\\
&+i\lambda\mu^{-1}\left(\underline{i\iota_{\delta z}\delta\bar{\underline{\varsigma}}^{\lrcorner}\theta^{\lrcorner}}_{16}+\underline{i\iota_{\delta z}\bar{\underline{\varsigma}}^{\lrcorner}\delta\theta^{\lrcorner}}_{17}\right)-\underline{i\delta(\lambda\mu^{-1})\iota_{\delta\xi}\bar{\varsigma}^{\lrcorner}\underline{\chi}^{\lrcorner}}_{18}\\
&-i\lambda\mu^{-1}\left(\underline{\iota_{\delta\xi}\delta\bar{\varsigma}^{\lrcorner}\underline{\chi}^{\lrcorner}}_{19}+\underline{\iota_{\delta\xi}\bar{\varsigma}^{\lrcorner}\delta\underline{\chi}^{\lrcorner}}_{20}\right)+i\delta(\lambda\mu^{-1})\left(\underline{\delta(\lambda\mu^{-1})\iota_z\bar{\varsigma}^{\lrcorner}\underline{\chi}}_{21}+\underline{\lambda\mu^{-1}\iota_z\delta\bar{\varsigma}^{\lrcorner}\underline{\chi}}_{22}\right)\\
&+i\delta(\lambda\mu^{-1})\left(\underline{\lambda\mu^{-1}\iota_z\bar{\varsigma}^{\lrcorner}\delta\underline{\chi}}_{23}+\underline{\lambda\mu^{-1}\iota_{\delta z}\bar{\varsigma}^{\lrcorner}\underline{\chi}}_{24}\right)+\delta(\lambda\mu^{-1})\left(\underline{\iota_{\delta\underline{z}}\bar{\varsigma}^{\lrcorner}\theta_{\lrcorner}}_{25}+\underline{\iota_{\underline{z}}\delta\bar{\varsigma}^{\lrcorner}\theta_{\lrcorner}}_{26}-\underline{\iota_{\underline{z}}\bar{\varsigma}^{\lrcorner}\delta\theta_{\lrcorner}}_{27}\right)\\
&+\frac{1}{3!}\delta(\lambda\mu^{-1})\left(\underline{\delta\underline{\underline{f}}^{\lrcorner}\bar{\psi}\gamma^3\varsigma^{\lrcorner}}_{28}+\underline{\underline{\underline{f}}^{\lrcorner}\delta\bar{\psi}\gamma^3\varsigma^{\lrcorner}}_{29}-\underline{\underline{\underline{f}}^{\lrcorner}\bar{\psi}\gamma^3\delta\varsigma^{\lrcorner}}_{30}+\underline{\delta e\bar{\varsigma}^{\lrcorner}\gamma^3\underline{\varsigma}^{\lrcorner}}_{31}+\underline{2e\bar{\varsigma}^{\lrcorner}\gamma^3\delta\underline{\varsigma}^{\lrcorner}}_{32}\right)\\
&-i\delta(\lambda\mu^{-1})\left(\underline{\delta(\lambda\mu^{-1})\iota_z\bar{\varsigma}^{\lrcorner}\underline{\chi}}_{33}+\underline{\lambda\mu^{-1}\iota_z\delta\bar{\varsigma}^{\lrcorner}\underline{\chi}}_{34}+\underline{\lambda\mu^{-1}\iota_z\bar{\varsigma}^{\lrcorner}\delta\underline{\chi}}_{35}+\underline{\lambda\mu^{-1}\iota_{\delta z}\bar{\varsigma}^{\lrcorner}\underline{\chi}}_{36}\right),
\end{aligned}
\tag{B.28}
$$

and

$$
\begin{aligned}
&\Phi_r^*\left(i\delta\underline{\bar{\psi}}_n\delta\psi^{\lrcorner}+i\delta\bar{\psi}\delta\underline{\psi}^{\lrcorner}_n+i\delta\bar{\chi}\delta\underline{\chi}^{\lrcorner}_n\right)\\
&=\frac{1}{3!}\delta\bar{\psi}\left(\underline{2\gamma^3\underline{\varsigma}^{\lrcorner}\delta\underline{e}}_{1}+\underline{\underline{e}\gamma^3\delta\underline{\varsigma}^{\lrcorner}}_{2}-\underline{\gamma^3\psi\delta\underline{\underline{f}}^{\lrcorner}}_{3}-\underline{\underline{\underline{f}}^{\lrcorner}\gamma^3\delta\psi}_{4}+\underline{\delta(\lambda\mu^{-1})f^{\lrcorner}\gamma^3\underline{\varsigma}^{\lrcorner}}_{5}-\underline{\lambda\mu^{-1}\delta f^{\lrcorner}\gamma^3\underline{\varsigma}^{\lrcorner}}_{6}\right)\\
&-\underline{\frac{1}{3!}\delta\bar{\psi}\lambda\mu^{-1}f^{\lrcorner}\gamma^3\delta\underline{\varsigma}^{\lrcorner}}_{7}+\frac{1}{3!}\delta(\lambda\mu^{-1})\bar{\varsigma}^{\lrcorner}\left(\underline{2\gamma^3\underline{\varsigma}^{\lrcorner}\delta\underline{e}}_{8}+\underline{2\underline{e}\gamma^3\delta\underline{\varsigma}^{\lrcorner}}_{9}-\underline{\gamma^3\psi\delta\underline{\underline{f}}^{\lrcorner}}_{10}-\underline{\underline{\underline{f}}^{\lrcorner}\gamma^3\delta\psi}_{11}+\underline{\delta(\lambda\mu^{-1})f^{\lrcorner}\gamma^3\underline{\varsigma}^{\lrcorner}}_{12}\right)\\
&-\frac{1}{3!}\delta(\lambda\mu^{-1})\bar{\varsigma}^{\lrcorner}\left(\underline{\lambda\mu^{-1}\delta f^{\lrcorner}\gamma^3\underline{\varsigma}^{\lrcorner}}_{13}-\underline{\lambda\mu^{-1}f^{\lrcorner}\gamma^3\delta\underline{\varsigma}^{\lrcorner}}_{14}\right)+\frac{1}{3!}\lambda\mu^{-1}\delta\bar{\varsigma}\left(\underline{2\gamma^3\underline{\varsigma}^{\lrcorner}\delta\underline{e}}_{15}+\underline{\underline{e}\gamma^3\delta\underline{\varsigma}^{\lrcorner}}_{16}-\underline{\gamma^3\psi\delta\underline{\underline{f}}^{\lrcorner}}_{17}\right)\\
&+\frac{1}{3!}\lambda\mu^{-1}\delta\bar{\varsigma}^{\lrcorner}\left(-\underline{\underline{\underline{f}}^{\lrcorner}\gamma^3\delta\psi}_{18}+\underline{\delta(\lambda\mu^{-1})f^{\lrcorner}\gamma^3\underline{\varsigma}^{\lrcorner}}_{19}\right)+i\delta\bar{\psi}\left(\underline{\iota_{\delta\underline{z}}\theta^{\lrcorner}}_{20}+\underline{\iota_{\underline{z}}\delta\theta^{\lrcorner}}_{21}\right)\\
&+i\delta(\lambda\mu^{-1})\bar{\varsigma}^{\lrcorner}\left(\underline{\iota_{\delta\underline{z}}\theta^{\lrcorner}}_{22}+\underline{\iota_{\underline{z}}\delta\theta^{\lrcorner}}_{23}\right)-i\lambda\mu^{-1}\delta\bar{\varsigma}^{\lrcorner}\left(\underline{\iota_{\delta\underline{z}}\theta^{\lrcorner}}_{24}+\underline{\iota_{\underline{z}}\delta\theta^{\lrcorner}}_{25}\right)+i\delta\bar{\psi}\left(\underline{\iota_{\delta\xi}\underline{\chi}^{\lrcorner}}_{26}+\underline{\iota_{\xi}\delta\underline{\chi}^{\lrcorner}}_{27}\right)\\
&+i\delta(\lambda\mu^{-1})\bar{\varsigma}^{\lrcorner}\left(\underline{\iota_{\delta\xi}\underline{\chi}^{\lrcorner}}_{28}+\underline{\iota_{\xi}\delta\underline{\chi}^{\lrcorner}}_{29}\right)-i\lambda\mu^{-1}\delta\bar{\varsigma}^{\lrcorner}\left(\underline{\iota_{\delta\xi}\underline{\chi}^{\lrcorner}}_{30}+\underline{\iota_{\xi}\delta\underline{\chi}^{\lrcorner}}_{31}\right)+\underline{i\delta\bar{\chi}\delta\underline{\chi}^{\lrcorner}}_{32}+\underline{i\delta(\lambda\mu^{-1})\iota_{\xi}\bar{\varsigma}^{\lrcorner}\delta\underline{\chi}^{\lrcorner}}_{33}\\
&-\underline{i\lambda\mu^{-1}\iota_{\delta\xi}\bar{\varsigma}^{\lrcorner}\delta\underline{\chi}^{\lrcorner}}_{34}-\underline{i\lambda\mu^{-1}\iota_{\xi}\delta\bar{\varsigma}^{\lrcorner}\delta\underline{\chi}^{\lrcorner}}_{35}+\underline{i\delta\underline{\bar{\epsilon}}\delta\theta^{\lrcorner}}_{36}-\underline{i\iota_{\delta\xi}\underline{\varsigma}^{\lrcorner}\delta\theta^{\lrcorner}}_{37}-\underline{i\iota_{\xi}\delta\underline{\bar{\varsigma}}^{\lrcorner}\delta\theta^{\lrcorner}}_{38}+\underline{i\delta(\lambda\mu^{-1})\iota_{\underline{z}}\bar{\varsigma}^{\lrcorner}\delta\theta^{\lrcorner}}_{39}\\
&-\underline{i\lambda\mu^{-1}\delta\iota_{\underline{z}}\bar{\varsigma}^{\lrcorner}\delta\theta^{\lrcorner}}_{40}-\underline{i\lambda\mu^{-1}\iota_{\delta\underline{z}}\bar{\varsigma}^{\lrcorner}\delta\theta^{\lrcorner}}_{41}-\underline{i\lambda\mu^{-1}\delta\iota_{\underline{z}}\delta\bar{\varsigma}^{\lrcorner}\delta\theta^{\lrcorner}}_{42}
\end{aligned}
\tag{B.29}
$$

We can then see that the following terms inside $\Phi_r^*(\varpi_{SG}^r)$ add up to ϖ_{SG}^{AKSZ}:

- $(B.28.1)+(B.29.1)=(B.27.1)$
- $(B.28.2)=(B.27.2)$
- $(B.29.3)=(B.27.4)$

- $(B.29.4) = (B.27.4)$
- $(B.29.2) = (B.27.5)$
- $(B.29.36) = (B.27.6)$
- $(B.29.32) = (B.27.7)$
- $(B.28.12) = (B.27.8)$
- $(B.29.38) = (B.27.9)$
- $(B.29.20) + (B.29.21) = (B.27.10) + (B.27.11)$
- $(B.29.26) + (B.29.27) = (B.27.12) + (B.27.13)$

while the remaining terms in $\Phi_r^*(\varpi_{SG}^r)$ add up to zero

- $(B.28.3) + (B.28.31) + (B.29.8) = 0$
- $(B.28.4) + (B.29.15) = 0$
- $(B.28.5) + (B.28.31) + (B.29.5) + (B.29.29) = 0$
- $(B.28.6) + (B.28.30) = 0$
- $(B.28.7) + (B.29.8) = 0$
- $(B.28.8) + (B.29.14) + (B.29.19) = 0$
- $(B.28.9) + (B.29.6) = 0$
- $(B.28.10) + (B.29.17) = 0$
- $(B.28.11) + (B.29.13) = 0$
- $(B.29.7) + (B.29.18) = 0$
- $(B.29.9) + (B.29.18) = 0$
- $(B.29.10) + (B.28.28) = 0$
- $(B.29.16) = 0$ because of (A.54) and the parity of $\delta\varsigma^{\lrcorner}$
- $(B.29.22) + (B.28.25) = 0$
- $(B.29.23) + (B.29.39) + (B.28.15) + (B.28.27) = 0$
- $(B.29.24) + (B.28.16) = 0$
- $(B.29.25) + (B.29.38) = 0$
- $(B.29.37) + (B.28.13) = 0$
- $(B.29.41) + (B.28.17) = 0$
- $(B.28.14) + (B.28.26) = 0$
- $(B.29.28) + (B.28.18) = 0$

- $(B.29.29) + (B.29.33) = 0$
- $(B.29.30) + (B.28.19) = 0$
- $(B.29.31) + (B.29.35) = 0$
- $(B.29.34) + (B.28.20) = 0$
- $(B.28.21) + (B.28.33) = 0$
- $(B.28.22) + (B.28.34) = 0$
- $(B.28.23) + (B.28.35) = 0$
- $(B.28.24) + (B.28.36) = 0$

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