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Renormalizability of Yang–Mills–Dirac Theory I:

Renormalizability on manifolds with zero boundary

Master's Thesis

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Abstract

The renormalizability of perturbative Yang–Mills theory coupled to Dirac spinors on Euclidean manifolds without boundary in the BV formalism is investigated, using the method of homotopic renormalization. The required mathematical background is recalled. A recollection of the BV formalism is given. Costello's homotopic renormalization is introduced. The classical BV data of Yang–Mills theory coupled to Dirac spinors is derived, and the homological calculations necessary for the proof of renormalizability are demonstrated. The conclusion is that Yang–Mills–Dirac theory is renormalizable on $\mathbb{R}^4$.

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Conventions and Notation

If not explicitly stated otherwise,

- All fields are assumed to have characteristic 0.
- A summation is understood for repeated upper and lower indices, e.g. $x^i y_i = \sum_i x^i y_i$.
- Manifolds are assumed to be smooth, orientable and equipped with an orientation. Therefore, no careful distinction is made between forms of top degree or Berezinians or densities.
- For a graded object x, $|x|$ denotes its total degree.
- For an n-tuple of variables $x = (x^1, \ldots, x^n)$ and a multi-index $\alpha \in \mathbb{N}^n$,

  $$x^\alpha = (x^1)^{\alpha_1} \cdots (x^n)^{\alpha_n}.$$

  If applied to a partial differential, it means

  $$\partial_x^\alpha = \left(\frac{\partial}{\partial x^1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x^n}\right)^{\alpha_n}.$$

- A decreasing filtration F of an object A is denoted with an upper index,

  $$\{0\} \subset \cdots \subset F^r A \subset \cdots \subset F^1 A \subset F^0 A = A.$$

  The associated graded gr of the decreasing filtration is,

  $$\mathrm{gr}^r A = F^r A / F^{r+1} A.$$

- One denotes by $R[x]$ the ring of polynomials over some object R in the variables $x = (x^1, \ldots, x^n)$. The polynomials up to order r are written as $F_r R[x] := R[x]/x^{r+1} R[x]$, homogeneous elements of order r as written as $\mathrm{gr}_r R[x]$.
- Analogously, one denotes by $R[\![x]\!]$ formal power series in the variables $x = (x^1, \ldots, x^n)$ over some object R. The formal power series truncated at some power r are written as $F_r R[\![x]\!] := R[\![x]\!]/x^{r+1} R[\![x]\!]$, homogeneous elements of order r as written as $\mathrm{gr}_r R[\![x]\!]$.
- The Hodge star operator $*$ on an n-dimensional Riemannian manifold (M, g) acts in local coordinates (x^i) on $\alpha \in \Omega^p(M)$ by

  $$*\alpha = \frac{\sqrt{\det g}}{(n-p)!(p!)} \alpha_{i_1,\ldots,i_p} g^{i_1 j_1} \cdots g^{i_p j_p} \epsilon_{j_1,\ldots,j_n} \mathrm{d}x^{j_{p+1}} \wedge \cdots \wedge \mathrm{d}x^{j_n}$$

Introduction

Among the most successful theories in modern physics are non-Abelian gauge theories of Yang–Mills type. However, problems of a twofold nature arise with their perturbative quantization. First, in the path integral formalism, the gauge invariance is responsible for the degeneracies of the classical action, obstructing a straightforward derivation of the perturbative integral. Second, once a formal expression for the path integral is found, divergences occur term by term in the high-energy (UV) limit that require renormalization.

A powerful approach for addressing the problem of gauge redundancy is the Batatalin-Vilkovisky (BV) formalism. It yields a formally well-defined path integral. However, in the setting of integrating over infinite-dimensional spaces of fields, some central objects of the formalism become ill-defined. On Euclidean manifolds, this can be overcome by a heat kernel regularization. For the remaining problem of renormalization, a method called homotopic renormalization has been proposed. It is based on an approach studying the equivalence between theories, finding that a well-behaved quantum theory is equivalent to a renormalizable quantum theory. The purpose of this thesis is to thoroughly review the above concepts and apply them to Yang–Mills–Dirac theory.

In Chapter 1, some mathematical tools underlying the following constructions are developed. Graded and supergeometric structures are introduced. A careful setup of functionals of global sections is provided, focusing on the notion of local action functionals and their emergence from local objects. In Chapter 2, a review of the BV formalism is provided. In particular, the BV construction is studied in the well-defined finite-dimensional case before the generalization to infinite dimensions is given. Then, examples of some classical BV theories are included. In Chapter 3, an introduction to homotopic renormalization, following [Cos11], is presented. It originates from considerations of effective field theories due to a Wilsonian renormalization approach. This is motivated in the first part of the chapter. The second part shows how objects in the quantum BV theory can be regularized in the language of effective field theories. In particular, homotopic equivalence between theories is discussed, giving rise to a notion of renormalizability. In Chapter 4, a concrete application of this method for proving renormalizability is presented. The classical BV theory of Yang–Mills–Dirac, i.e. Yang–Mills coupled to spinors, is set up in the first part, following a discussion of the compatibility with homotopic renormalization. Then, the core computation of the cohomology of local action functionals follows, after which the renormalizability of Yang–Mills–Dirac theory is concluded. Further, the deformations and symmetries of the action are analyzed, and the results are discussed and compared with related work. To conclude, the direction for the continuation of this project is outlined.

Chapter 1

Mathematical Tools

This chapter contains an overview of the various tools essential for the setup of the theory. An overview of symplectic supergeometry, following [Mne19], is given in sections 1.1 and 1.2. In section 1.3, jets, $\mathcal{D}_{\mathcal{M}}$-modules, local Lagrangians, and functionals of sections are discussed, following [Sau89], [Kas03], [Rab21] and [Cos11], [CG21], respectively.

1.1 Super and Graded Structures

The goal of this section is to introduce the basic objects of super and graded geometry that are needed to set up the BV formalism.

Definition 1.1.1 (Super Vector Space)**.** A *super vector space* V is the direct sum of two vector spaces V_0, V_1 over some field $\mathbb{K}$, $V = V_0 \oplus V_1$. An element $v \in V$ is said to have even parity if $v \in V_0$ and odd parity if $v \in V_1$.

Definition 1.1.2 (Parity shift)**.** Let V be a super vector space, then its *parity shift* is the super vector space ΠV with $(\Pi V)_0 = V_1$, $(\Pi V)_1 = V_0$.

Definition 1.1.3 ($\mathbb{Z}$-Graded Vector Space)**.** Let V_i, $i \in \mathbb{Z}$, be vector spaces over some field $\mathbb{K}$. A $\mathbb{Z}$-*graded vector space* is the direct sum of vector spaces $V = \bigoplus_{i\in\mathbb{Z}} V_i$. An element $v \in V_i$ is called *homogeneous of degree* $|v| = i$.

Definition 1.1.4 (Shift)**.** Let V be a graded vector space, then its k-*shift* is $V[k] = \bigoplus_{i\in\mathbb{Z}}(V_{i+k})$, sending the vector space of degree $i + k$ to degree i.

Remark. One way of constructing a $\mathbb{Z}$-graded vector space is by using only shifts of the field $\mathbb{K}$ and considering usual vector spaces $W_i \cong V_i$ concentrated in degree 0. Then, one can identify

$$V = \bigoplus_{i\in\mathbb{Z}} W_i \otimes \mathbb{K}[-i].$$

Remark. There is a more general notion of graded vector spaces over some index set I. In particular, a super vector space is just a $\mathbb{Z}_2$-graded vector space. However, in the scope of this chapter, graded is understood as $\mathbb{Z}$-graded, and super as $\mathbb{Z}_2$-graded.

Definition 1.1.5 (Dual of Graded Vector Spaces)**.** Let $V = \bigoplus_{i\in\mathbb{Z}} V_i$ be an $\mathbb{Z}$-graded $\mathbb{K}$-vector space. An element $\omega \in V^\vee$, where $V^\vee$ denotes the dual of V, is a linear map $\omega : V \to \mathbb{K}$.

One can write ω as a sum over restrictions to the subspaces of homogeneous degree, $\omega = \sum_{i\in\mathbb{Z}} \omega\,|_{V_i}$. If $\mathbb{K}$ is interpreted as the one-dimensional $\mathbb{K}$-vector space concentrated in degree 0, each $\omega\,|_{V_i}$ becomes a graded linear map of degree $-i$. Thus, there is an induced grading of the dual space, such that $(V_i)^\vee = (V^\vee)_{-i}$.

Definition 1.1.6 (Graded Algebra)**.** A *graded algebra* $A = \bigoplus_{i\in\mathbb{Z}} A_i$ is a $\mathbb{Z}$-graded vector space with a bilinear product

$$\cdot : A_i \times A_j \to A_{i+j}.$$

A is called *graded-commutative*, if for homogeneous elements $x, y \in A$, the Koszul sign rule holds:

$$x \cdot y = (-1)^{|x||y|} y \cdot x.$$

Example 1.1.1. Let V be a graded vector space, then the tensor algebra $\mathcal{T}(V) = \bigoplus_{i\in\mathbb{N}} V^{\otimes i}$ is a graded-commutative algebra in a natural way by setting $v \otimes w = (-1)^{|v||w|} w \otimes v$, $v, w \in V$.

Remark. In this case where one encounters multiple gradings, the commutation rule is the Koszul sign rule with respect to the total degree. This is easily seen by using the explicit construction of an element in bidegree (i, j) as $v \otimes k_i \otimes k_j$, where v does not carry a degree but $k_i \in \mathbb{K}[-i], k_j \in \mathbb{K}[-j]$.

Definition 1.1.7 (dg Vector Space)**.** A *differentially graded (dg) vector space* is a graded vector space $V = (V_i)_{i\in\mathbb{Z}}$ together with a graded linear map Q of degree 1 such that $Q^2 = 0$.

Definition 1.1.8 (dg Algebra)**.** A *differential graded algebra (dga)* is a graded algebra together with a graded linear operator Q that squares to zero and is a graded derivation of degree 1, i.e. fulfills the graded Leibniz rule

$$Q(a \cdot b) = (Qa) \cdot b + (-1)^{|a|} a \cdot (Qb).$$

The graded signature for permutations of vectors of a graded vector space V is denoted by $\chi(\sigma, v_1, \dots, v_n)$, for $\sigma \in S_n$ and $v_1, \dots, v_n \in V$, and it is determined by the equation

$$v_1 \otimes \cdots \otimes v_n = (-1)^{|\sigma|} \chi(\sigma, v_1, \dots, v_n)\, v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}.$$

Definition 1.1.9 (L_∞-Algebra)**.** Let L be a graded vector space and let $[\cdot, \dots, \cdot]_n : V^{\otimes n} \to V$ be a collection of multilinear maps of respective degree $n-2$ for every $n \in \mathbb{N}$, called the n-brackets. L is an L_∞ *algebra* if the n-brackets fulfill

(i) (graded skew symmetry). For all $n \in \mathbb{N}$ and $v_1, \dots, v_n \in L$,

$$[v_1, \dots, v_n]_n = \chi(\sigma, v_1, \dots, v_n)\, [v_{\sigma(1)}, \dots, v_{\sigma(n)}]_n.$$

(ii) (homotopy Jacobi identity). For all $n \in \mathbb{N}$ and $v_1, \dots, v_n \in L$,

$$\sum_{i+j=n} \sum_{\sigma\in\mathrm{Shuff}(i,j)} \chi(\sigma, v_1, \dots, v_n)\, (-1)^{i(j-1)} [[v_{\sigma(1)}, \dots, v_{\sigma(i)}]_i v_{\sigma(i+1)}, \dots, v_{\sigma(n)}] = 0,$$

where $\mathrm{Shuff}(i, j) \subset S_n$ are permutations that, acting on $(1, \dots, n)$ has the first i elements in ascending order, and then again the next j elements in ascending order.

An alternative definition, used e.g. by [KS], states that an L_∞ algebra is a pair (L, Q), where L is a graded vector space and Q a differential on the graded coalgebra $\mathrm{Sym}^\bullet(L[1])$. Note that the restriction to the cogenerators induces a map $Q : \mathrm{Sym}^\bullet(L[1]) \to L[2]$. One can decompose Q into $Q_n : \mathrm{Sym}^n(L[1]) \to L[2]$. By the décalage isomorphism $\mathrm{Sym}^n(L[1]) \cong \wedge^n(L)[n]$, Q_n gives rise to the n-bracket

$$[\cdot, \dots, \cdot]_n : \wedge^n L \to L[2-n].$$

It can be checked that the property of the differential $Q^2 = 0$ encodes the homotopy Jacobi identity.

Remark. If all brackets except for the 1-bracket vanish, an L_∞-algebra is just a dg vector space. If the brackets are only non-trivial for $n = 1, 2$, one obtains what is called a differential graded Lie algebra (dgLa), which is a dg vector space together with a Lie bracket and a Leibniz rule for applying the differential on the Lie bracket.

Before introducing the notions of a supermanifold and a graded manifold, respectively, the setup of super rings and graded rings is required. Their construction parallels that of graded-commutative algebras.

Definition 1.1.10 (Super Ring)**.** A *super ring* is a ring $(R, +, \cdot)$ with decomposition $R = R_0 \oplus R_1$ into additive Abelian groups R_i such that

$$\cdot : R_i \times R_j \to R_{i+j}.$$

Moreover, it is a *supercommutative ring* if, in addition, the multiplication obeys the Koszul rule.

Definition 1.1.11 (Graded Ring)**.** A *graded ring* is a ring $(R, +, \cdot)$ with decomposition $R = \bigoplus_{i\in\mathbb{Z}} R_i$ into additive Abelian groups R_i such that

$$\cdot : R_i \times R_j \to R_{i+j}.$$

Moreover, it is a *graded-commutative ring* if, in addition, the multiplication obeys the Koszul rule.

Example 1.1.2. There are two main examples of super rings and graded rings,

(i) Let $V = (V_0, V_1)$ be a super vector space with super coordinates (x^i, θ^I). One can consider the ring of polynomial functions on V, $\mathrm{Sym}(V^\vee)$, which is generated by the super coordinates. Imposing the Koszul rule according to parity, i.e.

$$x^i x^j = x^j x^i \qquad x^i\theta^I = \theta^I x^i \qquad \theta^J\theta^I = -\theta^I\theta^J,$$

naturally endows $\mathrm{Sym}(V^\vee)$ with the structure of a super-commutative ring.

(ii) Analogously, for a graded vector space $V = (V_i)_{i\in\mathbb{Z}}$ with coordinates (x^{I_i}) of each V_i, one obtains the graded-commutative ring of polynomial functions $\mathrm{Sym}(V^\vee)$ by setting

$$x^{I_i}x^{I_j} = (-1)^{i+j}\, x^{I_j}x^{I_i}.$$

Supermanifolds and graded manifolds are defined following the approach of [CS11]. Recall that a smooth manifold M can be viewed as a locally ringed space (M, C^∞_M), where C^∞_M is the sheaf of smooth functions on M. Then, a smooth map between manifolds M and N is precisely a morphism of locally ringed spaces $(f, \bar f)\colon (M, C^\infty_M) \to (N, C^\infty_N)$, consisting of a continuous map of topological spaces $f\colon M \to N$ and a morphism of sheaves $\bar f : C^\infty_N \to f_* C^\infty_M$, where f_* denotes the direct image functor.

More explicitly, the pair $(f, \bar f)$ in this definition corresponds to the family of morphisms $(f, \bar f_V)_{V\subset Y \text{ open}}$, where $\bar f_V : C^\infty_N(V) \to C^\infty_M(f^{-1}(V))$, that are compatible with restrictions, in the sense that the following diagram commutes for any $V_1 \subset V_2 \subset Y$:

$$\begin{array}{ccc} C^\infty_N(V_2) & \xrightarrow{\bar f_{V_2}} & C^\infty_M(f^{-1}(V_2)) \\ \downarrow & & \downarrow \\ C^\infty_N(V_1) & \xrightarrow{\bar f_{V_1}} & C^\infty_M(f^{-1}(V_1)) \end{array}$$

Definition 1.1.12 (Supermanifold)**.** A *supermanifold* of dimension $(n|m)$ over an n-dimensional smooth base manifold M is a locally super ringed space $\mathcal{M} = (M, C^\infty_{\mathcal{M}})$, where $C^\infty_{\mathcal{M}}$ is called the structure sheaf of $\mathcal{M}$, which on an open subset $U \subset M$ is locally isomorphic to $C^\infty(U) \otimes \wedge^\bullet V^\vee$, for $\wedge^\bullet V^\vee$ the algebra of polynomial functions on some real m-dimensional vector space V. A morphism of supermanifolds is a morphism of locally super-ringed spaces.

Example 1.1.3 (Odd (co)tangent bundle)**.** Given an n-dimensional manifold M, one can consider the $(n|n)$-dimensional supermanifolds of the odd tangent bundle ΠTM and the odd cotangent bundle ΠT^*M. They have respective sets of even and odd local coordinates (x^i, ∂_i) and $(x_i, \mathrm{d}x^i)$. Then the functions on the respective supermanifolds are

$$C^\infty(\Pi TM) = \Gamma(\bigwedge T^*M) = \Omega(M),$$
$$C^\infty(\Pi T^*M) = \Gamma(\bigwedge TM) = \mathfrak{X}(M),$$

where the differential forms $\Omega(M)$, as well as the algebra of multivector fields $\mathfrak{X}(M)$ are identified.

A graded manifold is defined in a similar way to supermanifolds.

Definition 1.1.13 (Graded Manifold)**.** A *graded manifold* over a smooth base manifold M is a locally graded ringed space $\mathcal{M} = (M, C^\infty_{\mathcal{M}})$, where $C^\infty_{\mathcal{M}}$ is called the structure sheaf of $\mathcal{M}$, which on an open subset $U \subset M$ is locally isomorphic to $C^\infty(U) \otimes \mathrm{Sym}^* V^\vee$ for $\mathrm{Sym}^* V^\vee$ the algebra of polynomial functions on some finite-dimensional graded vector space V. A morphism of graded manifolds is a morphism of locally graded-ringed spaces.

Example 1.1.4 (Shifted (co)tangent bundle)**.** Analogously to Example 1.1.3, one can define the graded manifolds of the 1-shifted tangent bundle $T[1]M$ and the (-1)-shifted cotangent $T^*[-1]M$. The structure sheaves are again differential forms and multivector fields, respectively.

Definition 1.1.14 (Differential forms)**.** By the construction of the odd tangent bundle and shifted tangent bundle, differential forms on supermanifolds and graded manifolds are defined as follows:

(i) A *k-form on a supermanifold* $\mathcal{M}$ is an element of $C^\infty(\Pi T\mathcal{M})$ of degree k.

(ii) A *k-form on a graded manifold* $\mathcal{M}$ is an element of $C^\infty(T[1]\mathcal{M})$ of degree k.

1.2 Symplectic Geometry

In this section, some fundamentals of symplectic geometry are collected and applied to set up tools for the BV formalism.

Definition 1.2.1 (Symplectic Vector Space)**.** Let V be a finite-dimensional vector space and $\omega \in \wedge^2 V^\vee$ such that the map $\omega^\flat : V \to V^\vee, v \mapsto \omega(v, \cdot)$ is an isomorphism. Then the pair (V, ω) is called a *symplectic vector space* and ω is called *symplectic form*.

The conditions on ω are equivalent to saying ω is an antisymmetric, non-degenerate bilinear form on V. For later discussion, there are four kinds of subspaces that are of particular interest:

Definition 1.2.2 (Isotropic, Coisotropic, Lagrangian, and symplectic subspaces)**.** Let (V, ω) be a symplectic vector space and $W \subset V$ a subspace. The symplectic orthogonal complement is defined as

$$W^\perp := \{v \in V |\, \omega(v, w) = 0 \;\forall w \in I\}.$$

(i) W is an *isotropic subspace* if $W \subset W^\perp$.

(ii) W is a *coisotropic subspace* if $W^\perp \subset W$.

(iii) W is *Lagrangian* if one of the following three equivalent statements hold: W is both isotropic and coisotropic, W is maximally isotropic, W is minimally coisotropic.

(iv) W is *symplectic* if $\omega|_{W\times W}$ is non-degenerate.

Often, calculations simplify if done in a convenient basis. The following proposition provides a suitable choice.

Proposition 1.2.1 (Darboux basis)**.** *Let* (V, ω) *be symplectic vector space of* $\dim V = 2n$*. There exists a basis* $(e_1, \ldots, e_n, f_1, \ldots, f_n)$ *of* V *such that* $\omega = e_i^\vee \wedge f_i^\vee$*. In particular,*

$$\omega(e_i, e_j) = \omega(f_i, f_j) = 0 \qquad \omega(e_i, f_j) = \delta_{ij}$$

Proof. Choose arbitrary nonzero $e_1, f_1 \in V$ such that $\omega(e_1, f_1) = 1$. Due to non-degeneracy, such vectors always exist if V is non-trivial, otherwise there is nothing to prove. Let W_1 be the subspace spanned by e_1, f_1, then $W_1 \cap W_1^\perp = \{0\}$. Indeed, suppose $v = ae_1 + bf_1 \in W_1^\perp$, then

$$0 = \omega(v, e_1) = -b \qquad 0 = \omega(v, f_1) = a,$$

therefore $v = 0$. Further, claim that $W_1 \oplus W_1^\perp = V$. This follows from the fact that one can decompose any $v \in V$ into a sum of

$$v = (\omega(v, f_1)\, e_1 - \omega(v, e_1)\, f_1) + (v - \omega(v, f_1)\, e_1 + \omega(v, e_1)\, f_1),$$

where the first and second term are clearly in W_1 and $W_1^\perp$, respectively. One now inductively defines two-dimensional subspaces W_{i+1} spanned by vectors $e_{i+1}, f_{i+1} \in W_i^\perp$ chosen such that $\omega(e_{i+1}, f_{i+1}) = 1$. Again, they exist due to non-degeneracy if $W_i^\perp$ is non-trivial. Moreover, one sets the vector space $W_{i+1}^\perp$ as the symplectic complement of W_{i+1} in $W_i^\perp$. This process is repeated until the decomposition

$$V = \bigoplus_{1 \leq i \leq n} W_i$$

is obtained. Thus, the set $(e_i, f_i)_{1 \leq i \leq n}$ is a basis. Since W_i are mutually orthogonal with respect to ω and $\omega(e_i, f_i) = 1$ by construction, this concludes the proof. □

Remark. In particular, the above proof implies that symplectic vector spaces have even dimension. Otherwise, one would at some point arrive at a one-dimensional relative symplectic complement $W_i^\perp$, which would contradict non-degeneracy.

Now, one can introduce more complicated geometric objects that have the fiber structure of a symplectic vector space. This can be done by defining a 2-form on a manifold M that, at a point $x \in M$, yields a symplectic form on the tangent space $T_x M$.

Definition 1.2.3 (Symplectic Manifold)**.** A *symplectic manifold* (M, ω) is a manifold M together with a non-degenerate closed 2-form ω. ω is called the symplectic form.

Definition 1.2.4 (Symplectomorphism)**.** Morphisms of *symplectic manifolds, symplectic diffeomorphisms or symplectomorphisms* are maps $f : (M, \omega) \to (M', \omega')$ such that $f : M \to M'$ is a diffeomorphism and ω is the pullback of ω' along f, $f^*\omega' = \omega$

The notion of isotropic, coisotropic, Lagrangian, and symplectic vector subspaces extends to isotropic, coisotropic, Lagrangian, and symplectic manifolds in the following way:

Definition 1.2.5 (Isotropic, Coisotropic, Lagrangian, and symplectic submanifolds)**.** Let (M, ω) be a symplectic manifold and $N \subset M$ a subsubmanifold.

(i) N is an *isotropic subsubmanifold* if the tangent space T_xN is an isotropic subspace of T_xM for all $x \in M$.

(ii) N is a *coisotropic subsubmanifold* if T_xN is a coisotropic subspace of T_xM for all $x \in M$.

(iii) N is *Lagrangian* if T_xN is a Lagrangian subspace of T_xM for all $x \in M$.

(iv) N is *symplectic* if $\omega|_N$ is a symplectic form on N.

Definition 1.2.6 (Hamiltonian Vector field)**.** Let X be a vector field on a symplectic manifold (M, ω). It is *Hamiltonian* if there exists a function $h_X \in C^\infty(M)$ such that $\iota_X\omega = \mathrm{d}h_X$. In this case, one calls h_X the *Hamiltonian* of X.

Remark. Since ω is non-degenerate, there exists a unique Hamiltonian vector field for all $h \in C^\infty(M)$.

One can further define symplectic structures on manifolds with graded fibers.

Definition 1.2.7 (Odd-Symplectic Supermanifold)**.** An *odd-symplectic supermanifold* $(\mathcal{M}, \omega)$ is a supermanifold $\mathcal{M}$ together with a non-degenerate, odd closed 2-form ω.

Remark. The fibers of $\mathcal{M}$ form a symplectic super vector space. If ω is odd, there must be a Darboux basis $(e_1, \ldots, e_n, f_1, \ldots f_n)$ for some n such that the e_i have even parity and f_i have odd parity. One calls local coordinates x^i, θ^i, where x^i is even and θ^i is odd, Darboux coordinates if $\mathrm{d}x^i, \mathrm{d}\theta^i$ furnish a Darboux basis on the fibers. In this case, $\omega = \omega_{ia}\mathrm{d}x^i \wedge \mathrm{d}\theta^a$. In particular, this forces $\mathcal{M}$ to be of dimension $(n|n)$.

This can be further generalized to k-symplectic graded manifolds by imposing that the symplectic form is of degree k. The notion of symplectomorphism extends to odd and k-symplectic manifolds as morphisms of the underlying manifold that are compatible with the respective symplective forms. For computation, it is always desirable to work in Darboux coordinates. The following result asserts that one can assume their existence and gives a well-known representative of odd-symplectic manifolds:

Theorem 1.2.1 (Schwarz [Sch93])**.** *Any odd-symplectic manifold $(\mathcal{M}, \omega)$ with body M admits Darboux coordinates locally. Further, there is global symplectomorphism to the odd cotangent bundle ΠT^*M with the canonical symplectic form $\omega_{\mathrm{can}} = \mathrm{d}x^i \wedge \mathrm{d}\xi_i$.*

Definition 1.2.8 (Anti-Bracket)**.** Let $(\mathcal{M}, \omega)$ be an odd-symplectic manifold. The *odd Poisson bracket* $\{\cdot, \cdot\} : C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ induced by the odd-symplectic form via $\{f, g\} = X_f g$, where X_f is the Hamiltonian vector field associated to f, is called the anti-bracket. In local Darboux coordinates, it takes the form

$$\{f, g\} = \frac{\partial f}{\partial x^i}\frac{\partial g}{\partial \xi_i} - (-1)^{(|f|+1)(|g|+1)}\frac{\partial g}{\partial x_i}\frac{\partial f}{\partial \xi^i}.$$

Lemma 1.2.1. *Let $f, g \in C^\infty(\mathcal{M})$ and X_f, X_g be their Hamiltonian vector fields. Then $\{f, g\} = \mathrm{const}$ is equivalent to $[X_f, X_g] = 0$.*

Proof. One has the graded Jacobi identity $\{f, \{g, h\}\} = \{\{f, g\}, h\} + (-1)^{(|f|+1)(|g|+1)}\{g, \{f, h\}\}$. Hence,

$$\begin{aligned} X_{\{f,g\}} &= \{\{f, g\}, \cdot\} = \{f, \{g, \cdot\}\} - (-1)^{(|f|+1)(|g|+1)}\{g, \{f, \cdot\}\} = X_f X_g - (-1)^{(|f|+1)(|g|+1)} X_g X_f \\ &= [X_f, X_g] \end{aligned}$$

By definition of the Hamiltonian vector field one can conclude. □

Definition 1.2.9 (BV Laplacian)**.** On an odd-symplectic manifold $(\mathcal{M}, \omega)$, the *BV Laplacian* is the second order differential operator $\Delta : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ that in local Darboux coordinates takes the form

$$\Delta = \frac{\partial}{\partial x^i} \frac{\partial}{\partial \xi_i}.$$

It is a degree 1 coboundary operator. In any coordinate system, the BV Laplacian is given by the contraction of derivatives with the inverse matrix of the symplectic form.

Remark. The anti-bracket from Definition 1.2.8 is the failure of the BV Laplacian to be a derivative, i.e.

$$\{f, g\} = \Delta(fg) - (\Delta f)g - (-1)^{|f|} f(\Delta g).$$

Example 1.2.1. Consider the (-1)-shifted cotangent bundle over an n-dimensional manifold M. Given a top-degree form ρ, one can define an isomorphism by contraction,

$$\rho : \mathfrak{X}^k(M) \longrightarrow \Omega^{n-k}(M),$$

where $\mathfrak{X}^k(M)$ denotes the k-multivector fields. An operator Δ that induces the Schouten-Nijenhuis bracket as in above remark can be constructed by the de Rham operator in the following way:

$$\begin{array}{ccc} \mathfrak{X}^k(M) & \xrightarrow{\Delta} & \mathfrak{X}^{k-1}(M) \\ \downarrow \cong & & \downarrow \cong \\ \Omega^k(M) & \xrightarrow{\mathrm{d_{dR}}} & \Omega^{k+1}(M) \end{array}$$

1.3 Local Analysis for Functionals

This section treats the setup of functionals on spaces of sections. In particular, the connection between differential operators, jets and local action functionals is studied.

For the purpose of this discussion, vector bundles $E \to M$ over a manifold M are identified with their locally free sheaves of sections. Therefore, both the vector bundle E and its sheaf of sections will be denoted by E. For vector bundles E, F over M, the bundle $E \times_M F$ corresponds to the sheaf $E \oplus F$ and the bundle $E \otimes F$ corresponds to the sheaf $E \otimes_{C^\infty_M} F$. The space of bundle morphisms $\mathrm{Hom}(E, F)$ identifies the space of sheaf morphisms $\mathrm{Hom}_{C^\infty_M}(E, F)$, which induces the Hom-sheaf $\mathcal{H}\mathrm{om}_{C^\infty_M}(E, F)$. In particular, the dual bundle E^* identifies with the sheaf $\mathcal{H}\mathrm{om}_{C^\infty_M}(E, C^\infty_M)$.

Jets and Jet Bundles

Let $\pi \colon E \to M$ be a finite-dimensional vector bundle. Jets can be constructed in a similar way to the construction of tangent vectors as equivalence class of curves that have equal first derivatives at a point.

Definition 1.3.1 (k-th Order Jet)**.** Let σ be a smooth local section around $p \in M$. Suppose a set of local coordinates (x^i, u^I) around $\sigma(p)$. For $k \in \mathbb{N}$, one defines an equivalence relation $\sim_{k,p}$ on local sections around p such that $\sigma_1 \sim_{k,p} \sigma_2$ if

$$\left. \frac{\partial^{|\alpha|} \sigma_1^I}{(\partial x)^\alpha} \right|_p = \left. \frac{\partial^{|\alpha|} \sigma_2^I}{(\partial x)^\alpha} \right|_p, \qquad \alpha \in \mathbb{N}^n, |\alpha| \leq k.$$

The *k-th order jet* of σ at p, denoted $j^k_p \sigma$, is the equivalence class with respect to $\sim_{k,p}$ of σ.

This definition can be compactly formulated using the interpretation of a k-jet as the equivalence class of sections with the same Taylor expansion up to order k at p. One can now study the manifold given by all possible jets.

Definition 1.3.2 (k-th Jet Manifold)**.** Let $J^k_p(E)$ be the set of all jets of order k at a point $p \in M$. Then, the *k-th jet manifold* $J^k(E)$ is defined as the union over all p.

Note that by this definition, the 0-th jet manifold is just the bundle E itself. For most calculations, it is useful to have a local coordinate system on jet bundles. This can be implemented by considering an open $U \subset M$ that admits local coordinates (x^i, u^I). There is an induced coordinate system on $J^k(E)$ defined by (x^i, u^I, u^I_α), where the multi-index is $\alpha \in \mathbb{N}^n$, $|\alpha| \leq k$. The coordinate u^I_α is then specified by

$$u^I_\alpha(j^k_p\sigma) = \left.\frac{\partial^{|\alpha|}\sigma}{(\partial x)^I}\right|_p .$$

Instead of using the multi-index notation, one can interpret u^I_α as a formal power series in x^i, corresponding to the derivatives taken. This yields the coordinates $(x^i, F_r u^I[\![x]\!])$. One can now establish that the jet manifolds are again vector bundles over M. Thus, one also calls the jet manifold the jet bundle. In particular, the jet bundle of order $k+1$ is also a bundle of the k-th jet manifold, induced by the projection $\pi^{k+1,k}\colon J^{k+1}(E) \to J^k(E)$. These projections may be composed to obtain all bundles of the form $\pi^{k+l,k}\colon J^{k+l}(E) \to J^k E$.

Proposition 1.3.1 (Proposition 6.2.8 [Sau89])**.** *For* $0 \leq l < k$, $\pi^{k,l}\colon J^k(E) \to J^l(E)$ *is a bundle.*

This induces the structure

$$\begin{array}{ccccccccc} E & \xleftarrow{\pi^{1,0}} & J^1(E) & \longleftarrow & \cdots & \longleftarrow & J^r(E) & \xleftarrow{\pi^{k+1,k}} & J^{k+1}(E) & \longleftarrow & \cdots \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi^1} & & & & \downarrow{\scriptstyle \pi^k} & & \downarrow{\scriptstyle \pi^{k+1}} & & \\ M & \xleftarrow[\mathrm{id}]{} & M & \longleftarrow & \cdots & \longleftarrow & M & \xleftarrow[\mathrm{id}]{} & M & \longleftarrow & \cdots \end{array} \tag{1.1}$$

It is now a natural question to ask what the projective limit of the first line of this diagram is. Intuitively, it is the equivalence class of all sections that have the same Taylor expansion. This leads to the following definition:

Definition 1.3.3 (The ∞-Jet bundle)**.** The *bundle of infinite order jets* is defined as the inverse limit

$$J^\infty(E) = \varprojlim_k J^k(E).$$

It naturally gives rise to a projection onto the base $\pi^\infty\colon J^\infty(E) \to M$ and the set of projections onto lower order jet bundles $\pi^{\infty,k}\colon J^\infty(E) \to J^k(E)$.

Now that one has constructed various new vector bundles related to E one can talk about sections of them. Particularly interesting are the sections that arise by lifting sections of E:

Definition 1.3.4 (k-th Jet Prolongation)**.** The *k-th jet prolongation* is the lift of a section σ of E to a section of $J^k(E)$ defined by

$$j^k\sigma\colon p \mapsto j^k_p\sigma.$$

Example 1.3.1. Let σ be a local section of E and assume local coordinates (x^i, u^I). Then, the second order jet in p is given as

$$j_p^2\sigma = (p, \sigma^I(p), \partial_i\sigma^I(p), \partial_i\partial_j\sigma^I(p))$$

Thus the second order jet prolongation of σ is

$$\begin{aligned} j^2\sigma(x) &= \sigma^I(x)u^I + \partial_i\sigma^I(x)\, u_i^I + \sum_{i\le j}\partial_i\partial_j\sigma^I(x)\, u_{ij}^I \\ &= \sigma^I(x)u^I + \partial_i\sigma^I(x)\, u^I\tilde{x}_i + \sum_{i\le j}\partial_i\partial_j\sigma^I(x)\, u^I\tilde{x}_i\tilde{x}_j, \end{aligned}$$

where the formal series representation of coordinates was used in the second line and the tilde was added to emphasize the distinction of variables and formal variables.

Differential Operators and D-Modules

Definition 1.3.5 (Differential Operator)**.** A *differential operator of order* k on M is an operator $P : C_M^\infty \to C_M^\infty$ that can be written in a local coordinate system $(x^1, \dots, x^n)$ on an open $U \subset M$ as

$$P = \sum_{\alpha\in\mathbb{N}^n} p_\alpha(x)\partial^\alpha$$

with $p_\alpha \in C^\infty(U)$ and $p_\alpha = 0$ for $|\alpha| > k$.

The sheaf of differential operators of order $\le k$ on M is denoted by $\mathcal{D}_M^k$. For $k \le l$, there is a natural inclusion $\mathcal{D}_M^k \subset \mathcal{D}_M^l$.

Definition 1.3.6 (Sheaf of Differential operators)**.** The *sheaf of differential operators* on M is the direct limit

$$\mathcal{D}_M = \lim_{k\longrightarrow} \mathcal{D}_M^k.$$

Note that, in particular, C_M^∞ is a subsheaf of $\mathcal{D}_M$. One can introduce the two categories $\mathrm{Mod}(\mathcal{D}_M)$ and $\mathrm{Mod}(\mathcal{D}_M^{\mathrm{op}})$ of left and right $\mathcal{D}_M$-modules. In particular, every vector bundle endowed with a flat connection is a $\mathcal{D}_M$ module.

Example 1.3.2. (i) The sheaf C_M^∞ is a left $\mathcal{D}_M$-module with the usual action of differential operators.

(ii) The sheaf Ω_M of differential forms is a right $\mathcal{D}_M$-module, where the right action of the generators is defined as

$$v(\omega) = -L_v\omega, \quad v \in \Gamma(TM), \omega \in \Omega_M,$$

where L denotes the Lie derivative. In particular, the forms of top degree Ω_M^n are a right $\mathcal{D}_M$-module, inducing a right $\mathcal{D}_M$-module structure on the sheaf of densities on M, Dens_M.

(iii) Consider the C_M^∞ module of smooth sections of ∞-jets $J^\infty(E)$ for a vector bundle $E \to M$. The Cartan distribution, which is the natural distribution spanned by all total derivative vector fields, yields a flat connection, that in a local trivialization takes the form

$$\begin{aligned} \nabla\colon C^\infty(U)\otimes\mathbb{R}[\![x^1,\dots,x^n]\!]\otimes E_0 &\longrightarrow \Omega^1(U)\otimes\mathbb{R}[\![x^1,\dots,x^n]\!]\otimes E_0 \\ f\otimes g\otimes e &\longmapsto \mathrm{d}f\otimes g\otimes v + f\mathrm{d}x^i\otimes\frac{\partial}{\partial x^i}g\otimes v, \end{aligned}$$

where E_0 denotes the fiber of E. This makes $J^\infty(E)$ into a left $\mathcal{D}_M$-module.

For later purposes, there are two tensor product functors of right and left $\mathcal{D}_M$-modules to be considered. The first is the tensor product over C^∞_M. For $E \in \mathrm{Mod}(\mathcal{D}^{\mathrm{op}}_M)$, $F \in \mathrm{Mod}(\mathcal{D}_M)$, one defines $(e \otimes f)v = ev \otimes f - e \otimes vf$ for all $e \in E, f \in F, v \in TM$. This gives rise to the functor

$$\cdot \otimes_{C^\infty_M} \cdot \colon \mathrm{Mod}(\mathcal{D}^{\mathrm{op}}_M) \times \mathrm{Mod}(\mathcal{D}_M) \to \mathrm{Mod}(\mathcal{D}^{\mathrm{op}}_M).$$

Second, one defines the tensor product over $\mathcal{D}_M$ itself. Since $\mathcal{D}_M$ is a noncommutative ring, this will yield a module over its center. It is easily seen that the constant functions $\mathbb{R}_M$ on M are the center of $\mathcal{D}_M$, such that this functor is

$$\cdot \otimes_{\mathcal{D}_M} \cdot \colon \mathrm{Mod}(\mathcal{D}^{\mathrm{op}}_M) \times \mathrm{Mod}(\mathcal{D}_M) \to \mathrm{Mod}(\mathbb{R}_M).$$

This is a right exact functor and one can form its left derived functor, the left derived tensor product on the bounded derived category $\mathrm{D}^b(\mathcal{D}_M)$ of $\mathcal{D}_M$-modules:

$$\cdot \overset{\mathbf{L}}{\otimes}_{\mathcal{D}_M} \cdot \colon \mathrm{D}^b(\mathcal{D}^{\mathrm{op}}_M) \times \mathrm{D}^b(\mathcal{D}_M) \to \mathrm{D}^b(\mathbb{R}_M)$$

Definition 1.3.7 (Differential Homomorphism)**.** Let $E, F \to M$ be vector bundles over M. A *differential homomorphism* of order k, or simply *differential operator* of order k, between E and F is an operator P such that, in local coordinates (x^i, e^I) and (x^i, f^J) of E and F, respectively, over an open $U \subset M$,

$$P(\sigma)(x) = P_{IJ}(\sigma^I(x))\, f^J = \sum_{\alpha \in \mathbb{N}^n, |\alpha| \le k} p_{IJ,\alpha}(x) \partial^\alpha(\sigma^I(x))\, f^J$$

for a finite number of differential operators $P_{IJ} = \sum_{\alpha \in \mathbb{N}^n, |\alpha| \le k} p_{IJ,\alpha}(x)\partial^\alpha$ up to order k.

The space of differential homomorphisms of order $\le k$ on M is denoted by $\mathrm{Diff}^k(E, F)$. For $k \le l$, there is a natural inclusion $\mathrm{Diff}^k(E, F) \subset \mathrm{Diff}^l(E, F)$.

Proposition 1.3.2. *There is an isomorphism between differential homomorphisms up to order k and morphisms from the k-th jet bundle,*

$$\mathrm{Diff}^k(E, F) \cong \mathrm{Hom}_{C^\infty_M}(J^k(E), F).$$

There are various approaches to proving this; a differential-geometric one is chosen here:

Proof. Suppose local coordinates (x^i, e^I) and (x^i, f^J) of E and F, respectively, over an open $U \subset M$. By definition, a differential homomorphism of order at most k, $P \in \mathrm{Diff}^k(E, F)$, can then be written as

$$P(\sigma) = P_{IJ}(\sigma^I(x))\, f^J = \sum_{\alpha \in \mathbb{N}^n, |\alpha| \le k} p_{IJ,\alpha}(x) \partial^\alpha(\sigma^I(x))\, f^J$$

for a finite number of differential operators $P_{IJ} = \sum_{\alpha \in \mathbb{N}^n, |\alpha| \le k} p_{IJ,\alpha}(x)\partial^\alpha$ up to order k. Comparing this with the k-th jet prolongation, this suggests the definition

$$\begin{aligned} \iota \colon \mathrm{Diff}^k(E, F) &\longrightarrow \mathrm{Hom}_{C^\infty_M}(J^k(E), F) \\ P &\longmapsto \iota(P) \end{aligned}$$

where the operator $\iota(P)$ acts on $\tau \in J^k(E)$ by

$$\iota(P) \colon \tau = \sum_{\alpha \in \mathbb{N}^n, |\alpha| \le k} \tau_{I,\alpha}(x) u^I\, \hat{x}^\alpha \longmapsto \sum_{\alpha \in \mathbb{N}^n, |\alpha| \le k} p_{IJ,\alpha}(x) \tau_{I,\alpha}(x)\, f^J$$

with coefficients $\tau_{I,\alpha} \in C^\infty(U)$. It is clear that the map is well-defined and injective. Surjectivity follows from the inverse construction: given an operator $Q \in \mathrm{Hom}_{C^\infty_M}(J^k(E), F)$, one can expand it as

$$Q(\tau) = \sum_{\alpha\in\mathbb{N}^n, |\alpha|\leq k} q_{IJ}(x)\tau_{I,\alpha} f^J$$

for some $p_{IJ} \in C^\infty(U)$. Then one defines the map

$$\begin{aligned} \kappa : \mathrm{Hom}_{C^\infty_M}(J^k(E), F) &\longrightarrow \mathrm{Diff}^k(E, F) \\ Q &\longmapsto \kappa(Q) \end{aligned}$$

by setting the differential operator

$$\kappa(Q)_{IJ} = \sum_{\alpha\in\mathbb{N}^n, |\alpha|\leq k} q_{IJ}(x)\partial^\alpha,$$

thereby fully defining the differential homomorphism. It is evident that this construction is injective. In particular, one finds that $\kappa = \iota^{-1}$, hence bijectivity is proven. Since all constructions were C^∞_M-linear, the claim follows. □

Definition 1.3.8. The *space of differential homomorphisms* from E to F is the direct limit,

$$\mathrm{Diff}(E, F) = \lim_{k\longrightarrow} \mathrm{Diff}^k(E, F)$$

Proposition 1.3.3. *Let $E, F \to M$ be vector bundles on a manifold M. The space of differential homomorphisms is isomorphic to C^∞_M-morpisms of the jet bundle,*

$$\mathrm{Diff}(E, F) \cong \mathrm{Hom}_{C^\infty_M}(J^\infty(E), F) \cong \mathrm{Hom}_{D_M}(J^\infty(E), J^\infty(F)).$$

Proof. By Proposition 1.3.2, one has

$$\mathrm{Diff}(E, F) = \lim_{k\longrightarrow} \mathrm{Diff}^k(E, F) \cong \lim_{k\longrightarrow} \mathrm{Hom}_{C^\infty_M}(J^k(E), F).$$

Note that the functor $\mathrm{Hom}_{C^\infty_M}(\cdot, F)$ acts exact on the category of smooth local sections of smooth finite-dimensional vector bundles over M. Indeed, since local sections are locally free, they are projective as an object in the category of C^∞_M-modules. Therefore, each $\mathrm{Ext}_{C^\infty_M}(E, C^\infty_M)$ vanishes. One concludes that the colimit can be pulled in, converting it to a limit, since $\mathrm{Hom}_{C^\infty_M}(\cdot, F)$ is contravariant. This yields

$$\lim_{k\longrightarrow} \mathrm{Hom}_{C^\infty_M}(J^k(E), F) = \mathrm{Hom}_{C^\infty_M}(\lim_{\longleftarrow k} J^k(E), F) = \mathrm{Hom}_{C^\infty_M}(J^\infty(E), F),$$

proving the first isomorphism. The second claim follows directly by taking the ∞-jet prolongation. □

Example 1.3.3. It is instructive to give an explicit demonstration of how the bijection from Proposition 1.3.3 works by giving the example for the first order differential operator $\mathrm{d} \in \mathrm{Diff}(\Omega^0(M), \Omega^1(M))$. Suppose local coordinates x^i on M and $f \in \Omega(M)$. The operator d acts by $f(x) \mapsto \partial_i f \, \mathrm{d}x^i$. The 1-jet prolongation of f is

$$j^1 f(x) = \partial_i f(x)\, \tilde{x}^i.$$

By the construction in the proof of Proposition 1.3.2, there is an induced map $j^1 f(x) \mapsto \partial_i f \, \mathrm{d}x^i$, extending to all sections $\tau \in J^1(\Omega(M))$ by

$$\tau = \tau_0(x) + \tau_i \tilde{x}^i \longmapsto \tau_i(x)\, \mathrm{d}x^i.$$

The left multiplication of the subring $C^\infty_M \subset \mathcal{D}_M$ naturally endows the space $\mathrm{Diff}(E,F)$ with the structure of a left C^∞_M-module. This then induces the sheaf of differential homomorphisms $\mathcal{D}\mathrm{iff}(E,F)$. Hence, the previous proposition can be extended to the sheaves:

Proposition 1.3.4. *The sheaf of differential homomorphisms is isomorphic to the sheaves of morphisms from the jet bundle,*

$$\mathcal{D}\mathrm{iff}(E,F) \cong \mathcal{H}\mathrm{om}(J^\infty(E),F) \cong \mathcal{H}\mathrm{om}_{C^\infty_M}(J^\infty(E),C^\infty_M) \otimes_{C^\infty_M} F.$$

Furthermore, by considering the right adjoint $J^\infty : \mathrm{Mod}(C^\infty_M) \to \mathrm{Mod}(\mathcal{D}_M)$ to the forgetful functor $|\cdot| : \mathrm{Mod}(\mathcal{D}_M) \to \mathrm{Mod}(C^\infty_M)$, $|\cdot| \dashv J^\infty$, one can add an extra isomorphism to Proposition 1.3.4,

$$\mathcal{D}\mathrm{iff}(E,F) \cong \mathcal{H}\mathrm{om}_{\mathcal{D}_M}(J^\infty(E),J^\infty(F)).$$

One can now introduce the sheaf of multidifferential homomorphisms:

Definition 1.3.9 (Multidifferential Homomorphism)**.** Let $E_1,\dots,E_k,F \to M$ be vector bundles over M. The *sheaf of multidifferential homomorphisms, or multidifferential operators,* $\mathcal{M}\mathrm{ulti}\mathcal{D}\mathrm{iff}(E_1\otimes\cdots\otimes E_k,F)$ is given by

$$\mathcal{D}\mathrm{iff}(E_1,C^\infty_M)\otimes_{C^\infty_M}\cdots\otimes_{C^\infty_M}\mathcal{D}\mathrm{iff}(E_k,C^\infty_M)\otimes_{C^\infty_M} F.$$

With this definition, the isomorphism following Proposition 1.3.4 naturally extends to multidifferential operators as follows:

Corollary 1.3.1. *For $E_1,\dots,E_k,F \to M$ vector bundles over M, there is an isomorphism*

$$\mathcal{M}\mathrm{ulti}\mathcal{D}\mathrm{iff}(E_1\otimes\cdots\otimes E_k,F) \cong \mathcal{H}\mathrm{om}_{\mathcal{D}_M}(J^\infty(E_1)\otimes_{C^\infty_M}\cdots\otimes_{C^\infty_M} J^\infty(E_k),J^\infty(F)).$$

Lagrangian Densities

An indispensable notion for field theory is the local action functional. The idea is that a local action functional is the integral of a Lagrangian density, which is given by the product of finitely many finite order differential operators acting on a section. Locally, a differential operator may be specified by a dual ∞-jet, i.e. an element in

$$J^\infty(E)^\vee = \mathrm{Hom}_{C^\infty_M}(J^\infty(E),C^\infty_M) = \mathrm{Diff}(E,C^\infty_M).$$

One can think of a product of differential operators acting on a section as polynomial in derivatives of the section. This motivates the definition

$$\mathcal{O}(J^\infty(E)) := \prod_{k\geq 0}\mathrm{Sym}^k_{C^\infty_M} J^\infty(E)^\vee \cong \prod_{k\geq 0}\mathrm{MultiDiff}(E^{\otimes k},C^\infty_M)^{S_k}.$$

Note that $J^\infty(E)^\vee$ is naturally a left $\mathcal{D}_M$-module. This then endows $\mathcal{O}(J^\infty(E))$ with a left $\mathcal{D}_M$-module structure. With this object, one can now form the sheaf of local Lagrangian densities

$$\mathrm{Dens}_M \otimes_{C^\infty_M} \mathcal{O}(J^\infty(E)).$$

An action functional is an equivalence class of Lagrangian densities up to total derivatives. By the discussion in [BD04], this quotient can be implemented by the homotopy functor

$$h\colon E\otimes_{C^\infty_M} F \longmapsto E\otimes_{\mathcal{D}_M} F \qquad E\in\mathrm{Mod}(\mathcal{D}^{\mathrm{op}}_M), F\in\mathrm{Mod}(\mathcal{D}_M).$$

Then, the sheaf of local action functionals is

$$\mathrm{Dens}_M \otimes_{\mathcal{D}_M} \mathcal{O}(J^\infty(E)).$$

Functionals of Global Sections

Let $\pi\colon E \to M$ be a vector bundle over M and $\mathcal{E} := \Gamma(M, E)$ the global sections of E, which form a locally convex vector space. One works in a convenient category of locally convex vector spaces, such that the tensor product $\otimes$ and the dual $\vee$ are well-behaved. One possible choice is the category of nuclear Fréchet spaces; for further details, refer to Appendix 2, [Cos11]. One denotes $C^\infty(M) = \Gamma(M, C^\infty_M)$ and $\mathrm{Dens}(M) = \Gamma(M, \mathrm{Dens}_M)$.

Definition 1.3.10 (Differential and Multidifferential Operators)**.** Let $E, F \to M$ be vector bundles and $\mathcal{E}, \mathcal{F}$ the respective spaces of global sections. The *differential operators between global sections* are defined as

$$\mathrm{Diff}(\mathcal{E}, \mathcal{F}) = \Gamma(M, \mathcal{D}\mathrm{iff}(E, F)).$$

Multidifferential operators between global sections are defined as

$$\mathrm{MultiDiff}(\mathcal{E}^{\otimes k}, \mathcal{F}) = \Gamma(M, \mathcal{M}\mathrm{ulti}\mathcal{D}\mathrm{iff}(E^{\otimes k}, F)).$$

One can now use global sections of the sheaf of local action functionals to construct what is usually regarded as a local action functional:

Definition 1.3.11 (Local action functional)**.** A *local action functional* S is the sum of integrals over a Lagrangian density that depends only on finite order jets of the section. Explicitly, S takes the form

$$S[\phi] = \sum_k S_k[\phi], \quad \phi \in \mathcal{E},$$

where every S_k may be written as

$$S_k[\phi] = \int_M (D_1\phi)(D_2\phi)\ldots(D_k\phi)\,\mathrm{dvol},$$

with $D_i \in \mathrm{Diff}(\mathcal{E}, C^\infty_M)$. The *space of local functionals* is denoted as $\mathcal{O}_{\mathrm{loc}}(\mathcal{E})$.

Lemma 1.3.1 (Lemma 6.2.1, Chapter 5 [Cos11])**.** *Let $E, F \to M$ be vector bundles and denote their sections with $\mathcal{E}, \mathcal{F}$. There is a bijection identifying*

$$\mathrm{Diff}(\mathcal{E}, \mathcal{F}) = \mathrm{Hom}_{\mathcal{D}_M}(J^\infty(\mathcal{E}), J^\infty(\mathcal{F})).$$

It can be generalized to multidifferential homomorphisms,

$$\mathrm{MultiDiff}(\mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_k, \mathcal{F}) = \mathrm{Hom}_{\mathcal{D}_M}(J^\infty(\mathcal{E}_1) \otimes_{C^\infty(M)} \cdots \otimes_{C^\infty(M)} J^\infty(\mathcal{E}_k), J^\infty(\mathcal{F})).$$

Proof. The argument is analogous to Propositions 1.3.3, with the modification that one uses the Theorem of Serre–Swan, stating that the smooth section functor Γ maps an object in the category of smooth vector bundles over M to an object in the category of projective $C^\infty(M)$-modules. This asserts that $\mathcal{E}$ is projective as $C^\infty(M)$-module. □

Definition 1.3.12 (Functional)**.** The *space of functionals* on $\mathcal{E}$ is defined as the direct product of S_n-invariant continuous linear maps

$$\mathcal{O}(\mathcal{E}) := \prod_{n \in \mathbb{N}} \mathrm{Hom}(\mathcal{E}^{\otimes n}, \mathbb{R})^{S_n}.$$

Here, $\otimes$ denotes the completed projective tensor product.

Proposition 1.3.5. *There is an inclusion*

$$\iota : \mathrm{Dens}(M) \otimes_{\mathcal{D}_M} \mathcal{O}(J^\infty(\mathcal{E})) \longrightarrow \mathcal{O}(\mathcal{E}).$$

An element $S \in \operatorname{im}\iota$ is of the form presented in Definition 1.3.11, i.e. $\operatorname{im}\iota = \mathcal{O}_{\mathrm{loc}}(\mathcal{E})$.

There is an obvious grading in n on the space of functionals. A homogeneous element $f \in \mathrm{Hom}(\mathcal{E}^{\otimes n}, \mathbb{R})^{S_n}$ can be interpreted as a formal monomial of degree n on $\mathcal{E}$, suggesting the interpretation of $f \in \mathrm{Sym}^n(\mathcal{E}^\vee)$, where $\mathcal{E}^\vee$ denotes the continuous dual of $\mathcal{E}$. This induces a natural algebra structure inherited by the symmetric product. Let $f \in \mathrm{Hom}(\mathcal{E}^{\otimes n}, \mathbb{R})^{S_n},\ g \in \mathrm{Hom}(\mathcal{E}^{\otimes m}, \mathbb{R})^{S_n}$. Via the identification of $\mathrm{Hom}(\mathcal{E}^{\otimes n}, \mathbb{R})^{S_n} = \mathrm{Sym}^n(\mathcal{E}^\vee)$, the map $\odot : f, g \mapsto f \odot g$ induces a product

$$\mathrm{Hom}(\mathcal{E}^{\otimes n}, \mathbb{R})^{S_n} \times \mathrm{Hom}(\mathcal{E}^{\otimes m}, \mathbb{R})^{S_n} \to \mathrm{Hom}(\mathcal{E}^{\otimes n+m}, \mathbb{R})^{S_{n+m}}.$$

This turns $\mathcal{O}(\mathcal{E})$ into an algebra, which is regarded as the completed symmetric algebra of $\mathcal{E}^\vee$, $\mathcal{O}(\mathcal{E}) = \widehat{\mathrm{Sym}}(\mathcal{E}^\vee)$. On the space of functionals, there is an action of the Lie algebra of derivations

$$\mathrm{Der}(\mathcal{O}(\mathcal{E})) := \mathcal{O}(\mathcal{E}) \otimes \mathcal{E} = \prod_{n \in \mathbb{N}} \mathrm{Hom}(\mathcal{E}^{\otimes n}, \mathcal{E})^{S_n}.$$

It is straightforward that these operators satisfy the Leibniz rule. Further, one defines the Lie subalgebra of local derivations as

$$\mathrm{Der}_{\mathrm{loc}}(\mathcal{O}(\mathcal{E})) := \prod_{n \in \mathbb{N}} \mathrm{MultiDiff}(\mathcal{E}^{\otimes n}, \mathcal{E})^{S_n}.$$

For the application to gauge theory, it is of particular interest how the functionals behave under transformations of a gauge group $\mathcal{G} = \mathrm{Map}(M, G)$, which denotes functions from the base manifold to a Lie group G that acts on fibers of E.

Definition 1.3.13 (Local Lie Algebra)**.** Let $L \to M$ be a vector bundle with smooth sections $\mathfrak{L}$. A *local Lie algebra* is defined as $(\mathfrak{L}, [\cdot,\cdot]_{\mathfrak{L}})$, where $[\cdot,\cdot]_{\mathfrak{L}} : \mathfrak{L} \otimes \mathfrak{L} \to \mathfrak{L}$ is an antisymmetric multidifferential operator that satisfies the Jacobi identity.

Definition 1.3.14 (Local Lie algebra action)**.** A local Lie algebra acts on the space of smooth sections $\mathcal{E}$ via a Lie algebra map

$$\mathfrak{L} \to \mathrm{Der}_{\mathrm{loc}}(\mathcal{O}(\mathcal{E})).$$

The action of $\mathcal{G}$ may then be reduced to the action of the local Lie algebra action of $\mathfrak{L}$, where L has fibers $\mathfrak{g} = \mathrm{Lie}(G)$. In this situation, one can consider the Chevalley–Eilenberg complex of the supermanifold $\mathfrak{L}[1] \oplus \mathcal{E}$, consisting of the cochains $C^\bullet(\mathfrak{L}[1] \oplus \mathcal{E}) = \mathrm{Sym}^\bullet(\mathfrak{L}[1] \oplus \mathcal{E})^\vee$. The Chevalley–Eilenberg differential is then the adjoint map of the sum of $[\cdot,\cdot]_{\mathfrak{L}}$ and the local Lie algebra action.

Remark. If $X \in \mathrm{Der}_{\mathrm{loc}}(\mathcal{E})$ is square-zero, it allows for the construction of a local L_∞ structure, following the mechanism described after Definition 1.1.9.

Lemma 1.3.2 (Lemma 6.6.1, Chapter 5 [Cos11])**.** *Let $\mathcal{E}$ be a differentially graded space of sections with differential X. Then there is an isomorphism of cochain complexes*

$$\mathcal{O}_{\mathrm{loc}}(\mathcal{E})/\mathbb{R} = \mathrm{Dens}(M) \overset{\mathbf{L}}{\otimes}_{D_M} (\mathcal{O}(J^\infty(\mathcal{E}))/C^\infty(M)).$$

Proof. By the definition of locality, one should interpret a local action functional as the integral of a Lagrangian density, which is an element of

$$\begin{aligned}\mathrm{Dens}(M)\otimes_{C^\infty(M)}\prod_{k\in\mathbb{N}}\Big(\mathrm{Sym}^k\mathrm{Diff}(\mathcal{E},C^\infty(M))\Big) &= \mathrm{Dens}(M)\otimes_{C^\infty(M)}\prod_{k\in\mathbb{N}}\Big(\mathrm{Sym}^k\mathrm{Hom}_{C^\infty(M)}(J^\infty(\mathcal{E}),\mathbb{R})\Big)\\ &= \mathrm{Dens}(M)\otimes_{C^\infty(M)}\prod_{k\in\mathbb{N}}\Big(\mathrm{Sym}^k(J^\infty(\mathcal{E})^\vee\Big)\end{aligned}$$

where Lemma 1.3.1 was used and $J^\infty(\mathcal{E})^\vee = \mathrm{Hom}(J^\infty(\mathcal{E}),\mathbb{R})$ was identified. This is a right D_M-module, as it is obtained from applying $\cdot\otimes_{C^\infty(M)}\cdot$ to the right D_M-module of densities and the left D_M-module of $\mathrm{Sym}^k(J^\infty(\mathcal{E}))^\vee$. As described in the discussion of local action functionals, one now needs to identify all equivalent Lagrangians by quotienting out total derivatives. This is again done by the application of the homotopy functor h. After reducing the complex to non-constant action functionals, this leaves

$$\mathcal{O}_{\mathrm{loc}}(\mathcal{E})/\mathbb{R} = \mathrm{Dens}(M)\otimes_{D_M}(\mathcal{O}(J^\infty(\mathcal{E}))/C^\infty(M)).$$

To prove the statement, it is left to show that one can replace the tensor product over D_M with the left derived one. It suffices to show that the second factor is flat, guaranteeing that the Tor functor vanishes (Proposition A.3.2). As described in the respective proof in [Cos11], this is easily done for $\mathcal{O}(J^\infty(\mathcal{E}))/C^\infty(M)$, as $J^\infty(\mathcal{E})^\vee$ can be identified locally with the D_M-module $E_0^\vee\otimes D_M$ for E_0 a fiber of E. Therefore, it is freely generated over D_M and, in particular, flat. All tensor powers may be decomposed in a similar way, proving flatness, and the claim follows. The differential on the right hand side is the one induced by X according to Lemma 1.3.1 plus a contribution of the differential of the projective resolution used to compute the left derived tensor product. $\square$

Chapter 2

The BV formalism

In this chapter, a short overview of the quantum BV formalism is given. It is widely considered to be the most general approach to the gauge fixing problem in the path integral formalism. In fact, the BV formalism even works for more general settings of an action invariant under any integrable distribution, but the main interest will remain in ordinary gauge theory. The discussion is based on [Mne19], [Cos11], and [CMR18].

First, the problem this formalism provides a solution for is outlined. The path integral is commonly defined as

$$Z = \int_{\mathcal{F}} \mathcal{D}\phi^a \; e^{iS_{\text{cl}}[\phi^a]/\hbar}, \tag{2.1}$$

where $S_{\text{cl}}[\phi^a]$ is some classical action functional of a physical theory, depending on a set of physical fields ϕ^a in the space of fields $\mathcal{F}$. However, before delving further into this subject, one should specify more precisely what kind of functionals S_{cl} are worked with. The application is to physical theories, but an arbitrary element in $\mathcal{O}(\mathcal{F})$ may fail to describe a physical theory. Two necessary conditions are given such that this is not the case.

Postulate 2.0.1. *A physical theory may be described by an action functional that is local according to Definition 1.3.11.*

Postulate 2.0.2. *At classical level modulo $\hbar$, that is, any physical theory is the perturbation of a free theory by some interaction. Explicitly, one has*

$$S_{\text{cl}}[\phi] = \int_M \phi D\phi + I[\phi],$$

where $I \in \mathcal{O}_{\text{loc}}(\mathcal{F})$ is at least cubic.

The issue arises when, as in most realistic situations, the space of fields is infinite-dimensional. For this case, it is proven that there exists no translation invariant integration measure $\mathcal{D}\phi^a$, therefore the path integral is ill-defined. And yet, such integrals can still be made sense of as series expansions, if one assumes the same asymptotics to hold in the infinite-dimensional setting as in the finite-dimensional case:

Theorem 2.0.1 (Stationary phase formula)**.** *Let M be an oriented n-manifold with a compactly supported top-degree form $\mu \in \Omega_c^n(M)$ and $f \in C^\infty(M)$ a smooth function such that the critical points on* supp μ*, $x_1, \dots, x_m$, are non-degenerate. Then the integral*

$$I(k) := \int_M \mu \; e^{ikf(x)}$$

behaves for $k \to \infty$ as

$$I(k) \sim \sum_{i=1}^{m} e^{ikf(x_i)} \left|\det f''(x_i)\right| \cdot e^{\frac{\pi i}{4}\operatorname{sign} f''(x_i)} \cdot \mu(x_i) + O(k^{-\frac{n}{2}-1}). \tag{2.2}$$

As the integrand in this theorem in fact has the same form as the one in (2.1), one may give a sensible meaning to the infinite-dimensional integral. The idea is that one expands perturbatively in the limit $\hbar \to 0$ around the critical points of the action, i.e. the classical solutions of the equations of motion. Of course this requires that the conditions for the stationary phase formula are formally satisfied. In particular, one needs non-degeneracy at the critical points.

2.1 The finite-dimensional case

Classical Data

For now, the case (2.1) is set aside and instead the integral over a finite-dimensional vector space F is considered. Even though one is ultimately more interested in the infinite-dimensional case, it is more instructive to develop the BV mechanism in finite dimensions. So, let

$$Z = \int_F e^{if/\hbar}, \tag{2.3}$$

where f is invariant under the action of a finite-dimensional Lie group G with Lie algebra $\mathfrak{g}$. As the action is constant along the orbits of symmetry, the Hessian is necessarily degenerate if one does not fix a gauge. One may attempt integrating over the quotient by the group action F/G, resulting in the integration on G-invariant functions of F. This is essentially the BRST construction. Instead of the G-invariants, one may take the $\mathfrak{g}$-invariants. Since the invariant functor may not be exact, one should use the derived invariants instead of the naive ones. From homological algebra, it is known that they are given by the Chevalley–Eilenberg cochains

$$C^\bullet(\mathfrak{g}, F) = \wedge^\bullet \mathfrak{g}^\vee \otimes \widehat{\operatorname{Sym}}(F) \cong \widehat{\operatorname{Sym}}(\mathfrak{g}[1] \oplus F), \tag{2.4}$$

with the Chevalley–Eilenberg differential d_{CE}. But this is nothing else than the algebra of functions on $\mathfrak{g}[1] \oplus F$. Therefore, one interprets the action as a cochain in homological degree 0 and replaces the integral by

$$Z = \int_{\mathfrak{g}[1]\oplus F} e^{if/\hbar} \tag{2.5}$$

This is, however, still degenerate, since there is no $\mathfrak{g}$-dependence in the integrand. To overcome this, one moves to the shifted cotangent bundle

$$E := T^*[-1](\mathfrak{g}[1] \oplus F) = \mathfrak{g}[1] \oplus F \oplus F^\vee[-1] \oplus \mathfrak{g}^\vee[-2]$$

which is a $\mathbb{Z}$-graded odd symplectic manifold with the canonical symplectic form ω induced by the pairing of the vector spaces with their respective duals. On this vector space, one can construct a new integral that is not degenerate. Let $\pi\colon E \to \mathfrak{g}[1] \oplus F$ denote the canonical projection. f is pulled back along π to a function on E which one continues to call f. Further, observe that the action of the Chevalley–Eilenberg differential on $\mathfrak{g}[1] \oplus F$ is the same as a degree 1 vector field X_{CE}, which fulfills $[X_{\mathrm{CE}}, X_{\mathrm{CE}}] = 0$, since it is a differential. One defines its lift $X := \pi^* X_{\mathrm{CE}}$ such that it is Hamiltonian. Since X_{CE} squares to 0 on $\mathfrak{g}[1] \oplus F$, X squares to 0 on E. Therefore, there exists a Hamiltonian function h_X on E that satisfies $\{h_X, h_X\} = 0$.

Definition 2.1.1 (BV manifold)**.** A *BV manifold* $(\mathcal{E}, \omega, S)$ is a triple consisting of an odd symplectic supermanifold $\mathcal{E}$ together with a degree 0 function S satisfying the classical master equation (CME)

$$\{S, S\} = 0.$$

In this case, one calls S the BV action or master action.

Remark. One may replace the action in the BV data by its Hamiltonian vector field Q, which is also referred to as the cohomological BV vector field. It is easy to see that it defines S up to constants that are disregarded due to Postulate 2.0.2, is of degree 1 and, by Lemma 1.2.1, fulfills $[Q, Q] = 0$.

Since f only depends on coordinates of $\mathfrak{g}[1] \oplus F$, but not on the shifted tangent vector coordinates, $\{f, f\} = 0$. Gauge invariance of f is equivalent to $X_{\mathrm{CE}} f = 0$, a statement that lifts to E and becomes $Xf = \{h_x, f\} = 0$. And, as remarked before, $\{h_X, h_X\} = 0$. One concludes that $S := f + h_X$ fulfills the CME and thus, (E, ω, S) defines a BV manifold. Furthermore, the action S does not have a trivially degenerate Hessian and one may recover the original function f by $\pi_* S = f$. It thus becomes clear that one should interpret Z as the integral

$$Z = \int_{\mathcal{L}_0 \subset E} e^{iS/\hbar}, \tag{2.6}$$

where $\mathcal{L}_0$ is the Lagrangian submanifold of E given by the obvious embedding of $\mathfrak{g} \oplus F$. For later convenience, it is useful to decompose this action into a quadratic part and an interaction part. There exists a linear vector field Q of cohomological degree 1, that is skew-selfadjoint with respect to the symplectic pairing, and an element $I \in \mathcal{O}_{\mathrm{loc}}(E)$, that is of at least cubic order, such that

$$S(e) = \frac{1}{2}\omega(e, Qe) + I(e).$$

One can insert this expression in the CME and notice that it splits into three parts, consisting of the BV bracket of the free part, the bracket of the interaction, and a mixed term,

$$\begin{aligned} 0 =& \left\{\frac{1}{2}\omega(e, Qe) + I(e), \frac{1}{2}\omega(e, Qe) + I(e)\right\} \\ =& \frac{1}{4}\{\omega(e, Qe), \omega(e, Qe)\} + \{I(e), I(e)\} + \{\omega(e, Qe), I(e)\}. \end{aligned}$$

Observe that the only quadratic term is the bracket of the quadratic part, thus it needs to vanish individually. Identifying $\frac{1}{2}\omega(e, Qe)$ as the Hamiltonian of Q, this is equivalent to $Q^2 = 0$. Using the definition of the BV bracket, one then finds the CME for the interaction

$$QI(e) + \frac{1}{2}\{I(e), I(e)\} = 0.$$

BV integration

To progress further, some results on integration over Lagrangian submanifolds are reviewed.

Definition 2.1.2 (Compatible Berezinian)**.** Let $(\mathcal{M}, \omega)$ be an $(n|n)$-dimensional odd symplectic manifold and $\mu \in \Gamma(M, \mathrm{Ber}(\mathcal{M}))$ a Berezinian, where $\mathrm{Ber}(\mathcal{M})$ denotes the Berezin line bundle. A Berezinian μ is *compatible* with ω if there exists an atlas of Darboux charts (x^i, ξ_i) such that $\mu = d^n x \mathcal{D}^n \xi$.

Note that this is equivalent to the half-density $\mu^{\frac{1}{2}}$ being in the kernel of the BV operator. This property becomes useful if one considers a class of μ-dependent BV operators defined as $\Delta_\mu : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ such that $\mu^{\frac{1}{2}}\Delta_\mu f = \Delta(\mu^{\frac{1}{2}} f)$. Then, for any compatible Berezinian μ, the canonical BV Laplacian and Δ_μ coincides in coordinates given by the atlas described in Definition 2.1.2. In particular, only then is Δ_μ also a coboundary operator.

Definition 2.1.3 (BV integral)**.** Let $(\mathcal{M}, \omega)$ an odd-symplectic manifold with compatible Berezinian μ. Then, a *BV integral* is an integral of the form

$$\int_{\mathcal{L}\subset\mathcal{M}} f\ \mu^{\frac{1}{2}},$$

where $\mathcal{L}$ is a Lagrangian submanifold and $f \in C^\infty(\mathcal{M})$ is such that $\Delta_\mu f = 0$.

Theorem 2.1.1 (Batalin-Vilkovisky [BV81], Schwarz [Sch93])**.** *On an odd-symplectic manifold $(\mathcal{M}, \omega)$ with compatible Berezinian μ, the following hold:*

(i) Let $f \in C^\infty(\mathcal{M})$ and $\mathcal{L} \subset \mathcal{M}$ a Lagrangian submanifold. Assuming convergence of the integral,

$$\int_{\mathcal{L}} (\Delta_\mu g)\ \mu^{\frac{1}{2}} = 0.$$

(ii) Let $\mathcal{L}_t$ be a continuous family of Lagrangian submanifolds parametrized by $t \in [0,1]$ and $f \in C^\infty(\mathcal{M})$ a Δ_μ-closed function. Assuming convergence of the integral,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{L}_t} f\ \mu^{\frac{1}{2}} = 0.$$

Theorem 2.1.2 (BV pushforward)**.** *Let $(\mathcal{M}, \omega)$ an odd symplectic manifold that factors as a direct product of odd symplectic manifolds $(\mathcal{M}', \omega'), (\mathcal{M}'', \omega'')$, such that $\mathcal{M} = \mathcal{M}' \times \mathcal{M}''$ and $\omega = \omega' + \omega''$. Then the space of half-densities factorizes as*

$$\mathrm{Dens}^{\frac{1}{2}}(\mathcal{M}) = \mathrm{Dens}^{\frac{1}{2}}(\mathcal{M}')\hat{\otimes}\mathrm{Dens}^{\frac{1}{2}}(\mathcal{M}'').$$

Moreover, for any Lagrangian submanifold $\mathcal{L} \subset \mathcal{M}''$, the map

$$\mathrm{id} \otimes \int_{\mathcal{L}} : \mathrm{Dens}^{\frac{1}{2}}(\mathcal{M}) \to \mathrm{Dens}^{\frac{1}{2}}(\mathcal{M}')$$

is well-defined and has the following two properties:

(i) Let $\xi \in \mathrm{Dens}^{\frac{1}{2}}(\mathcal{M})$, then $\int_{\mathcal{L}} \Delta\xi = \Delta' \int_{\mathcal{L}} \xi$.

(ii) Let $\mathcal{L}_t \subset \mathcal{M}''$ be a continuous family of Lagrangian submanifolds parametrized by $t \in [0,1]$ and let $\xi \in \mathrm{Dens}^{\frac{1}{2}}(\mathcal{M})$ be Δ-closed, then

$$\int_{\mathcal{L}_1} \xi - \int_{\mathcal{L}_0} \xi = \Delta'\Psi$$

for some $\Psi \in \mathrm{Dens}^{\frac{1}{2}}(\mathcal{M}')$.

Remark. In fact, one can weaken the assumptions in the previous theorem. Let $\mathcal{C} \subset \mathcal{M}$ be a coisotropic manifold with a smooth reduction $\underline{\mathcal{C}}$. Suppose a direct product $\underline{\mathcal{C}} \times \mathcal{M}'$, where $\mathcal{M}'$ is as above. Then there exists a pushforward of half densities on $\mathcal{M}$ to half densities on the smooth reduction by integration over a Lagrangian submanifold. In particular, properties (i) and (ii) still hold.

Quantization

In light of Theorem 2.1.1(i), it becomes apparent how the integral (2.6) can be interpreted in a meaningful manner. While the integration over the Lagrangian submanifold $\mathcal{L}_0$ may be problematic, one can now choose any other Lagrangian submanifold as the domain of integration, provided it is continuously deformable to $\mathcal{L}_0$ and

$$\Delta e^{iS/\hbar} = 0 \quad \Longleftrightarrow \quad \frac{1}{2}\{S, S\} - i\hbar\Delta S = 0. \tag{2.7}$$

This is known as the quantum master equation (QME). The choice of Lagrangian submanifold can then be seen as gauge-fixing. If the classical BV action solves the QME, one has the two separate conditions $\{S, S\} = 0$ and $\Delta S = 0$, which correspond to the CME and volume conservation in a generalized sense. However, in most applications, this is not the case. To overcome this, one makes an Ansatz for the quantum action as a formal power series in the parameter $\hbar$, $\Sigma = \sum_{i\in\mathbb{N}} \hbar^i S^{(i)} \in \mathcal{O}(E)[\![\hbar]\!]$, where the term of 0-th order is just the classical action S. Thus, the QME can be understood as an order-by-order in $\hbar$ condition on the quantum action, yielding

$$\begin{aligned} \left\{S^{(0)}, S^{(0)}\right\} &= 0 && \text{0th order} \\ \left\{S^{(0)}, S^{(1)}\right\} + i\Delta S^{(0)} &= 0 && \text{1st order} \\ \frac{1}{2}\left\{S^{(1)}, S^{(1)}\right\} + \left\{S^{(0)}, S^{(2)}\right\} + i\Delta S^{(1)} &= 0 && \text{2nd order} \\ &\vdots \end{aligned}$$

Quantization thus equates to finding a quantum action Σ such that its terms $S^{(i)}$ solve the above. Switching again to the formulation in terms of interactions, one writes the Ansatz as

$$\Sigma = \omega(e, Qe) + I', \quad I' = I + \sum_{i\in\mathbb{N}_+} \hbar^i I^{(i)}.$$

Note that the interactions of higher order need not necessarily be of higher than cubic order. The square part is a solution of the QME, since the only functions of quadratic order that are not in the kernel of the BV Laplacian are of degree 1. Thus, the QME becomes a condition for the interactions only, which takes the form

$$QI + \frac{1}{2}\{I, I\} - i\hbar\Delta I = 0$$

and may, as above, be expanded order by order. One can then construct a solution $I' \in \mathcal{O}(E)[\![\hbar]\!]$ to the QME iteratively by inserting the solution up to order n in $\hbar$. This will, in general, not solve the equation of order $n+1$, but will yield an obstruction on the right hand side of the equation. However, under the assumption that the obstruction is exact in the cohomology of $Q + \left\{I^{(0)}, \cdot\right\}$, one can deform the interaction by an element $I^{(n+1)}$ in the preimage of the obstruction under the differential to a solution up to order $n+1$.

Definition 2.1.4. Let $I_n \in \mathcal{O}(E)[\![\hbar]\!]/\hbar^{n+1}\mathcal{O}(E)[\![\hbar]\!]$ an action functional of order at most n in $\hbar$. An *obstruction* is an element in $\mathcal{O}(E)$ of degree 1 that is the failure of an interaction up to order n to be a solution to the QME up to order $n+1$. A *deformation* is an element $I^{(n+1)} \in \mathcal{O}(E)$ of degree 0, such that $I_n + \hbar^{n+1} I^{(n+1)}$ solves the QME up to order $n+1$.

Lemma 2.1.1. *If the cohomology class in $H^1(\mathcal{O}(E), Q + \left\{I^{(0)}, \cdot\right\})$ vanishes for all deformations, there exists a quantum master action that solves the QME at all orders.*

Remark. This is, in particular, the case if the first cohomology group is trivial. Additionally, one can measure the uniqueness of a deformation that lifts an interaction at order n to one at order $n+1$ by studying the cohomology group of deformations , $H^0(\mathcal{O}(E), Q + \{I^{(0)}, \cdot\})$, and the one of the symmetries, which is given by $H^{-1}(\mathcal{O}(E), Q + \{I^{(0)}, \cdot\})$.

Suppose now that the QME is solved and a Lagrangian submanifold $\mathcal{L} \subset E$, such that $\omega(e, Qe)$ is non-degenerate, is found. Then the integral

$$\int_{\mathcal{L}} e^{\frac{1}{2\hbar}\omega(e,Qe)+\frac{1}{\hbar}I'(e)}$$

is perturbatively well defined. Thus, the problem is solved, up to giving a continuous deformation $\mathcal{L}_0 \to \mathcal{L}$. However, there still remains the issue of the existence of such a submanifold.

Lemma 2.1.2. *A Lagrangian submanifold $\mathcal{L} \subset E$ such that $\omega(e, Qe)$ is non-degenerate exists if and only if $H^\bullet(E, Q) = 0$.*

Proof. "$\Rightarrow$" For $e \in \mathcal{L}$, e is not in $\ker Q$. For dimensional reasons, it is thus sufficient to show that the map $Q\colon \mathcal{L} \to E$ is injective. Indeed, for $f, e \in \mathcal{L}$, $Qe = Qf$ implies $\omega(e-f, Q(e-f)) = 0$, which, by the degeneracy condition, holds if and only if $e = f$.
"$\Leftarrow$" This direction follows by construction. First, note that $\ker Q = \operatorname{im} Q$, so due to the rank-nullity theorem one has $2\dim(\operatorname{im} Q) = \dim E = 2n$. Moreover, $\omega(Qe, Qf) = (-1)^{|e|-1}\omega(e, Q^2 f) = 0$, thus $\operatorname{im} Q$ is Lagrangian. One now picks a basis $(e_1, \ldots, e_n) \subset \operatorname{im} Q$ and vectors f_i that are in the preimage of the respective e_i. Denote the space spanned by the f_i as L and notice that $Q|_L\colon L \to \operatorname{im} Q$ is an isomorphism. One can claim that $\omega(f, Qf)$ is non-degenerate on L, which equates to the statement that $\omega(f_i, Qf_j)$ is an invertible matrix. Indeed, suppose one has a nonzero vector a in the kernel of M, the $a_i M_{ij} = \omega(a_i f_i, Qf_j) = 0$. Since $\operatorname{im} Q$ is Lagrangian and, in particular, isotropic, $a_i f_i \in \operatorname{im} Q$. Therefore, $Q(a_i f_i) = 0$, which contradicts $Q|_L$ being isomorphic. Making the subspace isotropic is then a simple exercise in symplectic geometry. □

In general, $H^\bullet(E, Q)$ does not vanish. However, one can still simplify the problem by using the BV pushforward. Observe that $\ker Q$ is a coisotropic submanifold. Indeed, by rank-nullity, one obtains that $\dim(\ker Q) + \dim(\operatorname{im} Q) = \dim E$, and since E is symplectic, $\dim(\ker Q) + \dim(\ker Q)^\perp = \dim E$. As seen in the proof of Lemma 2.1.2 and $\operatorname{im} Q \subset (\ker Q)^\perp$, they coincide by dimensional reasons. Using the same method as in the proof of Lemma 2.1.2, one can construct a symplectic subspace

$$\mathcal{L} \oplus \operatorname{im} Q,$$

such that $\mathcal{L}$ is Lagrangian in this subspace. Hence, one obtains a decomposition of E into the three subspaces

$$\mathcal{L} \oplus \operatorname{im} Q \;\oplus\; H^\bullet(E, Q).$$

This allows the application of Theorem 2.1.2 in its generalized form, according to the remark following the theorem.

Lemma 2.1.3. *Let $I \in \mathcal{O}(E)[\![\hbar]\!]$ be a solution of the QME. Then, there exists an isotropic subspace $\mathcal{L}$ such that the BV pushforward*

$$f := \int_{e\in\mathcal{L}} e^{\frac{1}{2\hbar}\omega(e,Qe)+\frac{1}{\hbar}I(e)}$$

is a well defined function on $H^\bullet(E, Q)$ that is in the kernel of the BV Laplacian on $H^\bullet(E, Q)$. In particular, it defines an effective interaction in cohomology

$$I^{\text{eff}} = \hbar \log f.$$

2.2 BV in Infinite Dimensions

The BV theory in the finite-dimensional case was measure-theoretically well-defined. Now, this construction is mirrored for an infinite-dimensional domain of integration. The rationale is that the BV construction formally meets the conditions of Theorem 2.0.1. Thus, one obtains a meaningful theory of integration as the perturbative expansion in the limit $\hbar \to 0$. The matter of quantization will be treated in the following chapter. Here, it is outlined how the classical data arise for infinite-dimensional spaces.

Classical Data

Consider the problem initially posed in this chapter, which is to make sense of an integral over a field space $\mathcal{F}$, consisting of the sections of a principal G-bundle $F \to M$ over a compact base manifold M for a Lie group G,

$$Z = \int_{\mathcal{F}} \mathcal{D}\phi^a \; e^{iS_{\mathrm{cl}}[\phi^a]/\hbar}. \tag{2.8}$$

Further, $\mathcal{F}$ is acted upon by the gauge group $\mathcal{G} = \mathrm{Map}(M, G)$ and $S_{\mathrm{cl}} \in \mathcal{O}(\mathcal{F})$ is an action functional that is invariant under the action of $\mathcal{G}$. In the case of non-compact manifolds, one may need to impose further regularity requirements on the space of sections.

One begins by constructing the infinite-dimensional BV manifold $(\mathcal{E}, S, \omega)$. To avoid redundancy, one should integrate over the quotient $\mathcal{F}/\mathcal{G}$, or rather $\mathcal{L}[1] \oplus \mathcal{F}$, by the BRST construction. Here, $\mathcal{L}$ are sections of a bundle over M with fibers $\mathfrak{g} := \mathrm{Lie}(G)$. Note that $\mathcal{L}$ has the structure of a local Lie algebra induced by the Lie bracket on fibers. Additionally, the group action of G on the fibers of F induces a local Lie algebra action. This is exactly the situation described in Section 1.3, where the Chevalley–Eilenberg complex $C^*(\mathcal{L}[1] \oplus \mathcal{F})$ was obtained, with the differential given by the degree 1 vector field X. For the BV setup, one now constructs a symplectic structure. However, there is a subtle distinction from the finite-dimensional case.

Definition 2.2.1. A degree -1 symplectic structure on the space of sections $\mathcal{E}$ of the graded vector bundle $E \to M$ is given by a symmetric, non-degenerate, fiber-wise map of degree -1 to the bundle of densities over M,

$$\phi_x : E_x \times E_x \to (\mathrm{Dens_M})_x.$$

This induces the symplectic form ω of degree -1 as the integration pairing

$$\omega : e_1 \otimes e_2 \mapsto \int_M \phi(e_1, e_2), \quad e_1, e_2 \in \mathcal{E}.$$

Therefore, the analog of the (-1)-shifted tangential bundle is not the naive construct, where one adds the shifted dual sections, but rather the following: Define for a vector bundle $U \to M$ the vector bundle $U^! := U^\vee \otimes \mathrm{Dens}_M$ and set

$$E := L[1] \oplus F \oplus F^![-1] \oplus L^![-2].$$

The symplectic space is thus given by the sections of E,

$$\mathcal{E} = \mathcal{L}[1] \oplus \mathcal{F} \oplus \mathcal{F}^![-1] \oplus \mathcal{L}^![-2].$$

From left to right, it is customary to refer to the direct summands as the ghosts, which are the generators of symmetry, the physical fields or just fields, the anti-fields, and the anti-ghosts.

Remark. In the literature on the BV formalism, the grading on $\mathcal{E}$ is usually such that ghosts, fields, antifields, and anti-ghosts are in degree $1, 0, -1, -2$, respectively. Notice that this construction, due to [Cos11], results in a flipped grading on the field space. The customary grading can be recovered in the context of functionals on $\mathcal{E}$. If viewed as elements in the dual of $\mathcal{E}$, the relation between the graded vector space and its dual, as described in Definition 1.1.5, yields the flipped grading. One can then consider functionals that are integrals over formal polynomials in $\mathcal{E}$. By assigning the degree to each field, which corresponds to the degree of the functional, one obtains the usual grading.

Thus far, $\mathcal{E}$ and ω have been provided, so it remains to give an action functional that satisfies the CME to obtain the infinite-dimensional BV manifold. First, however, a refinement of the definition of the BV bracket is needed, since an arbitrary $f \in \mathcal{O}(\mathcal{E})$ may fail to be a Hamiltonian.

Theorem 2.2.1 (Lemma 3.2.3 [Cos11]). *Every local action functional $f \in \mathcal{O}_{\mathrm{loc}}(\mathcal{E})$ is a Hamiltonian. There is a bijection of Hamiltonian functions up to constants and symplectic vector fields.*

Hence, one can define the BV bracket as for the finite-dimensional case, but restricted to local action functionals,

$$\begin{aligned} \{\cdot,\cdot\}\colon \mathcal{O}_{\mathrm{loc}}(\mathcal{E}) \times \mathcal{O}_{\mathrm{loc}}(\mathcal{E}) &\longrightarrow \mathcal{O}_{\mathrm{loc}}(\mathcal{E}) \\ (f,g) &\longmapsto \{f,g\} = X_f g, \end{aligned}$$

Since it was assumed that S_{cl} is local, this poses no problem. The remaining part of the procedure follows directly from the finite-dimensional case, i.e., one lifts the Chevalley–Eilenberg vector field to a symplectic vector field X' on $\mathcal{E}$ and obtains the master action as the sum

$$S = S_{\mathrm{cl}} + h_{X'}.$$

2.3 Examples of Classical BV theories

Classical Electrodynamics

Let (M, g) be an n-dimensional pseudo-Riemannian manifold. The classical, first order action of electrodynamics, which one identifies with $U(1)$ gauge theory, is given by

$$S_{\mathrm{ED,cl}} = \int_M B \wedge F_A + \frac{1}{2} B \wedge *B.$$

Here, F_A denotes the curvature form, which in the Abelian case simply becomes $\mathrm{d}A$. The field space consists of the physical fields of kind $A \in \Omega^1(M, \mathfrak{u}(1))$, which transforms like connection, and $B \in \Omega^{n-2}(M, \mathfrak{u}(1))$, transforming by conjugation. Thus, the space of physical fields is

$$\mathcal{F} = \Omega^1(M, \mathfrak{u}(1)) \oplus \Omega^{n-2}(M, \mathfrak{u}(1)).$$

There is a local $U(1)$ action on the physical fields, giving rise to ghost fields $c \in \Omega^0(M, \mathfrak{u}(1))$. Identifying $(\Omega^p(M))^! = \Omega^{n-p}(M)$, one finds the extended space of BV fields

$$\begin{aligned} \mathcal{E}_{\mathrm{ED}} = \Omega^0(M, \mathfrak{u}(1))[1] &\oplus \Omega^1(M, \mathfrak{u}(1))[0] \oplus \Omega^{n-2}(M, \mathfrak{u}(1))[0] \\ &\oplus \Omega^{n-1}(M, \mathfrak{u}(1))[-1] \oplus \Omega^2(M, \mathfrak{u}(1))[-1] \oplus \Omega^n(M, \mathfrak{u}(1))[-2]. \end{aligned}$$

Antifields are denoted with a $+$-superscript. Thus, in $\mathcal{E}_{\mathrm{ED}}$ from left to right, one has ghosts c of cohomological degree -1, the physical fields A, B of cohomological degree 0, the antifields A^+, B^+ in degree 1 and the antighost c^+ in degree 2.

Remark. This cohomological grading will also be referred to as the ghost number.

On this space of fields, the canonical symplectic form, agreeig with Definition 2.2.1, is given by

$$\omega_{\mathrm{ED}} = \int_M \delta A \wedge \delta A^+ + \delta B \wedge \delta B^+ + \delta c \wedge \delta c^+$$

To find the master action, one has to find a symplectic lift of the Chevalley–Eilenberg vector field, which in the case of $U(1)$ is just $X_{\mathrm{CE}} = \int_M \mathrm{d}c \frac{\delta}{\delta A}$. This is easily found by making an ansatz

$$X = \int_M \mathrm{d}c \frac{\delta}{\delta A} + e \frac{\delta}{\delta A^+} + f \frac{\delta}{\delta B^+} + g \frac{\delta}{\delta c^+}$$

with functionals $e, f, g \in \mathcal{O}_{\mathrm{loc}}(\mathcal{E}_{\mathrm{ED}})$ linear on $\Omega^0(M, \mathfrak{u}(1))[1] \oplus \Omega^1(M, \mathfrak{u}(1)) \oplus \Omega^{n-2}(M, \mathfrak{u}(1))$. Then one imposes

$$\iota_{X'}\omega = \int \mathrm{d}c \wedge \delta A^+ + (-1)^{n-1} e \wedge \delta A + (-1)^n f \wedge \delta B + (-1)^n g \wedge \delta c \overset{!}{=} \delta h_{X'}$$

and finds that $e = f = 0, g = dA^+$, with $h_{X'} = \int_M \mathrm{d}c \wedge A^+$. This results in the master action

$$S_{\mathrm{ED}} = \int_M B \wedge F_A + \frac{1}{2} B \wedge *B + A^+ \mathrm{d}c.$$

Yang–Mills Theory

Consider again an n-dimensional Riemannian manifold (M, g) and suppose a finite-dimensional gauge group G with semisimple Lie algebra $\mathfrak{g}$ and fix a symmetric, non-degenerate, conjugation invariant form on $\mathfrak{g}$. This form may be extended to a pairing $\Omega^p(M, \mathfrak{g}) \times \Omega^q(M, \mathfrak{g}) \to \Omega^{p+q}(M)$ in the obvious way. Since $\mathfrak{g}$ is semisimple, the Killing form meets all requirements. For the purpose of notation, it will be assumed that $\mathfrak{g}$ is a matrix Lie algebra, and that the Killing form is given by the trace. The classical, first order action of Yang–Mills theory is given by

$$S_{\mathrm{YM,cl}} = \int_M \mathrm{Tr}\left(B \wedge F_A + \frac{1}{2} B \wedge *B \right)$$

with the curvature form $F_A = \mathrm{d}A + \frac{1}{2}[A, A]$. The physical field space $\mathcal{F}_{\mathrm{YM}}$ now consists of Lie algebra valued forms, the connection $A \in \Omega^1(M, \mathfrak{g})$ and $B \in \Omega^{n-2}(M, \mathfrak{g})$, transforming by conjugation. The action of the gauge group is implemented by the ghost $c \in \Omega^0(M, \mathfrak{g})$. Again, using the identification $(\Omega^p(M, \mathfrak{g}))^! = (\Omega^{n-p}(M, \mathfrak{g})$, which pairs with $\Omega^p(M, \mathfrak{g})$ to a density by the extension of the Killing form, yields the space of BV fields

$$\mathcal{E}_{\mathrm{YM}} = \Omega^0(M, \mathfrak{g})[1] \oplus \Omega^1(M, \mathfrak{g})[0] \oplus \Omega^{n-2}(M, \mathfrak{g})[0] \oplus \Omega^{n-1}(M, \mathfrak{g})[-1] \oplus \Omega^2(M, \mathfrak{g})[-1] \oplus \Omega^n(M, \mathfrak{g})[-2].$$

The identification of ghosts, physical fields, antifields and antighosts with ghost number $-1, 0, 1, 2$, respectively, is analogous to that in electrodynamics. Further, one obtains the canonical BV form

$$\omega_{\mathrm{YM}} = \int_M \mathrm{Tr}\left(\delta A \wedge \delta A^+ + \delta B \wedge \delta B^+ + \delta c \wedge \delta c^+ \right) \tag{2.9}$$

To find the Chevalley–Eilenberg vector field, observe that the connection form A transforms as $A \mapsto A + [c, A] + \mathrm{d}c = A + \mathrm{d}_A c$, where the covariant derivative $\mathrm{d}_A = \mathrm{d} + [\cdot, A]$ was introduced. The field B changes as $B \mapsto B + [c, B]$. Thus,

$$X_{\mathrm{CE}} = \int_M \mathrm{Tr}\left((\mathrm{d}_A c \frac{\delta}{\delta A} + [c, B] \wedge \frac{\delta}{\delta B} + \frac{1}{2}[c, c] \wedge \frac{\delta}{\delta c} \right).$$

Proceeding as before, X_{CE} is lifted to the vector field on $\mathcal{E}_{\mathrm{YM}}$

$$X = \int_M \mathrm{Tr}\Big((\mathrm{d}_A c \wedge \frac{\delta}{\delta A} + [c,B] \wedge \frac{\delta}{\delta B} + \frac{1}{2}[c,c] \wedge \frac{\delta}{\delta c} + (-1)^n \big[c, A^+\big] \wedge \frac{\delta}{\delta A^+} + (-1)^n \big[c, B^+\big] \frac{\delta}{\delta B^+} + (-1)^n (\mathrm{d}_A A^+ + \big[B, B^+\big] + [c, c^+]) \frac{\delta}{\delta c+})\Big).$$

Adding the associated Hamiltonian to the classical action then results in the Yang–Mills classical master action

$$S_{\mathrm{YM}} = \int_M \mathrm{Tr}\left(B \wedge F_A + \frac{1}{2} B \wedge *B + \mathrm{d}_A c \wedge A^+ + [c,B] \wedge B^+ + \frac{1}{2}[c,c] \wedge c^+\right).$$

Free Massless Fermion Field

Now the BV data associated to massless fermions on an n-dimensional Riemannian manifold (m,g) is discussed. Suppose one has an n-dimensional Riemannian manifold (M,g) and a compactly supported section of the spinor bundle $\psi \in \Gamma_c(M, \Sigma M)$. There is a spin connection ∇^{Σ} associated to the Levi-Civita connection induced by the metric g.[1] Then, the free Dirac equation is given by

$$\gamma^j \cdot \nabla^{\Sigma}_j \psi = 0.$$

One can think of the γ-representation as a section of the tangent bundle $\gamma \in \Gamma(M, TM)$ which acts on the spinor section by local Clifford multiplication. In local coordinates of the tangent space $(v_1, \dots, v_n)$, the gamma representation can thus be written as $\gamma = \gamma^j v_j$. Denote the Dirac operator as $D_{\mathrm{Dirac}} = \gamma^j \cdot \nabla^{\Sigma}_j$. One finds the Dirac action yielding the Dirac equation

$$S_{\mathrm{Dirac}}[\psi] = \int_M \bar{\psi} \wedge *D_{\mathrm{Dirac}}\psi,$$

where the Dirac pairing between the Spinor field and its adjoint, $\bar{\psi}\psi = \bar{\psi}_A \psi^A$, was used. Indeed, this action produces the correct equations of motion:

$$\delta S_{\mathrm{Dirac}} = \int_M \delta\bar{\psi} \wedge *D_{\mathrm{Dirac}}\psi + *(D_{\mathrm{Dirac}}\bar{\psi}) \wedge \delta\psi,$$

where it was exploited that the Dirac operator is formally self adjoint according to Lemma A.1.4. The first term yields exactly the Dirac equation as stated above, while the second term is the Dirac adjoint of the Dirac equation.

Remark. Due to the berezinian construction of the fermionic path integral, one should consider the spinor field and its adjoint as separate fields. Thus, the action which will be worked with from now onward is

$$S_{\mathrm{Dirac}}[\psi, \bar{\psi}] = \int_M \bar{\psi} \wedge *D_{\mathrm{Dirac}}\psi,$$

This is now promoted to a BV structure. To this end, one re-interprets the sections of the spinor bundle and its Dirac adjoint as spinor bundle-valued differential 0-forms of cohomological degree 0, i.e. $\psi \in \Omega^0(M, \Sigma)[0]$, $\bar{\psi} \in \Omega^0(M, \bar{\Sigma})[0]$. These are the physical fields of the theory.

Remark. Bearing in mind that spinor coordinates should be Grassmann numbers, it might be more fitting to write explicitly $\psi \in \Omega^0(M, \Pi\Sigma), \Omega^0(M, \Pi\bar{\Sigma})$. However, this explicit shift is omitted, and $\Sigma, \bar{\Sigma}$ is understood as already odd.

[1] A detailed discussion of properties of spinors, the construction of the spinor bundle and the spin connection is in Appendix A.1.

Note that the free fermion field does not have any local symmetries, so the theory does not contain any ghosts. However, one still needs anti-fields, which are top-degree forms of cohomological degree -1, hence $\psi^+ \in \Omega^n(M,\Sigma)[1]$, $\bar{\psi}^+ \in \Omega^n(M,\bar{\Sigma})[1]$. One finds the canonical BV 2-form

$$\omega_{\text{Dirac}} = \int_M \delta\psi\,\delta\psi^+ + \delta\bar{\psi}\,\delta\bar{\psi}^+ \tag{2.10}$$

on the space of fields

$$\mathcal{E}_{\text{Dirac}} = \Omega^0(M,\Sigma)[0] \oplus \Omega^0(M,\bar{\Sigma})[0] \oplus \Omega^n(M,\Sigma)[1] \oplus \Omega^n(M,\bar{\Sigma})[1].$$

Since the Chevalley–Eilenberg differential is trivial because there is no local Lie-algebra action, the BV action is again the action S_{Dirac}.

Chapter 3

Renormalization

The content of this Chapter is split into two parts. Section 3.1 has a motivational character. It reviews the ideas of the effective action approach to renormalization. Then, the adaptation due to [Cos11] is presented with a recollection of the main results. In section 3.2, the framework of homotopic renormalization is presented. This provides an effective theoretical approach to the renormalization of BV theories and lays the foundation for Chapter 4.

3.1 Wilsonian Renormalization Picture

The Idea of Wilsonian Renormalization

Opposed to most other early approaches to renormalization, which are focused on explicitly investigating the high energy limit, isolating the divergences and regularizing them, the Wilsonian method is built around the dependence of a theory on an energy scale. Roughly speaking, a theory is renormalizable if it looks renormalizable at any energy scale. Given a physical theory consisting of an action S satisfying the postulates 2.0.1 and 2.0.2 on a space of fields $\mathcal{F}$, one can define an effective action at an energy scale Λ via

$$e^{S^{\text{eff}}_{\Lambda}} = \int_{\mathcal{F}_{>\Lambda}} e^{S/\hbar},$$

where $\mathcal{F}_{>\Lambda}$ denotes the subspace of fields with energy above a cutoff energy Λ. One speaks of the effective theory at scale Λ. This effective action can be specified in terms of a set of effective couplings $\{g_\alpha(\Lambda)\}_{\alpha\in A}$, where A is an index set containing labels of all types of interactions $\{I_\alpha\}_{\alpha\in A}$. The effective action is then recovered as

$$S^{\text{eff}}_{\Lambda} = g_\alpha(\Lambda) I_\alpha.$$

One now implements a change of energy scale on the theory. To this end, one switches to the dimensionless couplings G_α. If g_α has mass dimension d, i.e. units $[\text{mass}]^{d_\alpha}$, one sets $G_\alpha(\Lambda) := \Lambda^{-d_\alpha} g_\alpha(\Lambda)$. There exists a function F assigning to every set of couplings $G_\alpha(\Lambda)$ at scale Λ a set of couplings $G_\alpha(\Lambda')$ at scale Λ' that depends only on the ratio of the two cutoff energies and the set of initial couplings. The relation is given by "integrating out" the fields in the energy interval between the two cutoffs. Then one obtains the equation

$$G_\alpha(\Lambda') = F_\alpha(G(\Lambda), \Lambda'/\Lambda).$$

One applies $\frac{\mathrm{d}}{\mathrm{d}\Lambda'}\big|_{\Lambda'=\Lambda}$ and defines functions $\beta_\alpha(G) = \frac{\partial}{\partial z} F_i(G,z)\big|_{z=1}$ to obtain what is known as Wilson's renormalization group equation (RGE),

$$\frac{\mathrm{d}}{\mathrm{d}\log\Lambda} G_\alpha(\Lambda) = \Lambda \frac{\mathrm{d}}{\mathrm{d}\Lambda} G_\alpha(\Lambda) = \beta_\alpha(G(\Lambda)).$$

It contains the information of how the couplings behave under change of the energy scale. For the analysis, the couplings are divided into the relevant couplings G_a, which have non-negative scaling dimension, and the irrelevant couplings G_n. Note that there is only a finite number N of relevant couplings, since there are only finitely many interactions with mass dimension less than the dimension of the base manifold, while there are infinitely many irrelevant couplings.

Theorem 3.1.1. *Let $(G_a^0, G_n^0) := (G_a(\Lambda_0), G_n(\Lambda_0))$ be a set of couplings at a cutoff Λ_0 that lie on an N-dimensional initial surface Σ_0 of the manifold of all sets of couplings that may be parametrized by the relevant couplings. Then, for $\Lambda \ll \Lambda_0$, the couplings approach a fixed surface Σ that is independent of Λ_0 and Σ_0. Moreover, Σ is stable under flows induced by the RGE.* [1]

Proof. To prove the first part about the existence of the surface Σ, one shows that the irrelevant couplings in this regime depend only on the relevant ones. For that, the behavior of the couplings under small perturbations is investigated. According to the RGE, one finds

$$\Lambda \frac{\mathrm{d}}{\mathrm{d}\Lambda} \delta G_i = \frac{\partial \beta_i}{\partial G_j} \delta G_j \,.$$

This system of differential equations couples both relevant and irrelevant couplings. It is possible to decouple the irrelevant ones by defining

$$\xi_n := \delta G_n - \frac{\partial G_n}{\partial G_a^0} \left(\frac{\partial G}{\partial G^0} \right)^{-1}_{ab} \delta G_b \,,$$

Indeed, note that due to the chain rule

$$\Lambda \frac{\mathrm{d}}{\mathrm{d}\Lambda} \frac{\partial G_i}{\partial G_a^0} = \frac{\partial \beta_i}{\partial G_j} \frac{\partial G_j}{\partial G_a^0},$$

such that one obtains the differential equations

$$\Lambda \frac{\mathrm{d}}{\mathrm{d}\Lambda} \xi_n = \left(\frac{\partial \beta_n}{\partial G_m} - \frac{\partial G_n}{\partial G_a^0} \left(\frac{\partial G}{\partial G^0} \right)^{-1}_{ab} \frac{\partial \beta_b}{\partial G_m} \right) \xi_m . \tag{3.1}$$

Consider the theory at hand now as a perturbation of the free theory, which does not need a cutoff. In this model, the small couplings g_i can be assumed to be independent of the energy scale Λ, so the dimensionless couplings G_i scale as Λ^{-d_i}. One thus finds the approximations

$$\frac{\partial \beta_i}{\partial G_j} \approx -d_i \delta_{ij}, \quad \left(\frac{\partial \beta_n}{\partial G_m} - \frac{\partial G_n}{\partial G_a^0} \left(\frac{\partial G}{\partial G^0} \right)^{-1}_{ab} \frac{\partial \beta_b}{\partial G_m} \right) \approx -d_n \delta_{nm},$$

and can conclude that the irrelevant parameters ξ_n decay for $\Lambda \ll \Lambda_0$ as $(\Lambda/\Lambda_0)^p$ for some $p > 0$. Then, by the definition of the ξ_n, one obtains in the low energy limit the relation

$$\delta G_n = \frac{\partial G_n}{\partial G_a^0} \left(\frac{\partial G}{\partial G^0} \right)^{-1}_{ab} \delta G_b \,.$$

Therefore, the perturbations of the irrelevant couplings depend only on the perturbations of the relevant ones, but not on the choice of cutoff or the initial surface. Thus, for $\Lambda \to 0$, the G_n are functions of G_a only,

[1] This theorem is based on a rigorous statement in [Pol84], but follows the formulation in [Wei05]. So, the label "theorem" might be misleading, as the statement and proof are more conceptual than formal.

such that one again finds a surface Σ in the manifold of couplings that is dependent solely on the relevant couplings.
The stability of the surface can now be deduced by analyzing the trajectories of a point G_i near the surface. The evolution (3.1) implies that in the approximative approach, the irrelevant couplings change as a funcion of Λ in positive powers of Λ/Λ_0. So in the low energy limit, the flow takes values close to Σ. Since the trajectories in the limit $\Lambda \to 0$ tend to Σ, as one chooses initial points G_i closer to the surface, the trajectory also remains closer to the surface. In the limit, one finds that the flow leaves Σ invariant.
Lastly, the use of the perturbative limit is justified In fact, one obtains the same results as long as the evolution matrix on the right hand side of (3.1) is positive definite. This can be guaranteed by suitable restrictions on the rowth of the irrelevant couplings in Λ. □

With this theorem, one can now formulate the Wilsonian paradigm more explicitly. By integrating out high energy fields, one works with effective field theories. In these effective theories, there will, in general, be all types of interactions, i.e. all types of couplings. However, under the assumption that the irrelevant couplings do not diverge strongly in the high energy limit, the low energy limit of the effective field theory is only weakly dependent on the energy scale and is described solely by the set of relevant couplings.

Costello's Approach to Renormalization

In this subsection, the treatment of renormalization due to [Cos11] is discussed. It is based on the idea of the renormalization group flow. However, there are some modifications to the original Wilsonian approach. First, instead of working with the energy scale Λ, a formulation with the length scale L is used. This works better with the notion of locality, which is at the origin of UV divergences. Moreover, one works directly with interactions rather than couplings. For scalar fields on a compact Riemannian manifold, this yields the following:

Definition 3.1.1 (Propagator)**.** Let D be a generalized Laplacian[2] and $K_l(x, y)$ its heat kernel, i.e. a solution of the equation $(\partial_l + D_x)K_l(x, y) = 0$. The *full propagator* of the free scalar theory is defined as

$$P = \int_0^\infty e^{-lm^2} K_l(x, y)\, dl.$$

The *regularized propagator* for length scales $\epsilon, L > 0$ reads

$$P(\epsilon, L) = \int_\epsilon^L e^{-lm^2} K_l(x, y)\, dl.$$

Definition 3.1.2 (Renormalization group flow)**.** The *renormalization group flow* from scale ϵ to scale L on action functionals is defined via the contraction with the regularized propagator as

$$\begin{aligned} \mathcal{O}(C^\infty(M))[[\hbar]] &\longrightarrow \mathcal{O}(C^\infty(M))[[\hbar]] \\ I &\longmapsto W(P(\epsilon, L), I) := \hbar \log\left(e^{\partial_{\hbar P(\epsilon,L)}} e^{I/\hbar}\right), \end{aligned}$$

where ∂_K denotes the contraction with a kernel K.

Definition 3.1.3 (Perturbative Scalar Field Theory)**.** Let (M, g) be a compact n-dimensional Riemannian manifold and let $\mathcal{O}^+(C^\infty(M))[[\hbar]]$ be the functionals that are at least cubic modulo $\hbar$. A *perturbative quantum field theory* is a kinetic term $-\frac{1}{2}\int_M \phi(D + m)\phi$ together with a set of effective actions $\{I[L]\} \subset \mathcal{O}^+(C^\infty(M))[[\hbar]]$ for every $L \in (0, \infty]$ such that

[2] Throughout this chapter, the definition of the generalized Laplacian according to [BGV92] is used.

(i) (RGE). The RGE of the form $I[L] = W(P(\epsilon, L), I[\epsilon])$ holds for all $\epsilon \in (0, \infty]$.

(ii) (Asymptotic locality). For small L, there is an expansion $I[L] = \sum_k \hbar^i I_{i,k}[L]$, such that each $I_{i,k}[L]$ is a local action functional of order k in $\phi \in C^\infty(M)$.

For all generalized Laplacians the heat kernel has the same small l asymptotics of $l^{-n/2}$, which leads to divergences in Feynman graphs and, in particular, renders the limit

$$\text{“}\lim_{\epsilon \to 0} W(P(\epsilon, L), I[L])\text{”},$$

i.e. the full action of the theory, ill defined. This can be regularized by computing counterterms. Their construction involves the choice of a renormalization scheme $\mathbf{RS}$, which can be viewed as a choice of a certain set of purely singular functions in the limit $\epsilon \to 0$.

Theorem 3.1.2 (Existence of Local Counterterms)**.** *Given $I \in \mathcal{O}_{\text{loc}}(C^\infty(M))[\![\hbar]\!]$, there exists a series of local counterterms $I^{CT}_{i,k} \in I \in \mathcal{O}_{\text{loc}}(C^\infty(M))[\![\hbar]\!] \otimes \mathbf{RS}$ for all $i > 0, k \geq 0$ such that $I^{CT}_{i,k}$ is of degree k and, for all $L \in (0, \infty]$, the limit*

$$\lim_{\epsilon \to 0} W(P(\epsilon, L), I[L] - \sum \hbar^i I^{CT}_{i,k})$$

exists.

Conceptually, the proof works by inductively removing all singularities from Feynman graphs order by order. It is worth noting that this process always requires the unnatural choice of a renormalization scheme. However, this does not affect the underlying physical theory, as all observables are already defined by the effective theories, which remain unchanged. One can then use the existence of local counterterms to prove the following statement:

Theorem 3.1.3. *Let $\mathcal{T}^{(n)}$ denote the set of perturbative scalar field theories defined modulo $\hbar^{n+1}$. Then $\mathcal{T}^{(n+1)} \to \mathcal{T}^{(n)}$ is, in a canonical way, a torsor for the Abelian group $\mathcal{O}_{\text{loc}}(C^\infty(M))$. Further, $\mathcal{T}^{(0)}$ is canonically isomorphic to the space $\mathcal{O}^+_{\text{loc}}(C^\infty(M))$ of local action functionals that are at least cubic. The choice of a renormalization scheme $\mathbf{RS}$ yields a section of the torsor $\mathcal{T}^{(n+1)} \to \mathcal{T}^{(n)}$ for every n and thus an isomorphism $\mathcal{T}^{(\infty)} \cong \mathcal{O}^+_{\text{loc}}(C^\infty(M))[\![\hbar]\!]$.*

This is shown by noting that, given a local action functional I the renormalized RG flow from scale 0 to scale L,

$$I[L] := W^R(P(0, L), I) = \lim_{\epsilon \to 0} W(P(\epsilon, L), I[\epsilon] - I^{CT}[\epsilon])$$

defines a theory. Conversely, one can then inductively define local action functionals $I_{i,k}$ from a theory $\{I[L]\}$ by subtracting the non-local part due to the renormalization group flow of already local functionals, i.e.

$$I_{i,k} = I_{i,k} - W^R_{i,k}\left(P(0, L), \sum_{(r,s) \prec (i,k)} \hbar^r I_{r,s}\right).$$

This establishes a bijection of $\mathcal{T}^{(\infty)}$ and $\mathcal{O}^+(C^\infty(M))[\![\hbar]\!]$, and between $\mathcal{T}^{(n)}$ and $\mathcal{O}^+_{\text{loc}}(C^\infty(M))[\![\hbar]\!]/\hbar^{n+1}$. Further, for two theories $\{I[L]\}, \{J[L]\}$ that are defined up to order $n + 1$ in $\hbar$ and agree up to order n and $\{I[L]\} \in \mathcal{T}^{(0)}$, one observes that

$$I_0[L] + \frac{1}{\hbar^{n+1}} \delta(\{I[L]\}, \{J[L]\}) \in \mathcal{O}(C^\infty(M))$$

satisfies the RGE modulo δ^2. Thus, it defines an element in $T_{I_0[L]}\mathcal{T}^{(0)}$, which is canonically isomorphic to $\mathcal{O}_{loc}(C^\infty(M))$, yielding the theorem.

Moving on to the case of the non-compact manifold $\mathbb{R}^n$, one restricts the space of fields to Schwartz functions $\mathscr{S}(\mathbb{R}^n)$ and again characterizes a theory by sets $\{I[L]\} \subset \mathcal{O}^+(\mathscr{S}(\mathbb{R}^n))[\![\hbar]\!]$ that fulfill the RGE and asymptotic locality. However, the space of functionals is restricted by requiring the action functionals to be $\mathbb{R}^n$-translation invariant and their kernel to be of rapid decay away from the small diagonal. Under these conditions, there is an equivalent statement of Theorem 3.1.3. To investigate renormalizability, one defines the rescaling action, which acts in the case of the scalar field theory by

$$\begin{aligned} R_l\colon \mathscr{S}(\mathbb{R}^n) &\longrightarrow \mathscr{S}(\mathbb{R}^n) \\ \phi(x) &\longmapsto R_l(\phi)(x) = l^{n/2-1}\phi(lx). \end{aligned}$$

The choice of scaling factor is such that the kinetic term is invariant. The propagator is explicitly given as

$$P(\epsilon, L) = \int_\epsilon^L l^{-n/2} e^{-lm^2} e^{\|x-y\|^2}\, dl,$$

which can be shown to transform as $R_l P(\epsilon, L) = P(l^{-2}\epsilon, l^{-2}L)$. On the level of action functionals, the rescaling is defined as R_l^* such that

$$(R_l^* I)(R_l\phi) = I(\phi).$$

Definition 3.1.4 (Local renormalization group flow)**.** The *local renormalization group flow* is the action of $\mathbb{R}_{>0}$ on the space of theories by

$$\begin{aligned} \mathcal{RG}_l\colon \mathcal{T}^{(\infty)} &\longrightarrow \mathcal{T}^{(\infty)} \\ \{I[L]\} &\longmapsto \{R_l^* I[L]\}. \end{aligned}$$

Proposition 3.1.1. *Let $\{I[L]\}$ be a translation invariant theory on $\mathbb{R}^n$. Then, for all $L > 0$,*

$$\mathcal{RG}_l(I[L]) \in \mathcal{O}^+(\mathscr{S}(\mathbb{R}^n))[\![\hbar]\!] \otimes \mathbb{C}[l, l^{-1}, \log l].$$

This allows for classification of the action functionals according to their scaling behavior:

Definition 3.1.5. An action functional $I[L]$ is called

(i) *irrelevant*, if $\mathcal{RG}_l(I[L])$ varies as $l^k \log^r l$ for some $k < 0$ and $r \geq 0$,

(ii) *relevant*, if $\mathcal{RG}_l(I[L])$ varies as $l^k \log^r l$ for some $k, r \geq 0$,

(iii) *marginal*, if $\mathcal{RG}_l(I[L])$ varies as $\log^r l$ for some $r \geq 0$.

One extends these notions to theories by calling a theory $\{I[L]\}$ irrelevant, relevant, or marginal if $I[L]$ is irrelevant, relevant, or marginal for every $L > 0$. The set of relevant theories is denoted by $\mathcal{R}^{(\infty)} \subset \mathcal{T}^{(\infty)}$ and the set of marginal theories by $\mathcal{M}^{(\infty)} \subset \mathcal{T}^{(\infty)}$.

Remark. By this definition, a marginal functional is, in particular, relevant. Thus, one has the inclusion of sets of theories $\mathcal{M}^{(n)} \subset \mathcal{R}^{(n)}$ at every order of n, as well as $\mathcal{M}^{(\infty)} \subset \mathcal{R}^{(\infty)}$.

Definition 3.1.6 (Renormalizability)**.** A theory on $\mathbb{R}^n$ is *renormalizable* if it is relevant and, at every order of $\hbar$, only has finitely many relevant deformations, i.e. $T_{I[L]}\mathcal{R}^{(n)}$ is finite-dimensional at any order. It is strictly renormalizable if it is renormalizable and marginal. It is strongly renormalizable if it is strictly renormalizable with only marginal deformations, $T_{I[L]}\mathcal{R}^{(n)} = T_{I[L]}\mathcal{M}^{(n)}$ for $\{I[L]\} \in \mathcal{M}^{(n)}$.

Theorem 3.1.4. *Relevant theories have the canonical structure of a torsor $\mathcal{R}^{(n+1)} \to \mathcal{R}^{(n)}$ for the Abelian group $\mathcal{O}^+_{\text{loc},\geq}(\mathscr{S}(\mathbb{R}^n))$, the space of action functional cubic modulo $\hbar$ and of non-negative scaling. Further, $\mathcal{R}^{(0)}$ is canonically isomorphic to $\mathcal{O}^+_{\text{loc},\geq}(\mathscr{S}(\mathbb{R}^n))$.*
The analogous statement holds for marginal theories $\mathcal{M}^{(n)}$ and $\mathcal{O}^+_{\text{loc},0}(\mathscr{S}(\mathbb{R}^n))$, the space of action functional cubic modulo $\hbar$ and of at most logarithmic scaling.
The choice of renormalization scheme $\mathbf{RS}$ *induces a section of each of the torsors $\mathcal{R}^{(n+1)} \to \mathcal{R}^{(n)}$ and $\mathcal{M}^{(n+1)} \to \mathcal{M}^{(n)}$ and subsequently to bijections*

$$\mathcal{R}^{(\infty)} \cong \mathcal{O}^+_{\text{loc},\geq}(\mathscr{S}(\mathbb{R}^n)),$$
$$\mathcal{M}^{(\infty)} \cong \mathcal{O}^+_{\text{loc},0}(\mathscr{S}(\mathbb{R}^n)).$$

This theorem, as well as Theorem 3.1.4 may be generalized to vector bundle valued theories under certain conditions. One essential point is that the rescaling of the propagator is, as in the scalar case, $R_l P(\epsilon, L) = P(l^{-2}\epsilon, l^{-2}L)$.

3.2 Homotopic renormalization

In this section, it is discussed how infinite-dimensional BV theories can be treated with a method inspired by the paradigm of effective actions arising from Wilsonian renormalization, as proposed by [Cos11]. First, the approach is developed over compact manifolds and then extended to $\mathbb{R}^n$.

Given a free BV manifold $(\mathcal{E}, \omega, S = \omega(e, Qe))$ over a compact base manifold M together with an interaction I satisfying the QME, one is interested in the effective action on the cohomology $H^\bullet(\mathcal{E}, Q)$ satisfying the QME, which was defined in 2.1.3 as

$$I^{\text{eff}}(a) := \hbar \log \int_{\mathcal{L}} e^{\frac{1}{2\hbar}\omega(e,Qe) + \frac{1}{\hbar} I(e+a)}.$$

One way of specifying the isotropic submanifold $\mathcal{L}$ is by defining it as the image of a gauge fixing operator.

Definition 3.2.1 (Gauge Fixing operator)**.** A *gauge fixing operator* $Q^{GF} \colon \mathcal{E} \to \mathcal{E}$ on a BV manifold $(\mathcal{E}, \omega, S = \omega(e, Qe) + I(e))$ is an operator such that:

(i) Q^{GF} is of cohomological degree -1, square-zero and self-adjoint with respect to $\omega(\cdot, \cdot)$.

(ii) The commutator $D = \left[Q, Q^{GF}\right]$ is a generalized Laplacian.

The key problem to quantization is that in infinite dimensions the BV operator takes the form

$$\text{“}\Delta = \int_M \frac{\delta}{\delta \phi^i(x)} \frac{\delta}{\delta \phi^+_i(x)}\text{”},$$

where ϕ^i, ϕ^+_i are all fields and respective antifields of the theory. This operator is ill-defined, since it produces the same singularities as the one-loop diagrams, which extends to the QME being ill-defined. However, one can interpret it as the contraction with a kernel K_0 that is singular on the diagonal, which is assumed to correspond to the limit $L \to 0$. To circumvent this problem, a kernel at non-zero L order is considered, which gives rise to the scale L effective theory. For this construction, one introduces the convolution operator

$$\begin{aligned} \star \colon \mathcal{E} \otimes \mathcal{E} &\longrightarrow \operatorname{End}(\mathcal{E}) \\ K &\longmapsto K \star e = (-1)^{|K|}(\mathbb{1} \otimes \omega(\cdot, \cdot))(K \otimes e). \end{aligned}$$

Using this convolution operator, the heat kernel K_l of an operator D can be defined by the property

$$K_l \star e = e^{-lD}.$$

In particular, this operator exists if D is a generalized Laplacian. One chooses D to be the generalized Laplacian $D = \left[Q, Q^{GF}\right]$. This allows for the definition of the propagator of the theory as

$$P(\epsilon, L) = \int_\epsilon^L (Q^{GF} \otimes \mathbb{1}) K_l \, dl.$$

One can use the heat kernel to construct a well-defined scale-L BV Laplacian by contraction, $\Delta_L \coloneqq \partial_{K_L}$, where ∂ denotes the contraction operator.

Definition 3.2.2 (QME at Scale L)**.** Let $\{\cdot,\cdot\}_L$ denote the failure of the BV Laplacian at order L to be a derivation, i.e.

$$\{I, J\}_L \coloneqq \Delta_L(IJ) - (\Delta_L I)J - (-1)^{|I|} I(\Delta_L J).$$

Then, the *scale-L QME* of interactions at scale L is defined as

$$(Q + \Delta_L)e^{I[L]/\hbar} = 0 \qquad \Longleftrightarrow \qquad QI[L] + \frac{1}{2}\{I[L], I[L]\}_L + \hbar\Delta_L I[L] = 0.$$

Lemma 3.2.1. *The effective action $I[\epsilon]$ solves the QME at scale ϵ if and only if $I[L] \coloneqq W(P(\epsilon, L), I[\epsilon])$ solves the QME at scale L.*

Definition 3.2.3 (Pre-Theories and Theories)**.** A *pre-theory* over a free BV manifold $(\mathcal{E}, \omega, Q)$ is a collection of effective interactions $\{I[L]\}$ such that the following hold:

(i) Each $I[L] \in \mathcal{O}(\mathcal{E})[\![\hbar]\!]$ is of degree 0 and at least cubic modulo $\hbar$.

(ii) The RGE is satisfied: $I[L] = W(P(\epsilon, L), I[\epsilon]) = \hbar \log\left(e^{\hbar\partial_{(P(\epsilon,L))}} e^{I/\hbar}\right)$.

(iii) In the expansion $I[L] = \sum \hbar^i I_{i,k}[L]$, each $I_{i,k}[L]$ has small L asymptotic expansion in local action functionals.

If a pre-theory additionally satisfies the QME, in the sense that all $I[L]$ solve the scale-L QME, it is called a theory. The set of pre-theories and theories will be denoted by $\widetilde{\mathcal{T}}^{(\infty)}$ and $\mathcal{T}^{(\infty)}$, respectively. Correspondingly, for pre-theories and theories up to order n in $\hbar$, $\widetilde{\mathcal{T}}^{(n)}$ and $\mathcal{T}^{(n)}$ is used.

Upon the choice of a renormalization scheme, a correspondence between $\{I[L]\} \in \widetilde{\mathcal{T}}^{(\infty)}$ and $I \in \mathcal{O}^+_{\mathrm{loc}}(\mathcal{E})[\![\hbar]\!]$ is established. The condition that $\{I[L]\}$ is a theory, i.e. fulfills the QME, can then be viewed as a condition on the corresponding local action functional I. This condition is interpreted as the renormalized QME.

One can now study the equivalences between theories. In the finite-dimensional case, the most intuitive way of thinking of two equivalent theories is a change in coordinates by a symplectomorphism. It can be shown that this is equivalent to the following notion of homotopy between theories:

Definition 3.2.4. Let $F \in \mathcal{O}(\mathcal{E}) \otimes \Omega([0,1])$. F is a homotopy of theories if it satisfies the homotopy QME

$$(\mathrm{d}_{dR} + \Delta)e^{F/\hbar} = 0.$$

Writing $F(t, \mathrm{d}t) = A(t) + B(t)\mathrm{d}t$, this can be split in the two conditions

$$\frac{1}{2}\{A(t), A(t)\} + \hbar\Delta A(t) = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}A(t) + \{A(t), B(t)\} + \hbar\Delta B(t) = 0$$

In the infinite-dimensional setting, the interpretation of a change of coordinates is problematic, since it induces a change of measure which itself is not well-defined. However, the notion of equivalence as homotopy persists. In particular, the homotopy QME can be extended to the homotopy QME of scale L. Homotopies correspond to 1-simplices of theories. This can be generalized to homotopies between homotopies represented by 2-simplices, and so on, which yields an enrichment of theories by simplicial sets. By the effective field theory paradigm, of particular interest are the homotopies of effective actions. One obtains a homotopy between effective actions either by a homotopy of the original theory or by a homotopy of the domain of integration $\mathcal{L}$, i.e. by varying the gauge fixing. This extends to simplicial sets in the following way. Let $\mathcal{QME}(\mathcal{E}, Q)$ be the simplicial set of solutions to the homotopy QME and $\mathcal{GF}(\mathcal{E}, Q)$ the simplicial set of gauge fixings.

Lemma 3.2.2. *There is a canonical map of simplicial sets*

$$\begin{aligned} \mathcal{QME}(\mathcal{E}, Q) \times \mathcal{GF}(\mathcal{E}, Q) &\longrightarrow \mathcal{QME}(H(\mathcal{E}, Q), 0) \\ (I, Q^{GF}) &\longmapsto I^{\text{eff}} = \hbar \log \int_{\operatorname{im} Q^{GF}} e^{\frac{1}{2\hbar}\omega(e, Qe) + \frac{1}{\hbar} I(e+a)}. \end{aligned}$$

Let $\{I[L]\} \in \mathcal{T}^{(n)}(\mathcal{E}, Q)[k]$ be a k-simplex of theories up to order n. Choosing an arbitrary lift $\{\widetilde{I}[L]\} \in \widetilde{\mathcal{T}}^{(n+1)}(\mathcal{E}, Q)[k]$, one defines the obstruction to be the failure of $\widetilde{I}[L]$ to be a solution of the QME up to order $n+1$ at scale L

$$O_{n+1}[L] := \hbar^{-n-1} \left(Q\widetilde{I}[L] + \frac{1}{2}\left\{\widetilde{I}[L], \widetilde{I}[L]\right\} + \hbar \Delta_L \widetilde{I}[L] \right).$$

Lemma 3.2.3. *Let ϵ be a parameter of square-zero and cohomological degree -1, and let $I^{(0)}[L]$ be $I[L]$ modulo $\hbar$. Then $I^{(0)}[L] + \epsilon O_{n-1}[L]$ satisfies both the scale-L CME and the RGE. Therefore, $I^{(0)}[L] + \epsilon O_{n-1}[L]$ defines a classical BV theory.*
The set of lifts of $\{I[L]\}$ to $\mathcal{T}^{(n+1)}(\mathcal{E}, Q)[k]$ is the set of elements $J[L] \in \mathcal{O}_{\text{loc}}(\mathcal{E}, \Omega(\Delta^k))$ such that

$$QJ[L] + \left\{I^0, J[L]\right\} = O_{n+1}$$

and $I^{(0)}[L] + \delta J[L]$ satisfies the RGE and the asymptotic locality postulate up to order δ^2.

So, given a theory $\{I[L]\} \in \mathcal{T}^{(n)}(\mathcal{E}, Q)[k]$, the obstruction is an element $O_{n+1} \in \mathcal{O}_{\text{loc}}(\mathcal{E}, \Omega(\Delta^k))$ which is a closed degree 1 element in the cochain complex $\left(\mathcal{O}_{\text{loc}}(\mathcal{E}, \Omega(\Delta^k)), Q + \left\{I^0, \cdot\right\}\right)$. A lift of $\{I[L]\}$ to $\mathcal{T}^{(n+1)}(\mathcal{E}, Q)[k]$ is then given by elements $J \in \mathcal{O}_{\text{loc}}(\mathcal{E}, \Omega(\Delta^k))$ that make the obstruction exact.

Corollary 3.2.1. *There is a homotopy fiber diagram of simplicial sets*

$$\begin{array}{ccc} \mathcal{T}^{(n+1)}(\mathcal{E}, Q) & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \mathcal{T}^{(n)}(\mathcal{E}, Q) & \xrightarrow{O_{n+1}} & \mathcal{O}_{\text{loc}}(\mathcal{E})[1] \end{array}$$

Proof. Suppose $P\mathcal{O}_{\text{loc}}(\mathcal{E})[k]$ is the simplicial set of pairs $\alpha, \beta \in \mathcal{O}_{\text{loc}}(\mathcal{E}, \Omega(\Delta^k))$ such that

$$(Q + \mathrm{d}_{dR})\alpha + \left\{I^{(0)}, \alpha\right\} = \beta.$$

In particular, there is a fibration $p\colon P\mathcal{O}_{\text{loc}}(\mathcal{E})[k] \to \mathcal{O}_{\text{loc}}(\mathcal{E})[k],\ (\alpha, \beta) \mapsto \beta$. By Lemma 3.2.3 it becomes apparent that

$$\mathcal{T}^{(n+1)}(\mathcal{E}, Q) = \mathcal{T}^{(n)}(\mathcal{E}, Q) \times_{\mathcal{O}_{\text{loc}}(\mathcal{E})[1]} P\mathcal{O}_{\text{loc}}(\mathcal{E})[1].$$

The fibration p makes this a homotopy fiber product. Observing that $P\mathcal{O}_{\mathrm{loc}}(\mathcal{E})[1]$ is contractible, one finds that theories of order $n+1$ are the homotopy fiber product

$$\mathcal{T}^{(n+1)}(\mathcal{E},Q) = \mathcal{T}^{(n)}(\mathcal{E},Q) \times_{\mathcal{O}_{\mathrm{loc}}(\mathcal{E})[1]} \{0\}$$

and the statement follows. □

Therefore, a theory of order n is equivalent to a theory of order $n+1$ by homotopy if the obstruction is exact. Another important observation to be made is that the notion of a pre-theory and a theory always requires a choice of gauge fixing, since it is involved in the construction of the propagator that enters in the RGE. This implies that $\mathcal{T}^{(n+1)}(\mathcal{E},Q)$ is in fact a product of $\mathcal{GF}(\mathcal{E},Q)$ with the simplicial set consisting of elements in $\mathcal{O}_{\mathrm{loc}}(\mathcal{E},\Omega(\Delta^k))$.

Corollary 3.2.2. *There are fibrations of simplicial sets*

$$\begin{aligned}\mathcal{T}^{(n+1)}(\mathcal{E},Q) &\longrightarrow \mathcal{T}^{(n)}(\mathcal{E},Q)\\ \mathcal{T}^{(\infty)}(\mathcal{E},Q) &\longrightarrow \mathcal{GF}(\mathcal{E},Q).\end{aligned}$$

In particular, if $\mathcal{GF}(\mathcal{E},Q)$ is contractible, the theory is independent of the choice of gauge fixing conditions.

Turning to the case of $\mathbb{R}^n$, there is some additional data to keep track of due to rescaling.

Definition 3.2.5 (Free BV theory)**.** A *free BV theory* over $\mathbb{R}^n$ is the collection of the following data:

(i) A bigraded space of fields $\mathcal{E}$ consisting of the Schwartz sections of the trivial vector bundle with fibers E for some graded vector space E, i.e. $\mathcal{E} \cong E \times \mathscr{S}(\mathbb{R}^n)$. The first grading is the one induced by the grading on E. The second one defines a rescaling action R_l of $\mathbb{R}_{>0}$ on the space fields. Suppose $f(x) = \omega(x)e \in \mathcal{E}$ is of second degree j with $\omega \in \mathscr{S}(\mathbb{R}^n)$ and $e \in E$. Then let

$$R_l(f(x)) = \omega(lx)\, l^j e.$$

(ii) A degree -1 anti-symmetric and scaling-invariant pairing of fields $\langle\langle\cdot,\cdot\rangle\rangle : \mathcal{E}\times\mathcal{E} \to \mathbb{R}$, which is induced by a non-degenerate pairing $\langle\langle\cdot,\cdot\rangle\rangle_0 : E \times E \to \mathrm{Dens}_{\mathbb{R}^n}$. Thus, the pairing of fields takes the explicit form

$$\langle\langle f_1, f_2\rangle\rangle = \int_{\mathbb{R}^n} \omega_1(x)\omega_2(x)\langle\langle e_1,e_2\rangle\rangle_0\, \mathrm{d}x$$

for fields $f_1 = \omega_1(x)e_1,\ f_2 = \omega_2(x)e_2 \in \mathcal{E}$.

(iii) A differential operator $Q : \mathcal{E} \to \mathcal{E}$ that is translation invariant, of first degree one, preserves scaling dimension, squares to zero, and is graded adjoint with respect to the pairing of fields.

Remark. The first grading is the usual cohomological grading in the BV formalism. The second grading is understood simply as the scaling dimension of the fields.

Definition 3.2.6 (Gauge fixing operators)**.** For a free BV theory $(\mathcal{E},Q)$, a *family of gauge fixing operators* on $\mathcal{E}$ parametrized by $\Omega(\Delta^m)$ is an $\Omega(\Delta^m)$-linear differential operator

$$Q^{GF} : \mathcal{E}\otimes\Omega(\Delta^m) \to \mathcal{E}\otimes\Omega(\Delta^m)$$

with the following properties:

(i) Q^{GF} is of bidegree $(-1,-2)$, translation invariant, squaring to zero, and self-adjoint with respect to $\langle\!\langle \cdot,\cdot \rangle\!\rangle$.

(ii) There is a decomposition of the commutator

$$D = \left[Q + \mathrm{d}_{dR}, Q^{GF}\right] = D' + D''$$

such that D' is the tensor product of the Laplacian on $\mathbb{R}^n$ with the identity on E, and D'' is a nilpotent operator commuting with D'.

With this definition of gauge fixing, it is again possible to write down a well-defined heat kernel K_l that regularizes the scale L QME and defines the propagator of the theory as

$$P(\epsilon, L) = \int_\epsilon^L (Q^{GF} \otimes \mathbb{1}) K_l \, dl,$$

which is constructed such that it behaves as $R_l\, P(\epsilon, L) = P(l^{-2}\epsilon, l^{-2})$. Thus, there is an $\mathbb{R}_{>0}$-action by the local renormalization group flow on the space of pre-theories $\widetilde{\mathcal{T}}^{(\infty)}(\mathcal{E}, Q)$, such that

$$\mathcal{RG}_l(I[L]) \in \mathcal{O}^+(\mathcal{E})[\![\hbar]\!] \otimes \mathbb{C}[l, l^{-1}, \log l].$$

One can now again classify the relevant and marginal pre-theories, denoted by $\widetilde{\mathcal{R}}^{(\infty)}(\mathcal{E}, Q)$ and $\widetilde{\mathcal{M}}^{(\infty)}(\mathcal{E}, Q)$, respectively:

$$\begin{aligned}
&\mathcal{RG}_l(I[L]) \in \mathcal{O}^+(\mathcal{E})[\![\hbar]\!] \otimes \mathbb{C}[l, \log l], \qquad && \{I[L]\} \in \widetilde{\mathcal{R}}^{(\infty)}(\mathcal{E}, Q) \subset \widetilde{\mathcal{T}}^{(\infty)}(\mathcal{E}, Q),\\
&\mathcal{RG}_l(I[L]) \in \mathcal{O}^+(\mathcal{E})[\![\hbar]\!] \otimes \mathbb{C}[\log l], \qquad && \{I[L]\} \in \widetilde{\mathcal{M}}^{(\infty)}(\mathcal{E}, Q) \subset \widetilde{\mathcal{T}}^{(\infty)}(\mathcal{E}, Q).
\end{aligned}$$

This generalizes to simplicial sets of theories in the obvious way. As before, the notions of theories $\mathcal{T}^{(\infty)}(\mathcal{E}, Q)$, relevant theories $\mathcal{R}^{(\infty)}(\mathcal{E}, Q)$, and marginal theories $\mathcal{M}^{(\infty)}(\mathcal{E}, Q)$ are given by the subsets of $\widetilde{\mathcal{T}}^{(\infty)}(\mathcal{E}, Q)$, $\widetilde{\mathcal{R}}^{(\infty)}(\mathcal{E}, Q)$, and $\widetilde{\mathcal{M}}^{(\infty)}(\mathcal{E}, Q)$, respectively, that satisfy the QME in the renormalized sense. One can now pursue an analog of obstruction theoretic analysis for these sets, as was given on compact manifolds. One finds the following:

Lemma 3.2.4. *Let $\{I[L]\} \in \mathcal{T}^{(n)}(\mathcal{E}, Q)$. The obstruction O_{n+1} is*

- *relevant, if $\{I[L]\} \in \mathcal{R}^{(n)}(\mathcal{E}, Q)$ and the lift to calculate the obstruction is $\{\widetilde{I}[L]\} \in \widetilde{\mathcal{R}}^{(n+1)}(\mathcal{E}, Q)$.*
- *marginal, if $\{I[L]\} \in \mathcal{M}^{(n)}(\mathcal{E}, Q)$ and the lift to calculate the obstruction is $\{\widetilde{I}[L]\} \in \widetilde{\mathcal{M}}^{(n+1)}(\mathcal{E}, Q)$.*

Corollary 3.2.3. *There are homotopy fiber diagrams of simplicial sets for theories, relevant theories, and marginal theories,*

$$\begin{array}{ccc}
\mathcal{T}^{(n+1)}(\mathcal{E}, Q) & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
\mathcal{T}^{(n)}(\mathcal{E}, Q) & \xrightarrow{O_{n+1}} & \mathcal{O}_{\mathrm{loc}}(\mathcal{E})[1]
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{R}^{(n+1)}(\mathcal{E}, Q) & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
\mathcal{R}^{(n)}(\mathcal{E}, Q) & \xrightarrow{O_{n+1}} & \mathcal{O}^{\geq 0}_{\mathrm{loc}}(\mathcal{E})[1]
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{M}^{(n+1)}(\mathcal{E}, Q) & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
\mathcal{M}^{(n)}(\mathcal{E}, Q) & \xrightarrow{O_{n+1}} & \mathcal{O}^{0}_{\mathrm{loc}}(\mathcal{E})[1]
\end{array}$$

Here $\mathcal{O}^{\geq 0}_{\mathrm{loc}}(\mathcal{E})[1]$ and $\mathcal{O}^{0}_{\mathrm{loc}}(\mathcal{E})[1]$ are the 1-simplices of local action functionals of non-negative scaling and scaling-invariants, respectively.

Proof. In the light of 3.2.4, one can use the statement of Lemma 3.2.3 and the argument is parallel to the one given in the proof of Corollary 3.2.1. □

Chapter 4

Renormalizability of Yang–Mills coupled to Spinors

In this chapter, a concrete application of homotopic renormalization, namely for Yang–Mills theory coupled to the spinor field, which will be referred to as Yang–Mills–Dirac theory (YMD), is given.

Theorem 4.0.1. *Let $\mathfrak{g}$ be a finite-dimensional, semisimple Lie algebra. Then, Yang–Mills–Dirac theory with coefficients in $\mathfrak{g}$ is renormalizable on $\mathbb{R}^4$.*

In Section 4.1, the BV data for the theory is presented. In Section 4.2, the compatibility with the approach in [Cos11] is discussed. Then a lengthy cohomological computation follows in Section 4.3. Lastly, these results are used in Section 4.4 to prove Theorem 4.0.1, and the deformation and symmetry terms are analyzed.

4.1 The BV Structure of Yang–Mills coupled to Spinors

First, the theory that will be investigated is specified, beginning with some remarks on the fermionic sector. One works on $\mathbb{R}^4$ with Euclidean metric, thus the Levi-Civita connection is trivial. Therefore, the Dirac operator $D_{\text{Dirac}} = \gamma^i \nabla_i^{\Sigma}$ becomes $\not{d} := \gamma^i \partial_i = \gamma \mathrm{d}$. The action of the gauge group can be implemented by considering any unitary representation

$$\rho\colon G \to GL(V)$$

for some finite-dimensional vector space V. One can then define the action of G on spinors with values in V, $\psi \in \Omega^0(\mathbb{R}^4, \Sigma\mathbb{R}^4 \otimes V)$ by trivial action on $\Sigma\mathbb{R}^4$ and by ρ on V.

Remark. A physical example for this is quantum chromodynamics, where one has a collection of three spinor fields ψ_1, ψ_2, ψ_3 acted on by $SU(3)$. This corresponds to the setup for the fundamental representation $\rho\colon SU(3) \to SU(3) \subset GL(3)$ with the representation space $\mathbb{C}^3$ and spinors $\psi \in \Omega^0(\mathbb{R}^4, \Sigma\mathbb{R}^4 \otimes \mathbb{C}^3)$.

For invariance of the action, one additionally needs a covariant derivative. As usual, this can be constructed using the gauge field A that transforms as a connection. The action of an element in the Lie algebra $\mathfrak{g}$ of the gauge group is obtained by the corresponding Lie algebra representation to ρ given by differentiation, $\rho_*(X) := \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \rho(\exp(tX))$. Therefore, the covariant derivative becomes

$$\not{d}_A := \not{d} + \gamma(\rho_*(A)) = \not{d} + \gamma^i(\rho_*(A_i)).$$

To write down an invariant classical action, one further fixes a non-degenerate bilinear pairing $\langle \cdot, \cdot \rangle$ on V that is invariant under the G-action. Then, the classical action is

$$S_{\text{YMD,cl}} = \int_{\mathbb{R}^4} \operatorname{Tr}\left(B \wedge F_A + \frac{1}{2} B \wedge *B\right) + \langle \bar{\psi}, *\not{d}_A \psi \rangle.$$

One refers to the terms corresponding to pure Yang–Mills as bosonic part and to the terms involving spinors as the fermionic part. To obtain the classical master action one proceeds as in Section 2.3. In the bosonic sector, the Chevalley–Eilenberg differential is the same as for pure Yang–Mills and its lift yields the BV terms as discussed. In the fermionic sector, one picks up additional terms that encapsulate the Lie algebra action on the spinor fields. The resulting BV action is

$$\begin{aligned} S_{\mathrm{YMD}} = \int_{\mathbb{R}^4} \mathrm{Tr}\left(B \wedge F_A + \frac{1}{2} B \wedge *B + \mathrm{d}_A c \wedge A^+ + [c, B] \wedge B^+ + \frac{1}{2}[c, c] \wedge c^+\right) \\ + \langle \bar{\psi}, *\not{d}_A \psi \rangle + \langle \rho_*(c)\bar{\psi}, \bar{\psi}^+ \rangle + \langle \psi^+, \rho_*(c)\psi \rangle. \end{aligned} \tag{4.1}$$

The field space $\mathcal{E}_{\mathrm{YMD}}$ is now a bigraded vector space in the ghost number and an additional fermionic $\mathbb{Z}_2$-grading.

$$\begin{array}{llcccc} & \text{ghost number} & -1 & 0 & 1 & 2 \\ \text{bosonic} & \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right. & \Omega^0(\mathbb{R}^4, \mathfrak{g}) & \begin{array}{c} \Omega^1(\mathbb{R}^4, \mathfrak{g}) \\ \oplus \\ \Omega^2(\mathbb{R}^4, \mathfrak{g}) \end{array} & \begin{array}{c} \Omega^2(\mathbb{R}^4, \mathfrak{g}) \\ \oplus \\ \Omega^3(\mathbb{R}^4, \mathfrak{g}) \end{array} & \Omega^4(\mathbb{R}^4, \mathfrak{g}) \\ & & & \oplus & \oplus & \\ \text{fermioic} & \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right. & & \begin{array}{c} \Omega^0(\mathbb{R}^4, \Sigma\mathbb{R}^4 \otimes V) \\ \oplus \\ \Omega^0(\mathbb{R}^4, \bar{\Sigma}\mathbb{R}^4 \otimes V) \end{array} & \begin{array}{c} \Omega^4(\mathbb{R}^4, \bar{\Sigma}\mathbb{R}^4 \otimes V) \\ \oplus \\ \Omega^4(\mathbb{R}^4, \Sigma\mathbb{R}^4 \otimes V) \end{array} & \end{array} \tag{4.2}$$

More precisely, one should restrict the field space to Schwartz forms to make the theory well-behaved. Next, one constructs a suitable symplectic form on $\mathcal{E}_{\mathrm{YMD}}$. Since there is a direct sum decomposition into bosonic and fermionic sectors, it suffices to give a symplectic form on each of them and take their sum. In the bosonic sector, one can use the symplectic form of pure Yang–Mills theory ω_{YM} as defined in equation (2.9) without modifications. For the fermionic sector, one has to slightly adjust the form ω_{Dirac} given for the free fermion in equation (2.10), since the spinor fields are now charged. Therefore, one needs to contract component-wise with the pairing on V. The resulting symplectic form on $\mathcal{E}_{\mathrm{YMD}}$ reads

$$\omega_{\mathrm{YMD}} = \int_{\mathbb{R}^4} \mathrm{Tr}\left(\delta A \wedge \delta A^+ + \delta B \wedge \delta B^+ + \delta c \wedge \delta c^+\right) + \langle \delta\psi^+, \delta\psi \rangle + \langle \delta\bar{\psi}, \delta\bar{\psi}^+ \rangle. \tag{4.3}$$

To find the cohomological vector field $\{S_{\mathrm{YMD}}, \cdot\}$, one first varies the action:

$$\begin{aligned} \delta S_{\mathrm{YMD}} = \int_{\mathbb{R}^4} \mathrm{Tr}\Big(& \delta B \wedge F_A + B \wedge [A, \delta A] - \mathrm{d}B \wedge \delta A + *B \wedge \delta B + [c, \delta A] \wedge A^+ \\ & + \mathrm{d}_A \, \delta c \wedge A^+ + \mathrm{d}_A c \wedge \delta A^+ + [\delta c, B] \wedge B^+ + [c, \delta B] \wedge B^+ + [c, B] \wedge \delta B^+ \\ & + [c, \delta c] \wedge c^+ + \frac{1}{2}[c, c]\, \delta c \Big) \\ & + \langle \delta\bar{\psi}, *\not{d}_A \psi \rangle + \langle *\not{d}_A \bar{\psi}, \delta\psi \rangle + \langle \bar{\psi}, *(\gamma\, \delta A)\psi \rangle + \langle \rho_*(\delta c)\bar{\psi}, \bar{\psi}^+ \rangle + \langle \rho_*(c)\, \delta\bar{\psi}, \bar{\psi}^+ \rangle \\ & + \langle \rho_*(c)\bar{\psi}, \delta\bar{\psi}^+ \rangle + \langle \delta\psi^+, \rho_*(c)\psi \rangle + \langle \psi^+, \rho_*(\delta c)\psi \rangle + \langle \psi^+, \rho_*(c)\, \delta\psi \rangle \\ = \int_{\mathbb{R}^4} \mathrm{Tr}\Big(& \big([c, A^+] - \mathrm{d}_A B\big) \wedge \delta A + \langle \bar{\psi}, *(\gamma\, \delta A)\psi \rangle + \big(F_A + *B + [c, B^+]\big)\, \delta B \\ & + \big(\mathrm{d}_A A^+ + [B, B^+] + [c, c^+] + \langle \psi^+, \rho_*(\cdot)\psi \rangle + \langle \bar{\psi}, \rho_*(\cdot)\bar{\psi}^+ \rangle\big)\, \delta c \\ & + \mathrm{d}_A c \wedge \delta A^+ + [c, B] \wedge \delta B^+ + \frac{1}{2}[c, c] \wedge \delta c^+ \Big) \\ & + \langle *\not{d}_A \bar{\psi} - \rho_*(c)\psi^+, \delta\psi \rangle + \langle \delta\bar{\psi}, *\not{d}_A \psi + \rho_*(c)\bar{\psi}^+ \rangle + \langle \delta\psi^+, \rho_*(c)\psi \rangle - \langle \rho_*(c)\bar{\psi}, \delta\bar{\psi}^+ \rangle. \end{aligned} \tag{4.4}$$

Here, extensive use of Stokes' theorem and the graded invariance of the Killing form was made to manipulate the terms. From the Hamiltonian relation $\delta S = \iota_{\{S,\cdot\}}\omega$ one can derive

$$\begin{aligned}\{S_{\mathrm{YMD}},\cdot\} = \int_{\mathbb{R}^4} \mathrm{Tr}\Big(&\mathrm{d}_A c \wedge \frac{\delta}{\delta A} + [c,B]\wedge\frac{\delta}{\delta B} + \frac{1}{2}[c,c]\wedge\frac{\delta}{\delta c} \\ &+ \big([c,A^+] - \mathrm{d}_A B\big)\wedge\frac{\delta}{\delta A^+} + \langle\bar{\psi}, *(\gamma\frac{\delta}{\delta A^+})\psi\rangle + \big(F_A + *B + [c,B^+]\big)\frac{\delta}{\delta B^+} \\ &+ \big(\mathrm{d}_A A^+ + [B,B^+] + [c,c^+]\big)\wedge\frac{\delta}{\delta c^+} + \big(\langle\psi^+,\rho_*(\cdot)\psi\rangle + \langle\bar{\psi},\rho_*(\cdot)\bar{\psi}^+\rangle\big)\frac{\delta}{\delta c^+}\Big) \\ &+ \langle *\not{d}_A\bar{\psi} - \rho_*(c)\psi^+, \frac{\delta}{\delta\psi^+}\rangle + \langle\frac{\delta}{\delta\bar{\psi}^+}, *\not{d}_A\psi + \rho_*(c)\bar{\psi}^+\rangle \\ &+ \langle\frac{\delta}{\delta\psi},\rho_*(c)\psi\rangle - \langle\rho_*(c)\bar{\psi},\frac{\delta}{\delta\bar{\psi}}\rangle.\end{aligned} \tag{4.5}$$

In particular, this vector field can be split into a linear part due to the quadratic part of the action and a second part due to the interaction, $\{S_{\mathrm{YMD}},\cdot\} = Q + \{I^{(0)},\cdot\}$. The action of the linear part can be represented by the following diagram:

$$\begin{array}{ccccccc} -1 & & 0 & & 1 & & 2 \\ \Omega^0(\mathbb{R}^4,\mathfrak{g}) & \xrightarrow{\mathrm{d}} & \Omega^1(\mathbb{R}^4,\mathfrak{g}) & \xrightarrow{\mathrm{d}} & \Omega^2(\mathbb{R}^4,\mathfrak{g}) & & \Omega^4(\mathbb{R}^4,\mathfrak{g}) \\ & & & \nearrow^{*} & & \nearrow^{\mathrm{d}} & \\ & & \Omega^2(\mathbb{R}^4,\mathfrak{g}) & \xrightarrow{\mathrm{d}} & \Omega^3(\mathbb{R}^4,\mathfrak{g}) & & \end{array} \tag{4.6}$$

$$\Omega^0(\mathbb{R}^4,\Sigma\mathbb{R}^4\otimes V) \xrightarrow{*\not{d}} \Omega^4(\mathbb{R}^4,\Sigma\mathbb{R}^4\otimes V)$$

$$\Omega^0(\mathbb{R}^4,\bar{\Sigma}\mathbb{R}^4\otimes V) \xrightarrow{*\not{d}} \Omega^4(\mathbb{R}^4,\bar{\Sigma}\mathbb{R}^4\otimes V)$$

With this definition, one can decompose the action into the quadratic free part and the interaction

$$S_{\mathrm{YMD}} = \omega(e,Qe) + I^{(0)}(e), \quad e\in\mathcal{E}_{\mathrm{YMD}}, \tag{4.7}$$

with the interaction at classical level $I^{(0)}$. The vector field due to this interaction also admits further decomposition. Of particular relevance for the later calculation is the part that describes the action of the local Lie algebra and induces the Chevalley–Eilenberg differential on the Q-cohomology:

$$\begin{aligned}X = \int_{\mathbb{R}^4}\mathrm{Tr}\Big(&[c,A]\wedge\frac{\delta}{\delta A} + [c,B]\wedge\frac{\delta}{\delta B} + \frac{1}{2}[c,c]\wedge\frac{\delta}{\delta c} + [c,A^+]\wedge\frac{\delta}{\delta A^+} + [c,B^+]\frac{\delta}{\delta B^+}\Big) \\ &+ \langle\frac{\delta}{\delta\psi},\rho_*(c)\psi\rangle + \langle\rho_*(c)\bar{\psi},\frac{\delta}{\delta\bar{\psi}}\rangle.\end{aligned} \tag{4.8}$$

This completes the classical BV data $(\mathcal{E}_{\mathrm{YMD}},\omega_{\mathrm{YMD}},S_{\mathrm{YMD}})$, the differential graded BV manifold of Yang–Mills–Dirac theory. In particular, a decomposition of the action into its free and interacting parts was given, and the respective emerging vector fields were discussed. The next step is to fit this data into the framework of homotopic renormalization.

4.2 Admissibility to Homotopic Renormalization

To prove that the method of homotopic renormalization is applicable to YMD, one has to show two things. First, one needs to show that the free part of the theory fits the framework given by Definition 3.2.5.

Definition 4.2.1 (Free YMD)**.** The theory of free YMD is the triple $(\mathcal{E}, \langle\!\langle \cdot, \cdot \rangle\!\rangle, Q)$, where

1. $\mathcal{E}$ is the space $\mathcal{E}_{\mathrm{YMD}}$, as defined before. It consists of the Schwartz sections of the bundle $E \to \mathbb{R}^4$ with fibers at any $x \in \mathbb{R}^4$

$$\begin{array}{lccccccc} E_x = & \wedge^0\mathbb{R}^4[-1]\otimes\mathfrak{g} & \oplus & \wedge^1\mathbb{R}^4\otimes\mathfrak{g} & \oplus & \wedge^2\mathbb{R}^4[1]\otimes\mathfrak{g} & \oplus & \wedge^4\mathbb{R}^4[2]\otimes\mathfrak{g} \\ & & & \oplus & & \oplus & & \\ & & & \wedge^2\mathbb{R}^4\otimes\mathfrak{g} & & \wedge^3\mathbb{R}^4[1]\otimes\mathfrak{g} & & \\ & & & \oplus & & \oplus & & \\ & & & \wedge^0\mathbb{R}^4\otimes\mathbb{C}^4_{\Sigma}\otimes V & & \wedge^4\mathbb{R}^4\otimes\mathbb{C}^4_{\bar{\Sigma}}[1]\otimes V & & \\ & & & \oplus & & \oplus & & \\ & & & \wedge^0\mathbb{R}^4\otimes\mathbb{C}^4_{\bar{\Sigma}}\otimes V & & \wedge^4\mathbb{R}^4\otimes\mathbb{C}^4_{\Sigma}[1]\otimes V & & \end{array} \tag{4.9}$$

Here, the notation $\mathbb{C}^4_{\Sigma}$ and $\mathbb{C}^4_{\bar{\Sigma}}$ for the fibers of the spinor bundles $\Sigma\mathbb{R}^4$ and $\bar{\Sigma}\mathbb{R}^4$ was introduced.[1] The cohomological degree is the one induced by the shifts, which gives rise to the ghost number on $\mathcal{E}$ as indicated in (4.2). The scaling dimension of the fields $\mathcal{E}_{\mathrm{YMD}}$ is set to be the same as the form degree in the bosonic sector. In the fermionic sector, the fields are assigned a scaling dimension of $l = 3/2$, and the antifields $l = 5/2$. [2]

2. $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ is the contraction of the BV 2-form with the tangent fields associated to the field under the isomorphism of vector spaces $T_e\mathcal{E} \cong \mathcal{E}$, $e \in \mathcal{E}$.

3. Q is given by the operator defined in the diagram (4.6).

Proposition 4.2.1. *The free theory of YMD, as above, defines a free BV theory in the sense of Definition 3.2.5.*

Proof. 3.2.5(i) is fulfilled by definition. However, it is worth noting that E and $\mathcal{E}$, respectively, have even richer graded structure.
For 3.2.5(ii), it suffices to check that the pairing due to the BV form is a degree -1 anti-symmetric, scaling invariant pairing. Scaling invariance follows from the choice of scaling dimensions in the definition. For degree -1 anti-symmetry, it suffices to consider pairs of fields and their anti-fields, since the pairing between other combinations is trivially 0. Further, since the statement is about a graded property, one should consider homogeneous elements. The computation, which is trivial, is demonstrated for the field-antifield pair $A \in \Omega^1(\mathbb{R}^4, \mathfrak{g})$, $A^+ \in \Omega^3(\mathbb{R}^4, \mathfrak{g})$, but it follows analogously for other pairs.

$$\langle\!\langle A, A^+ \rangle\!\rangle = \int_{\mathbb{R}^4} \mathrm{Tr}\left(A \wedge A^+\right) = -(-1)^{|A||A^+|-|A|-|A^+|}\langle\!\langle A^+, A \rangle\!\rangle = \langle\!\langle A^+, A \rangle\!\rangle,$$

since $|A||A'^+|$ is even and $|A| + |A^+|$ is odd, as desired.

Lastly, one has to show that the operator Q is as in 3.2.5(iii). Translation invariance follows from integration on $\mathbb{R}^4$ being translation invariant. That Q has cohomological degree 1 on $\mathcal{E}$ is apparent from (4.6). To prove graded adjointness, one can consider the bosonic and fermionic sectors separately, since they

[1] The fibers are isomorphic to $\mathbb{C}^4$, but one needs to keep in mind that they transform differently.

[2] The definition parallels the usual argument in physics that the fermion field has mass dimension $3/2$. This leaves no choice for the anti-fields, since the symplectic form needs to be scaling invariant.

do not pair. The cohomological vector field in the bosonic sector differs from [Cos11] only by constants, so his argument can be used. For the fermionic sector, one computes

$$\begin{aligned}
\langle\!\langle (\psi,\bar\psi,\psi^+,\bar\psi^+), Q(\psi',\bar\psi',\psi'^+,\bar\psi'^+)\rangle\!\rangle &= \langle\!\langle (\psi,\bar\psi), Q(\psi',\bar\psi')\rangle\!\rangle \\
&= \frac{1}{2}\int_{\mathbb{R}^4} \langle \bar\psi, *\not{d}\psi'\rangle + \langle *\not{d}\bar\psi', \psi\rangle \\
&= \frac{1}{2}\int_{\mathbb{R}^4} \langle *\not{d}\bar\psi, \psi'\rangle + \langle \bar\psi', *\not{d}\psi\rangle \\
&= \langle\!\langle Q(\psi,\bar\psi), (\psi',\bar\psi')\rangle\!\rangle,
\end{aligned}$$

by formal adjointness of the Dirac operator. Since the physical fermion field is of cohomological degree 0, this proves the claim. ☐

The second condition is that the theory admits a gauge fixing operator satisfying Definition 3.2.6. To this end, one defines the Hodge-de Rham operator $\mathrm{d}^* : \Omega^k(\mathbb{R}^4) \to \Omega^{k-1}(\mathbb{R}^4),\ \omega \mapsto (-1)^k * \mathrm{d} * \omega$ and considers the diagram

$$\begin{array}{ccccccc}
-1 & & 0 & & 1 & & 2 \\
\Omega^0(\mathbb{R}^4,\mathfrak{g}) & \xleftarrow[\mathrm{d}^*]{} & \Omega^1(\mathbb{R}^4,\mathfrak{g}) & \xleftarrow[2\mathrm{d}^*]{} & \Omega^2(\mathbb{R}^4,\mathfrak{g}) & & \Omega^4(\mathbb{R}^4,\mathfrak{g}) \\
 & & & & & \swarrow_{\mathrm{d}^*} & \\
 & & \Omega^2(\mathbb{R}^4,\mathfrak{g}) & \xleftarrow[2\mathrm{d}^*]{} & \Omega^3(\mathbb{R}^4,\mathfrak{g}) & & \\
 & & \Omega^0(\mathbb{R}^4,\mathbb{C}^4_\Sigma\otimes\mathfrak{g}) & \xleftarrow[\not{d}*]{} & \Omega^4(\mathbb{R}^4,\mathbb{C}^4_\Sigma\otimes\mathfrak{g}) & & \\
 & & \Omega^0(\mathbb{R}^4,\mathbb{C}^4_{\bar\Sigma}\otimes\mathfrak{g}) & \xleftarrow[\not{d}*]{} & \Omega^4(\mathbb{R}^4,\mathbb{C}^4_{\bar\Sigma}\otimes\mathfrak{g}) & &
\end{array} \tag{4.10}$$

Proposition 4.2.2. *The operator $Q^{GF} : \mathcal{E} \to \mathcal{E}$ as defined in diagram (4.10) defines a gauge fixing for YMD.*

Remark. Since the theory is independent of the choice of family of gauge fixing operators, one may choose a constant one, which is done here.

Proof. By the diagrams (4.6) and (4.10) it becomes clear that the gauge fixing in the bosonic and fermionic sectors can be analyzed separately, since there is no map between them. Additionally, it was already checked by [Cos11] that the gauge fixing in the bosonic sector defines a gauge fixing. It remains to prove this for the fermionic sector. It thus needs to be shown that Q^{GF} has the properties of Definition 3.2.6. For (i), one needs to show that Q^{GF} has cohomological degree -1, which directly follows from the definition, has scaling dimension -2, and is self-adjoint with respect to $\langle\!\langle\cdot,\cdot\rangle\!\rangle$. For self-adjointness, one computes

$$\langle\!\langle Q^{GF}\psi^+, \bar\psi^+\rangle\!\rangle = \int_{\mathbb{R}^4} \langle \not{d} * \psi^+, \bar\psi^+\rangle = \int_{\mathbb{R}^4} \langle \psi^+, \not{d} * \bar\psi^+\rangle = \langle\!\langle \psi^+, Q^{GF}\bar\psi^+\rangle\!\rangle.$$

For (ii), one uses the fact that the Dirac-Laplacian on $\mathbb{R}^n$ coincides with the normal Laplacian, $D = \not{d}^2$, and observes that the commutator of Q and Q^{GF} acts by

$$\begin{aligned}[Q, Q^{GF}] \colon \psi &\longmapsto (\not{d}*)(*\not{d})\psi \\ \bar{\psi} &\longmapsto (\not{d}*)(*\not{d})\bar{\psi} \\ \psi^+ &\longmapsto (*\not{d})(\not{d}*)\psi^+ \\ \bar{\psi}^+ &\longmapsto (*\not{d})(\not{d}*)\bar{\psi}^+.\end{aligned}$$

For $\psi, \bar{\psi}$, one directly obtains the Laplacian since the hodge operator squares to the identity. For the anti-fermions, consider the action of $(*\not{d})(\not{d}*) = *D*$ on any top degree Schwartz form on $\mathbb{R}^4$, $\alpha \in \Omega^4(\mathbb{R}(4))$. Let ω be the canonical volume element, then

$$*D * \alpha = *D * \alpha(x)\omega = *(D\alpha(x)) = (D\alpha(x))\omega.$$

One can conclude that $[Q, Q^{GF}]$ acts like the Laplacian on $\mathbb{R}^4$ and as the identity on fibers. $\square$

4.3 Homological Calculation

To prove renormalizability, it must be shown that at any length scale, YMD up to order n may be lifted to a relevant theory of order $n+1$. This is done by establishing that the cohomology of translation invariant action functionals with respect to $Q + \left\{I^{(0)}, \cdot\right\}$ in degree one, i.e. the obstruction group, vanishes. Further, one can classify deformations and symmetries by computing the cohomology in degree 0 and -1. In particular, since YMD is marginal at the classical level and the differential conserves scaling dimension, it suffices to calculate the cohomology at the marginal level, i.e. at scaling dimension 0.

Following the procedure in [Cos11], one defines for the purpose of calculation the auxiliary spaces of fields

$$\mathcal{Y} := Y \otimes \mathscr{S}(\mathbb{R}^4), \qquad \mathcal{S} := S \otimes \mathscr{S}(\mathbb{R}^4),$$

consisting of Schwartz sections of translation invariant fields Y and S that represent the bosonic and fermionic sectors, respectively,

$$\begin{array}{lcccc} \text{cohomological degree} & 0 & 1 & 2 & 3 \\[6pt] Y := \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right. & \begin{array}{c} \bar{\Omega}^0(\mathbb{R}^4) \\ \\ \\ \end{array} & \begin{array}{c} \bar{\Omega}^1(\mathbb{R}^4) \\ \oplus \\ \bar{\Omega}^2(\mathbb{R}^4) \end{array} & \begin{array}{c} \bar{\Omega}^2(\mathbb{R}^4) \\ \oplus \\ \bar{\Omega}^3(\mathbb{R}^4) \end{array} & \begin{array}{c} \bar{\Omega}^4(\mathbb{R}^4) \\ \\ \\ \end{array} \\[6pt] S := \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right. & & \begin{array}{c} \bar{\Omega}^0(\mathbb{R}^4, \mathbb{C}^4_{\Sigma}) \\ \oplus \\ \bar{\Omega}^0(\mathbb{R}^4, \mathbb{C}^4_{\bar{\Sigma}}) \end{array} & \begin{array}{c} \bar{\Omega}^4(\mathbb{R}^4, \mathbb{C}^4_{\bar{\Sigma}}) \\ \oplus \\ \bar{\Omega}^4(\mathbb{R}^4, \mathbb{C}^4_{\Sigma}) \end{array} & \end{array} . \tag{4.11}$$

Here, $\bar{\Omega}$ denotes the space of translation invariant forms. One then obtains the space of the BV fields of the Yang–Mills–Dirac theory as the semi-direct product

$$\mathcal{Y} \otimes \mathfrak{g}[1] \ltimes \mathcal{S} \otimes V[1],$$

where the fermionic sector is acted on by the $\mathfrak{g}$-component of the bosonic sector via ρ_*, according to equation (4.8). Further, the formal completions of $\mathcal{Y}, \mathcal{S}$ at 0 are defined as

$$\widehat{\mathcal{Y}} := Y[\![x]\!] \qquad \widehat{\mathcal{S}} := S[\![x]\!].$$

With regard to the discussion of jets in Section 1.3, one identifies these spaces as the ∞-jets of Y, S at $0 \in \mathbb{R}^4$. For renormalizability and classification of deformations, the following theorem is needed:

Theorem 4.3.1. *For a semisimple Lie algebra* $\mathfrak{g}$*, the cohomology of local action functionals on* $\mathcal{Y}$ *in scaling dimension 0 fulfills*

$$H^i\left(\mathcal{O}_{\mathrm{loc}}(\mathcal{Y}\otimes\mathfrak{g}[1]\ltimes\mathcal{S}\otimes V[1])^{\mathbb{R}^4}, Q+\left\{I^{(0)},\cdot\right\}\right)^{\mathrm{Spin}(4)} \cong \begin{cases} 3\ \mathrm{Sym}^2 V^\vee \oplus 4\ \wedge^2 V^\vee & i=-1 \\ \mathrm{Sym}^2\mathfrak{g}^\vee \oplus 3\ \mathrm{Sym}^2 V^\vee & i=0 \\ H^5(\mathfrak{g}) & i=1 \end{cases}$$

Before this is proven, some tools in the form of six lemmas and a theorem are developed.

Lemma 4.3.1 (Lemma 6.7.1, Chapter 5 [Cos11])**.** *Suppose the base manifold is* $\mathbb{R}^n$*. Then there is a canonical quasi-isomorphism of complexes*

$$(\mathcal{O}_{\mathrm{loc}}(\mathcal{E})/\mathbb{R})^{\mathbb{R}^n} = \mathbb{R}\overset{\mathbf{L}}{\otimes}_{\mathbb{R}[\partial]} C^\bullet_{\mathrm{red}}(J^\infty_0(E)[-1]),$$

where $C^\bullet_{\mathrm{red}}(J^\infty_0(E)[-1])$ *denotes the reduced Chevalley–Eilenberg cochains of the Lie algebra* $J^\infty_0(E)[-1]$*.*

Proof. This is shown by taking the translation invariants on the right-hand side of the isomorphism in 1.3.2. First, one determines the $\mathbb{R}^n$-invariants of $D_{\mathbb{R}^4} = C^\infty(\mathbb{R}^4)[\partial]$. This yields the differential operators with constant coefficients $\mathbb{R}[\partial]$, as desired. The densities on $\mathbb{R}^n$ are isomorphic to sections of the bundle of determinants. The translation invariants are, therefore, constant determinants, which can be identified with $\mathbb{R}$. Notice that $\mathbb{R}$ still has the right $\mathbb{R}[\partial]$-module structure of $\mathrm{Dens}(\mathbb{R}^n)$, which acts trivially. Lastly, one needs to determine the translation invariants of $\mathcal{O}(J^\infty(\mathcal{E}))$. By decomposition into the symmetric powers, one finds that

$$\mathcal{O}(J^\infty(\mathcal{E}))^{\mathbb{R}^4} = \mathrm{Sym}^k(J^\infty_0(\mathcal{E}))^\vee,$$

where $J^\infty_0(\mathcal{E})^\vee$ are the translation invariants of $J^\infty(\mathcal{E})^\vee$ that one can explicitly identify with the sections of the ∞-jet bundle evaluated at $0 \in \mathbb{R}^n$ or equivalently the fiber E_0 of E. The action of $\mathbb{R}[\partial]$ on $J^\infty(\mathcal{E})^\vee$ extends to $J^\infty_0(\mathcal{E})^\vee$ in the obvious way. Moreover, the differential X extends to a differential on $J^\infty_0(\mathcal{E})$. Recall that, by the treatment following Definition 1.1.9, an L_∞-algebra can be interpreted as a pair (L, Q) of a vector space and a differential of the cofree cocommutative coalgebra $\mathrm{Sym}^\bullet L[1]$. Applying this to $J^\infty_0(\mathcal{E})$, one obtains a L_∞-algebra $(J^\infty_0(\mathcal{E})[-1], X)$, with the n-brackets induced by X. In particular, the algebra of functions on $J^\infty_0(\mathcal{E})$ takes the form of Chevalley–Eilenberg cochains of $J^\infty_0(\mathcal{E})$ with respect to the induced Lie bracket. □

Remark. In the context above where one views the Chevalley–Eilenberg cochains as functions on an L_∞-algebra, the Chevalley–Eilenberg cohomology is understood as the cohomology induced by Q, i.e. with respect to all brackets.

Lemma 4.3.2. *There is an isomorphism of complexes*

$$\mathcal{O}_{\mathrm{loc}}(\mathcal{Y}\otimes\mathfrak{g}[1]\ltimes\mathcal{S}\otimes V[1])^{\mathbb{R}^4} = \mathbb{R}\overset{\mathbf{L}}{\otimes}_{\mathbb{R}[\partial]} C^\bullet_{\mathrm{red}}(\widehat{\mathcal{Y}}\otimes\mathfrak{g}\ltimes\widehat{\mathcal{S}}\otimes V),$$

where $\mathbb{R}[\partial] := \mathbb{R}[\partial_1,\partial_2,\partial_3,\partial_4]$ *is the ring of differential operators with constant coefficients.* $\mathbb{R}$ *is a trivial right* $\mathbb{R}[\partial]$*-module and* $C^\bullet_{\mathrm{red}}(\widehat{\mathcal{Y}}\otimes\mathfrak{g})$ *has a left* $\mathbb{R}[\partial]$*-module structure induced by the left* $\mathbb{R}[\partial]$*-module structure of jets introduced in Example 1.3.2. In addition,* $\mathcal{O}_{\mathrm{loc}}(\mathcal{Y}\otimes\mathfrak{g}[1])\ltimes(\mathcal{S}\otimes V[1])^{\mathbb{R}^4}(k)$ *corresponds to* $C^\bullet_{\mathrm{red}}(\widehat{\mathcal{Y}}\otimes\mathfrak{g}\ltimes\widehat{\mathcal{S}}\otimes V)(k-4)$*.*

Proof. This is a modification of Lemma 4.3.1 using the identification of $J_0^\infty(\mathcal{Y} \otimes \mathfrak{g}[1] \ltimes \mathcal{S} \otimes V[1])[-1]$ as $\widehat{\mathcal{Y}} \otimes \mathfrak{g} \ltimes \widehat{\mathcal{S}} \otimes V$. Moreover, observe that the constant densities denoted as the trivial right $\mathbb{R}[\partial]$-module $\mathbb{R}$ are of scaling dimension 4, inducing the shift of scaling dimension between $\mathcal{O}_{\mathrm{loc}}(\mathcal{Y} \otimes \mathfrak{g}[1]) \ltimes (\mathcal{S} \otimes V[1])^{\mathbb{R}^4}$ and $C^\bullet_{\mathrm{red}}(\widehat{\mathcal{Y}} \otimes \mathfrak{g} \ltimes \mathcal{S} \otimes V)$. ☐

Definition 4.3.1 (Koszul Complex)**.** Let R be a commutative ring and $\phi : R^r \to R$ an R-linear map. The *Koszul complex* $K(R^r, \phi)$ is defined as

$$\wedge^r R^r \longrightarrow \wedge^{r-1} R^r \longrightarrow \cdots \longrightarrow \wedge^1 R^r \longrightarrow \wedge^0 R^r \simeq R.$$

The arrows are given by the maps

$$\wedge^k R^r \ni \alpha_1 \wedge \cdots \wedge \alpha_k \longmapsto \sum_i (-1)^{i+1} \phi(\alpha_i) \alpha_1 \wedge \cdots \wedge \widehat{\alpha}_i \cdots \wedge \alpha_k,$$

where the hat indicates that the factor is omitted.

Lemma 4.3.3. *The Koszul complex $K(\mathbb{R}[\partial]^4, \phi)$, where $\phi : \mathbb{R}[\partial]^4 \to \mathbb{R}[\partial]$ is the map given by the $\mathbb{R}[\partial]$-linear extension of $e_i \mapsto \partial_i$, is a projective resolution of the trivial right $\mathbb{R}[\partial]$-module $\mathbb{R}$.*

Proof. It is a well established fact in the literature[3] that, for a commutative ring R and any R-regular sequence $(x_1 \ldots, x_r)$, the Koszul complex $K(R^r, \phi)$ is an R-free resolution of $R/(x_1 \ldots, x_r)R$, where ϕ is the contraction map with $(x_1, \ldots, x_r)$. Since the resolution is free, it is, in particular, projective. Note that the trivial right $\mathbb{R}[\partial]$-module $\mathbb{R}$ can be identified as

$$\mathbb{R}[\partial]/\mathbb{R}[\partial](\partial_1, \partial_2, \partial_3, \partial_4).$$

The ring of differential operators with constant coefficients is commutative on $\mathbb{R}^4$, therefore the proof reduces to showing that $(\partial_1, \partial_2, \partial_3, \partial_4)$ is, in fact, a regular sequence, i.e. that ∂_i is not a zero-divisor in $\mathbb{R}[\partial]/\mathbb{R}[\partial](\partial_1, \ldots, \partial_{i-1})$. This is immediate, because none of the ∂_i are zero-divisors of $\mathbb{R}[\partial]$ and the only element mapped to 0 by taking the quotient is $0 \in \mathbb{R}[\partial]$. ☐

Lemma 4.3.4. *Let A be a dg vector space that is degree-wise finite, concentrated in positive degree, and bounded, and V a finite-dimensional vector space. Then*

$$H^\bullet\left((\mathrm{Sym}^r(A \otimes V))^\vee\right) \cong \left(\mathrm{Sym}^r\left(H^\bullet(A) \otimes V\right)\right)^\vee.$$

Proof. Note that by assumption, $\mathrm{Sym}^r(A \otimes V)$ is degree-wise finite and therefore projective, and bounded. In particular, $H^n(\mathrm{Sym}^r(A \otimes V))$ is a projective object. By Proposition A.3.3, the n-th Ext-functor of $\mathrm{Sym}^r(A \otimes V)$, $\mathrm{Ext}^n(\mathrm{Sym}^r(A \otimes V), \cdot)$, is the zero functor. Thus, by Theorem A.3.3, the $\mathrm{Hom}(\cdot, \mathbb{R})$-functor commutes with the cohomology functor H. One can now calculate

$$\begin{aligned} H^\bullet\left((\mathrm{Sym}^r(A \otimes V))^\vee\right) &\cong \left(H^\bullet\left(\mathrm{Sym}^r(A \otimes V)\right)\right)^\vee \\ &\cong \left(H^\bullet\left(\left((A \otimes V)^{\otimes r}\right)^{S_r}\right)\right)^\vee \\ &\cong \left(\left(H^\bullet(A)^{\otimes r} \otimes V^{\otimes r}\right)^{S_r}\right)^\vee \\ &\cong \left(\mathrm{Sym}^r\left(H^\bullet(A) \otimes V\right)\right)^\vee, \end{aligned}$$

where it was used in the step from the second to the third line that the cohomology functor commutes with taking invariants of finite groups and applied the Künneth formula. ☐

Lemma 4.3.5. *The following isomorphisms of vector spaces hold:*

[3] See e.g. Theorem 16.5 in [Mat87].

(i) $H^\bullet(\widehat{\mathcal{Y}}(0)) \cong \mathbb{R}$

(ii) $H^\bullet(\widehat{\mathcal{Y}}(1)) \cong 0$

(iii) $H^\bullet(\widehat{\mathcal{S}}(\frac{3}{2})) \cong (\mathbb{C}^4_\Sigma \oplus \mathbb{C}^4_{\bar{\Sigma}})[-1]$

(iv) $H^\bullet(\widehat{\mathcal{Y}}(2)) \cong \wedge^2\mathbb{R}^4[-1]$

(v) $H^\bullet(\widehat{\mathcal{S}}(\frac{5}{2})) \cong \ker\ *\not{d}\big|_{\mathrm{gr}_1(\mathbb{C}^4_\Sigma\oplus\mathbb{C}^4_{\bar{\Sigma}})[x]}[-1]$

Proof. (i) This is trivial, since $\widehat{\mathcal{Y}}(0) = \bar{\Omega}^0 \cong \mathbb{R}$ is concentrated in degree 0.

(ii) The complex $\widehat{\mathcal{Y}}(1)$ takes the form

$$0 \longrightarrow \mathrm{gr}_1\bar{\Omega}^0[x] \cong \mathrm{gr}_1\mathbb{R}[x] \xrightarrow{\ \mathrm{d}\ } \bar{\Omega}^1 \longrightarrow 0$$

Note that $\mathrm{gr}_1\bar{\Omega}^0[x]$ is spanned by the elements $x^i \in \mathrm{gr}_1\bar{\Omega}^0[x]$ and the translation invariant 1-forms are spanned by $\mathrm{d}x^i$. The differential maps x^i to $\mathrm{d}\big(x^i\big) = \mathrm{d}x^i$. Thus, d is an isomorphism, the sequence is exact and the cohomology vanishes.

(iii) Since $\widehat{\mathcal{Y}}(\frac{3}{2}) = \bar{\Omega}^0 \otimes (\mathbb{C}^4_\Sigma \oplus \mathbb{C}^4_{\bar{\Sigma}}) \cong \mathbb{C}^4_\Sigma \oplus \mathbb{C}^4_{\bar{\Sigma}}$ is concentrated in degree 1, taking the cohomology yields the complex itself.

(iv) One needs to calculate the cohomology of the complex

$$0 \longrightarrow \mathrm{gr}_2\bar{\Omega}^0[x] \xrightarrow{\ \mathrm{d}_0\ } \mathrm{gr}_1\bar{\Omega}^1[x] \oplus \bar{\Omega}^2 \xrightarrow{\ \mathrm{d}_1\ } \bar{\Omega}^2 \longrightarrow 0$$

with differentials $\mathrm{d}_0 = \mathrm{d}$ and $\mathrm{d}_1 = \mathrm{d} \oplus *$. First the differential at $\mathrm{gr}_2\bar{\Omega}^0[x]$ is examined. A general element in this vector space can be written as $c_{ij}x^ix^j \in \mathrm{gr}_2\bar{\Omega}^0[x]$, $c_{ij} \in \mathbb{R}$. Applying the differential to this element yields

$$\mathrm{d}_0(c_{ij}x^ix^j) = c_{ij}(x^i\mathrm{d}x^j + x^j\mathrm{d}x^i),$$

which is non-zero, so the sequence is exact at $\mathrm{gr}_2\bar{\Omega}^0[x]$ and the cohomology in degree 0 vanishes. To obtain the cohomology in degree 1, it is useful to make the decomposition

$$\mathrm{gr}_1\bar{\Omega}^1[x] \oplus \bar{\Omega}^2 \cong \wedge^2\mathbb{R}^4 \oplus \mathrm{Sym}^2\mathbb{R}^4 \oplus \bar{\Omega}^2.$$

The respective terms have bases given by elements $x^i\mathrm{d}x^j - x^j\mathrm{d}x^i - \mathrm{d}x^i \wedge \mathrm{d}x^j$ in the case of $\wedge^2\mathbb{R}^4$, $x^i\mathrm{d}x^j + x^j\mathrm{d}x^i$ for $\mathrm{Sym}^2\mathbb{R}^4$ and $\mathrm{d}x^i \wedge \mathrm{d}x^j$ for $\bar{\Omega}^2$. It is immediate that the set $\big(x^i\mathrm{d}x^j + x^j\mathrm{d}x^i\big)_{ij}$ is also a basis of the image of d_0. Further, by the properties of the Hodge operator, the subspace $\bar{\Omega}^2$ is isomorphic to the target under the restriction of the differential $\mathrm{d}_1|_{\bar{\Omega}^2}$. A simple calculation shows that all elements in $\wedge^2\mathbb{R}^4$ are in the kernel of d_1. Thus,

$$H^1(\widehat{\mathcal{Y}}(2)) = \ker \mathrm{d}_1/\mathrm{im}\,\mathrm{d}_0 \cong \wedge^2\mathbb{R}^4$$

is obtained. Since it was already established that d_1 is surjective, the cohomology in degree 2 vanishes again.

(v) The complex $\widehat{\mathcal{S}}(\frac{5}{2})$ is the sequence

$$0 \longrightarrow \mathrm{gr}_1\bar{\Omega}^0 \otimes (\mathbb{C}^4_\Sigma \oplus \mathbb{C}^4_{\bar{\Sigma}})[x] \xrightarrow{\ *\not{d}\ } \bar{\Omega}^4 \otimes (\mathbb{C}^4_\Sigma \oplus \mathbb{C}^4_{\bar{\Sigma}}) \longrightarrow 0. \tag{4.12}$$

The isomorphism in the statement follows directly from the definition of cohomology and the fact that $*\not{d}$ is surjective in above sequence.

□

Lemma 4.3.6. *The following statements are true:*

(i) $\left(\mathrm{Sym}^2(\wedge^2\mathbb{R}^4)^\vee\right)^{\mathrm{Spin}(4)}$ *is a two-dimensional subspace spanned by the Euclidean metric pairing and the Hodge pairing of elements* $A, B \in \wedge^2\mathbb{R}^4$ *identified as antisymmetric rank 2 tensors,*

$$\langle A, B\rangle = \frac{1}{2}\delta^{ik}\delta^{jl}A_{ij}B_{kl}, \qquad \langle A, B\rangle_* = \frac{1}{2}A_{ij}B_{ij}\,\epsilon^{ijkl}.$$

(ii) $\left((\mathbb{R}^4 \otimes \mathbb{C}^4_\Sigma \otimes \mathbb{C}^4_\Sigma)^\vee\right)^{\mathrm{Spin}(4)}$ *is a two-dimensional vector space spanned by the Dirac term and the pseudo-Dirac term as given by Proposition A.1.6.*

Proof. To show these statements, the representation theory of $SU(2)$ as presented in Appendix A.2 will be utilized. In particular, one needs to find trivial representations of $\mathrm{Spin}(4) \cong SU(2)_L \times SU(2)_R$, which correspond to invariants.

(i) First, the irreducible representations of $\wedge^2\mathbb{R}^4$ are determined by decomposing the vector space $(\mathbb{R}^4)^{\otimes 2}$ into irreducible representations. Since $\mathbb{R}^4$ transforms under the representation $(1,1)$, one finds

$$(1,1) \otimes (1,1) = (0,0) \oplus (2,0) \oplus (0,2) \oplus (2,2).$$

With both the symmetric and antisymmetric products being invariant subspaces, they must be a sum of irreducible representations. Further, observe that $\wedge^2\mathbb{R}^4$ has dimension 6. The only sum of irreducible representations with dimension 6 in the decomposition of $(\mathbb{R}^4)^{\otimes 2}$ is $(2,0) \oplus (0,2)$, which therefore must be the representation of $\wedge^2\mathbb{R}^4$. One can now continue to decompose the vector space $(\wedge^2\mathbb{R}^4)^{\otimes 2}$ in the same fashion:

$$\begin{aligned}[(2,0) \oplus (0,2)]^{\otimes 2} &= (2,0)^{\otimes 2} \oplus (2,0) \otimes (0,2) \oplus (0,2) \otimes (2,0) \oplus (0,2)^{\otimes 2} \\ &= 2(0,0) \oplus (2,0) \oplus (0,2) \oplus (0,4) \oplus (4,0) \oplus 2(2,2)\end{aligned}$$

Thus, one concludes that there are two $\mathrm{Spin}(4)$-invariants. Therefore, to prove the claim, it is sufficient to check that the proposed elements are, in fact, linearly independent, symmetric, and $\mathrm{Spin}(4)$-invariant. Linear independence is trivial, as is symmetry for the metric pairing. For the Hodge pairing, this follows by $\epsilon^{ijkl} = \epsilon^{klij}$. To show invariance, one exploits the fact that the $(1,1)$ representation acts as the fundamental $SO(4)$ representation on $\mathbb{R}^4$. Thus, one computes for $\Lambda \in SO(4)$

$$\begin{aligned}\langle \Lambda A, \Lambda B\rangle &= \frac{1}{2}\left(\Lambda^{j_1}_{i_1}\Lambda^{j_2}_{i_2}A_{j_1j_2}\right)\left(\Lambda^{j_3}_{i_1}\Lambda^{j_4}_{i_2}B_{j_3j_4}\right) = \frac{1}{2}\delta^{j_1j_3}\delta^{j_2j_4}A_{j_1j_2}B_{j_3j_4} = \langle A, B\rangle, \\ \langle \Lambda A, \Lambda B\rangle_* &= \frac{1}{2}\left(\Lambda^{j_1}_{i_1}\Lambda^{j_2}_{i_2}A_{j_1j_2}\right)\left(\Lambda^{j_3}_{i_3}\Lambda^{j_4}_{i_4}B_{j_3j_4}\right)\left(\Lambda^{k_1}_{i_1}\Lambda^{k_2}_{i_2}\Lambda^{k_3}_{i_3}\Lambda^{k_4}_{i_4}\epsilon_{k_1k_2k_3k_4}\right) \\ &= \delta^{j_1k_1}\delta^{j_2k_2}\delta^{j_3k_3}\delta^{j_4k_4}A_{j_1j_2}B_{j_3j_4}\epsilon_{k_1k_2k_3k_4} \\ &= \langle A, B\rangle_*,\end{aligned}$$

as desired.

(ii) One can proceed as for item (i), only that now there is one vector component transforming under the $(1,1)$-representation and two spinor factors, each transforming under $(1,0) \oplus (0,1)$. The decomposition into irreducible representations yields

$$\begin{aligned}(1,1) \otimes [(1,0) \oplus (0,1)]^{\otimes 2} =&(1,1) \otimes [2(0,0) \oplus (2,0) \oplus (0,2) \oplus 2(1,1)] \\ =&2(0,0) \oplus 2(2,0) \oplus 2(0,2) \oplus 4(1,1) \oplus (3,1) \oplus (1,3) \oplus 2(2,2).\end{aligned}$$

So, this vector space also admits two invariants. That the proposed elements are, in fact, invariant is already established by Proposition A.1.6. Thus, it will not be checked again here.

□

Theorem 4.3.2. *The reduced Chevalley–Eilenberg cohomology of* $C^\bullet_{\mathrm{red}}(\widehat{\mathcal{Y}} \otimes \mathfrak{g} \ltimes \widehat{\mathcal{S}} \otimes V)$ *is*

$$H^\bullet_{\mathrm{red}}(\widehat{\mathcal{Y}}\otimes\mathfrak{g}\ltimes\widehat{\mathcal{S}}\otimes V)(k) = \begin{cases} H^\bullet_{\mathrm{red}}(\mathfrak{g}) & k=0 \\ 0 & k=-1 \\ 0 & k=-2 \\ H^\bullet(\mathfrak{g}, \mathrm{Sym}^2(H^\bullet(\widehat{\mathcal{S}}(\tfrac{3}{2}))\otimes V[1])^\vee) & k=-3 \\ H^\bullet(\mathfrak{g}, \mathrm{Sym}^2(\wedge^2\mathbb{R}^4\otimes\mathfrak{g})^\vee \oplus (H^\bullet(\widehat{\mathcal{S}}(\tfrac{3}{2}))\otimes H^\bullet(\widehat{\mathcal{S}}(\tfrac{5}{2}))\otimes \mathrm{Sym}^2 V[1])^\vee) & k=-4. \end{cases}$$

Proof. One first computes the cohomology of the non-reduced cochains. Note that the differential conserves the scaling dimension. Thus, one can decompose the cochain complex $C^\bullet(\widehat{\mathcal{Y}} \otimes \mathfrak{g} \ltimes \widehat{\mathcal{S}} \otimes V)$ by scaling dimensions,

$$C^\bullet(\widehat{\mathcal{Y}} \otimes \mathfrak{g} \ltimes \widehat{\mathcal{S}} \otimes V) = \prod_{k\leq 0} C^\bullet(\widehat{\mathcal{Y}} \otimes \mathfrak{g} \ltimes \widehat{\mathcal{S}} \otimes V)(k).$$

The direct product is over non-positive integers only since all fields scale with positive scaling dimension, i.e. for $k>0$ one has $\widehat{\mathcal{Y}}(-k)=0$, $\widehat{\mathcal{S}}(-k)=0$ and therefore $\widehat{\mathcal{Y}}^\vee(k)=\widehat{\mathcal{Y}}(-k)^\vee=0$, $\widehat{\mathcal{S}}^\vee(k)=\widehat{\mathcal{S}}(-k)^\vee=0$. By the definition of the Chevalley–Eilenberg complex, one can write

$$C^\bullet(\widehat{\mathcal{Y}} \otimes \mathfrak{g} \ltimes \widehat{\mathcal{S}} \otimes V)(k) = \mathrm{Sym}^\bullet(\widehat{\mathcal{Y}} \otimes \mathfrak{g}[1] \ltimes \widehat{\mathcal{S}} \otimes V[1])^\vee(k)$$

This complex can be filtered by total symmetric power,

$$F^pC(\widehat{\mathcal{Y}} \otimes \mathfrak{g} \ltimes \widehat{\mathcal{S}} \otimes V)(k) = \mathrm{Sym}^{\geq p}(\widehat{\mathcal{Y}} \otimes \mathfrak{g}[1] \ltimes \widehat{\mathcal{S}} \otimes V[1])^\vee(k).$$

It corresponds to a splitting of the total degree, which is the ghost number of the action functional, into the degree due to the shift, accounted for by the symmetric power, and the complementary degree due to contributions from the the grading on $\widehat{\mathcal{Y}}$ and $\widehat{\mathcal{S}}$. The filtration is finite since in any fixed scaling dimension k, $\widehat{\mathcal{Y}}(k)$ and $\widehat{\mathcal{S}}(k)$ are finite-dimensional. One can thus use this filtration to compute a spectral sequence that converges to cohomology. On the 0th page, one obtains the bigraded complex with one degree given by the filtration and a second degree by the ghost number, which is indicated by a superscript (q). Explicitly, the 0th page is

$$E_0^{p,q}(k) = \mathrm{gr}^pC(\widehat{\mathcal{Y}} \otimes \mathfrak{g}[1] \ltimes \widehat{\mathcal{S}} \otimes V[1])(k)^{(q)} = \mathrm{Sym}^p(\widehat{\mathcal{Y}} \otimes \mathfrak{g}[1] \ltimes \widehat{\mathcal{S}} \otimes V[1])^\vee(k)^{(q)}.$$

The differential $d_0 : E_0^{p,q}(k) \to E_0^{p,q+1}(k)$ is therefore the part of the differential that increases only the ghost number, i.e. the one induced by Q. To obtain the first term, one takes cohomology,

$$\begin{aligned} E_1^{p,q}(k) &= H^q(E_0^{p,\bullet}, d_0) \\ &= H^q\left(\mathrm{Sym}^p(\widehat{\mathcal{Y}} \otimes \mathfrak{g}[1] \ltimes \widehat{\mathcal{S}} \otimes V[1])^\vee(k), d_0\right) \\ &= \mathrm{Sym}^p\left(H^\bullet(\widehat{\mathcal{Y}}) \otimes \mathfrak{g}[1] \ltimes H^\bullet(\widehat{\mathcal{S}}) \otimes V[1]\right)^\vee(k)^{(q)}. \end{aligned}$$

Here, Lemma 4.3.4 was applied to go from the second to the third line. According to this expression, one can identify the first term as the Chevalley–Eilenberg cochains on the Q-cohomology of the fields. The differential on this term is a map $d_1 : E_1^{p,q}(k) \to E_1^{p+1,q}(k)$ that only increases the symmetric power. Hence, it is the one given by the Lie algebra structure induced by $\left\{I^{(0)}, \cdot\right\}$. Thus, the second page is given by

$$E_2^{p,q}(k) = H^p(H^\bullet(\widehat{\mathcal{Y}}) \otimes \mathfrak{g} \ltimes H^\bullet(\widehat{\mathcal{S}}) \otimes V)(k)^{(q)},$$

viewed as Lie algebra cohomology. This can be computed using the Hochschild–Serre spectral sequence (Theorem A.4.2) with coefficients in the trivial module $\mathbb{R}$. To apply this theorem, notice that $H^\bullet(\widehat{\mathcal{S}}) \otimes V$ is a Lie ideal. Indeed, this can be read off the differential $\left\{I^{(0)}, \cdot\right\}$ acting on cohomology, given by (4.8). This property descends to any restriction to finite dimensions. Hence, one obtains

$$E_2^{p,q}(k) = \bigoplus_{r+s=p} H^r(H^\bullet(\widehat{\mathcal{Y}}) \otimes \mathfrak{g}, H^s(H^\bullet(\widehat{\mathcal{S}}) \otimes V))(k)^{(q)}.$$

Any further differential $d_i : E_i^{p,q}(k) \to E_i^{p+i,q-i+1}(k)$ for $i > 1$ must vanish since there are no higher brackets, i.e. no maps that could increase symmetric power by more than one. Thus, the spectral sequence converges on the second page, and one finds the cohomology

$$H^\bullet(\widehat{\mathcal{Y}} \otimes \mathfrak{g} \ltimes \widehat{\mathcal{S}} \otimes V)(k) = H^\bullet(H^\bullet(\widehat{\mathcal{Y}}) \otimes \mathfrak{g}, \mathrm{Sym}^\bullet(H^\bullet(\widehat{\mathcal{S}}) \otimes V[1])^\vee)(k),$$

where it was used that, on the fermionic Lie subalgebra, the Lie bracket is trivial. Thus, the cohomology is just given by the cochains. One can now investigate what this complex is in the interesting scaling dimensions $0, -1, \ldots, -4$ that might result in marginal terms. For that, it is useful to write

$$\begin{aligned} H^\bullet(\widehat{\mathcal{Y}} \otimes \mathfrak{g} \ltimes \widehat{\mathcal{S}} \otimes V) = H^\bullet(H^\bullet(\widehat{\mathcal{Y}}) \otimes \mathfrak{g}) \oplus H^\bullet(H^\bullet(\widehat{\mathcal{Y}}) \otimes \mathfrak{g}, \mathrm{Sym}^2(H^\bullet(\widehat{\mathcal{S}}) \otimes V[1])^\vee) \\ \oplus \text{ terms of lower or half-integer scaling dimension.} \end{aligned} \tag{4.13}$$

The terms in this decomposition are analyzed separately, starting with $H^\bullet(H^\bullet(\widehat{\mathcal{Y}}) \otimes \mathfrak{g})$. Define $\widehat{\mathcal{Y}}_+ \subset \widehat{\mathcal{Y}}$ as the subcomplex with positive scaling dimension. Then, this Lie algebra cohomology is the cohomology of the cochain complex

$$\begin{aligned} \mathrm{Sym}^\bullet(H^\bullet(\widehat{\mathcal{Y}}) \otimes \mathfrak{g}[1])^\vee &= \mathrm{Sym}^\bullet(H^\bullet(\widehat{\mathcal{Y}}(0)) \otimes \mathfrak{g}[1])^\vee \otimes \mathrm{Sym}^\bullet(H^\bullet(\widehat{\mathcal{Y}}_+) \otimes \mathfrak{g}[1])^\vee \\ &= \mathrm{Sym}^\bullet \mathfrak{g}[1]^\vee \otimes \mathrm{Sym}(H^\bullet(\widehat{\mathcal{Y}}_+ \otimes \mathfrak{g}[1])^\vee, \end{aligned}$$

where in the second line, Lemma 4.3.5 (i) was used to replace $H^\bullet(\widehat{\mathcal{Y}}(0)) = \mathbb{R}$, and the trivial tensor factor $\mathbb{R}$ was omitted. As pointed out in the proof of lemma 7.0.2 in chapter 6 of [Cos11], its cohomology can be interpreted as the Lie algebra cohomology of $\mathfrak{g}$ with coefficients in the $\mathfrak{g}$-module $\mathrm{Sym}(H^\bullet(\widehat{\mathcal{Y}}_+ \otimes \mathfrak{g}[1])^\vee$. This complex can be decomposed as follows:

$$H^\bullet(\mathfrak{g}, \mathrm{Sym}(H^\bullet(\widehat{\mathcal{Y}}_+) \otimes \mathfrak{g}[1])^\vee)(\geq -4) = H^\bullet(\mathfrak{g}) \oplus H^\bullet(\mathfrak{g}, (H^\bullet(\widehat{\mathcal{Y}})_+ \otimes \mathfrak{g}[1])^\vee) \oplus H^\bullet(\mathfrak{g}, \mathrm{Sym}^2(H^\bullet(\widehat{\mathcal{Y}}_+) \otimes \mathfrak{g}[1])^\vee).$$

Here, it was used that the complex $H^\bullet(\widehat{\mathcal{Y}}_+ \otimes \mathfrak{g}[1])$ is of scaling dimension ≤ -2, since $H^\bullet(\widehat{\mathcal{Y}}(1))$ vanishes by Lemma 4.3.5 (ii). Therefore, the only term in the symmetric product $\mathrm{Sym}^2(H^\bullet(\widehat{\mathcal{Y}}_+ \otimes \mathfrak{g}[1])^\vee)$ is the one due to $H^\bullet(\widehat{\mathcal{Y}}(2))$, which can be identified with $\wedge^2\mathbb{R}^4[1]$, according to Lemma 4.3.5 (iii). Additionally, the second term in the decomposition simplifies since there is no $\mathfrak{g}$-action on $\mathcal{Y}$:

$$H^\bullet(\mathfrak{g}, H^\bullet(\widehat{\mathcal{Y}}_+) \otimes \mathfrak{g}[1]) = H^\bullet(\mathfrak{g}, \mathfrak{g}[1]^\vee) \otimes H^\bullet(\widehat{\mathcal{Y}}_+)^\vee.$$

This vanishes since, for finite-dimensional semisimple Lie algebras, one has

$$H^\bullet(\mathfrak{g}, \mathfrak{g}) = H^\bullet(\mathfrak{g}, \mathfrak{g}^\vee) = 0.$$

Putting everything together, one obtains in integer-scaling dimension ≥ -4

$$H^\bullet(\mathfrak{g}, \mathrm{Sym}(H^\bullet(\widehat{\mathcal{Y}}_+) \otimes \mathfrak{g}[1])^\vee) = H^\bullet(\mathfrak{g}) \oplus H^\bullet(\mathfrak{g}, \mathrm{Sym}^2(\wedge^2\mathbb{R}^4 \otimes \mathfrak{g}[1])^\vee),$$

where the first term is of scaling dimension 0 and the second term of scaling dimension -4. One can now turn to the second term in equation 4.13. First, one decomposes the symmetric product:

$$\mathrm{Sym}^2(H^\bullet(\widehat{\mathcal{S}})\otimes V[1])^\vee(\geq -4) = \mathrm{Sym}^2(H^\bullet(\widehat{\mathcal{S}}(\frac{3}{2}))\otimes V[1])^\vee \oplus H^\bullet(\widehat{\mathcal{S}}(\frac{3}{2}))^\vee \otimes H^\bullet(\widehat{\mathcal{S}}(\frac{5}{2}))^\vee \otimes \mathrm{Sym}^2 V[1]^\vee,$$

which yields one term of scaling dimension -3 and one term of scaling dimension -4. Therefore, the Lie algebra cohomology $H^\bullet(H^\bullet(\widehat{\mathcal{Y}})\otimes\mathfrak{g}, \mathrm{Sym}^2(H^\bullet(\widehat{\mathcal{S}})\otimes V[1])^\vee)$ reduces in scaling dimension ≥ 4 to the cohomology with coefficients in $\mathrm{Sym}^2(H^\bullet(\widehat{\mathcal{S}})\otimes V[1])^\vee$ of the Lie subalgebras $H^\bullet(\widehat{\mathcal{Y}})(0)\otimes\mathfrak{g} = \mathfrak{g}$ and $H^\bullet(\widehat{\mathcal{Y}})(\leq 1)\otimes\mathfrak{g} = \mathfrak{g}$. The expression for the respective subalgebras is obtained by application of Lemma 4.3.5(i) and (ii). This yields

$$\begin{aligned} H^\bullet(H^\bullet(\widehat{\mathcal{Y}})\otimes\mathfrak{g}, \mathrm{Sym}^2(H^\bullet(\widehat{\mathcal{S}})\otimes V[1])^\vee)(\geq -4) &= H^\bullet(\mathfrak{g}, \mathrm{Sym}^2(H^\bullet(\widehat{\mathcal{S}})(\frac{3}{2})\otimes V[1])^\vee) \\ &\oplus H^\bullet(\mathfrak{g}, H^\bullet(\widehat{\mathcal{S}}(\frac{3}{2}))^\vee\otimes H^\bullet(\widehat{\mathcal{S}}(\frac{5}{2}))^\vee\otimes \mathrm{Sym}^2 V[1]^\vee), \end{aligned}$$

with summands in scaling dimension -3 and -4, respectively. With these terms, all contributions to the non-reduced Chevalley–Eilenberg cohomology have been collected, which therefore reads

$$H^\bullet(\widehat{\mathcal{Y}}\otimes\mathfrak{g}\ltimes\widehat{\mathcal{S}}\otimes V)(k) = \begin{cases} H^\bullet(\mathfrak{g}) & k=0 \\ 0 & k=-1 \\ 0 & k=-2 \\ H^\bullet(\mathfrak{g}, \mathrm{Sym}^2(H^\bullet(\widehat{\mathcal{S}}(\frac{3}{2}))\otimes V[1])^\vee) & k=-3 \\ H^\bullet(\mathfrak{g}, \mathrm{Sym}^2(\wedge^2\mathbb{R}^4\otimes\mathfrak{g})^\vee \oplus (H^\bullet(\widehat{\mathcal{S}}(\frac{3}{2}))\otimes H^\bullet(\widehat{\mathcal{S}}(\frac{5}{2}))\otimes \mathrm{Sym}^2 V[1])^\vee) & k=-4. \end{cases}$$

The reduced Lie algebra cohomology is then obtained via the augmentation map $C^\bullet(\mathfrak{g}, M)\to M$, where M is the $\mathfrak{g}$ module. The only term affected by this is $H^\bullet(\mathfrak{g})$, which should be replaced by $H^\bullet_{\mathrm{red}}(\mathfrak{g})$. Thereby, the statement follows. □

Equipped with all of these results, one can finally prove the main theorem of this section.

Proof of Theorem 4.3.1. By Lemma 4.3.2, one has the isomorphism of complexes

$$\mathcal{O}_{\mathrm{loc}}((\mathcal{Y}\otimes\mathfrak{g}[1])\ltimes(\mathcal{S}\otimes V[1]))^{\mathbb{R}^4} \cong \mathbb{R}\overset{\mathbf{L}}{\otimes}_{\mathbb{R}[\partial]} C^\bullet_{\mathrm{red}}(\widehat{\mathcal{Y}}\otimes\mathfrak{g}\ltimes\mathcal{S}\otimes V).$$

To explicitly compute the left derived tensor product, a projective resolution of either of the factors is needed. Lemma 4.3.3 provides a projective resolution of the factor $\mathbb{R}$, given by the Koszul complex. Therefore, one obtains

$$\begin{aligned} \mathcal{K} := \mathbb{R}\overset{\mathbf{L}}{\otimes}_{\mathbb{R}[\partial]} C^\bullet_{\mathrm{red}}(\widehat{\mathcal{Y}}\otimes\mathfrak{g}\ltimes\widehat{\mathcal{S}}\otimes V) &= K(\mathbb{R}^4[\partial],\phi)\otimes_{\mathbb{R}[\partial]} C^\bullet_{\mathrm{red}}(\widehat{\mathcal{Y}}\otimes\mathfrak{g}\ltimes\widehat{\mathcal{S}}\otimes V) \\ &= \bigoplus_{i\geq 0}\wedge^i\mathbb{R}^4[\partial][i]\otimes_{\mathbb{R}[\partial]} C^\bullet_{\mathrm{red}}(\widehat{\mathcal{Y}}\otimes\mathfrak{g}\ltimes\widehat{\mathcal{S}}\otimes V) \\ &= \bigoplus_{i\geq 0}\Big(\wedge^i\mathbb{R}^4\otimes_{\mathbb{R}} C^\bullet_{\mathrm{red}}(\widehat{\mathcal{Y}}\otimes\mathfrak{g}\ltimes\widehat{\mathcal{S}}\otimes V)\Big)[i]. \end{aligned}$$

The differential is the one induced by the bijection of multidifferential operators and homomorphisms on the jet bundle on $C^\bullet_{\mathrm{red}}(\widehat{\mathcal{Y}}\otimes\mathfrak{g}\ltimes\widehat{\mathcal{S}}\otimes V)$ plus the action of the map ϕ, yielding the action of derivatives ∂_i on $C^\bullet_{\mathrm{red}}(\widehat{\mathcal{Y}}\otimes\mathfrak{g}\ltimes\widehat{\mathcal{S}}\otimes V)$ from the left in the obvious way. Note that since $\wedge^i\mathbb{R}^4$ acts as i derivatives, each reducing jet order by one, it contributes a scaling dimension of $-i$. Thus, the scaling dimension of the complex is $4 - i - k$, where k is the scaling of the cochains. To compute the cohomology, one uses the spectral sequence of the filtration starting at -4 [4]

$$F^p\mathcal{K} = \bigoplus_{i \geq p} \left(\wedge^{-i}\mathbb{R}^4 \otimes_{\mathbb{R}} C^\bullet_{\mathrm{red}}(\widehat{\mathcal{Y}} \otimes \mathfrak{g} \ltimes \widehat{\mathcal{S}} \otimes V)\right)[-i].$$

This filtration is manifestly finite, as $\wedge^i\mathbb{R}^4$ is nonzero only for integers 0 to 4. It results in a bigraded complex in the first degree given by the filtration, and in the second degree given by the cochain degree. On the 0th page, one finds

$$E_0^{p,\bullet} = \mathrm{gr}^p\mathcal{K}^q = \left(\wedge^{-p}\mathbb{R}^4 \otimes_{\mathbb{R}} C^\bullet_{\mathrm{red}}(\widehat{\mathcal{Y}} \otimes \mathfrak{g} \ltimes \widehat{\mathcal{S}} \otimes V)\right)[-p].$$

So the first differential d_0 is the one on the cochain complex, and one obtains the first term of the spectral sequence

$$E_1^{p,\bullet} = \left(\wedge^{-p}\mathbb{R}^4 \otimes_{\mathbb{R}} H^\bullet_{\mathrm{red}}(\widehat{\mathcal{Y}} \otimes \mathfrak{g} \ltimes \widehat{\mathcal{S}} \otimes V)\right)[-p].$$

Since the reduced Chevalley–Eilenberg cohomology $H^\bullet_{\mathrm{red}}(\widehat{\mathcal{Y}} \otimes \mathfrak{g} \ltimes \widehat{\mathcal{S}} \otimes V)$ in the interesting scaling dimensions was already computed in Theorem 4.3.2, one can explicitly write down the first page in the relevant scaling dimension

$$\begin{aligned} E_1^{\bullet,\bullet}(\geq 0) = &\bigoplus_{i\geq 0} \left(\wedge^i\mathbb{R}^4 \otimes H^\bullet_{\mathrm{red}}(\mathfrak{g})\right)[i] \\ &\oplus \wedge^1\mathbb{R}^4 \otimes H^\bullet(\mathfrak{g}, \mathrm{Sym}^2(H^\bullet(\widehat{\mathcal{S}}(\tfrac{3}{2}))[1] \otimes V)^\vee)[1] \oplus H^\bullet(\mathfrak{g}, \mathrm{Sym}^2(H^\bullet(\widehat{\mathcal{S}}(\tfrac{3}{2}))[1] \otimes V)^\vee) \\ &\oplus H^\bullet(\mathfrak{g}, \mathrm{Sym}^2(\wedge^2\mathbb{R}^4 \otimes \mathfrak{g})^\vee \oplus (H^\bullet(\widehat{\mathcal{S}}(\tfrac{3}{2}))[1] \otimes H^\bullet(\widehat{\mathcal{S}}(\tfrac{5}{2}))[1] \otimes \mathrm{Sym}^2 V)^\vee), \end{aligned} \tag{4.14}$$

where the identification $\wedge^0\mathbb{R}^4 \cong \mathbb{R}$ with constant densities was made, and the trivial tensor product was omitted. The further differentials of the spectral sequence, $d_k^{p,q} : E_k^{p,q} \to E_k^{p+k,q-k+1}$, are maps

$$d_k^{p,\bullet} : \wedge^{-p}\mathbb{R}^4 \otimes H^\bullet_{\mathrm{red}}(\widehat{\mathcal{Y}} \otimes \mathfrak{g} \ltimes \widehat{\mathcal{S}} \otimes V)[-p] \longrightarrow \wedge^{-(p+k)}\mathbb{R}^4 \otimes H^\bullet_{\mathrm{red}}(\widehat{\mathcal{Y}} \otimes \mathfrak{g} \ltimes \widehat{\mathcal{S}} \otimes V)[-(p+k)],$$

which will now be computed explicitly. This is done in the following way: One applies the differential due to the Koszul complex; then, one constructs a coboundary of the result in the Chevalley–Eilenberg complex. This is repeated until an element of the correct bidegree is found. The method can be represented by the following diagram:

$$\begin{array}{ccccccc} E^{p,q} & \longrightarrow & E^{p+1,q} & & & & \\ & & \downarrow & & & & \\ & & E^{p+1,q-1} & \longrightarrow & E^{p+2,q-1} & & \\ & & & & \downarrow & & \\ & & & & E^{p+2,q-2} & \longrightarrow & \dots \end{array} \tag{4.15}$$

[4] This may seem unnatural, but it is the choice that yields the correct correspondence between the cohomological degree on $\mathcal{K}$ and the total degree of the spectral sequence. One could shift the filtration index by 4, but then one would need to be more careful in identifying the degrees of the spectral sequence and the complex $\mathcal{K}$.

Since again, all differentials preserve scaling dimension, and one is only interested in marginal terms, one can restrict the computation to scaling dimension 0, where one finds

$$\begin{aligned} E_1^{\bullet,\bullet}(0) = &\left(\wedge^4\mathbb{R}^4 \otimes H^\bullet_{\mathrm{red}}(\mathfrak{g})\right)[4] \\ &\oplus \left(\wedge^1\mathbb{R}^4 \otimes H^\bullet(\mathfrak{g}, \mathrm{Sym}^2(H^\bullet(\widehat{\mathbb{S}}(\frac{3}{2}))[1] \otimes V)^\vee)\right)[1] \\ &\oplus H^\bullet(\mathfrak{g}, \mathrm{Sym}^2(\wedge^2\mathbb{R}^4 \otimes \mathfrak{g})^\vee \oplus (H^\bullet(\widehat{\mathbb{S}}(\frac{3}{2}))[1] \otimes H^\bullet(\widehat{\mathbb{S}}(\frac{5}{2}))[1] \otimes \mathrm{Sym}^2 V)^\vee). \end{aligned} \tag{4.16}$$

The first differential is

$$\begin{array}{c} \mathbb{R}^4 \otimes H^\bullet(\mathfrak{g}, \mathrm{Sym}^2(H^\bullet(\widehat{\mathbb{S}}(\frac{3}{2}))[1] \otimes V)^\vee) \\ \Big\downarrow d_1^{-1,\bullet} \\ H^\bullet(\mathfrak{g}, \mathrm{Sym}^2(\wedge^2\mathbb{R}^4 \otimes \mathfrak{g})^\vee \oplus (H^\bullet(\widehat{\mathbb{S}}(\frac{3}{2}))[1] \otimes H^\bullet(\widehat{\mathbb{S}}(\frac{5}{2}))[1] \otimes \mathrm{Sym}^2 V)^\vee). \end{array}$$

Recall that one is only interested in symmetries, deformations, and obstructions. They correspond to the cohomology in degrees -1,0, and 1. Hence, the calculations are only needed for differentials attached to these total degrees. Note that, since the Lie algebra cohomology at hand is due to a finite-dimensional semisimple Lie algebra with coefficients in a finite-dimensional $\mathfrak{g}$-module, one can apply Whitehead's lemmas (Theorem A.3.4). Therefore, the Lie-Algebra cohomologies vanish in degrees 1 and 2, and the only relevant differential that is not trivially 0 is $d_1^{-1,0}$. Moreover, recall that one is only interested in $\mathrm{Spin}(4)$-invariants. One should compute them before calculating the differential to simplify the calculations. In degree 0, the Lie algebra cohomology is just the module. Thus, one needs to calculate

$$\left(\mathrm{Sym}^2(\wedge^2\mathbb{R}^4 \otimes \mathfrak{g})^\vee \oplus (H^\bullet(\widehat{\mathbb{S}}(\frac{3}{2})) \otimes H^\bullet(\widehat{\mathbb{S}}(\frac{5}{2})) \otimes \mathrm{Sym}^2 V[1])^\vee\right)^{\mathrm{Spin}(4)}.$$

Since the $\mathrm{Spin}(4)$-action on $\mathfrak{g}$ and V is trivial, one can directly apply Lemma 4.3.6 (i) to find

$$\left(\mathrm{Sym}^2(\wedge^2\mathbb{R}^4 \otimes \mathfrak{g})^\vee\right)^{\mathrm{Spin}(4)} \cong 2\,\mathrm{Sym}^2\mathfrak{g}^\vee,$$

where one copy of $\mathrm{Sym}^2\mathfrak{g}^\vee$ corresponds to the subspace spanned by the metric pairing tensored with $\mathrm{Sym}^2\mathfrak{g}^\vee$, and the other to the subspace due to Hodge pairing. In order to find the invariants of the spinor term $(H^\bullet(\widehat{\mathbb{S}}(\frac{3}{2})) \otimes H^\bullet(\widehat{\mathbb{S}}(\frac{5}{2})) \otimes \mathrm{Sym}^2 V[1])^\vee$, first consider $\widehat{\mathbb{S}}(\frac{3}{2})$ and $\widehat{\mathbb{S}}(\frac{5}{2})$ in degree 1 before taking cohomology. Other degrees can be discarded since the cohomology groups vanish. One obtains

$$\begin{aligned} \widehat{\mathbb{S}}(\frac{3}{2})[1] \otimes \widehat{\mathbb{S}}(\frac{5}{2})[1] \cong &(\mathbb{C}^4_\Sigma \oplus \mathbb{C}^4_{\bar{\Sigma}}) \otimes \mathrm{gr}_1(\mathbb{C}^4_\Sigma \oplus \mathbb{C}^4_{\bar{\Sigma}})[x] \\ \cong &\left((\mathbb{C}^4_\Sigma \otimes \mathrm{gr}_1\mathbb{C}^4_{\bar{\Sigma}}[x]) \oplus (\mathbb{C}^4_\Sigma \otimes \mathrm{gr}_1\mathbb{C}^4_\Sigma[x]) \oplus (\mathbb{C}^4_{\bar{\Sigma}} \otimes \mathrm{gr}_1\mathbb{C}^4_\Sigma[x]) \oplus (\mathbb{C}^4_{\bar{\Sigma}} \otimes \mathrm{gr}_1\mathbb{C}^4_{\bar{\Sigma}}[x])\right). \end{aligned} \tag{4.17}$$

After taking the dual and identifying the first order jet as a contribution of a tensor factor $\mathbb{R}^4$, this implies, together with Lemma 4.3.6(ii), that each of the summands has exactly a Dirac term and a pseudo-Dirac term that is $\mathrm{Spin}(4)$-invariant. However, by Lemma 4.3.5 (v), taking cohomology results in the Dirac term vanishing as $H^\bullet(\widehat{\mathbb{S}}(\frac{5}{2}))$ is the kernel of the Dirac operator. One thus obtains

$$\left((H^\bullet(\widehat{\mathbb{S}}(\frac{3}{2})) \otimes H^\bullet(\widehat{\mathbb{S}}(\frac{5}{2})) \otimes \mathrm{Sym}^2 V[1])^\vee\right)^{\mathrm{Spin}(4)} \cong 4\,\mathrm{Sym}^2 V^\vee.$$

For $\mathbb{R} \otimes \mathrm{Sym}^2(H^\bullet(\widehat{\mathbb{S}}(\frac{3}{2})) \otimes V[1])^\vee$, first, the symmetric product is decomposed,

$$\mathrm{Sym}^2(H^\bullet(\widehat{\mathbb{S}}(\frac{3}{2})) \otimes V[1])^\vee \cong \left(\mathrm{Sym}^2 H^\bullet(\widehat{\mathbb{S}}(\frac{3}{2})[1])^\vee \otimes \mathrm{Sym}^2 V^\vee\right) \oplus \left(\wedge^2 H^\bullet(\widehat{\mathbb{S}}(\frac{3}{2})[1])^\vee \otimes \wedge^2 V^\vee\right).$$

Using that, by Lemma 4.3.5 (iii), $H^\bullet(\widehat{\mathbb{S}}(\frac{3}{2}))[1] = \mathbb{C}^4_\Sigma \oplus \mathbb{C}^4_{\bar\Sigma}$, one finds for the terms on the right hand side

$$\left(\mathrm{Sym}^2\mathbb{C}^4_{\bar\Sigma} \oplus \mathrm{Sym}^2\mathbb{C}^4_\Sigma \oplus \mathbb{C}^4_\Sigma \otimes \mathbb{C}^4_{\bar\Sigma}\right)^\vee \otimes \mathrm{Sym}^2 V^\vee \tag{4.18}$$

and

$$\left(\wedge^2\mathbb{C}^4_{\bar\Sigma} \oplus \wedge^2\mathbb{C}^4_\Sigma \oplus \mathbb{C}^4_\Sigma \otimes \mathbb{C}^4_{\bar\Sigma}\right)^\vee \otimes \wedge^2 V^\vee.$$

Tensoring with $\mathbb{R}^4$, one observes that each term of the symmetric part of the decomposition admits one invariant corresponding to the Dirac term. Additionally, $(\mathbb{C}^4_{\bar\Sigma} \otimes \mathbb{C}^4_\Sigma \otimes \mathbb{R}^4)^\vee$ allows a pseudo-Dirac term. In the antisymmetric part, only the tensor product yields the Dirac and Pseudo-Dirac term, whereas each of the exterior products permits only the Pseudo-Dirac. Thus,

$$\left(\mathbb{R} \otimes \mathrm{Sym}^2(H^\bullet(\widehat{\mathbb{S}}(\frac{3}{2})) \otimes V[1])^\vee\right)^{\mathrm{Spin}(4)} \cong 4\,\mathrm{Sym}^2 V^\vee \oplus 4 \wedge^2 V^\vee.$$

One can now compute the map $d_1^{-1,0}$. The calculation is demonstrated for only one component, e.g. the $(\mathbb{R}^4 \otimes \mathbb{C}^4_\Sigma \otimes \mathbb{C}^4_{\bar\Sigma} \otimes \mathrm{Sym}^2 V)^\vee$-term, but it works analogously for the other components. Let $s_i \in (\mathbb{C}^4_\Sigma)^\vee$ be the dual standard basis, $\langle\!\langle\cdot,\cdot\rangle\!\rangle \in \mathrm{Sym}^2 V^\vee$ some symmetric pairing, and let e^i denote the standard basis of $\mathbb{R}^4$. The space before taking $\mathrm{Spin}(4)$-invariants is then generated by elements

$$e^k \otimes \langle\!\langle s_i \otimes v, \bar s_j \otimes w\rangle\!\rangle, \quad v, w \in V.$$

Under the differential $d_1^{-1,0}$, e^k is acted on by the map $\phi : e^k \to \partial_k$ of the Koszul complex. The derivative then acts on the jet factor. Hence, one finds

$$e^k \otimes \langle\!\langle s_i \otimes v, \bar s_j \otimes w\rangle\!\rangle \longmapsto \langle\!\langle s_i \otimes v, \bar s_j(\partial_k\cdot) \otimes w\rangle\!\rangle,$$

which is an element in $(H^\bullet(\widehat{\mathbb{S}}(\frac{3}{2})) \otimes H^\bullet(\widehat{\mathbb{S}}(\frac{5}{2})) \otimes \mathrm{Sym}^2 V[1])^\vee$ since $\bar s_j(\partial_k\cdot) \in (\mathrm{gr}_1\mathbb{C}^4_{\bar\Sigma}[x])^\vee$. One can now write down the elements corresponding to the Dirac and pseudo-Dirac terms, respectively, and apply $d_1^{-1,0}$:

$$\begin{aligned} d_1^{-1,0}: \quad & (\gamma_k)^{ij} e^k \otimes \langle\!\langle s_i \otimes v, \bar s_j \otimes w\rangle\!\rangle \longmapsto (\gamma_k)^{ij}\langle\!\langle s_i \otimes v, \bar s_j(\partial_k\cdot) \otimes w\rangle\!\rangle \\ & (\gamma_5\gamma_k)^{ij} e^k \otimes \langle\!\langle s_i \otimes v, \bar s_j \otimes w\rangle\!\rangle \longmapsto (\gamma_5\gamma_k)^{ij}\langle\!\langle s_i \otimes v, \bar s_j(\partial_k\cdot) \otimes w\rangle\!\rangle. \end{aligned}$$

Here, the Dirac and pseudo-Dirac terms are identified again. The former vanishes in cohomology, as remarked before. The latter is not trivially zero. However, note that in total degree 0 all terms in the spinor sector on the first page have a symmetric pairing of V. Therefore, all terms in total degree -1 with antisymmetric pairing on V must be mapped to 0. With this, one can now form the $\mathrm{Spin}(4)$-invariant contributions to the second page of the spectral sequence. Since $d_1^{0,0} = 0$, the entire space $E_1^{0,0}$ is in the kernel. Taking $\mathrm{Spin}(4)$-invariants leaves

$$2\,\mathrm{Sym}^2\mathfrak{g}^\vee \oplus 4\,\mathrm{Sym}^2 V^\vee,$$

corresponding to metric and Hodge pairing, and four pseudo-Dirac terms. After taking the quotient by the image of $d_1^{-1,0}$, the pseudo-Dirac term corresponding to $\mathbb{C}^4_\Sigma \otimes \mathrm{gr}_1\mathbb{C}^4_{\bar\Sigma}[x]$ vanishes in cohomology. Further, one obtains all terms corresponding to the Dirac term in the kernel of $d_1^{-1,0}$, as well as the pseudo-Dirac term with antisymmetric pairing on V. In summary, one finds

$$(E_2^{0,0})^{\mathrm{Spin}(4)} = 2\,\mathrm{Sym}^2\mathfrak{g}^\vee \oplus 3\,\mathrm{Sym}^2 V^\vee,$$
$$(E_2^{-1,0})^{\mathrm{Spin}(4)} = 3\,\mathrm{Sym}^2 V^\vee \oplus 4\,\wedge^2 V^\vee.$$

For other p, q, the $\mathrm{Spin}(4)$-invariant part of $E_2^{p,q}$ stays unchanged compared to the first page term. On the second page of the spectral sequence, the differential vanishes since there is no transition of two in the exterior power in (4.16). Therefore, the third page is equal to the second one. On the third page, there is a possible map

$$d_3^{-4,\bullet} : \left(\wedge^4\mathbb{R}^4 \otimes H^\bullet_{\mathrm{red}}(\mathfrak{g})\right)[4] \longrightarrow \Big(\wedge^1\mathbb{R}^4 \otimes H^\bullet(\mathfrak{g}, \mathrm{Sym}^2(H^\bullet(\widehat{\mathbb{S}}(\frac{3}{2})) \otimes V[1])^\vee)\Big)[1].$$

However, again using Whitehead's lemmas, the only possible non-zero differential is $d_3^{-4,5}$. Thus, one attempts to calculate this map by constructing an element of $\Big(\wedge^1\mathbb{R}^4 \otimes H^\bullet(\mathfrak{g}, \mathrm{Sym}^2(H^\bullet(\widehat{\mathbb{S}}(\frac{3}{2})) \otimes V)^\vee)\Big)[1]$ from an element in $\left(\wedge^4\mathbb{R}^4 \otimes H^\bullet_{\mathrm{red}}(\mathfrak{g})\right)[4]$. This should work by the mechanism described above, applying the Koszul differential and constructing coboundaries. Note that the final element needs to contain two spinors. Looking at equation (4.5), it becomes apparent that any cochain bounded by a coboundary containing spinors must have contained spinors in the first place, since there is no term of the vector field mapping fermions to purely bosonic terms. From that one can conclude that the differential must vanish, and the fourth page of the spectral sequence is again the same as the second. The next differential is, after making the usual exclusion due to Whitehead's lemmas,

$$d_4^{-4,3} : \left(\wedge^4\mathbb{R}^4 \otimes H^3_{\mathrm{red}}(\mathfrak{g})\right)^{\mathrm{Spin}(4)}[4] \longrightarrow (E_2^{0,0})^{\mathrm{Spin}(4)}.$$

One can write a generator of $H^3_{\mathrm{red}}(\mathfrak{g})$ as $\langle [x,y], z\rangle$, where $x, y, z \in \mathfrak{g}$ are arguments and $\langle\cdot,\cdot\rangle$ is some symmetric pairing on $\mathfrak{g}$. The canonically $\mathrm{Spin}(4)$-invariant elements of $\wedge^4\mathbb{R}^4 \otimes H^3_{\mathrm{red}}(\mathfrak{g})$ are

$$\sum \epsilon_{ijkl} e_i \wedge e_j \wedge e_k \wedge e_l \otimes \langle [x,y], z\rangle.$$

Applying the first Koszul differential yields

$$\sum \epsilon_{ijkl} e_i \wedge e_j \wedge e_k e \otimes \langle [x,y], z(\partial_l \cdot)\rangle,$$

where $(\partial_l \cdot)$ is understood as an element in the dual of first order jets $\mathrm{gr}_1\bar{\Omega}^0[x]$. Using the Chevalley–Eilenberg differential on $\mathfrak{g}$, i.e. the adjoint of the Lie bracket, this can be bounded by

$$\sum \epsilon_{ijkl} e_i \wedge e_j \wedge e_k \otimes \langle x, z(\partial_l \cdot)\rangle.$$

The next term is

$$\sum \epsilon_{ijkl} e_i \wedge e_j \otimes \langle x(\partial_k \cdot), z(\partial_l \cdot)\rangle.$$

Due to the map $\mathrm{d}\colon \mathrm{gr}_1\bar{\Omega}^0[x] \to \bar{\Omega}^1$, this has the boundary

$$\sum \epsilon_{ijkl} e_i \wedge e_j \otimes \langle x\partial_k, z(\mathrm{d}x^l)^\vee\rangle.$$

Applying the next Koszul map, one finds

$$\sum \epsilon_{ijkl} \wedge e_i \otimes \langle x(\partial_k \cdot), z(\mathrm{d}x^l)^\vee(\partial_j \cdot)\rangle.$$

Here, $(\mathrm{d}x^l)^\vee(\partial_j\cdot)$ is interpreted as an element in $(\mathrm{gr}_1\bar{\Omega}^1[x])^\vee$. Using the same map as before, this is bounded by

$$\sum \epsilon_{ijkl} e_i \otimes \langle x(\mathrm{d}x^k)^\vee, z(\mathrm{d}x^l)^\vee(\partial_j\cdot)\rangle.$$

With the last Koszul map, the final element is obtained

$$\sum \epsilon_{ijkl} \langle x(\mathrm{d}x^k)^\vee(\partial_i\cdot), z(\mathrm{d}x^l)^\vee(\partial_j\cdot)\rangle.$$

This is a non-zero element in cohomology. It evidently corresponds to the $\mathrm{Spin}(4)$-invariant subspace spanned by the Hodge pairing of $\mathrm{Sym}^2(\wedge^2\mathbb{R}^4 \otimes \mathfrak{g})^\vee$. Therefore, after taking the quotient with the image of d_4, one finds

$$(E_4^{0,0})^{\mathrm{Spin}(4)} = \mathrm{Sym}^2\mathfrak{g}^\vee \oplus 3\ \mathrm{Sym}^2 V^\vee,$$

where $\mathrm{Sym}^2\mathfrak{g}^\vee$ corresponds to the subspace spanned by the metric pairing. Further, since a general element is mapped to a non-zero element, one can conclude that $d_4^{-4,3}$ is injective and thus $\wedge^4\mathbb{R}^4 \otimes H^3_{\mathrm{red}}(\mathfrak{g})[4]$ vanishes in cohomology. The spectral sequence collapses on the fourth page since there are no higher differentials. Therefore, the case of scaling dimension 0 is concluded. To summarize, on the last page of the spectral sequence, the remaining terms are

$$\begin{aligned}(E_4^{-4,5})^{\mathrm{Spin}(4)} &\cong H^5(\mathfrak{g})\\ (E_4^{0,0})^{\mathrm{Spin}(4)} &\cong \mathrm{Sym}^2\mathfrak{g}^\vee \oplus 3\ \mathrm{Sym}^2 V^\vee\\ (E_4^{-1,0})^{\mathrm{Spin}(4)} &\cong 3\ \mathrm{Sym}^2 V^\vee \oplus 4\ \wedge^2 V^\vee.\end{aligned}$$

which proves the statement.

□

Naturally, the question arises what relevant lifts in non-zero scaling dimension there are. Starting with scaling dimension 1, from equation (4.14) one directly obtains

$$E_1^{\bullet,\bullet}(1) = \left(\wedge^3\mathbb{R}^4 \otimes H^\bullet_{\mathrm{red}}(\mathfrak{g})\right)[3] \oplus \left(H^\bullet(\mathfrak{g}, \mathrm{Sym}^2(H^\bullet(\widehat{\mathbb{S}}(\frac{3}{2})) \otimes V)[1])^\vee\right).$$

The first term on the right-hand side does not admit any $\mathrm{Spin}(4)$-invariants. Due to Whitehead's Lemmas, the second term needs only to be analyzed in degree 0, i.e. the only non-trivial cohomology group is the one of deformations. This can be again be decomposed as in equation (4.18) and following, and yields two $\mathrm{Spin}(4)$-invariants for each term containing a tensor product or symmetric product. They correspond to the Dirac and pseudo-Dirac pairing introduced in Proposition A.1.6. For dimensions 2 and 3, there are no $\mathrm{Spin}(4)$-invariants. In scaling dimension 4, one finds

$$E_1^{\bullet,\bullet}(4) = H^\bullet_{\mathrm{red}}(\mathfrak{g}),$$

which vanishes in degree 0 and 1. This concludes the classification of all relevant terms of the BV cohomology in degree -1,0 and 1.

4.4 Proof of Renormalizability and Interpretation

Now, one can prove Theorem 4.0.1, stating that Yang–Mills–Dirac theory is renormalizable. It is useful to recall the statement of 4.3.1, so it is restated here:

Theorem. *For a semisimple Lie algebra $\mathfrak{g}$, the cohomology of local action functionals on $\mathcal{Y}$ in the scaling dimension 0 fulfills*

$$H^i\left(\mathcal{O}_{\mathrm{loc}}(\mathcal{Y}\otimes\mathfrak{g}[1])\ltimes(\mathcal{S}\otimes V[1])^{\mathbb{R}^4}, Q+\left\{I^{(0)},\cdot\right\}\right)^{\mathrm{Spin}(4)}\cong\begin{cases}3\ \mathrm{Sym}^2V^\vee\oplus 4\ \wedge^2 V^\vee & i=-1\\ \mathrm{Sym}^2\mathfrak{g}^\vee\oplus\ 3\ \mathrm{Sym}^2V^\vee & i=0\\ H^5(\mathfrak{g}) & i=1\end{cases}.$$

Since the obstruction class is the same as in pure Yang–Mills theory, the argument follows exactly [Cos11], relying on the "apparently fortuitous vanishing of certain Lie algebra cohomology groups".[5]

Proof. Following the conclusions of chapter 3, in order to prove renormalizability, it is sufficient to show that all obstructions vanish. In the case at hand, the obstruction class is $H^5(\mathfrak{g})$, which may not trivially 0 for arbitrary semisimple Lie algebras. In particular, this does not hold for semisimple Lie algebras with simple factors $\mathfrak{su}(n)$, $n\geq 3$. However, one can impose further symmetries on the quantization that where already part of the classical theory. In particular, the classical theory of Yang–Mills–Dirac possesses a symmetry under the group of outer automorphisms that conserve the decomposition into simple factors,

$$\mathfrak{g}=\mathfrak{g}_1\oplus\cdots\oplus\mathfrak{g}_k,$$

which will be denoted $\mathcal{G}\subset\mathrm{Out}(\mathfrak{g})$. Then, one can conclude by the following lemma:

Lemma 4.4.1 (Chapter 6 Lemma 5.1.1 [Cos11])**.** *For any semisimple Lie algebra $\mathfrak{g}$,*

$$H^5(\mathfrak{g})^{\mathcal{G}}=0.$$

□

Further, it is interesting to analyze the meaning of the deformations that were obtained. Starting with the deformation spanned by $\mathrm{Sym}^2\mathfrak{g}$, as shown in the proof of theorem 4.3.1, this corresponds to the metric pairing on $\wedge^2\mathbb{R}^4$. This, in turn, is spanned by degree 0 elements $x^i\mathrm{d}x^j-x^j\mathrm{d}x^i-\mathrm{d}x^i\wedge\mathrm{d}x^j$. Recalling that these represent the jets of fields, one recovers that these correspond to $\partial_iA_j-\partial_jA_i-B_{ij}$, or rather $\mathrm{d}A-B$, where $A\in\Omega^1(\mathbb{R}^4,\mathfrak{g}), B\in\Omega^2(\mathbb{R}^4,\mathfrak{g})$. Now, one wants to construct the functional corresponding to the metric pairing $C_{ij}\delta^{ik}\delta^{jl}D_{kl}$ for two antisymmetric tensors. Using the Hodge star operator with the explicit definition, one finds for $\alpha,\beta\in\Omega^2(\mathbb{R}^4)$ that

$$\begin{aligned}\int_{\mathbb{R}^4}\alpha\wedge *\beta=&\int_{\mathbb{R}^4}\alpha_{ij}\mathrm{d}x^i\wedge\mathrm{d}x^j\wedge\left(\frac{1}{2}\beta_{kl}\delta^{kr}\delta^{ls}\epsilon_{rsmn}\mathrm{d}x^m\wedge\mathrm{d}x^n\right)\\ =&2\int_{\mathbb{R}^4}\alpha_{ij}\delta^{ik}\delta^{jl}\beta_{kl}\omega,\end{aligned}$$

where ω denotes the volume form. This is exactly the functional corresponding to the metric contraction of two antisymmetric tensor valued functions. Therefore, the deformation is

$$\int_{\mathbb{R}^4}\mathrm{Tr}\left((\mathrm{d}A-B)\wedge *(\mathrm{d}A-B)\right)_\hbar,$$

where $\mathrm{Tr}(\cdot)_\hbar$ is a deformation of the Killing form by symmetric bilinear invariant pairings of $\mathfrak{g}$. The second kind of deformation, given by $\mathrm{Sym}^2V^\vee$, was already interpreted as a pseudo-Dirac term. Explicitly, by identifying jets with fields again, one obtains

$$\langle\bar{s}_i,(\gamma_5\gamma_k)^{ij}s_jx^k\rangle\quad\longleftrightarrow\quad\langle\bar{\psi},\gamma_5\not{d}\psi\rangle,$$

[5] This was formulated too perfectly to not quote it directly.

resulting in the action functionals

$$\int_{\mathbb{R}^4} \langle \bar{\psi}, *\gamma_5 \not{d}\psi \rangle_\hbar, \qquad \int_{\mathbb{R}^4} \langle \bar{\psi}, *\gamma_5 \not{d}\bar{\psi}^\dagger \rangle_\hbar, \qquad \int_{\mathbb{R}^4} \langle \psi^\dagger, *\gamma_5 \not{d}\psi \rangle_\hbar,$$

where $\langle \cdot, \cdot \rangle_\hbar$ denotes a deformation of the chosen pairing of V by symmetric bilinear invariant pairings.

Moving on to the symmetries, one finds three that correspond to Dirac-type actions

$$\int_{\mathbb{R}^4} \langle \psi^\dagger, *\not{d}\psi \rangle, \qquad \int_{\mathbb{R}^4} \langle \bar{\psi}, *\not{d}\bar{\psi}^\dagger \rangle, \qquad \int_{\mathbb{R}^4} \langle \bar{\psi}, *\not{d}\psi \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes a symmetric bilinear pairing on V. Furthermore, there are four symmetries with antisymmetric pairing, corresponding to one Dirac and three pseudo-Dirac actions, respectively. It is noteworthy that there are relevant deformations that give rise to the possibility of creating further marginal terms by introducing scaling dependent couplings (e.g. a "mass"). As described at the end of the previous section, one finds a Dirac pairing term and a pseudo-Dirac pairing between the spinor and adjoint spinor fields. The corresponding actions are

$$\int_{\mathbb{R}^4} \langle \bar{\psi}, \psi \rangle, \qquad \int_{\mathbb{R}^4} \langle \bar{\psi}, \gamma^5 \psi \rangle,$$

with the form on V being symmetric. Moreover, there are four other pairing terms with the form on V being symmetric, corresponding to Dirac and pseudo-Dirac pairings for both spinor-spinor and adjoint spinor-adjoint spinor. Additionally, one obtains a Dirac and pseudo-Dirac pairing between the spinor and the adjoint spinor, which has an antisymmetric pairing on V.

Conclusion and Outlook

It was proven that Yang–Mills–Dirac theory is renormalizable. Further, the Lagrangians in the deformation classes and symmetry classes were explicitly identified. To conclude, an interpretation of these terms is discussed, a comparison to other work on this topic is provided, and further questions are stated.

To put the deformations into a physical context, they will be compared with the counterterms arising in the renormalization of Yang–Mills–Dirac theory in particle physics. In standard particle physics literature, such as [PS95], Yang–Mills–Dirac theory is treated in the second order formalism of Yang–Mills theory. Using the Faddeev–Popov method for gauge fixing, the path integral is computed and the diverging Feynman diagrams are regularized. Counterterms for the action are added to remove the singularities. Suppose a coupling constant g, such that $\not{d}_A = \not{d} + ig\gamma^i(\rho_*(A_i))$ and the curvature form is set $F_A = \mathrm{d}A + \frac{1}{2}g[A, A]$. Then, this procedure results in the correction

$$\begin{aligned} S_{\text{c.t.}} = \int_{\mathbb{R}^4} \operatorname{Tr}\Big(-\frac{1}{4}\delta^1 \mathrm{d}A \wedge *\mathrm{d}A - g\delta^2 \mathrm{d}A \wedge *[A, A] - \frac{1}{4}g^2\delta^3[A, A] \wedge *[A, A]\Big) \\ + \langle\bar{\psi}, *(i\not{d} + \delta_m)\psi\rangle + g\delta^4\langle\bar{\psi}, *\gamma(A)\psi\rangle. \end{aligned} \tag{4.19}$$

The δ^i correspond to the respective counterterms. Faddeev–Popov ghost terms were omitted. One can compare this to the terms obtained using the BV approach. First, the bosonic sector may be decomposed into three parts:

$$\int_{\mathbb{R}^4} \operatorname{Tr}\Big(\mathrm{d}A - B) \wedge *(\mathrm{d}A - B)\Big)_\hbar = \int_{\mathbb{R}^4} \operatorname{Tr}\Big((\mathrm{d}A \wedge *\mathrm{d}A - 2\mathrm{d}A \wedge *B + B \wedge *B\Big)_\hbar$$

The first term can be directly identified with its counterpart in (4.19). For the next two terms, there seems to be no clear correspondence in general. However, restricting to on-shell configurations with a vanishing anti-field B^+, a relation may be inferred. One recovers $*B = -F_A$ and subsequently

$$\begin{aligned} \mathrm{d}A \wedge *B &= -\mathrm{d}A \wedge \mathrm{d}A - \frac{1}{2}g\, \mathrm{d}A \wedge [A, A] \\ &= \mathrm{d}(A \wedge \mathrm{d}A) - \frac{1}{2}g\, \mathrm{d}A \wedge [A, A], \\ *F_A \wedge F_A &= \mathrm{d}A \wedge *\mathrm{d}A + g\, \mathrm{d}A \wedge [A, A] + \frac{1}{4}g^2[A, A] \wedge *[A, A]. \end{aligned}$$

Up to a topological term, every term has been identified with a term in the bosonic sector of (4.19).
Moving on to the many possible counterterms for the actions in the fermionic sector, recall that the aim is to consider a theory of path integration. Thus, one should discard all Lagrangians containing only one kind of spinor field, since they are incompatible with Berezin fermionic path integration. This leaves a deformation by a pseudo-Dirac term. Further, by introducing a dimensional “constant” that behaves under the rescaling action with a scaling factor -1, agreeing with the usual notion of a mass operator m. Thus, the fermionic

deformations can be presented as

$$\int_{\mathbb{R}^4} \langle \bar{\psi}, *\gamma_5 \not{d}\psi \rangle_\hbar, \qquad \int_{\mathbb{R}^4} \langle \bar{\psi}, m\psi \rangle, \qquad \int_{\mathbb{R}^4} \langle \bar{\psi}, \gamma^5 m\psi \rangle.$$

One should point out that the pseudo-terms may be sorted out by imposing further symmetries. In particular, the pseudo-terms are not parity-invariant. After excluding terms incompatible with Berezin integration, the remaining BV symmetry yields

$$\int_{\mathbb{R}^4} \langle \bar{\psi}, *\not{d}\psi \rangle.$$

Remark. All terms of the particle physics Lagrangian can be related to a deformation or symmetry of the BV theory, except for the spinor coupling to the gauge field. This vanishes in cohomology. However, note that the Dirac term and pairing in (4.19) might correspond to pseudo-Dirac counterparts. This is due to the fact that the calculations were done with a Euclidean signature. The transition from Minkowski to Euclidean space might result in an extra γ^5-factor, see [vNW96].

Remark. The terms with an antisymmetric pairing on V were dismissed as unphysical in general. If the pairing is degenerate, the degeneracy can be removed by identifying the corresponding constraints. If it is non-degenerate, it is symplectic. Thus, it is only invariant under the symplectic group. In the cases relevant for gauge theory, this is too restrictive and in conflict with ρ-invariance.

There has been another study reporting that the presence of spinors does not affect deformations and symmetries, see [EWY18]. There, a different field space with only one spinor field was used. In [EWY18, Lemma 4.7] a version of Theorem 4.3.2 was stated, agreeing with the statement in this work in scaling dimension $0, -1, -2$ and -4, but it was claimed that $H^\bullet_{\text{red}}(\widehat{\mathcal{Y}} \otimes \mathfrak{g} \ltimes \widehat{\mathcal{S}} \otimes V)(-3) = 0$. However, after a fruitful exchange with the authors, an agreement has been reached that it should be non-vanishing. In particular, it takes the form presented in Theorem 4.3.2. According to the method presented, this should affect the steps in the proof of Theorem 4.3.1 in such a way that Dirac and Pseudo-Dirac symmetries are found. Further, mass and pseudo-mass terms should be admitted. On this point, consensus has not yet been established.

It is desirable to extend the result of renormalizability to manifolds with boundary. Our aim is to make use of the BV-BFV formalism that was proposed in [CM20]. In [CCFRT24], the boundary data of Yang–Mills–Dirac has already been studied. The final goal is to prove the renormalizability of a theory integrating fermions, i.e. Yang–Mills–Dirac theory on the half-space $\mathbb{H}^4_+$. One can divide this into three intermediate steps. An approach to dealing with the renormalizability of boundary systems, built upon [Cos11], [CG21], is outlined in [Rab21]. Building upon the study of heat kernel renormalization in the presence of a boundary of [Alb20], a method involving the doubling of the manifold and gluing it along the boundary is presented. This "doubling trick" allows treatment as a manifold without boundary. One needs to prove that a BV-BFV theory fulfills the formal criteria to be amenable to the doubling trick. Secondly, after this has been done, one can treat the renormalizability of pure Yang-Mills theory. For this, one considers the field space $\mathcal{E} = \mathcal{Y} \otimes \mathfrak{g}$ on the half space. If this yields positive results, the last step is to again couple the system to fermions, considering $\mathcal{Y} \otimes \mathfrak{g} \ltimes \mathcal{S} \otimes V$ on $\mathbb{H}^4_+$.

Appendix A

Appendix

A.1 The Clifford Algebra and Spin bundles

In this section, some basic results of spin geometry necessary for defining spinors on a manifold are collected, following [BLM89], [FR25].

Clifford Algebras

Definition A.1.1 (Clifford map)**.** Let (V, η) be a $\mathbb{K}$-vector space with $\mathbb{K}$ either $\mathbb{R}$ or $\mathbb{C}$, equipped with a symmetric bilinear form and A an associative algebra. A map $\phi : V \to A$ is called a *Clifford map* if

$$\phi(u)\phi(u) = -\eta(u,u)\mathbb{1}_A \quad \forall u \in V.$$

Definition A.1.2 (Clifford Algebra)**.** A *Clifford algebra* $\mathrm{Cl}(V, \eta)$ of a pair (V, η) as above is a unital associative algebra together with a Clifford map $j\colon V \to \mathrm{Cl}(V, \eta)$ such that any Clifford map of V may be factored through $\mathrm{Cl}(V, \eta)$. That is, for any Clifford map $\phi\colon V \to A$, there exists a unique algebra homomorphism Φ that makes the diagram

$$\begin{array}{ccc} V & \xrightarrow{\;j\;} & \mathrm{Cl}(V,\eta) \\ & \searrow^{\phi} & \big\downarrow \Phi \\ & & A \end{array} \tag{A.1}$$

commute. This is the universal property of the Clifford algebra.

Remark. The universal property reflects the fact that a Clifford algebra of V is the smallest algebra such that a Clifford map exists.

Proposition A.1.1. *To any pair (V, η) with V of finite dimension d, there exists a Clifford algebra $\mathrm{Cl}(V, \eta)$ that is unique up to isomorphisms.*

Proof. First, one proves uniqueness: Suppose there are two Clifford algebras $\mathrm{Cl}(V, \eta), \mathrm{Cl}(V, \eta)'$ with respective Clifford maps j, j'. By the universal property, there exist homomorphisms of algebras $\Phi\colon \mathrm{Cl}(V, \eta) \to \mathrm{Cl}(V, \eta)'$ and $\Psi\colon \mathrm{Cl}(V, \eta)' \to \mathrm{Cl}(V, \eta)$ such that $j = \Psi \circ j'$ and $j' = \Phi \circ j$. By inserting the former into the latter and vice versa, one finds that Φ, Ψ are each others inverse, hence isomorphisms.
To prove existence, one explicitly constructs the Clifford algebra. Let $\mathcal{T}(V) = \bigoplus_{i\in\mathbb{N}} V^{\otimes i}$ be the tensor algebra of V and $\mathcal{I}$ the ideal spanned by elements of the form $v \otimes v + \eta(v, v)$. One sets $\mathrm{Cl}(V, \eta) = \mathcal{T}(V)/\mathcal{I}$

and denotes by π the corresponding projection.
The injection $i\colon V \hookrightarrow \mathcal{T}(V)$ composed with the projection defines a Clifford map: Let $j \coloneqq \pi \circ i$, then

$$j(u) \otimes j(u) = \pi(i(u)) \cdot \pi(i(u)) = \pi(i(u) \otimes i(u)) = \pi(-\eta(u,u)) = \eta(u,u) \cdot \mathbb{1},$$

so j indeed a Clifford map. It remains to show the universal property. Suppose a Clifford map $\phi\colon V \to A$ and define the homomorphism $\Phi\colon \mathrm{Cl}(V,\eta) \to A, \pi(v_1 \otimes \cdots \otimes v_k) \mapsto \phi(v_1) \cdots \phi(v_k)$, which is thus uniquely determined and makes the diagram (A.1) commute. It is left to show that Φ exists. For that, notice that by the universal property of the tensor product, one has a unique homomorphism $\varphi\colon \mathcal{T} \to A$ with $\phi = \varphi \circ i$ that is explicitly given as $\varphi(v_1 \otimes \cdots \otimes v_k) = \phi(v_1) \cdots \phi(v_k)$. To show that Φ is well defined and thus exists, φ must vanish on the ideal $\mathcal{I}$. Indeed,

$$\varphi(u \otimes u + \eta(u,u) \cdot \mathbb{1}) = \varphi(i(u) \otimes i(u) + \eta(u,u) \cdot \mathbb{1}) = \phi(u)\phi(u) + \eta(u,u) = 0,$$

since ϕ is a Clifford map. □

Remark. Instead of the relation $u \otimes u = \eta(u,u)$ one could have equivalently used $v \otimes w + w \otimes v = -2\eta(v,w)$ to generate the ideal $\mathcal{I}$, as can be seen by setting $u = v + w$. As a result, for any $\alpha, \beta \in \mathrm{Cl}(V,\eta)$, one has $\{\alpha, \beta\} = -2\eta(\alpha,\beta)\mathbb{1}$, where $\{\cdot,\cdot\}$ denotes the anti-commutator.

One now has the Clifford algebra as the concrete object $\mathcal{T}(V)/\mathcal{I}$ and can therefore look for a basis. Suppose a basis $(v_1, \ldots, v_d) \subset V$, then $\mathrm{Cl}(V,\eta)$ is generated by the elements

$$\mathbb{1} \qquad \alpha_{ab} \coloneqq v_a v_b,\ (a < b) \qquad \alpha_{abc} \coloneqq v_a v_b v_c,\ (a < b < c) \qquad \ldots \qquad \alpha_* \coloneqq v_1 v_2 \cdots v_d. \tag{A.2}$$

This result is simply obtained by writing down the naive basis of $\mathcal{T}(V)$ and reducing by the relation in above remark.

Proposition A.1.2. *There exists a canonical isomorphism of vector spaces between* $\mathrm{Cl}(V,\eta)$ *and* $\wedge^\bullet V$.

Proof. Suppose again a basis $\{v_a\} \subset V$ and additionally a dual basis $\{\nu^a\} \subset V^\vee$ such that $\nu^a(v_b) = \delta^a_b$. One defines a map

$$\phi : V \to \mathrm{End}(\wedge^\bullet V), \quad \phi(u)(\alpha) = u \wedge \alpha + \iota_{u^*}\alpha, \tag{A.3}$$

where $u^* \coloneqq \eta_{ab} u^b \nu^a$ is the dual of the vector $u = u^a v_a$ and ι denotes the contraction. A straightforward calculation then shows $\phi(u)\phi(u)(\alpha) = \eta(u,u)\alpha$. Hence, ϕ is a Clifford map and factorizes as $\phi = \Phi \circ j$ with a homomorphism $\Phi : \mathrm{Cl}(V,\eta) \to \mathrm{End}(\wedge^\bullet V)$. As composition of homomorphisms the evaluation of Φ at the identity of $\wedge^\bullet V$,

$$i \coloneqq \Phi(\cdot)(\mathbb{1})\colon \mathrm{Cl}(V,\eta) \to \wedge^\bullet V$$

is a homomorphism, too. Since the contraction of $\mathbb{1}$ with any dual vector vanishes, by (A.3), one has $i(u_1 \cdots u_k) = u_1 \wedge \cdots \wedge u_k$. Therefore, the basis of $\mathrm{Cl}(V,\eta)$ is sent to a basis of $\wedge^\bullet V$ and i is an isomorphism. □

Remark. This implies that, as the exterior algebra, the Clifford algebra has dimension 2^d.

Pin and Spin group

The discussion will continue with the treatment of some substructures of the Clifford algebra. Since in the scope of this work, one is ultimately interested in finite-dimensional vector spaces with η non-degenerate, from now on this assumption is made if not explicitly stated otherwise. Due to similarity, one may also assume that the bilinear form only takes values ± 1 on the diagonal. Therefore, η will be specified only by its signature (r, s), meaning it takes the form

$$\eta = \eta_{r,s} \coloneqq \mathrm{diag}(\underbrace{1, \ldots, 1}_{r \text{ times}}, \underbrace{-1, \ldots, -1}_{s \text{ times}}).$$

It is thus sufficient to label the Clifford algebra by signature only, which is written as $\mathrm{Cl}_{r,s} = \mathrm{Cl}(\mathbb{R}^{r,s}, \eta)$. The results for $\mathrm{Cl}_{r,s}$ may then be translated by suitable isomorphisms to any real vector space of dimension $d \coloneqq r + s$ and a symmetric bilinear form with signature (r, s). Observe that there is a natural $\mathbb{Z}_2$-grading on $\mathrm{Cl}_{r,s}$:

Definition A.1.3 (Grading map)**.** Let $\mu \coloneqq -j \colon V \to \mathrm{Cl}_{r,s},\ v \mapsto -v$ the sign-inverted Clifford map of $\mathrm{Cl}_{r,s}$. It is extended to $\mathrm{Cl}_{r,s}$ as an algebra homomorphism.

Note that μ is well-defined on $\mathrm{Cl}_{r,s}$, since it becomes the identity on $\mathcal{I}$. Also, μ squares to identity, thus one can decompose $\mathrm{Cl}_{r,s}$ into the eigen spaces with eigenvalue ± 1. Using the basis (A.2), it is easily seen that the positive eigenvalue subspace is generated by $\{\alpha_{i_1 \ldots i_k} |\ k \text{ even}\}$. Conversely, the negative eigenvalue subspace is generated by $\{\alpha_{i_1 \ldots i_k} |\ k \text{ odd}\}$.

Definition A.1.4 (Odd and even Clifford algebra)**.** The *odd Clifford algebra* $\mathrm{Cl}^-_{r,s} = \mathrm{Span}\{\alpha_{i_1 \ldots i_k} |\ k \text{ odd}\}$ is the eigen space of μ with eigenvalue -1 , the *even Clifford algebra* $\mathrm{Cl}^+_{r,s} = \mathrm{Span}\{\alpha_{i_1 \ldots i_k} |\ k \text{ even}\}$ is the eigen space with eigenvalue 1.

Remark. The odd and even Clifford algebra define a $\mathbb{Z}_2$-grading on $\mathrm{Cl}_{r,s}$. It can be viewed as the remainder of the $\mathbb{Z}$-grading of the tensor algebra after taking the quotient with $\mathcal{I}$, consisting of elements of inhomogeneous degree. It is also worth mentioning that only the even part is a (sub-)algebra.

One can now investigate some multiplicative substructures of the Clifford algebra. Let the group $\mathrm{Cl}^* \subset \mathrm{Cl}$ be the subset of invertible elements with respect to multiplication.

Definition A.1.5 (Transpose)**.** Suppose $\alpha = u_1 u_2 \ldots u_k \in \mathrm{Cl}_{r,s}$. The *transposition operator* t is defined by the action $\alpha^t \coloneqq u_k u_{k-1} \ldots u_1$ and is extended to $\mathrm{Cl}_{r,s}$ by linearity.

The transpose is a useful tool to construct inverses. Take for example α as above, then $\alpha^{-1} = \prod_i \eta(u_i, u_i)^{-1} \cdot \alpha^t$. Now the question is which group substructures can be found in $\mathrm{Cl}_{r,s}$, or rather $\mathrm{Cl}^*_{r,s}$.

Definition A.1.6 (Clifford group)**.** The *Clifford group* is defined as

$$P(V) \coloneqq \{S \in \mathrm{Cl}^*_{r,s} | \forall u \in V : \mu(S) u S^{-1} \in V\}.$$

One calls the action in the condition the twisted adjoint representation, which is given by

$$\widetilde{\mathrm{Ad}} : \Gamma(V) \to \mathrm{Aut}(V), \widetilde{\mathrm{Ad}}(S)(u) = \mu(S) u S^{-1}.$$

Remark. It is easily seen that $\widetilde{\mathrm{Ad}}$ is in fact a representation, as it is a homomorphism due to the adjoint representation Ad and the grading map μ both being homomorphisms, and it is well-defined since $u \in V$ is, in particular, in $\mathrm{Cl}^*_{r,s}$.

Lemma A.1.1 (without proof)**.** *The twisted adjoint representation fulfills*

(i) $\widetilde{\mathrm{Ad}}(\mu(S) = \widetilde{\mathrm{Ad}}(S), \quad \forall S \in P(V).$

(ii) For $u \in V$*,* $\eta(u,u) = \pm 1$*, the map* $\widetilde{\mathrm{Ad}}(u)$ *is a reflection about the plane orthogonal to it.*

(iii) $\ker(\widetilde{\mathrm{Ad}} = \mathbb{R}^*)$.

Corollary A.1.1. *There is a short exact sequence*

$$1 \longrightarrow \mathbb{R}^* \longrightarrow P(V) \longrightarrow O(V) \longrightarrow 1.$$

Proof. Exactness at $\mathbb{R}^*$ is trivial. At $P(V)$, it is a consequence of Lemma A.1.1(iii). For exactness at $O(V)$, one needs to show that every $O(V)$ can be found in the image of $\widetilde{\mathrm{Ad}}$. But with (ii) of Lemma A.1.1 this follows directly by the theorem of Cartan-Dieudonne, stating that any orthogonal transformation of a finite d-dimensional vector space can be written as product of at most d reflections. □

One can now contemplate why $\widetilde{\mathrm{Ad}}$ was chosen in the definition of $P(V)$ instead of Ad. Only due to the twist one obtains orientation reversing maps, which are essential to the construction. With this foundation, main objects of this paragraph are defined:

Definition A.1.7 (Pin and Spin group)**.** Let $S(V) \subset \mathrm{Cl}_{r,s}(V)$ be the subgroup where inverses of elements $s \in S(V)$ are proportional to their transpose, $s^t \propto s^{-1}$. The *pin group* is defined as the subgroup of $S(V)$ generated by unit vectors, The *spin group* is the subgroup of $\mathrm{Pin}(V)$ given by the intersection with the even Clifford algebra,

$$\mathrm{Pin}(V) := \{u_1 \cdots u_n | u_i \in \mathrm{Cl}_{r,s}, u_i^2 = \pm 1\}.$$

$$\mathrm{Spin}(V) := \mathrm{Pin}(V) \cap \mathrm{Cl}_{r,s}^{+} = \{u_1 \cdots u_{2n} | u_i \in \mathrm{Cl}_{r,s}, u_i^2 = \pm 1\}.$$

Corollary A.1.2. *The restrictions of* $\widetilde{\mathrm{Ad}}$ *to the pin and spin groups yield short exact sequences*

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Pin}(V) \longrightarrow O(V) \longrightarrow 1 \qquad 1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(V) \rightarrow SO(V) \longrightarrow 1.$$

One concludes that there is a double cover of Lie groups from the groups $\mathrm{Pin}(V)$ and $\mathrm{Spin}(V)$ to $O(V)$ and $SO(V)$, respectively.

Classification of Clifford Algebras

Before moving on to determine all irreducible representations, it pays to establish some facts about the general structure of a Clifford algebra with signature (r,s). As it turns out, real Clifford algebras follow some decomposition rules into Clifford algebras of lower dimensions, which make it possible to obtain any of them as the tensor product of factors. The concrete form of these building blocks, as well as the decomposition rules, is given by the following two lemmas:

Lemma A.1.2. *Denote by* $\mathbb{K}(n)$ *the algebra of* $n \times n$*-matrices over the fields* $\mathbb{K} = \mathbb{R}, \mathbb{C}$ *or the quarternions* $\mathbb{H}$*, then*

(i) $\mathrm{Cl}_{1,0} \simeq \mathbb{C}$,

(ii) $\mathrm{Cl}_{0,1} \simeq \mathbb{R} \oplus \mathbb{R}$,

(iii) $\mathrm{Cl}_{2,0} \simeq \mathbb{H}$,

(iv) $\mathrm{Cl}_{1,1} \simeq \mathrm{Cl}_{0,2} \simeq \mathbb{R}(2)$.

Proof. (i)-(iii) are straightforward checks with the obvious bases. One introduces the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma_4 = \mathbb{1}_2. \tag{A.4}$$

(iv) then follows by considering the Clifford algebra over the vector space spanned by $(\sigma_1, i\sigma_2)$ for a signature (1,1) and (σ_1, σ_3) for the case of (0,2). □

Lemma A.1.3. *There exist isomorphisms*

(i) $\mathrm{Cl}_{d,0} \otimes \mathrm{Cl}_{0,2} \simeq \mathrm{Cl}_{0,d+2}$ *(ii)* $\mathrm{Cl}_{0,d} \otimes \mathrm{Cl}_{2,0} \simeq \mathrm{Cl}_{d+2,0}$ *(iii)* $\mathrm{Cl}_{r,s} \otimes \mathrm{Cl}_{1,1} \simeq \mathrm{Cl}_{r+1,s+1}$.

Proof. All items are just a matter of constructing a basis of the algebra on the right hand side from tensor products of bases on the left hand side. So suppose two such bases $\{v_a\}$, $a = 1, \ldots, d$ and $\{w_\alpha\}$, $\alpha = 1, 2$. In case (i), this yields anti-commutation relations $\{v_a, v_b\} = -2\delta_{ab}$ and $\{w_\alpha, w_\beta\} = 2\delta_{\alpha\beta}$. One defines a new linearly independent set $\{u_A\}$, $A = 1, \ldots, d+2$ as

$$u_A := \begin{cases} v_A \otimes w_1 \cdot w_2 & 1 \leq A \leq d \\ \mathbb{1} \otimes w_{A-d} & \text{otherwise} \end{cases}.$$

It is straightforward to check that this new basis fulfills the anti-commutation relations $\{u_A, u_B\} = 2\delta_{AB}$. Thus, $\{u_A\}$ generates $\mathrm{Cl}_{0,d+2}$. Case (ii) works completely analogous with opposite signs for the initial anti-commutators. For item (iii), one initially has $\{v_a, v_b\} = -2(\eta_{r,s})_{ab}$ and $\{w_\alpha, w_\beta\} = -2(\eta_{1,1})_{\alpha\beta}$. The basis for the right hand side here takes the form

$$u_A := \begin{cases} v_A \otimes w_1 \cdot w_2 & 1 \leq A \leq r \\ \mathbb{1} \otimes w_1 & A = r+1 \\ v_{A-1} \otimes w_1 \cdot w_2 & r+1 \leq A \leq r+s+1 \\ \mathbb{1} \otimes w_2 & A = r+s+2 \end{cases}.$$

Again, by a simple calculation it is confirmed that this results in $\{u_A, u_B\} = 2(\eta_{r+1,s+1})_{AB}$, as desired. □

One can also relate the even Clifford algebra to the full one, which will be particularly useful when investigating the irreducible representations of the spin group.

Proposition A.1.3. *The even Clifford subalgebra* $\mathrm{Cl}^+_{r,s}$ *is isomorphic to the full one with signature* $s, r-1$ *and* $r, s-1$, $\mathrm{Cl}^+_{r,s} \simeq \mathrm{Cl}_{s,r-1} \simeq \mathrm{Cl}_{r,s-1}$. *Further, there is an isomorphism of even Clifford algebras with opposite signatures,* $\mathrm{Cl}^+_{r,s} \simeq \mathrm{Cl}^+_{s,r}$.

Before tending to represetations of the Clifford algebra, a short discussion of complexifications of Clifford algebras. Note that, thus far, one has considered the Clifford algebra of a real vector space V. Instead, now $V_\mathbb{C} := V \otimes_\mathbb{R} \mathbb{C}$ will be investigated. One can then define the map $j_\mathbb{C} : V \otimes_\mathbb{R} \mathbb{C} \to \mathrm{Cl}(V, \eta) \otimes_\mathbb{R} \mathbb{C}$, $v \otimes z \mapsto j(u) \otimes z$. This turns out to be a Clifford map:

$$j_\mathbb{C}(v \otimes z) j_\mathbb{C}(v \otimes z) = j(v) j(v) \otimes z^2 = -\eta(v, v)\mathbb{1} \otimes z^2 = -\eta(v \otimes z, v \otimes z)\mathbb{1}.$$

Thus, one observes that the complexification of the Clifford algebra is nothing else than the Clifford algebra over the complexified vecor space,

$$\mathrm{Cl}(V, \eta)_\mathbb{C} = \mathrm{Cl}(V, \eta) \otimes_\mathbb{R} \mathbb{C} = \mathrm{Cl}(V_\mathbb{C}, \eta).$$

Using the theory developed for the real case, one can easily prove a few properties for the complex one, summarized in the following corollary. As before, up to isomorphisms, all information is contained in considering metrics $\eta_{r,s}$ and the vector space $\mathbb{C}^d$.

Corollary A.1.3. *For complex Clifford algebras, the following statements are true:*

(i) Up to isomorphisms, there exists a unique complex Clifford algebra $\mathbb{C}\mathrm{l}_d$ *of dimension* d*,*

$$\mathbb{C}\mathrm{l}_d \simeq (\mathrm{Cl}_{d,0})_{\mathbb{C}} \simeq (\mathrm{Cl}_{d-1,1})_{\mathbb{C}} \simeq \cdots \simeq (\mathrm{Cl}_{0,d})_{\mathbb{C}}.$$

(ii) For the complex even Clifford algebra, $\mathbb{C}\mathrm{l}_d^+ \simeq \mathbb{C}\mathrm{l}_{d-1}$ *holds.*

(iii) There exists an isomorphism $\mathbb{C}\mathrm{l}_{n+2} \simeq \mathbb{C}\mathrm{l}_n \otimes \mathbb{C}\mathrm{l}_2$*, in particular,* $\mathbb{C}\mathrm{l}_{2k} \simeq \mathbb{C}(2^k)$ *and* $\mathbb{C}\mathrm{l}_{2k+1} \simeq \mathbb{C}(2^k) \oplus \mathbb{C}(2^k)$.

Proof. The first item is a consequence of existence and uniqueness of the algebras $\mathrm{Cl}_{r,s}$ (Proposition A.1.1) and the fact that, over $\mathbb{C}$, any of the metrics $\eta_{r,s}$ may be diagonalized to an Euclidean one. One may define $\mathbb{C}\mathrm{l}_d := (\mathrm{Cl}_{d,0})_{\mathbb{C}}$ and find isomorphisms to other signatures by diagonalizing the metrics. (ii) is then just the application of (i) to Proposition A.1.3 after complexification of both sides. For (iii), one uses (i) on the complexification of the isomorphisms in Lemma A.1.3 to obtain $\mathbb{C}\mathrm{l}_{n+2} \simeq \mathbb{C}\mathrm{l}_n \otimes \mathbb{C}\mathrm{l}_2$. The concrete expressions follow from this recursion relation and the complexified initial conditions given by Lemma A.1.2. □

Representations of Pin and Spin Group

One can now turn to classifying the representations of the pin and spin group. These representations, or rather the objects transforming according to these representations are what is called pinor and spinor, respectively. Recall the following basic definitions from representation theory:

Definition A.1.8 (Group Representation)**.** Let G be a group and W a $\mathbb{K}$-vector space. A *group* $\mathbb{K}$*-representation* of G is a group homomorphism

$$\rho\colon G \longrightarrow GL(W).$$

It is called finite-dimensional if W is finite-dimensional.

Since both Pin and Spin groups are obtained as subgroups of Clifford algebras, one can find their representations by restriction of a representation of the Clifford algebra.

Definition A.1.9 (Algebra representation)**.** Let A be a k-algebra and W a $\mathbb{K}$-vector space such that $k \subseteq \mathbb{K}$. An *algebra* $\mathbb{K}$*-representation* of A is a homomorphism of algebras

$$\bar{\rho}\colon A \longrightarrow \mathrm{End}(W).$$

It is called finite-dimensional if W is finite-dimensional.

Remark. Given a representation of an algebra, V may be viewed as an A-module. Accordingly, if the representation is clear from context, the representation will be omitted and the action of an element of the algebra is denoted by multiplication, i.e. $\bar{\rho}(a)(v) = a \cdot v$ for $a \in A,\ v \in W$ and $\bar{\rho}$ a representation of A on V. In case of the Clifford algebra, one refers to this as Clifford multiplication.

In the following, the fields of interest are $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ and $\mathbb{K}$ will be assumed to be either one of them.

Definition A.1.10 (Reducibility)**.** A representation on W is called *irreducible* if it admits no non-trivial invariant subspaces, otherwise it is *reducible*. It is called *completely reducible* if there is a decomposition $W = \bigoplus_i W_i$ such that every W_i is invariant.

Proposition A.1.4. *Every* $\mathbb{K}$*-representation of a Clifford algebra* $\mathrm{Cl}_{r,s}$ *is completely reducible.*

Proof. It is well known from basic representation theory that a finite-dimensional, real, in particular also complex and quarternionic representation of a compact group is completely reducible. One defines the finite Clifford group as $F_{r,s} \subset \mathrm{Cl}_{r,s}$ as the group generated by an orthonormal basis of $\mathbb{R}^{r,s}$. Note that the Clifford algebra is related to the group algebra $\mathbb{R}F_{r,s}$ of $F_{r,s}$ by $\mathrm{Cl}_{r,s} \simeq \mathbb{R}F_{r,s}/\mathbb{R} \cdot \{(-1), +1\}$ if $r > 0$, otherwise they are isomorphic. Hence, a representation of the Clifford algebra can be viewed as a linear extension of a finite Clifford group-representation. This implies that the reducibility properties of the representation of $\mathrm{Cl}_{r,s}$ are the same as those for the underlying $F_{r,s}$-representation. But since $F_{r,s}$ is a finite group, its representation is completely reducible and the claim follows. □

Therefore, all $\mathrm{Cl}_{r,s}$-representations of interest may be decomposed into irreducible representations. The further discussion is aimed at at classifying them. Additionally, in the previous subsections, it was ascertained that Clifford algebras are isomorphic to $\mathbb{K}(n)$ or a direct sum of two summands of $\mathbb{K}(n)$. Thus, irreducible representations of $\mathrm{Cl}_{r,s}$ can be constructed from irreducible representations of $\mathbb{K}(n)$.

Theorem A.1.1. *(i) $\mathbb{K}(n)$ has, up to equivalence, a unique real irreducible representation given by the natural representation $\bar{\rho}$ of $\mathbb{K}(n)$ acting on $\mathbb{K}^n$.*

(ii) $\mathbb{K}(n) \oplus \mathbb{K}(n)$ has, up to equivalence, exactly two real irreducible representations $\bar{\rho}_1(\varphi_1, \varphi_2) = \bar{\rho}(\varphi_1)$ and $\bar{\rho}_2(\varphi_1, \varphi_2) = \bar{\rho}(\varphi_2)$, where $\bar{\rho}$ denotes the natural representation.

Proof. For $n = 1$, the statement is trivial. Otherwise, $\mathbb{K}(n)$ is semisimple and one may apply Theorem 4.3 and 4.4 from [Lan02]. To conclude, it suffices to decompose $\mathbb{K}(n) = \bigoplus_{i=1}^{n} L_i$, where L_i are the simple left ideals spanned by matrices non-zero only in the i-th column, which are all isomorphic to $\mathbb{K}^n$. □

Remark. To construct complex representations, recall that a complex vector space can be thought of as a real vector space W with a real linear map $J \colon W \to W$ such that $J^2 = -\mathbb{1}_W$. Thus, given such a map J, a complex representation is a real representation ρ on W, if $\rho(\varphi) \circ J = J \circ \rho(\varphi)$, corresponding to $\mathbb{C}$-linearity. A similar statement holds for quarternionic representations.

Spinors on $\mathbb{R}^4$

With the setup above one can now construct the spinor representation for the case of the Clifford algebra $\mathrm{Cl}_{4,0}$ over the Euclidean space $\mathbb{R}^4$. By Lemma A.1.3 and Lemma A.1.3, one has

$$\mathrm{Cl}_{4,0} \simeq \mathrm{Cl}_{0,2} \otimes \mathrm{Cl}_{2,0} \simeq \mathbb{R}(2) \otimes \mathbb{H} \simeq \mathbb{H}(2), \tag{A.5}$$

$$\mathrm{Cl}^{+}_{4,0} \simeq \mathrm{Cl}_{3,0} \simeq \mathrm{Cl}_{0,1} \otimes \mathrm{Cl}_{2,0} \simeq \mathbb{H} \oplus \mathbb{H}. \tag{A.6}$$

Thus, by Theorem A.1.1, the real irreducible representation of $\mathrm{Cl}_{r,s}$ is $\mathbb{H}^2$. For $\mathrm{Cl}^{+}_{r,s}$, one has two copies of $\mathbb{H}$. From this, one can also construct the complex irreducible representations. Note that $\mathbb{C}(2)$ is the complexification of $\mathbb{H}$ viewed as a real vector space. Therefore, the complex irreducible representation of $\mathrm{Cl}_{4,0}$ is the natural representation induced by $\mathrm{Cl}_{4,0} \hookrightarrow \mathbb{C}\mathrm{l}_4 \simeq \mathbb{C}(4)$ on $\mathbb{C}^4$, according to Theorem (A.1.1). In the case of the even Clifford algebra, one gets $\mathrm{Cl}^{+}_{4,0} \hookrightarrow \mathbb{C}\mathrm{l}_3 \simeq \mathbb{C}(2) \oplus \mathbb{C}(2)$, and thus two irreducible representations, given by the natural action on $\mathbb{C}^2$.

Proposition A.1.5. *Let $(e^1, \dots, e^4) \subset \mathbb{R}^4$ denote the standard basis. The representation $\gamma \colon \mathbb{C}\mathrm{l}_4 \to \mathbb{C}(4)$ defined by the set of anti-hermitian matrices*

$$\gamma^i := \gamma(e^i) = i\sigma_2 \otimes \sigma_i, \quad 1 \leq i \leq 3 \qquad \text{and} \qquad \gamma^4 := \gamma(e^4) = i\sigma_1 \otimes \sigma_4 \tag{A.7}$$

is the irreducible representation of $\mathbb{C}\mathrm{l}_4$. It restricts to $\mathbb{C}\mathrm{l}^{+}_4$ as the direct sum of the two irreducible $\mathbb{C}(2)$-representations.

Remark. The objects transforming according to this representation are called spinors. If not stated otherwise, $\psi \in \mathbb{C}^4$ will denote such an object.

To explore how the spinors transform it is useful to define

$$\gamma^5 \coloneqq \gamma^1\gamma^2\gamma^3\gamma^4 = \sigma_3 \otimes \sigma_4.$$

A simple calculation shows that γ^5 anti-commutes with the γ-representation of the odd Clifford algebra and commutes with the even Clifford algebra. Therefore, they have joint invariant subspaces on $\mathrm{Cl}^+_{4,0}$. Explicitly, one classifies two parts of the spinor ψ, transforming with eigenvalue 1 and -1, respevtively:

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \qquad \gamma^5\psi = \begin{pmatrix} \psi_L \\ -\psi_R \end{pmatrix}, \qquad \psi_{R/L} \in \mathbb{C}^2,$$

which are called right- and left-handed component, respectively. Note that there is also a decomposition of the even Clifford algebra $\mathrm{Cl}^+_{4,0} \simeq \mathbb{C}(2)_L \oplus \mathbb{C}(2)_R$ inducing the identification of the spin subgroup as $\mathrm{Spin}(4) = SU(2)_L \times SU(2)_R$. The right- and left-handed spinor components are exactly those parts of the spinor that transform under the corresponding part of the spin group: Suppose $S = (\sigma_L, \sigma_R) \in \mathrm{Spin}(4)$, then

$$S\psi_L \coloneqq \sigma_L\psi_L, \quad S\psi_R \coloneqq \sigma_R\psi_R \qquad \Longrightarrow \qquad S\psi = \begin{pmatrix} \sigma_L\psi_L \\ \sigma_R\psi_R \end{pmatrix}.$$

This corresponds to the $(1,0) \oplus (0,1)$-representation of $SU(2)$. Further, for vectors $v \in \mathbb{R}^4$, corresponding to the $(1,1)$-representation, one has the transformation rule $v \mapsto SvS^{-1}$, with the transformation law explicitly given by the identification $v = v^i e_i \equiv \hat{v} = v^i\sigma_i$ and $S\hat{v}S^{-1} = v^i\sigma_L\sigma_i\sigma_R^{-1}$.

Definition A.1.11. Let $\psi, \phi \in \mathbb{C}^4$ be spinors. One defines the Dirac adjoint of a spinor as $\bar{\psi} \coloneqq \psi^\dagger$, the Dirac pairing as

$$\mathbb{C}^4 \times \mathbb{C}^4 \longrightarrow \mathbb{C},\ (\psi, \phi) \longmapsto \bar{\psi}\phi,$$

and the Dirac operator $D_{\mathrm{Dirac}} \coloneqq \gamma^i\partial_i$.

Remark. In the Euclidean case, there is no difference between the Dirac adjoint and the Hermitian adjoint. However, the notation is kept in accordance with the general case, which incidentally also helps with the distinction from the anti-field.

Proposition A.1.6. *There are two* $\mathrm{Spin}(4)$*-invariant pairings of spinors given by the Dirac pairing, as defined before, and the pseudo-Dirac pairing, given by*

$$\mathbb{C}^4 \times \mathbb{C}^4 \longrightarrow \mathbb{C},\ (\psi, \phi) \longmapsto \bar{\phi}\gamma^5\psi.$$

Further, there are two $SO(4)$*-invariants that pair two spinors with one vector, given by the Dirac term,*

$$\mathbb{C}^4 \times \mathbb{C}^4 \longrightarrow \mathbb{C},\ (\psi, \phi) \longmapsto \bar{\psi}D_{\mathrm{Dirac}}\phi,$$

and pseudo-Dirac term,

$$\mathbb{C}^4 \times \mathbb{C}^4 \longrightarrow \mathbb{C},\ (\psi, \phi) \longmapsto \bar{\psi}\gamma^5 D_{\mathrm{Dirac}}\phi.$$

Spinors on Manifolds

To introduce spinor fields on a manifold, recall how to construct vector bundles on manifolds using the example of the tangent bundle. The discussion is restricted to an orientable n-dimensional Riemannian manifold (M, g), but may be easily generalized to the pseudo-Riemannian case.

Definition A.1.12 (Frame bundle)**.** Let the *oriented orthonormal frame bundle* $\pi\colon P_{SO}(M)\to M$ be the bundle over M with fibers $P_{SO,x}(M)$ in $x\in M$ consisting of the orientation preserving isometries $h\colon \mathbb{R}^n\to T_xM$ from $\mathbb{R}^n$ to the tangent space in x.

There is a right $SO(n)$-action on $P_{SO}(M)$ defined by $P_{SO,x}(M)\times SO(n)\ni(h,A)\to h\circ A\in P_{SO,x}(M)$. This action obviously preserves the fiber, and is additionally free and transitve due to A being an isomorphism on $\mathbb{R}^n$. Therefore, $P_{SO}(M)$ is a principal $SO(n)$-bundle over M.

Definition A.1.13 (Associated vector bundle)**.** Let G be a topological group, $\pi\colon P\to M$ a principal G-bundle and (V,ρ) a representation of G. The *associated vector bundle* is

$$P\times_\rho V\coloneqq qP\times V/\sim,$$

with $(h,v)\sim(h',v')\iff \exists A\in G\colon (h,v)=(h'\circ A^{-1},\rho(A)v')$ for $h,h'\in P$ and $v,v'\in V$.

There is a well-defined induced projection $\tilde{\pi}\colon P\times_\rho V\to M$ given by $\tilde{\pi}(h,v)_\sim\coloneqq\pi(h)$. It inherits the vector space structure from V on the fibers $\tilde{\pi}^{-1}(x)$ by setting $\alpha(h,v)_\sim+\beta(h,v')_\sim=(h,\alpha v+\beta v')_\sim$. In the case of $P=P_{SO}(M)$ with the standard representation of $SO(n)$ on $\mathbb{R}^n$, there is a canonical isomorphism $P_{SO}(M)\otimes_\rho\mathbb{R}^n\to TM,\ (h,v)_\sim\mapsto h(v)$. One now applies this principle of thinking of vector bundles with fiber V as associated vector bundles of a representation (V,ρ) to spinors. However, there is need for a little bit more care in the definition of the principal bundle of the spin group due to its relation to $SO(n)$.

Definition A.1.14 (Spin Structure)**.** The tuple $(P_{\text{Spin}}(M),\xi)$, where $P_{\text{Spin}}(M)$ is a principal $\text{Spin}(n)$-bundle over M and $\xi: P_{\text{Spin}}(M)\to P_{SO}(M)$ is a two-fold covering, is a *spin structure* if the following diagram is commutative:

$$\begin{array}{ccccc} P_{\text{Spin}}(M)\times\text{Spin}(n) & \longrightarrow & P_{\text{Spin}}(M) & & \\ \Big\downarrow{\scriptstyle \xi\times\varphi} & & \Big\downarrow{\scriptstyle \xi} & \searrow & \\ & & & & M \\ & & & \nearrow & \\ P_{SO}(M)\times SO(n) & \longrightarrow & P_{SO}(M) & & \end{array}, \tag{A.8}$$

where $\varphi:\text{Spin}(n)\to SO(n)$ is the two-fold cover induced by Corollary $A.1.2$

Definition A.1.15 (spinnable manifold)**.** An oriented Riemannian manifold is called *spin* if it admits a spin structure. In this case, one speaks of a *Riemannian spin manifold*.

Remark. The requirements for a manifold to be spin will not be covered here. One may refer to chapter 2 of [BLM89] on this matter. However, it will be used that $\mathbb{R}^4$ is a Riemannian spin manifold.

Given this definition, one can now define the spinor bundle, provided one has a representation (Σ,γ) of the spin group as the associated vector bundle

$$\Sigma M\coloneqq P_{\text{Spin}}(M)\times_\gamma\Sigma.$$

A spinor field is thus just a smooth section $\sigma\in\Gamma(\Sigma M)$. To define the Dirac operator on manifolds, one still needs a notion of covariant differentiation on ΣM. Given a connection on the tangent bundle, e.g. the Levi-Civita connection ∇^{LC}, this is easily defined by a pullback along ξ. Recall that every linear connection ∇ on TM induces a unique connection one form $\omega\in\Omega^1(P_{SO},\mathfrak{so}(n))$ determined by the parallel transport of frames according to ∇. This 1-form may be now pulled back along the double cover ξ to define a connection 1-form on the spin bundle, $\tilde{\omega} := \xi_*^{-1}(\xi^*\omega) \in \Omega^1(P_{\mathrm{Spin},\mathfrak{spin}(n)})$, where the isomorphism $\xi_*^{-1} : \mathfrak{so}(n) \to \mathfrak{spin}(n)$ was used. The resulting action of the covariant derivative on a section $\sigma \in \Gamma(\Sigma M)$ is

$$\nabla^\Sigma \sigma_a = \mathrm{d}\sigma + \frac{1}{2}\sum_{i<j} \tilde{\omega}_{ij} \otimes e_i e_j \cdot \sigma_a, \tag{A.9}$$

where $(e_1, \ldots, e_n)$ is a local orthonormal basis of the tangent bundle. The object ∇^Σ is called the spin connection. In particular, it can be easily checked by a local calculation that it fulfills the product rule

$$\nabla^\Sigma(X \cdot \psi) = (\nabla^{LC} X) \cdot \psi + X \cdot (\nabla^\Sigma \psi)$$

for sections $X \in \Gamma(M, TM)$ and $\psi \in \Gamma(M, \Sigma M)$

Definition A.1.16 (Dirac operator on manifolds)**.** Let $(e_1, \ldots, e_n) \subset T_x M$ be an orthonormal frame and define γ_i as the action by Clifford multiplication with e_i. The *Dirac operator* $D_{\mathrm{Dirac}} : \Gamma(\Sigma M) \to \Gamma(\Sigma M)$ is the first order differential operator such that at $x \in M$

$$D_{\mathrm{Dirac}}\sigma = \gamma_j \nabla^\Sigma_j \sigma = e_j \cdot \nabla^\Sigma_j \sigma$$

for and σ. Its square D^2_{Dirac} is called Dirac Laplacian.

Lemma A.1.4. *The Dirac operator is formally self-adjoint with respect to the integration pairing induced by the pointwise spinor pairing, i.e* $\int_M (D_{\mathrm{Dirac}}\phi, \psi) = \int_M (D_{\mathrm{Dirac}}\phi, D_{\mathrm{Dirac}}\psi)$ *for all compactly supported sections of the spinor bundle* ϕ, ψ.

Proof. Suppose an orthonormal frame $(e_i, \ldots, e_n)$ around $x \in M$. Define a section $X \in \Gamma_c(M, TM \otimes \mathbb{C})$ by imposing locally

$$g(X, Y) = (\phi, Y \cdot \psi), \quad \forall Y \in T_x M.$$

One can compute the divergence of this section:

$$\begin{aligned}
\operatorname{div} X &= g(\nabla^{LC}_i X, e_i) \\
&= \partial_i g(X, e_i) - g(X, \nabla^{LC}_i e_i) \\
&= \partial_i (e_i \cdot \phi, \psi) - (\nabla^{LC}_i e_i \cdot \phi, \psi) \\
&= (\nabla^\Sigma_i (e_i \cdot \phi), \psi) + (e_i \cdot \phi, \nabla^\Sigma_i) - (\nabla^{LC}_i e_i \cdot \phi, \psi) \\
&= (e_i \nabla^\Sigma \phi + (\nabla^{LC}_i e_i), \psi) - (\phi, e_i \cdot \nabla^\Sigma_i \psi) - (\nabla^{LC}_i e_i \cdot \phi, \psi) \\
&= (D_{\mathrm{Dirac}}\phi, \psi) - (\phi, D_{\mathrm{Dirac}}\psi).
\end{aligned}$$

The third line follows from the definition of X. In the fourth line, the product rule as well as the orthonormality of the frame was applied, and in the fifth line, the product rule for the spin connection, along with the anti-hermitianity of the γ-matrices, was used. The statement now follows by applying the integral over M on both sides, where the term of the divergence vanishes since X is compactly supported. □

Lemma A.1.5. *The principal symbol of the Dirac Laplacian for any* $\zeta \in T^*M$ *is*

$$\sigma_\zeta(D^2_{\mathrm{Dirac}}) = \|\zeta\|^2.$$

Proof. One can work in a local trivialization around a fixed $x \in M$ with $(e_1, \ldots, e_n) \subset T_x M$ an orthonormal basis. One has $\sigma_\zeta(D^2_{\mathrm{Dirac}}) = \sigma_\zeta(D_{\mathrm{Dirac}})^2$. From (A.9), one immediately sees that

$$\nabla^\Sigma_i = \partial_i + \text{zero-order differential}.$$

Thus, the symbol of the Dirac operator is $\sigma_\zeta(D_{\mathrm{Dirac}}) = i\gamma_i\zeta_j$, and the claim follows. □

A.2 Basic Representation Theory of $SU(2)$

This appendix recalls the basic representation theory of $SU(2)$ and follows the classical reference [FH04]. One can classify all finite-dimensional complex irreducible $SU(2)$-representations:

Theorem A.2.1 (Irreducible representations of $SU(2)$)**.** *Up to equivalence, for every $n = 0, 1 \ldots,$ there exists a unique $(n+1)$-dimensional irreducible representation ρ_n of $SU(2)$.*

Remark. One representative of the equivalence class of ρ_n is a representation τ_n acting on the space of homogeneous polynomials of degree n in two complex variables $z = (z_1, z_2)$, which acts by

$$\tau_n(A)(p(z)) = p(A^{-1}z), \quad A \in SU(2).$$

Definition A.2.1 (Outer Tensor Product)**.** Let G, H be groups and V, W finite-dimensional $\mathbb{K}$-vector spaces. The *outer tensor product* of representations $\rho\colon G \to GL(V)$, $\tau\colon H \to GL(W)$ is defined as

$$\begin{aligned} \rho \boxtimes \tau\colon G \times H &\longrightarrow GL(V \otimes W) \\ (g, h) &\longmapsto \rho(g) \otimes \tau(h). \end{aligned}$$

Proposition A.2.1. *For irreducible $\mathbb{C}$-representations of $SU(2)$ ρ, τ, the outer tensor product $\rho \boxtimes \tau$ is irreducible.*

Definition A.2.2 (Inner Tensor product)**.** Let G be a group and V, W finite-dimensional $\mathbb{K}$-vector spaces. The *inner tensor product* of representations $\rho\colon G \to GL(V)$, $\tau\colon G \to GL(W)$ is defined as

$$\begin{aligned} \rho \otimes \tau\colon G &\longrightarrow GL(V \otimes W) \\ g &\longmapsto \rho(g) \otimes \tau(g). \end{aligned}$$

In general, the inner tensor product of irreducible representations fails to be irreducible. Finding the irreducible representations of such tensor products is known as Clebsh-Gordan problem, which is well-studied for the case of $SU(2)$:

Theorem A.2.2. *Let ρ_n, ρ_m be the unique $(n+1)$-dimensional and $(m+1)$-dimensional irreducible $SU(2)$ representations. Then, the inner tensor product may be decomposed in irreducible representations by the formula*

$$\rho_n \otimes \rho_m = \rho_{n+m} \oplus \rho_{n+m-2} \oplus \cdots \oplus \rho_{|n-m|}.$$

Corollary A.2.1. *Let ρ_i denote the $(i+1)$-dimensional complex irreducible $SU(2)$-representation. The inner direct product of of $SU(2) \times SU(2)$-representations $\rho_k \boxtimes \rho_l$ and $\rho_m \boxtimes \rho_n$ has the decomposition*

$$(\rho_k \boxtimes \rho_l) \otimes (\rho_m \boxtimes \rho_n) = \bigoplus_{i,j}^{\min(k,m),\min(l,n)} \rho_{n+k-2i} \boxtimes \rho_{l+n-2i}.$$

Proposition A.2.2. $\mathrm{Spin}(4)$*-representations are outer tensor product representations of $SU(2)$.*

This proposition follows from the decomposition of $\mathrm{Spin}(4)$ into a left- and right-handed part. One can make the notation more compact, which serves the purpose of uncluttering calculations. This is done by representing an $SU(2)$-irreducible ρ_n just by the number n. Extending this to the outer tensor product, one identifies $\rho_n \boxtimes \rho_m$ with the tupel (n, m). This allows to rewrite above corolarry as

$$(k, l) \otimes (n, m) \quad \bigoplus_{i,j}^{\min(k,m),\min(l,n)} \quad (n + k - 2i, l + n - 2i).$$

A.3 Elements of Homological Algebra

Many results of homological algebra are useful for the calculations in this work. In this section, the relevant definitions and results are discussed, following [Wei94].

Definition A.3.1 (Projective Object)**.** Let $\mathcal{A}$ be an Abelian category and $P \in \mathcal{A}$ an object. P is said to be *projective* if it satisfies the universal lifting property: For every epimorphism $g\colon B \to C$ and a morphism $\gamma\colon P \to C$, there exists a morphism $\beta\colon P \to C$ such that $\gamma = g \circ \beta$. That is, for such g, γ, there is a commuting triangle

$$\begin{array}{ccc} & & P \\ & \swarrow^{\exists\beta} & \downarrow^{\gamma} \\ B & \xrightarrow{g} & C \end{array}$$

$\mathcal{A}$ is said to have enough projectives if for any $A \in \mathcal{A}$ there is a surjection $P \to A$ with P projective.

Proposition A.3.1. *An R-module is projective if and only if it is a direct summand of a free R-module.*

Definition A.3.2 (Chain Complex of Projectives)**.** Let $P \in \mathbf{Ch}$ an object in the category of chain complexes. P is said to be a *chain complex of projectives* if every P_n is projective.

Definition A.3.3 (Projective Resolution)**.** Let M be an object of $\mathcal{A}$. A *projective resolution* of M is a complex of projectives P with $P_i = 0$ for $i < 0$, together with an augmentation map $\epsilon : P_0 \to M$ such that

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \xrightarrow{\epsilon} M \longrightarrow 0$$

is an exact complex.

Definition A.3.4 (Left Derived Functor)**.** Let $F : \mathcal{A} \to \mathcal{B}$ be a right-exact functor between Abelian categories. Suppose $\mathcal{A}$ has enough projectives. Moreover, suppose $A \in \mathcal{A}$ and fix a projective resolution $P \to A$. The *i-th left derived functor* $\mathrm{L}_i F$, $i \geq 0$, is defined via

$$\mathrm{L}_i F(A) = H_i(F(P)).$$

The total left derived functor is denoted $\mathbf{L}$.

Remark. $\mathrm{L}_i F$ is well-defined, that is, for two projective resolutions P, Q of A, there is a cannonical isomorphism $H_i(P) \cong H_i(Q)$, see [Wei94, Lemma 2.4.1].

Remark. For the 0-th left derived functor one recovers $\mathrm{L}_0 F(A) \cong F(A)$. This is due to the fact that

$$F(P_1) \longrightarrow F(P_0) \longrightarrow F(A) \longrightarrow 0$$

is exact due to right-exactness of F.

Of particular interest is the left derived functor of the tensor product functor over a ring R,

$$\cdot \otimes_R \cdot \colon \mathrm{Mod}(R^{\mathrm{op}}) \times \mathrm{Mod}(R) \longrightarrow \mathbf{Ab},$$

where $\mathrm{Mod}(R^{\mathrm{op}}), \mathrm{Mod}(R)$ denotes the category of right and left R-modules, respectively, and $\mathbf{Ab}$ is the category of Abelian groups. For A, B right and left R-modules, respectively, one can show that the functors $T : \mathrm{Mod}(R^{\mathrm{op}}) \to \mathbf{Ab}, T(A) = A \otimes_R B$ and $G : \mathrm{Mod}(R) \to \mathbf{Ab}, G(B) = A \otimes_R B$ are right-exact.

Definition A.3.5 (Tor Functor)**.** The i-th Tor functor, $i \geq 0$, is defined via

$$\mathrm{Tor}_i^R(A, B) = \mathrm{L}_i T(A).$$

Theorem A.3.1. *There is an isomorphism*

$$\mathrm{L}_i(G(B)) \cong \mathrm{L}_i(F(A)) = \mathrm{Tor}_i^R(A, B).$$

The notion of left derived functors can be extended to the category of bounded form above complexes of $\mathcal{A}$, given that $\mathcal{A}$ has enough projectives.

Proposition A.3.2. *For a left R module B, the following statements are equivalent:*

(i) B *is flat.*

(ii) $\mathrm{Tor}_n^R(A, B) = 0$ *for all* $n \neq 0$ *and* $A \in \mathrm{Mod}(\mathrm{R}^{\mathrm{op}})$.

(iii) $\mathrm{Tor}_1^R(A, B) = 0$ *for all* $A \in \mathrm{Mod}(\mathrm{R}^{\mathrm{op}})$.

Definition A.3.6 (Left derived tensor product)**.** Let $A \in \mathrm{D}^b(R^{\mathrm{op}}), B \in \mathrm{D}^b(R)$ be objects in the bounded derived category of R-modules, and denote the total complex of tensor products $\mathrm{Tot}(A \otimes_R B)$. The *left derived tensor product* is defined as

$$A \overset{\mathbf{L}}{\otimes}_R B = \mathbf{L}\mathrm{Tot}(A \otimes_R \cdot)(B).$$

One can also introduce right derived functors. Observe that this is the dual construction of the left derived functors above.

Definition A.3.7 (Injective Object)**.** Let $\mathcal{A}$ be an Abelian category and $I \in \mathcal{A}$ an object. I is said to be *injective* if for every monomorphism $g\colon B \to C$ and a morphism $\gamma\colon B \to I$, there exists a morphism $\beta\colon C \to I$ such that $\gamma = \beta \circ g$. That is, for such g, γ, there is a commuting triangle

$$\begin{array}{ccc} B & \xrightarrow{g} & C \\ \downarrow{\scriptstyle \gamma} & \swarrow{\scriptstyle \exists\beta} & \\ I & & \end{array}$$

$\mathcal{A}$ is said to have enough injectives if for any $A \in \mathcal{A}$ there is an injection $A \to I$ with I injective.

Definition A.3.8 (Chain Complex of Injectives)**.** Let $I \in \mathbf{Ch}$ an object in the category of chain complexes. I is said to be a *chain complex of injectives* if every I_n is injective.

Definition A.3.9 (Injective Resolution)**.** Let M be an object of $\mathcal{A}$. An *injective resolution* of M is a complex of injectives I with $I_i = 0$ for $i < 0$, together with a map $\epsilon\colon M \to I_0$ such that

$$0 \longrightarrow M \xrightarrow{\ \epsilon\ } I_0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow \cdots$$

is an exact complex.

Definition A.3.10 (Right Derived Functor)**.** Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left-exact functor between Abelian categories. Suppose $\mathcal{A}$ has enough injectives. Moreover, suppose $A \in \mathcal{A}$ and fix an injective resolution $A \to I$. The *i-th right derived functor* $\mathrm{R}_i F$, $i \geq 0$, is defined via

$$\mathrm{R}^i F(A) = H^i(F(I)).$$

The total right derived functor is denoted $\mathbf{R}$.

Remark. For the 0-th right derived functor, one again recovers $\mathrm{R}_0 F(A) \cong F(A)$.

An important example is the right derived functor of the Hom-functor over a ring R,

$$\mathrm{Hom}_R(\cdot,\cdot)\colon \mathrm{Mod}(R)^{\mathrm{op}} \times \mathrm{Mod}(R) \longrightarrow \mathbf{Ab}.$$

For $A \in \mathrm{Mod}(R)^{\mathrm{op}}, B \in \mathrm{Mod}(R)$, one can show that the functor $T : \mathrm{Mod}(R)^{\mathrm{op}} \to \mathbf{Ab}$ and the functor $G : \mathrm{Mod}(R) \to \mathbf{Ab}$ given by $T(A) = \mathrm{Hom}_R(A, B)$ and $G(B) = \mathrm{Hom}_R(A, B)$, respectively, are left-exact.

Definition A.3.11 (Ext Functor)**.** The i-th Ext functor, $i \geq 0$, is defined via

$$\mathrm{Ext}^i_R(A, B) = \mathrm{R}^i T(A).$$

Theorem A.3.2. *There is an isomorphism*

$$\mathrm{R}_i(G(B)) \cong \mathrm{R}_i(F(A)) = \mathrm{Ext}^i_R(A, B).$$

Proposition A.3.3. *Let X be a projective object in some commutative category $\mathcal{A}$. Then, for all $n \geq 1$,*

$$\mathrm{Ext}^n(X, \cdot) = 0$$

is the zero functor.

Theorem A.3.3 (Universal Coefficient Theorem for Cohomology 3.6.5 [Wei94])**.** *Let P be a chain complex of projective R-modules such that $d(P_n)$ is also projective. Then for every n and every R-module M, there exists a spit exact sequence*

$$0 \longrightarrow \mathrm{Ext}^1_R(H_{n-1}(P), M) \longrightarrow H^n(\mathrm{Hom}_R(P, M)) \longrightarrow \mathrm{Hom}_R(H_n(P), M) \to 0.$$

Definition A.3.12 (Universal Enveloping Algebra)**.** For a Lie algebra $\mathfrak{g}$ over a field $\mathbb{K}$, the *universal enveloping algebra* $U\mathfrak{g}$ is the quotient of the tensor algebra $\mathcal{T}(\mathfrak{g})$ by the two-sided ideal $\mathcal{I}$ generated by relations

$$i([x, y]) = i(x)i(y) - i(y)i(x), \quad x, y \in \mathfrak{g},$$

where $i : \mathfrak{g} \to \mathcal{T}(\mathfrak{g})$ denotes the canonical inclusion.

Definition A.3.13 (Lie Algebra cohomology)**.** The *i-th Lie algebra cohomology with coefficients in a $\mathfrak{g}$-module M* is defined as

$$H^i(\mathfrak{g}, M) \coloneqq q\mathrm{Ext}^i_{U\mathfrak{g}}(\mathbb{K}, M).$$

Definition A.3.14 (Chevalley–Eilenberg complex)**.** The *Chevalley–Eilenberg cochain complex* is the complex $\mathrm{Hom}_{\mathbb{K}}(\wedge^\bullet \mathfrak{g}, M)$. The differential given by

$$\begin{aligned}(df)(x_1, \ldots, x_{n+1}) = &\sum_i (-1)^{i+1} x_i \cdot f(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1})\\ &\sum_{i<j} (-1)^{i+j} f([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{n+1}).\end{aligned}$$

Proposition A.3.4. *For a $\mathfrak{g}$-module M, the i-th Lie algebra cohomology with coefficients in M is the i-th Chevalley–Eilenberg cohomology.*

Theorem A.3.4 (Whitehead's Lemmas)**.** *Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra and M a finite-dimensional $\mathfrak{g}$-module. Then the following holds:*

(i) $H^1(\mathfrak{g}, M) = 0.$ *(ii)* $H^2(\mathfrak{g}, M) = 0.$

A.4 Spectral Sequences

The spectral sequence is a useful tool to compute cohomologies. This appendix gives an overview over spectral sequences due to a filtration, as it can be found e.g. in [McC00].

Definition A.4.1 (Differential Bigraded Module)**.** A *differential bigraded module over a ring* R is a collection of R-modules $\{E^{p,q}\}_{p,q\in\mathbb{Z}}$, together with an R-linear map $d : E^{\bullet,\bullet} \to E^{\bullet,\bullet}$ that is of total bidegree $(s, 1-s)$ for some integer s and fulfills $d \circ d = 0$.

Definition A.4.2 (Spectral Sequence)**.** A *spectral sequence* is a collection of differential bigraded R-modules $\{E_r^{\bullet,\bullet}, d_r\}_{r\in\mathbb{N}_+}$, such that d_r has bidegree $(r, 1-r)$ and for all p, q, r there is an isomorphism

$$E_{r+1}^{p,q} = H^{p,q}(E_r^{\bullet,\bullet}, d_r).$$

The differential graded module $E^{\bullet,\bullet}$ is called the r-th term or page of the spectral sequence. Suppose a differential graded R-module A that has a filtration $F^\bullet$.

Definition A.4.3 (Filtered Differential Graded Module)**.** A *filtered differential graded module* is a differential graded module (A, d) together with a filtration $F^\bullet$ such that the differential is compatible with the filtration, that is, $d : F^pA \to F^pA$.

Definition A.4.4 (Convergence)**.** A spectral sequence of a filtered differential graded module is *convergent* if

$$E_\infty^{p,q} \cong E_0^{p,q}(F^\bullet H^\bullet(A)).$$

Here, $E_\infty^{p,q}$ denotes the limit term of the spectral sequence. If the spectral sequence is *convergent*, one writes $E_r^{p,q} \Rightarrow H^{p+q}(A, d)$.

In the case that at some r, all further differentials d_s, $s \geq r$, vanish, the spectral sequence is said to collapse on the r-th page. In particular, one finds $E_r^{p,q} = E_\infty^{p,q}$.

Theorem A.4.1. *Given a filtered differential graded module* $(A, d, F^\bullet)$*, it determines a spectral sequence* $\{E_r^{\bullet,\bullet}, d_r\}_{r\in\mathbb{N}_+}$*, where* d_r *is of bidegree* $(r, 1-r)$ *and*

$$E_1^{p,q} \cong H^{p+q}(\mathrm{gr}^pA).$$

If the filtration is bounded, that is, $F^\bullet$ *is a finite filtration for* A^n*,* $n \in \mathbb{Z}$*, then the spectral sequence converges to* $H(A, d)$*,*

$$E_r^{p,q} \Rightarrow H^{p+q}(A, d).$$

Theorem A.4.2 (Hochschild–Serre [HS53])**.** *Let* $\mathfrak{g}$ *be a finite-dimensional Lie algebra,* M *a* $\mathfrak{g}$*-module and* $\mathfrak{h}$ *an ideal of* $\mathfrak{g}$*. Then there is a spectral sequence converging to cohomology* $H(\mathfrak{g}, M)$ *that collapses on the second page,*

$$E_2^{p,q} = H^p(\mathfrak{g}/\mathfrak{h}, H^q(\mathfrak{h}, M)) \Rightarrow H^{p,q}(\mathfrak{g}, M).$$

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