EQUIDISTRIBUTION OF CONTINUOUS LOW-LYING PAIRS OF HOROCYCLES VIA RATNER

CLAIRE BURRIN

Abstract. We record an alternative proof of a recent joint equidistribution result of Blomer and Michel, based on Ratner's topological rigidity theorem. This approach has the advantage of extending to general cofinite Fuchsian groups.

Blomer and Michel recently proved the following joint equidistribution result [BM23, Thm 1.3].

Theorem 1. *Let* $X = \mathrm{SL}_2(\mathbf{Z})\backslash\mathbf{H}$*,* $T > 1$*,* $y \in [1,2]$ *and write* $y = a/q + O(1/qQ)$ *for positive coprime integers* a, q *with* $q \leq Q := T^{0.99}$*. Let* $I \subseteq \mathbf{R}$ *be a fixed non-empty interval. Then*

$$\left\{ \left(\frac{x+i}{T}, \frac{xy+i}{T} \right) | x \in I \right\} \subseteq X \times X$$

equidistributes as $T \to \infty$ *for pairs* (y, T) *with* $q \to \infty$*.*

The argument of proof is based on the estimation of Weyl sums via a shifted convolution problem, Sato-Tate and a sieving argument, and crucially the measure classification theorem of Einsiedler–Lindenstrauss [EL19, Thm 1.4]. The purpose of this note is to record the following version of their result, using exclusively homogeneous dynamics. It would be very interesting to have such an argument for the sparse equidistribution analogue of Theorem 1, which is the main result of [BM23].

Write $a_t = \begin{pmatrix} \sqrt{t} & \\ & \sqrt{t}^{-1} \end{pmatrix}$ for $t \in \mathbf{R}_{>0}$ and $u_x = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$ for $x \in \mathbf{R}$.

Theorem 2. *Let* $G = \mathrm{SL}_2(\mathbf{R})$*,* $\Gamma = \mathrm{SL}_2(\mathbf{Z})$*,* $L = G \times G$*,* $\Lambda = \Gamma \times \Gamma$*. Let* $T > 1$*, let* y *a positive irrational number, and let* $I \subseteq \mathbf{R}$ *be a fixed non-empty interval. Then*

$$\left\{ \left(a_T^{-1} u_x, a_T^{-1} u_{xy} \right) : x \in I \right\} \subseteq \Lambda \setminus L$$

equidistributes as $T \to \infty$*.*

Proof. Let π be the composition of the smooth immersion $\iota_y : G \to L$, $\iota_y(g) = (g, a_y g a_y^{-1})$ and the canonical projection $L \to \Lambda \setminus L$. For any test function $\varphi \in C_b(\Lambda\backslash L)$, a result of Shah [Sha96] asserts that

$$\lim_{T\to\infty} \int_0^1 \varphi(\pi(a_{1/T} u_x))\, dx = \lim_{T\to\infty} \int_0^1 \varphi(a_{1/T} u_x, a_{1/T} u_{xy})\, dx = \int_{\Lambda\backslash L} \varphi,$$

where for the first identity we use the standard fact that the expansion (or contraction) of closed horocycles by the geodesic flow is algebraically expressed by $a_y u_x a_y^{-1} = u_{xy}$, and the second identity holds if $\overline{\pi(G)} = \Lambda\backslash L$ [Sha96, Theorem 1.4]. Let $\Delta(G)$ denote the diagonal embedding of G in L. By Ratner's topological rigidity theorem [Rat91], we know that $\overline{\pi(G)} = \pi(H)$ for some closed connected subgroup $\Delta(G) < H < L$ with $H \cap \Lambda$ a lattice in H. We may introduce a conjugation

Thanks are due to Alex Gorodnik for organizing the workshop *Distribution of orbits: Arithmetics and Dynamics* in the Swiss Alps, where Valentin Blomer's nice talk motivated this note. Following his presentation, Amir Mohammadi and Andreas Strömbergsson both independently also pointed out the Ratner-based argument presented in this note. The computation that follows also benefitted from separate discussions with Andreas Strömbergsson. The author acknowledges the support of the Swiss National Science Foundation, Grant No. 201557.

so that $L = G \times G$, $\Lambda = \Gamma \times a_y^{-1}\Gamma a_y$ and π corresponds to the diagonal embedding. Then since $\Delta(G)$ is maximal connected in L the only options are that $H = L$ (in which case $\overline{\pi(G)} = \Lambda\backslash L$ and equidistribution follows) or that $\Gamma_y := \Gamma \cap a_y^{-1}\Gamma a_y$ is a lattice. Since the commensurator of the modular group is $\mathrm{Comm}(\Gamma) = \mathbf{R}^\times \cdot \mathrm{GL}_2^+(\mathbf{Q})$, this is only satisfied if y is a positive rational. □

It follows from the proof that Theorem 2 holds true for general cofinite Fuchsian groups Γ, provided that the condition $y \notin \mathbf{Q}$ is replaced by $a_y \notin \mathrm{Comm}(\Gamma)$. It also holds for finitely many factors, provided $a_{y_i/y_j} \notin \mathrm{Comm}(\Gamma)$ for $i \neq j$; compare to [Str04, Theorem 4].

When $y \in \mathbf{Q}$ equidistribution fails, but the limit can be expressed explicitly as a product of interesting arithmetic and dynamical factors. Let $y = \frac{p}{q}$ with $(p,q) = 1$. Since $a_{p/q} \in \mathrm{Comm}(\Gamma)$, the group $\Gamma_{p/q} = a_{p/q}^{-1}\Gamma a_{p/q} \cap \Gamma$ is a finite index subgroup of Γ, in particular $\Gamma_{p/q}$ is a cofinite subgroup of G. Let $\varphi \in C_b(\Lambda\backslash L)$. Then $\varphi \circ \iota_{p/q} \in C_b(\Gamma_{p/q}\backslash G)$, and the equidistribution of pieces of expanding closed horocycles on $\Gamma_{p/q}\backslash G$ [Str04] yields

$$\lim_{T\to\infty} \int_I \varphi(a_T^{-1}u_x, a_T^{-1}u_{xp/q})\, dx = \frac{1}{\mu(\Gamma_{p/q}\backslash G)} \int_{\Gamma_{p/q}\backslash G} \varphi(g, a_{p/q} g a_{p/q}^{-1})\, d\mu(g),$$

where the Haar measure μ is normalized so that $\mu(\Gamma\backslash G) = 1$.

From here on we will assume that φ is right $K \times K$-invariant with $\varphi = f_1 \otimes f_2$, where f_1 and f_2 are Hecke eigenforms. By abuse of notation, we will write $f(g) = f(gi)$ as a function on G that is left Γ-invariant and right K-invariant. We next show that the RHS can be expressed in terms of arithmetic factors including the Hecke eigenvalue $\lambda_{f_2}(pq)$ of f_2 and the matrix coefficient $\langle f_1, a_{p/q}.f_2\rangle_{L^2(\Gamma\backslash G)}$.

First we recall some notation connected to the construction of Hecke operators (see, e.g., [Miy89]). For each element $\alpha \in \mathrm{Comm}(\Gamma)$ we have

$$\Gamma\alpha\Gamma = \sqcup_{m=1}^M \Gamma\alpha h_m$$

where the h_m's are the coset representatives $\Gamma = \sqcup_{m=1}^M \Gamma_\alpha h_m$, with $\Gamma_\alpha = \Gamma \cap \alpha^{-1}\Gamma\alpha$. For every function f on $\Gamma\backslash G$ we have $(f|\Gamma\alpha\Gamma)(g) = \sum_{m=1}^M f(\alpha h_m g)$. Further, if $\alpha \in \mathrm{GL}_2^+(\mathbf{Z})$ there is a uniquely determined pair of positive integers l, m such that $l \mid m$ and

$$\Gamma\alpha\Gamma = \Gamma \begin{pmatrix} l & \\ & m \end{pmatrix} \Gamma =: T(l,m)$$

and we set

$$T(n) = \sum_{\substack{lm=n\\ l|m}} T(l,m).$$

The Hecke operator arising from the double coset $\Gamma\alpha\Gamma$ is given by $T_n f = n^{-1/2}(f|T(n))$.

For $\alpha = a_{p/q}$ we find $\Gamma\alpha\Gamma = \delta_{\sqrt{pq}}^{-1} T(pq)$, where $\delta_x = \left(\begin{smallmatrix} x & \\ & x \end{smallmatrix}\right)$, and $M = \mu(\Gamma_{p/q}\backslash G) = (p+1)(q+1)$ [Miy89, Lemma 4.5.6]. Let $\mathcal{F}$ be a fundamental domain for $\Gamma\backslash G$; then $\sqcup_{m=1}^M h_m\mathcal{F}$ is a fundamental domain for $\Gamma_{p/q}\backslash G$ and

$$\begin{aligned} \frac{1}{\mu(\Gamma_{p/q}\backslash G)} \int_{\Gamma_{p/q}\backslash G} f_1(g) f_2(a_{p/q} g a_{p/q}^{-1})\, d\mu(g) &= \frac{1}{M} \sum_{m=1}^M \int_{h_m\mathcal{F}} f_1(g) f_2(a_{p/q} g a_{p/q}^{-1})\, d\mu(g) \\ &= \frac{1}{M} \int_{\mathcal{F}} f_1(g) \sum_{m=1}^M f_2(a_{p/q} h_m g a_{p/q}^{-1})\, d\mu(g) \\ &= \frac{\sqrt{pq}\,\lambda_{f_2}(pq)}{(p+1)(q+1)} \int_{\Gamma\backslash G} f_1(g) f_2(g a_{p/q}^{-1})\, d\mu(g). \end{aligned}$$

References

[BM23] V. Blomer, Ph. Michel, The unipotent mixing conjecture. JAMA **151** (2023), 25–57.
[EL19] M. Einsiedler, E. Lindenstrauss, *Joinings of higher rank torus actions on homogeneous spaces*, Publ. Math. IHES **129** (2019), 83–127.
[Miy89] T. Miyake, Modular forms. Springer, 1989.
[Rat91] M. Ratner, *Raghunathan's topological conjecture and distributions of unipotent flows*, Duke Math. J. **63** (1991), 235–280.
[Sha96] N. Shah, *Limit distribution of expanding translates of certain orbits on homogeneous spaces*, Proc. Indian Acad. Sci. (Math. Sci.) **106** (1996), 105–125.
[Str04] A. Strömbergsson, *On the uniform equidistribution of long closed horocycles*, Duke Math. J. **123** (2004), 507–547.

Institute of Mathematics, University of Zurich, Switzerland
Email address: claire.burrin@math.uzh.ch