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# An exposition of Ozawa's property (TTT)

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Master thesis
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July 2012

Department of Mathematics

ETH Zurich

Academic year 2011–2012

CONTENTS

## 1. INTRODUCTION

The purpose of this Master's thesis is to expose the results of N. Ozawa's recent article "Quasi-homomorphism rigidity with noncommutative targets" [Oza] in a most concise and accessible manner. There, the reader is introduced to property (TTT). This is a further step in the strengthening of Kazhdan's property (T), already undertaken with the introduction of property (TT) by M. Burger and N. Monod.

A locally compact group is said to admit property (T) if, for any unitarily action of the group that has "almost invariant" vectors in a Hilbert space, there exists a non-zero invariant vector. Property (T) is a rigidity property in the sense that it singles out groups which satisfy a given property (existence of a non-trivial invariant vector) under weaker requirements (existence of almost invariant vector). Alternatively, one could say that the given property is thought of as rigid because it resists small "deformations" (here, replacing invariant vectors by almost invariant vectors). Property (T) admits several equivalent characterizations and a broad range of applications in, among others, representation theory, operator algebras, ergodic theory, geometric group theory and expander graphs. A nice historical account can be found in [BHV]. As a consequence, several variations around property (T) were introduced, among which property (TT).

Property (TT) is the plain quasification of a cohomological characterization of property (T), its focus drifting from cocycles to *quasi*-cocycles. A quasi-cocycle on a locally compact group $G$ with coefficients in a unitary representation $(\pi, \mathscr{H})$ is a continuous map $b : G \to \mathscr{H}$ that satisfies the quasi-cocycle identity

$$\|b(gh) - b(g) - \pi(g)b(h)\|_{\mathscr{H}} < +\infty$$

for all $g, h \in G$. A group $G$ has property (TT) whenever every quasi-cocycle on $G$ is bounded. Prominent examples of groups with both properties (T) and (TT) are the algebraic groups $\mathrm{SL}_n(\mathbb{K})$ (for $n \geq 3$) over a local field $\mathbb{K}$ (e.g. $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}_p$) and their lattices.

By analogy, we call a locally compact group $H$ a-TT-menable if there exists a *proper* quasi-cocycle $c : H \to \mathscr{H}$, that is, a quasi-cocycle under which the preimage of every compact set is relatively compact in $H$. If the attention is restricted to cocycles, this is nothing other than the definition of a-T-menability. It makes sense to consider quasification here because the class of a-TT-menable groups encompasses not only all a-T-menable groups (e.g. amenable groups, free groups over a finite number of generators, $\mathrm{SL}_2(\mathbb{Z})$) but also all Gromov hyperbolic groups. Note that such groups are generic among finitely presented groups in the sense that, if we pick a finitely presented group at random, then it will be Gromov hyperbolic.

An attentive reading of the definitions yields the following conclusion on the compatibility of dynamics of property (TT) groups against a-TT-menability.

**Proposition 1.** *Any continuous homomorphism* $\varphi : G \to A$ *from a locally compact group G with property* (TT) *to an a-TT-menable group A has relatively compact image.*

In fact, the proper quasi-cocycle on $A$ may be pulled back to a quasi-cocycle on $G$ that is uniformly bounded by some $R \geq 0$. The relatively compact subset induced in $A$ by the ball of radius $R$ in $\mathscr{H}$ then contains $\varphi(G)$.

$$ G \xrightarrow{\ \varphi\ } \underset{\varphi(G) \subseteq c^{-1}(B_R)}{A} \xrightarrow{\ c\ } \mathscr{H} $$

Consider the following rigidity experiment in the setting of Proposition 1. Let us weaken the requirement $\varphi(gh) = \varphi(g)\varphi(h)$ to the condition that the set $\{\varphi(gh)^{-1}\varphi(g)\varphi(h) : g, h \in G\}$ be relatively compact in $A$. In other words, we want to consider more generally *quasi*-homomorphisms. However, without an actual homomorphism, it is not possible to pull back quasi-cocycles to $G$ anymore. In fact, a quasi-cocycle is defined with respect to a representation, whose composition with something else than an actual homomorphism is no representation anymore. The definition of a quasi-cocycle is thus too restrictive to allow a generalization of Proposition 1 to quasi-homomorphisms.

Here, Ozawa introduces a weaker form of quasi-cocycles. Let a wq-cocycle on a locally compact group $G$ be a continuous map $b : G \to \mathscr{H}$ that satisfies the quasi-cocycle identity with respect to a pair $(\pi, \mathscr{H})$ where $\pi : G \to \mathscr{U}(\mathscr{H})$ is now only required to be a continuous map. Consistently, a group is said to have property (TTT) whenever all wq-cocycles on it are bounded. Proposition 1 may then be refined to a quasi-homomorphism rigidity statement.

**Proposition 2.** *Any continuous quasi-homomorphism* $\varphi : G \to A$ *from a locally compact group G with property* (TTT) *to an a-TT-menable group A has relatively compact image.*

At this point, the reader is entitled to ask whether there actually are groups with property (TTT) for which this generalization would make sense. The main object of Ozawa's paper is to show that, for $n \geq 3$ and $\mathbb{K}$ any local field, the group $\mathrm{SL}_n(\mathbb{K})$ and its lattices have property (TTT).

As a final remark, we note that quasi-homomorphisms with target $\mathbb{R}$ or $\mathbb{C}$ have already been extensively studied but that, in view of Proposition 2, it might be interesting to look out also for examples of quasi-homomorphisms with non-commutative targets. The thesis is organized

as follows. Part 1 discusses in more depth the notions presented in this introduction within the framework of bounded cohomology and with a special emphasis placed on Gromov hyperbolic groups. The second part of this thesis presents the proof of property (TTT) for $SL_n(\mathbb{K})$ ($n \geq 3$) and its lattices and, finally, Part 3 regroups some secondary facts that are needed throughout the thesis.

**Part** 1. **Quasi-homomorphism rigidity**

**In Part 1, all locally compact groups will be assumed second countable and Hausdorff, all maps continuous, all Hilbert spaces separable. Note that by the definitions, a locally compact second countable Hausdorff group is $\sigma$-compact.**

**Definition 1.1.** *Let $G$ be a locally compact group. A **continuous Banach $G$-module** is a Banach space $E$ on which $G$ acts continuously by linear isometries. Equivalently, a continuous Banach $G$-module may be expressed as the data $(\pi, E)$ consisting of a Banach space $E$ and a continuous representation $\pi : G \to \mathrm{Iso}_{\mathscr{L}}(E)$, where $\mathrm{Iso}_{\mathscr{L}}(E)$ denotes the group of all linear isometries on $E$.*

Let $G$ denote a locally compact group. We collect the classes of examples of continuous Banach $G$-modules we will work with.

1.  The real line with trivial representation $(\mathrm{id}, \mathbb{R})$, i.e. $\mathrm{id}(g)x = x$ for all $g \in G$ and any $x \in \mathbb{R}$, is a continuous Banach $G$-module.

2.  More generally, any unitary representation $(\pi, \mathscr{H})$ of $G$ is a continuous Banach $G$-module. Recall that a unitary representation $(\pi, \mathscr{H})$ is the data consisting of a homomorphism $\pi : G \to \mathscr{U}(\mathscr{H})$ into the group of all unitary operators on a Hilbert space $\mathscr{H}$, i.e.

    $$\mathscr{U}(\mathscr{H}) \;=\; \{T : \mathscr{H} \to \mathscr{H} : T \text{ is an invertible bounded linear isometry}\}$$

    such that the assignment $g \mapsto \pi(g)\xi$ is continuous for every $\xi \in \mathscr{H}$.

3.  Fix $1 \le p < +\infty$. The left regular representation $(\lambda_G, L^p(G))$ of $G$, given by

    $$(\lambda_G(g)f)(x) = f(g^{-1}x) \qquad \text{for all } g \in G, f \in L^p(G),$$

    is a continuous Banach $G$-module. If $p = 2$, it is also a unitary representation. If $p = \infty$, then $(\lambda_G, L^\infty(G))$ is in general not continuous, unless $G$ is discrete.

Continuous Banach $G$-modules will play the rôle of coefficients in the ordinary resp. bounded continuous group cohomology of $G$. We recall here the non-homogeneous bar-resolution definition of bounded cohomology of a locally compact group $G$, which is more appropriate for our computations in low degrees. Note that by discarding all boundedness assumptions, this coincides with the analogous definition for ordinary cohomology.

**Definition 1.2.** *Let $(E, \pi)$ be a Banach $G$-module. The **bounded cohomology of $G$ with coefficients in $E$**, $\mathrm{H}_b^*(G, E)$, is the homology of the chain complex*

$$0 \longrightarrow C_b(G^0, E) \xrightarrow{\; d_1^\pi \;} C_b(G, E) \xrightarrow{\; d_2^\pi \;} C_b(G^2, E) \xrightarrow{\; d_3^\pi \;} C_b(G^3, E) \xrightarrow{\; d_4^\pi \;} \dots$$

*where, for each $n \geq 0$,*

$$C_{\mathrm{b}}(G^n, E) := \{f : G^n \to E : f \text{ is continuous and } \|f\|_\infty < +\infty\}$$

*with the supremum norm defined by*

$$\|f\|_\infty := \sup_{g_1,\dots,g_n \in G} \|f(g_1,\dots,g_n)\|_E$$

*and where the coboundary operator $d_n^\pi : C_{\mathrm{b}}(G^{n-1}, E) \to C_{\mathrm{b}}(G^n, E)$ is given by*

$$(d_n^\pi f)(g_1,\dots,g_n) = \pi(g_1)f(g_2,\dots,g_n) + \sum_{j=1}^{n-1}(-1)^j f(g_1,\dots,g_j g_{j+1},\dots,g_n) + (-1)^n f(g_1,\dots,g_{n-1}).$$

By convention, $C_{\mathrm{b}}(G^0, E) = E$. For every $n \geq 0$, let

$$\mathscr{Z}C_{\mathrm{b}}(G^n, E) = \ker(d_{n+1}^\pi : C_{\mathrm{b}}(G^n, E) \to C_{\mathrm{b}}(G^{n+1}, E))$$

be the space of all **bounded continuous cocycles of degree** $n$ and, since $d_{n+1}^\pi d_n^\pi = 0$,

$$\mathscr{B}C_{\mathrm{b}}(G^n, E) = \mathrm{im}(d_n^\pi : C_{\mathrm{b}}(G^{n-1}, E) \to C_{\mathrm{b}}(G^n, E))$$

is the subspace of all **bounded continuous coboundaries of degree** $n$. Then the bounded cohomology of $G$ with coefficients in $E$ of degree $n$ is given by

$$\mathrm{H}_{\mathrm{b}}^n(G, E) = \frac{\mathscr{Z}C_{\mathrm{b}}(G^n, E)}{\mathscr{B}C_{\mathrm{b}}(G^n, E)}.$$

In particular, the space of bounded 1-cocycles with respect to a unitary representation $(\pi, \mathscr{H})$ is

$$\mathscr{Z}C_{\mathrm{b}}(G, \mathscr{H}) = \{c \in C_{\mathrm{b}}(G, \mathscr{H}) : c(gh) = c(g) + \pi(g)c(h) \text{ for all } g, h \in G\}$$

while the subspace of its bounded 1-coboundaries with respect to $(\pi, \mathscr{H})$ is given by

$$\mathscr{B}C_{\mathrm{b}}(G, \mathscr{H}) = \{c \in C_{\mathrm{b}}(G, \mathscr{H}) : c(g) = \pi(g)v - v \text{ for some } v \in \mathscr{H}\}.$$

## 2. PROPERTIES (T) AND (TT)

In his seminal paper [Kaz] of 1967, D. Kazhdan introduced property (T) (also sometimes referred to as Kazhdan's property) in the language of representation theory to distinguish locally compact groups for which the trivial representation is (in some sense) isolated in the unitary dual. The "T" in fact comes from Trivial. In this short paper, Kazhdan proved that countable discrete groups with property (T) are finitely generated, and that a lattice in a locally compact group $G$ has property (T) if and only if the ambient group $G$ has property (T). The trivial class of examples of property (T) groups consists of the compact groups, and Kazhdan went on to show that simple algebraic groups of rank $\geq 2$ over a local field (e.g. $\mathrm{SL}_n(\mathbb{K})$ for $n \geq 3$) also possess the property, thus establishing the finite generation of a large class of lattices.

Property (T) admits a variety of equivalent definitions, including the one stated informally in the introduction. We will concentrate on the cohomological definition of property (T) while keeping in mind the results of Kazhdan cited above.

2.1. **Cohomological definition of property** (T). In the late 1970s, P. Delorme and A. Guichardet proved that property (T) is equivalent to a fixed point property of J.P. Serre (property (FH)) that requires every continuous group action by affine isometries on a real Hilbert space to have a fixed point, or, equivalently, to have bounded orbits (Prop. 2.2.9, Thm. 2.12.4, [BHV]).

A (real resp. complex) **affine Hilbert space** is a non-empty set $\mathscr{H}$ on which there is a simply transitive action of an additive abelian group $(\mathscr{H}^0, +)$, where $\mathscr{H}^0$ is a (real resp. complex) Hilbert space. The group action in question is denoted by $\mathscr{H}^0 \times \mathscr{H} \to \mathscr{H}$, $(\xi, v) \mapsto \xi + v$ and $(\mathscr{H}^0, +)$ is called the group of translations. Since for any two points $v, w \in \mathscr{H}$, there is exactly one vector $w - v$ in $\mathscr{H}^0$, an affine Hilbert space may be endowed with a metric $\mathrm{d}(v, w) = \|w - v\|_{\mathscr{H}^0}$.

In particular, note that by letting $(\mathscr{H}^0, +)$ act on itself by translation, we recover a structure of affine Hilbert space on $\mathscr{H}^0$. Conversely, if we fix an origin point $0 \in \mathscr{H}$, the simply transitive action of $\mathscr{H}^0$ on $\mathscr{H}$ defines an isomorphism $\mathscr{H}^0 \cong \mathscr{H}$ so that $\mathscr{H}$ is endowed with a linear structure.

If $\mathscr{H}$ is a real affine Hilbert space, then, by the Mazur-Ulam Theorem[1], all isometries on $\mathscr{H}$ are affine. We denote by $\mathrm{Isom}(\mathscr{H})$ the group of all isometries on $\mathscr{H}$. The map $p$ from $\mathrm{Isom}(\mathscr{H})$ into

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[1] The Mazur-Ulam Theorem states that a surjective isometry between normed vector spaces over $\mathbb{R}$ is an affine map. A proof can be found in Ch. XI, §3, Théorème 2 [Ban]. Actually, the statement here can also be proven directly. In fact, let $f : \mathscr{H} \to \mathscr{H}$ be an isometry, fix an origin $0 \in \mathscr{H}$ and set $g(\cdot) := f(\cdot) - f(0)$. Then $g$ is an isometry with $g(0) = 0$ and

$$\langle g(v), g(w) \rangle = \frac{1}{2} \left[ \|g(v)\|^2 + \|g(w)\|^2 - \|g(v) - g(w)\|^2 \right] = \langle v, w \rangle$$

the group $\mathscr{O}(\mathscr{H}^0)$ of all invertible linear isometries on $\mathscr{H}^0$ given by

$$p(\phi)\xi \; = \; \phi(v+\xi) - \phi(v) \qquad (\xi \in \mathscr{H}^0),$$

where $v \in \mathscr{H}$ is an arbitrary point, defines a surjective homomorphism of kernel $(\mathscr{H}^0, +)$. If we fix an origin $0 \in \mathscr{H}$, then the restriction of the canonical projection $\mathrm{Isom}(\mathscr{H}) \to \mathrm{Isom}(\mathscr{H})/\mathscr{H}^0$ to $\mathscr{O}(\mathscr{H}^0)$ is an isomorphism so that we have a semi-direct product decomposition

$$\mathrm{Isom}(\mathscr{H}) = \mathscr{O}(\mathscr{H}^0) \ltimes \mathscr{H}^0.$$

Let $G$ be a locally compact group and let $\mathscr{H}$ be a real affine Hilbert space. A **continuous affine isometric** $G$**-action on** $\mathscr{H}$ is a homomorphism $T : G \to \mathrm{Isom}(\mathscr{H})$ such that $g \mapsto T(g)v$ is continuous for every $v \in \mathscr{H}$. In view of the last observation, any continuous affine isometric $G$-action $g \mapsto T(g)$ on $\mathscr{H}$ has a unique decomposition

$$T(g)v \; = \; \pi(g)v + c(g), \quad \text{where} \ \ \pi(g) \in \mathscr{O}(\mathscr{H}^0) \text{ and } c(g) \in \mathscr{H}^0,$$

for all $g \in G$ and all $v \in \mathscr{H}$. We call $\pi$ the linear part of $T$, and $c$ the translation part of $T$. Because $T$ is a homomorphism, one can easily verify that $(\pi, \mathscr{H}^0)$ defines an orthogonal representation of $G$ and $c$ a continuous 1-cocycle on $G$ with respect to $(\pi, \mathscr{H}^0)$. Note that $c$ is a bounded cocycle if and only if $T$ has bounded orbits.

Consider the case where $c$ is a coboundary, i.e. $c(g) = \pi(g)\xi - \xi$ for some vector $\xi \in \mathscr{H}^0$. Then the associated affine isometric action $T$ is conjugated to $\pi$ by translation of the vector $\xi$, i.e.

$$\xi + T(g)v \; = \; \pi(g)(\xi + v) \qquad \qquad \text{for all } g \in G, \ v \in \mathscr{H}.$$

Observe that here $-\xi$ is a fixed point of $T$. Conversely, if $-\xi$ would be a fixed point of a continuous affine isometric action $T = \pi + c$, then

$$c(g) \; = \; T(g)(-\xi) - \pi(g)(-\xi) \; = \; \pi(g)\xi - \xi$$

for all $g \in G$. This allows to express property (T) as a cohomological vanishing property. We summarize these equivalent cohomological characterizations in the following definition of property (T). We come back to using $\mathscr{H}$ to denote a Hilbert space.

**Definition 2.1.** *A locally compact group $G$ has **property** (T) if it satisfies any one of the following equivalent properties.*

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for any $v, w \in \mathscr{H}$. Pick $\{e_i\}_{i\in I}$ an orthonormal basis of $\mathscr{H}$, then each point $g(v)$ has a unique representation with respect to the orthonormal basis $\{g(e_i)\}_{i\in I}$ given by $g(v) = \sum \langle g(v), e_i \rangle g(e_i) = \sum \langle v, e_i \rangle g(e_i)$. It follows that $g$ is linear and $f$ affine. If $\mathscr{H} = \mathbb{C}$, a counter-example to the statement is given by $z \mapsto \bar{z}$. A generalization of the Mazur-Ulam Theorem to $\delta$-isometries between Banach spaces (i.e. maps between Banach spaces $f : E \to F$ such that $| \, \|f(x) - f(y)\| - \|x - y\| \, | < \delta$ for all $x, y \in E$) is the object of [BS].

(a) *Any continuous affine isometric G-action on a Hilbert space has a fixed point.*

(b) *Any continuous affine isometric G-action on a Hilbert space has bounded orbits.*

(c) *Any cocycle on G with respect to an orthogonal representation $(\pi, \mathcal{H})$ is bounded.*

(d) *Any cocycle on G with respect to an orthogonal representation $(\pi, \mathcal{H})$ is a coboundary.*

(e) *For any orthogonal representation $(\pi, \mathcal{H})$, $\mathrm{H}^1(G, \mathcal{H}) = 0$.*

The monograph of Bekka, de la Harpe and Valette [BHV] is a great source on property (T). We refer to it for a precise exposition of what is discussed above, further examples of groups with property (T) and much more.

2.2. **Property** (TT) **: A quasification of** (T). In considering continuous affine isometric actions $T = \pi + c$ of $G$ on an affine Hilbert space $\mathcal{H}$, say we relax the constraint that $T$ is a group homomorphism to the condition

$$\sup_{v \in \mathcal{H}} \|T(gh)v - T(g)T(h)v\|_{\mathcal{H}^0} < +\infty \tag{2.1}$$

for all $g, h \in G$. We then call $T$ a **quasi-action**. Observe that, for any $t \in \mathbb{R}$ and all $g, h \in G$,

$$T(gh)(tv) - T(g)T(h)(tv) = (\pi(gh)(tv) - \pi(g)\pi(h)(tv)) + (c(gh) - c(g) - \pi(g)c(h)).$$

Since, by assumption, this last term is bounded in norm independently of $t$, it follows that $\pi$ is still a homomorphism while $c$ must not be a cocycle anymore, but merely satisfy $\left\|d_\pi^2 c(g,h)\right\|_{\mathcal{H}^0} < +\infty$ for all $g, h \in G$. Such a map is called a **quasi-cocycle**. Again, a quasi-cocycle $c$ is bounded if and only if the associated quasi-action $T = \pi + c$ has bounded orbits. We examine quasi-cocycles more attentively in the setting of bounded cohomology.

Let $G$ be a locally compact group and let $E$ be a Banach $G$-module. A continuous **quasi-$n$-cocycle** is a continuous map $f : G^n \to E$ that satisfies

$$\sup_{g_1, \ldots, g_{n+1} \in G} \left\|(d_{n+1}^\pi f)(g_1, \ldots, g_{n+1})\right\|_E < +\infty.$$

As a $(n+1)$-coboudary, $d_{n+1}^\pi f$ is a bounded representative of a trivial class in $\mathrm{H}^{n+1}(G, E)$. A natural question that arises is whether $d_{n+1}^\pi f$ then also represents a trivial class in bounded cohomology ? That is, we want to know what goes on in

$$\mathrm{EH}_{\mathrm{b}}^{n+1}(G, E) := \ker\left(\mathrm{H}_{\mathrm{b}}^{n+1}(G, E) \xrightarrow{c^{n+1}} \mathrm{H}^{n+1}(G, E)\right),$$

where $c^{n+1}$ is the map induced in cohomology by inclusion, and is called the comparison map[2].

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[2] We point out that the comparison map is in general neither injective nor surjective. For example, it will be made clear after the definition of property (TT) that $\mathrm{EH}_{\mathrm{b}}^2(\Gamma, \mathcal{H}) = 0$ where $\Gamma$ denotes a higher rank lattice while it was

By definition, $[d_{n+1}^{\pi}f] = 0$ in $H_b^{n+1}(G,E)$ if only if there exists a map $b \in C_b(G^n,E)$ such that $d_{n+1}^{\pi}f = d_{n+1}^{\pi}b$. If we assume this is the case, then it follows that the difference $c := f - b$ defines a continuous $n$-cocycle, and

$$f = c + b, \qquad \text{where } c \in \mathscr{Z}C(G^n,E), b \in C_b(G^n,E). \tag{2.2}$$

In other words, $[d_{n+1}^{\pi}f]$ is trivial in bounded cohomology if and only if $f$ is a bounded perturbation of an actual $n$-cocycle. We denote by $\mathscr{Q}(G^n,E)$ the space of all continuous quasi-$n$-cocycles and by $\mathscr{P}(G^n,E)$ the subspace of continuous perturbed $n$-cocycles of the form described by 2.2, also called **trivial quasi-$n$-cocycles**. It is now easy to verify the following statement.

**Proposition 2.2.** [Mo06] *The assignment* $\mathscr{Q}(G^n,E) \to \mathrm{EH}_b^{n+1}(G,E)$, $f \mapsto [d_{n+1}^{\pi}f]$ *induces, for each* $n \geq 0$, *a vector space isomorphism*

$$\mathrm{EH}_b^{n+1}(G,E) \cong \frac{\mathscr{Q}(G^n,E)}{\mathscr{P}(G^n,E)}.$$

Property (TT) was first introduced by Burger and Monod as a refinement of property (T). As such, it is also an obstruction for a locally compact group to admit non-trivial quasi-cocycles. In fact, one can check from the above discussion the following equivalences (or refer to Prop. 13.2.5, [Mon]).

**Definition 2.3.** *A locally compact group $G$ has **property** (TT) if it satisfies any one of the following equivalent properties.*

  (a) *Any continuous affine isometric quasi-$G$-action on a Hilbert space has bounded orbits.*
  (b) *Any quasi-cocycle on $G$ with respect to an orthogonal representation $(\pi, \mathscr{H})$ is bounded.*
  (c) *For any orthogonal representation $(\pi, \mathscr{H})$, $\mathrm{H}^1(G, \mathscr{H}) = 0$ and $\mathrm{EH}_b^2(G, \mathscr{H}) = 0$.*

In particular, (heavier) cohomological machinery allows to prove that if $\Gamma \leqslant G$ is a uniform lattice in a locally compact group $G$, then $\Gamma$ has property (TT) if and only if $G$ has property (TT) (Prop. 13.2.6 in [Mon]). This is similar to what happens for property (T) and is of importance as simple algebraic groups with rank $\geq 2$ have property (TT) (Theorem 13.4.1 [Mon]). We very

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proved by I. Mineyev that for any Gromov hyperbolic group $\Gamma$, $H_b^n(\Gamma,E) \twoheadrightarrow H^n(\Gamma,E)$ for all Banach $\Gamma$-modules in degree $n \geq 2$ (Theorem 11, [Min]).

roughly sketch why property (TT) passes to uniform lattices. Let $(\pi, \mathscr{H})$ be an orthogonal representation of $\Gamma \leqslant G$ and consider the commutative diagram

$$
\begin{array}{ccc}
\mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathscr{H}) & \xrightarrow{\;\;L^2\mathrm{i}\;\;} & \mathrm{H}^2_{\mathrm{b}}(G, L^2(\Gamma \backslash G, \mathscr{H})) \\
c \downarrow & & \downarrow c \\
\mathrm{H}^2(\Gamma, \mathscr{H}) & \xrightarrow[\;\;\mathrm{i}\;\;]{} & \mathrm{H}^2(G, L^2(\Gamma \backslash G, \mathscr{H}))
\end{array}
$$

where $c$ is the comparison map defined above and i resp. $L^2$i denote the induction resp. $L^2$ induction maps. We do not introduce these two maps other than to say that in this setting, the first one is an isomorphism in continuous cohomology and the second one is injective. We already mentioned that property (T) passes to lattices so that by definition of property (T), $\mathrm{H}^1(\Gamma, \mathscr{H}) = 0$. By the definition of property (TT), the comparison map on the right must be injective and forces the comparison map on the left on the commutative diagram to be injective so that $\mathrm{EH}^2_{\mathrm{b}}(\Gamma, \mathscr{H}) = 0$. By generality, $\Gamma$ has property (TT).

By definition, property (TT) yields property (T). The converse is however not true, so that we have an occurrence of a non-trivial rigidity phenomenon. In the next subsection, we discuss the class of Gromov hyperbolic groups, which contains particular examples of groups with property (T) but without (TT) and of groups with $\dim(\mathrm{H}^2_{\mathrm{b}}(G, \mathscr{H})) \neq \dim(\mathrm{H}^2(G, \mathscr{H}))$.

2.3. **Case study : Gromov hyperbolic groups.** It is known that lattices of the quaternionic hyperbolic spaces

$$
\mathrm{Sp}(n, 1) = \{g \in \mathrm{M}(n+1, \mathbb{H}) : {}^t\overline{g}Jg = J\} \quad \text{where} \quad J = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}
$$

admit property (T) if $n \geq 2$ (cf. Chapter 3.3 [BHV]) and are Gromov hyperbolic[3]. We will later see (Prop. 3.4) that Gromov hyperbolic groups never have property (TT).

Let $G$ be a finitely generated group. Let $S$ be a finite generating set, which we assume to be symmetric, i.e. $S = S^{-1}$, and not to contain the unit. To $(G, S)$ we can associate the Cayley graph $\Gamma(G, S)$ with vertex set $G$ and edge set $\{(g, gs) : g \in G, s \in S\}$. For example, the Cayley graph of $\mathbb{F}_2$, the free group over two generators $a$ and $b$, with finite generating set $\{a^{\pm 1}, b^{\pm 1}\}$ has the form

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[3] Keep in mind that lattices of a locally compact group with property (T) are finitely generated (cf. [Kaz]).
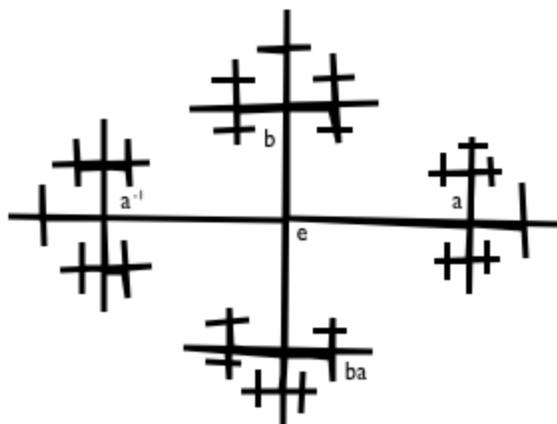
More generally, the Cayley graph of any finitely generated free group is a tree, that is, a connected graph without cycles.

By identifying each edge to the closed unit interval isometrically and then taking the infinimum over the length of all paths joining a pair of vertices, we endow $\Gamma(G,S)$ with a path metric. Now, if $d_S$ denotes the word metric on $G$, i.e. $d_S(v,w)$ returns the minimal number of generators required to write $v^{-1}w$, there is an isometric embedding of $(G,d_S)$ in $\Gamma(G,S)$. Cayley graphs of a finitely generated group $G$ of distinct generating sets $S$ and $S'$ are equivalent in the sense that they are quasi-isometric (cf. [GH]).

A metric space $(X,d)$ is called **geodesic** if, for any two points $x,y \in X$, there exists an isometry $\gamma : [0,d(x,y)] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. Here, $\gamma([0,d(x,y)])$ is called a **(geodesic) segment** and denoted by $[x,y]$. A Cayley graph of a finitely generated group is a geodesic metric space. Observe that the if $G$ is not freely generated by $S$, its Cayley graph contains a cycle (i.e. a closed path), so that pairs of vertices may admit several geodesic segments and the isometry may not be unique.

A geodesic metric space $X$ is called **hyperbolic** if there exists some $\delta \geq 0$ such that, for any geodesic triangle $\triangle = [x,y] \cup [y,z] \cup [z,x]$ and for any point $u$ of one of the three segments (e.g. $u \in [y,z]$), the distance of $u$ to the other two segments is bounded by $\delta$ (e.g. $d(u,[x,y] \cup [z,x]) \leq \delta$).
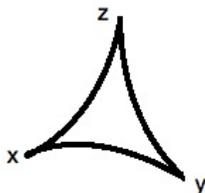
**Definition 2.4.** *A **Gromov hyperbolic group** is a finitely generated group whose Cayley graph is hyperbolic.*

Finite and infinite cyclic groups are obviously Gromov hyperbolic. A **virtually cyclic group** is a group that contains a cyclic subgroup of finite index. This class of groups obviously encompasses both previous examples. There is an established terminology distinguishing such trivial cases from more intricate constructions of hyperbolic groups. Virtually cyclic groups are called **elementary Gromov hyperbolic groups** while any other type of Gromov hyperbolic group will be designated as non-elementary.

### Elementary Gromov hyperbolic groups are amenable

Let $G$ be a locally compact group. A **Følner sequence** is a sequence of subsets $(F_n)_{n \in \mathbb{N}} \subset G$ with finite non-zero Haar measure, that are moreover asymptotically translation invariant, i.e.

$$\lim_{n \to \infty} \frac{|gF_n \cap F_n|}{|F_n|} = 1 \tag{2.3}$$

on compact subsets of $G$. Equivalently (and more usually), one can replace 2.3 by the condition

$$\lim_{n \to \infty} \frac{|gF_n \triangle F_n|}{|F_n|} = 0,$$

where $A \triangle B = (A \setminus B) \cup (B \setminus A)$ denotes the symmetric difference.

An **invariant mean** on $L^\infty(G)$ is a linear functional $M : L^\infty(G) \to \mathbb{C}$ satisfying the following set of properties.

1. $M(f) \geq 0$ for every $f \geq 0$.
2. $M(1_G) = 1$ where $1_G$ is the constant function on $G$ of value 1.
3. $M(g.f) = M(f)$ for all $g \in G$ and all $f \in L^\infty(G)$.

**Definition 2.5.** *A locally compact group $G$ is called **amenable** if it admits a Følner sequence or, equivalently, if there exists an invariant mean on $L^\infty(G)$.*

Amenability admits a number of further equivalent characterizations, for which we refer to [BHV]. In particular, it plays a rôle in the following characterization of compact groups : a group is compact if and only if it is both amenable and has property (T) (Thm. 1.1.6, [BHV]). If we consider only infinite groups, amenability is then the negation of property (T). Note that compact groups are in fact amenable with the normalized Haar measure playing the rôle of the invariant mean. As an easy exercise, one can show that both finite groups and the infinite cyclic group $\mathbb{Z}$ admit Følner sequences.

More generally, we show that in fact every virtually cyclic group is amenable. Let $\Gamma$ be a virtually cyclic group with cyclic subgroup of finite index $\Lambda$ and consider the short exact sequence

$$0 \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow \Gamma/\Lambda \longrightarrow 0$$

given by the inclusion and projection maps. By definition, $\Lambda$ is a cyclic group and $\Gamma/\Lambda$ is a finite group, hence both are amenable. Because amenable groups behave nicely under short exact sequences, the center term, the virtually cyclic group $\Gamma$, is then amenable (Prop. G.2.2 in [BHV]).

### On the second bounded cohomology of non-elementary hyperbolic groups

Free groups $\mathbb{F}_n$ over $n \geq 2$ of generators are non-elementary Gromov hyperbolic groups. In fact, the Cayley graph of a finitely generated free group is a tree and any geodesic triangle in a tree is a tripode, that is each edge of the triangle is contained in the union of the other two. Hence free groups over a finite number of generators are hyperbolic metric spaces (with $\delta = 0$).

Free groups over $\geq 2$ generators do not have property (T) (this will be the content of Prop. 3.3). Here we present a construction of P. Rolli [Rol] to show that $H_b^2(\mathbb{F}_2, \mathbb{R})$ is infinite-dimensional. Beforehand, we make some remarks. First, it is known that in ordinary cohomology $H^2(\mathbb{F}_2, \mathbb{R}) = 0$. Rolli's construction offers a (much simpler) alternative to a well-known construction of R. Brooks and both constructions actually hold for free groups over $n$ generators, where $n \geq 2$. More generally, it was shown by D.B.A. Epstein and K. Fujiwara that $H_b^2(\Gamma, \mathbb{R})$ is infinite-dimensional for any non-elementary hyperbolic group (see [Rol] for all references).

By definition, any non-trivial $x \in \mathbb{F}_2$ has a unique shortest factorization into powers of the two generators, so that it can be written

$$w = a^{k_0} b^{k_1} a^{k_2} b^{k_3} \cdots a^{k_{m-1}} b^{k_m} \qquad \text{where } k_i \in \mathbb{Z}.$$

Pick arbitrarily a sequence $\sigma$ in $\ell^\infty(\mathbb{N}_0)$, the real vector space of real-valued bounded sequences. There is an embedding of $\ell^\infty(\mathbb{N}_0)$ in $\ell^\infty(\mathbb{Z})$ via

$$(a_n)_{n \in \mathbb{N}_0} \mapsto (\widetilde{a}_n)_{n \in \mathbb{Z}} \quad \text{where} \quad \widetilde{a}_n = \begin{cases} a_n & \text{if } n \geq 0 \\ -a_{-n} & \text{if } n < 0 \end{cases}$$

so that $\sigma$ is extended to an odd function on $\mathbb{Z}$. Consider the assignment $g_\sigma : \mathbb{F}_2 \to \mathbb{R}$ given by

$$w = a^{k_0} b^{k_1} \cdots a^{k_{m-1}} b^{k_m} \quad \mapsto \quad g_\sigma(w) = \sum_{i=0}^m \sigma(k_i).$$

If $w$ is the trivial word, then clearly $g_\sigma(w) = \sigma(0)$. One can check that $g_\sigma$ defines a quasi-cocycle on $\mathbb{F}_2$ with respect to the trivial representation (or see Prop. 2.1 [Rol]).

**Proposition 2.6** (Prop. 2.2 [Rol]). *The linear map $\ell^\infty(\mathbb{N}_0) \to \mathrm{H}^2_{\mathrm{b}}(\mathbb{F}_2, \mathbb{R})$, $\sigma \mapsto [d^\pi_2 g_\sigma]$ is injective. In particular, $\mathrm{H}^2_{\mathrm{b}}(\mathbb{F}_2, \mathbb{R})$ is infinite-dimensional.*

*Proof.* Suppose that $[d^\pi_2 g_\sigma] = 0$. Since $\mathrm{H}^2(\mathbb{F}_2, \mathbb{R}) = 0$, this class lies in $\mathrm{EH}^2_{\mathrm{b}}(\mathbb{F}_2, \mathbb{R})$ and must be of the form $g_\sigma = \varphi + b$, where $\varphi$ is a continuous real-valued homomorphism on $G$ and $b \in C_{\mathrm{b}}(G, \mathbb{R})$.

In a first step, observe that $g_\sigma(a^k) = \sigma(k) = k\varphi(a) + b(a^k)$ for any $k \in \mathbb{Z}$. Letting $k$ tend to infinity, the boundedness assumption on $\sigma$ and $b$ yields $\varphi = 0$.

In a second step, let $k, l \in \mathbb{Z}$. Then the same argument with $g_\sigma(\underbrace{(a^k b^k)(a^k b^k) \cdots (a^k b^k)}_{l \text{ times}}) = 2l\sigma(k)$

yields $\sigma(k) = 0$, and by generality, that $\sigma = 0$. $\qquad\qquad\square$

## 3. Antitheses

By definition, property (T), and *a fortiori* property (TT), impose a strong constraint on affine isometric actions on Hilbert spaces. More generally, let us consider a locally compact group $G$ with property (T) together with a continuous homomorphism $\varphi : G \to H$ into a locally compact target group $H$.

### 3.1. A-T-menability, a-TT-menability.
The dichotomy between property (T) and amenability (cf. Def. 2.5) offers a first example of a strong constraint on the homomorphism.

**Proposition 3.1** (Coro. 1.3.5, [BHV]). *If $G$ has property* (T) *and $A$ is amenable, then any continuous homomorphism $\varphi : G \to A$ has relatively compact image in $A$.*

*Proof.* Let $(\pi, \mathcal{H})$ be an orthogonal representation of $A$ into a Hilbert space $\mathcal{H}$ and consider a cocycle $c : A \to \mathcal{H}$. Restricting $c$ to the subgroup $\varphi(G)$, the composition $c \circ \varphi$ defines a cocycle on $G$ with respect to the orthogonal representation $(\pi \circ \varphi, \mathcal{H})$, which must be bounded by assumption. By continuity, the cocycle $c$ is then bounded on $\overline{\varphi(G)}$. Then $\overline{\varphi(G)} \subseteq A$ is both an amenable and a property (T) group, and must thus be compact.

$\square$

For every $n \geq 1$, observe that the sequence $(F_k)_{k \geq 0}$ defined by $F_k := [-k, \dots, k]^n$ is a Følner sequence in $\mathbb{R}^n$. To see this, fix $k \in \mathbb{N}_0$ and an arbitrary element $x \in \mathbb{R}^n$. A simple sketch of the symmetric difference $x F_k \triangle F_k$ hints that while the sets $F_n$ grow polynomially as $n$ tends to infinity, the symmetric difference may only have linear growth. Thus $\mathbb{R}^n$ is amenable and if $\varphi : G \to \mathbb{R}^n$ is a continuous homomorphism on a group $G$ with property (T), then $\varphi = 0$. Yet more general is the concept of a-T-menability.
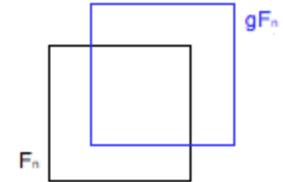


Let $(X, d)$ be a metric space on which $G$ acts by continuous affine isometries. Fix $x_0 \in X$. We call the $G$-action **(metrically) proper** if, for any $R > 0$, the subset $\{g \in G : d(x_0, gx_0) \leq R\}$ is relatively compact in $G$.

**Definition 3.2.** *A locally compact group $G$ is said to be **a-T-menable** if it satisfies one of the two following equivalent properties.*

(a) *There exists a proper affine isometric $G$-action on a Hilbert space $\mathcal{H}$.*

(b) *There exists a proper cocycle c on G with respect to an orthogonal representation $(\pi, \mathscr{H})$, that is, for any $R > 0$, the subset $\{g \in G : \|c(g)\|_{\mathscr{H}} \leq R\}$ is relatively compact in G.*

Traditionally, property (a) is called the Haagerup property, while property (b) defines a-T-menability. A notable feature of a-T-menable groups is that they satisfy the Baum-Connes conjecture, see [HK].

It follows from the definitions that a-T-menability comes as a negation of property (T), the only objects capable of having concurrently the two properties being the compact groups. As a consequence, any continuous homomorphism from a group with property (T) to an a-T-menable group has relatively compact image. Examples of a-T-menable groups include amenable groups (cf. [CJV]) and free groups.

**Proposition 3.3.** *Fix $n \geq 2$. The free group over n generators $\mathbb{F}_n$ is a-T-menable. In particular, $\mathbb{F}_n$ does not have property* (T).

In fact, the proof given here implies that any group acting properly on a tree is a-T-menable. The construction can be found, e.g., in [BHV].

*Proof.* Let $S$ denote a generating set of $\mathbb{F}_n$ and let $\Gamma := \Gamma(\mathbb{F}_n, S)$ be its Cayley graph, which is a tree. The left action by concatenation of $\mathbb{F}_n$ on itself induces a metrically proper $\mathbb{F}_n$-action on $\Gamma$. More generally, note that any finitely generated group acts properly on its Cayley graphs.

We denote by $\mathscr{E}$ the set of oriented edges in $\Gamma$ so that each edge will come up twice in $\mathscr{E}$, the second time with opposite orientation. Because trees do not contain cycles, the geodesic segment $[x, y]$ is uniquely defined for every $x, y \in \mathbb{F}_n$. We will take particular care to denote $[x, y]$ the geodesic segment with initial point $x$ and terminal point $y$ so that an edge within the segment is called positively oriented if it has the same orientation as the segment. To every pair $(x, y) \in \mathbb{F}_n^2$, we assign a map $c_{[x,y]} \in \ell^2(\mathscr{E})$, defined by

$$c_{[x,y]}(e) := \begin{cases} +1 & e \text{ is positively oriented in } [x, y] \\ -1 & e \text{ is negatively oriented in } [x, y] \\ 0 & e \text{ is not in } [x, y]. \end{cases}$$

Fix an origin $x_0 \in \mathbb{F}_n$. We claim that $b(x) := c_{[x_0, xx_0]}$ defines a proper cocycle on $\mathbb{F}_n$ with respect to the left regular representation $(\lambda_{\mathbb{F}_n}, \ell^2(\mathscr{E}))$.

Since any geodesic triangle $\triangle = [x, y] \cup [y, z] \cup [z, x]$ in a tree is a tripode, then for every edge $e \in \Gamma$, $c_{[x,y]}(e) + c_{[y,z]}(e) = c_{[x,z]}(e)$.
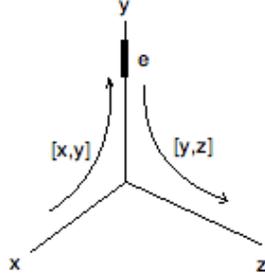
It follows that

$$b(xy) \;=\; c_{[x_0,xyx_0]} \;=\; c_{[x_0,xx_0]} + c_{[xx_0,xyx_0]} \;=\; b(x) + \lambda_{\mathbb{F}_n}(x)b(y)$$

and that any cocycle $b'$ defined for a distinct origin point $x_0' \neq x_0$ is cohomologous to $b$. From the illustration above, one sees immediately that $\|c(x,y)\|_{\ell^2(\mathscr{E})}^2 = 2\mathrm{d}(x,y)$, where d denotes the path metric on $\Gamma$. Since the action of $\mathbb{F}_n$ on $\Gamma$ is proper, the cocycle $b : \mathbb{F}_n \to \ell^2(\mathscr{E})$ is proper.

$\square$

Gromov hyperbolic groups are in general not a-T-menable (e.g. $\mathrm{Sp}(n,1)$). However a construction of I. Mineyev yields the following result. We follow the exposition given in the proof of Theorem 7.13 [MS].

**Proposition 3.4.** *Every Gromov hyperbolic group admits a proper quasi-cocycle with respect to the left regular representation.*

*Proof.* Let $G$ be a Gromov hyperbolic group with finite generating set $S$ and endowed with word metric $\mathrm{d}_S$. Denote by $\Gamma := \Gamma(G,S)$ its Cayley graph and by $\mathscr{E}$ the set of oriented edges in $\Gamma$ as in the proof of the previous proposition. A **bicombing** is a function that assigns to every pair of vertices $x,y \in G$ an oriented path of initial vertex $x$ and terminal vertex $y$ in $\Gamma$. Formally, a bicombing takes values in the space $C_{00}(\mathscr{E})$ of functions on $\mathscr{E}$ with finite support. Let $\|\cdot\|_1$ denote the $\ell^1$ norm on $C_{00}(\mathscr{E})$. In particular, note that for any pair of vertices $x,y \in G$, if $q$ is a bicombing, then $\|q(x,y)\|_1 \geq \mathrm{d}_S(x,y)$. Mineyev constructed a bicombing $q$ with the following properties (Thm. 10, [Min]) :

    (a) $q$ is quasigeodesic, i.e. there exists a constant $R \geq 0$ such that for any pair $x,y \in G$, the support of $q(x,y)$ is contained in the tubular neighborhood $N([x,y],R)$ of any geodesic segment $[x,y]$ ;

    (b) $q$ is $G$-equivariant and anti-symmetric, i.e. $q(x,y) = -q(y,x)$ for any $x,y \in G$ ;

(c) there exists a constant $C \geq 0$ such that, for any $x, y, z \in G$, $\|q(x,y) + q(y,z) + q(z,x)\|_1 \leq C$.

By (a), the support of $q(x,y)$ is contained in a union of "balls" in $\mathscr{E}$ centered each at a vertex on the path $q(x,y)$ and all of same (large enough) radius $R$. Therefore, the number of edges in the support of $q(x,y)$ is bounded by $K \cdot d_S(x,y)$ where $K \geq 1$ is the maximal number of elements in any such ball. Clearly, $K$ depends only on $R$ and the number of generators in $S$. Hölder's inequality then yields

$$d_S(x,y) \leq \|q(x,y)\|_1 \leq \|q(x,y)\|_2 (K\, d_S(x,y))^{1/2}$$

so that $\|\cdot\|_2 \leq \|\cdot\|_1$ and we may view all $q(x,y)$ as maps in $\ell^2(\mathscr{E})$. From (b) and (c) and since the action of $G$ on its Cayley graphs is metrically proper, $b(\cdot) := q(1,\cdot)$ defines a proper quasi-cocycle on $G$ with respect to the left regular representation $(\lambda_G, \ell^2(\mathscr{E}))$.

$\square$

In particular, infinite Gromov hyperbolic groups never have property (TT). Given a definition of a-TT-menability, where a locally compact group $G$ is a-TT-menable whenever it admits a proper quasi-cocycle with respect to an orthogonal representation $(\pi, \mathscr{H})$, any continuous homomorphism on a locally compact group with property (TT) and target an a-TT-menable group is trivial in the sense that it has relatively compact image. Recall that this was Proposition 1 in the introduction.

3.2. **Quasi-homomorphisms, property** (TTT) **and a-TTT-menability.** Let $G$ and $H$ be locally compact groups. We call a map $q: G \to H$ a **quasi-homomorphism** if the defect $\{q(gh)^{-1}q(g)q(h) : g, h \in G\}$ is relatively compact in $H$. If we endow $H$ with a metric $d_H$, then this is equivalent to require that for some $\delta > 0$,

$$\sup_{g,h \in G} d_H(q(gh), q(g)q(h)) < \delta.$$

If the above holds, then $q$ is also called a $\delta$-**homomorphism**. When $H = \mathbb{R}$, then $q$ is nothing else than a quasi-cocycle with trivial coefficient and is more usually called a quasimorphism. The map $g_\sigma$ constructed by Rolli in the context of Proposition 2.6 is an example of quasimorphism. A variant of this construction, also from [Rol], gives examples of quasi-homomorphisms with non-commutative target. Fix $d \geq 1$ and let $\mathscr{U}(d)$ denote the unitary group of $d \times d$ matrices. We endow $\mathscr{U}(d)$ with the Hilbert-Schmidt inner product $\langle A, B \rangle_{\mathrm{HS}} := \mathrm{tr}(AB^*)$ where $B^*$ denotes the conjugate transpose of $B$. This scalar product induces a metric on $\mathscr{U}(d)$ which we normalize so that the

identity matrix has norm one :

$$d_{HS}(A,B) \; = \; \|B-A\|_{HS} \; := \; \frac{1}{\sqrt{d}} [\operatorname{tr}(B-A)(B-A)^*)]^{1/2} \qquad (A,B \in \mathscr{U}(d)). \qquad (3.1)$$

Observe that this metric is bi-invariant. Take $\sigma \in \ell^\infty(\mathbb{Z}, \mathscr{U}(d))$ to satisfy $\sigma(-k) = \sigma(k)^{-1}$ for all $k \in \mathbb{Z}$, fix $n \geq 2$ and define $g_\sigma : \mathbb{F}_n \to \mathscr{U}(d)$ by

$$x = x_0^{k_0} \cdots x_m^{k_m} \; \mapsto \; g_\sigma(x) = \prod_{i=0}^m \sigma(k_i), \qquad (3.2)$$

where we should read on the left hand side the unique shortest factorization into powers of $x$. Note from the proof of the next proposition that if we choose $\sigma \in \ell^\infty(\mathbb{Z}, \mathscr{U}(d))$ to map into the ball of radius $\delta/3$, then $g_\sigma$ is a $\delta$-homomorphism, more appropriately called in this case a **unitary $\delta$-representation**. In [BOT], the authors show that $g_\sigma$ is $(2 - \frac{\delta}{3})$-away from any actual unitary representation.

**Proposition 3.5** (Prop. 5.1, [Rol]). *Fix $d \geq 1$, $n \geq 2$. The map $g_\sigma : \mathbb{F}_n \to \mathscr{U}(d)$ defined by 3.2 is a quasi-homomorphism.*

*Proof.* Let $x,y \in \mathbb{F}_n$ with shortest factorizations into powers

$$x = x_0^{k_0} \cdots x_m^{k_m} \quad \text{and} \quad y = y_0^{l_0} \cdots y_n^{l_n},$$

where $x_i, y_j$ are generators with $x_i \neq x_{i+1}$ and $y_j \neq y_{j+1}$, and $k_i, l_j \in \mathbb{Z}$. Then, in considering the concatenation $xy$, either parts of the two words cancel out and $xy$ has shortest factorization into powers

$$xy = x_0^{k_0} \cdots x_{m-r}^{k_{m-r}} y_r^{l_r} \cdots y_n^{l_n} \quad \text{for some } r \in \{0,\dots,\min(m,n)\}$$

or, at some point, some two letters do not cancel each other completely and we are left with

$$xy = x_0^{k_0} \cdots x_{m-r-1}^{k_{m-r-1}} y_r^{(k_{m-r}+l_r)} y_{r+1}^{l_{r+1}} \cdots y_n^{l_n} \quad \text{for some } r \in \{0,\dots,\min(m,n)\}$$

(where $x_{m-i}^{k_{m-i}} = y_i^{-l_i}$ for all $i \in \{0,\dots,r-1\}$). The first case is trivial and in the second case, it follows that

$$
\begin{aligned}
d_{HS}\left(g_\sigma(xy), g_\sigma(x)g_\sigma(y)\right) \; &= \; d_{HS}\left(\sigma(k_{m-r}+l_r), \sigma(k_{m-r})\sigma(k_{m-r+1})\cdots\sigma(k_m)\sigma(l_0)\cdots\sigma(l_r)\right) \\
&= \; d_{HS}\left(\sigma(k_{m-r}+l_r), \sigma(k_{m-r})\sigma(l_{r-1})^{-1}\cdots\sigma(l_0)^{-1}\sigma(l_0)\cdots\sigma(l_r)\right) \\
&= \; d_{HS}\left(\sigma(k_{m-r}+l_r), \sigma(k_{m-r})\sigma(l_r)\right) \\
&\leq \; 3 \, \|\sigma\|_\infty
\end{aligned}
$$

which is bounded by definition.

$\square$

Recall the discussion in the introduction where property (TTT) is introduced to allow for the generalization in Proposition 2. Part of the conditions on quasi-cocycles (continuity, taking values in a real Hilbert space) are motivated by the framework of bounded cohomology. However, in the case of a wq-cocycle, coefficients do not have a Banach $G$-module structure so the whole makes little sense in cohomology. Property (TTT) may thus be defined more generally.

**Definition 3.6.** *A locally compact group $G$ has **property** (TTT) if every wq-cocycle on $G$ is bounded. Here, a **wq-cocycle** is a Borel locally bounded map $b : G \to \mathscr{H}$, where $\mathscr{H}$ denotes a Hilbert space, together with a Borel locally bounded map $\pi : G \to \mathscr{U}(\mathscr{H})$ such that*

$$\sup_{g,h \in G} \|b(gh) - b(g) - \pi(g)b(h)\|_{\mathscr{H}} < +\infty.$$

Cocycles and quasi-cocycles are trivially wq-cocycles so that property (TTT) yields property (TT). In analogy, a locally compact group is called **a-TTT-menable** if it admits a proper wq-cocycle. It is not clear whether there are locally compact groups which are a-TTT-menable but not a-TT-menable.

**Theorem 3.7** (Theorem A, [Oza]). *Any quasi-homomorphism from a locally compact group with property* (TTT) *into an a-TTT-menable locally compact group $H$ has relatively compact image.*

*Proof.* Let $q : G \to H$ be a quasi-homomorphism and let $b$ be a proper wq-cocycle on $H$. Then the composition $b \circ q$ is a wq-cocycle on $G$. Set $R := \sup_{g \in G} \|b \circ q(g)\| < +\infty$. Since $b$ is proper, the image set $\{q(g) : g \in G\} \subseteq \{h \in H : \|b(h)\| \leq R\}$ is relatively compact in $H$. $\square$

**Part** 2. **Introducing property** (TTT)

**In Part 2, all locally compact groups will be assumed second countable and Hausdorff, all maps Borel and locally compact (i.e. bounded on compact sets), all Hilbert spaces separable. Note that by the definitions, a locally compact second countable Hausdorff group is $\sigma$-compact.**

## 4. DEFINITION OF PROPERTY (TTT)

We shortly recall the setting for property (TTT) presented in Part 1. Given a unitary representation $(\pi, \mathcal{H})$ of a locally compact group $G$, recall that a map $b: G \to \mathcal{H}$ is called a **cocycle** if

$$b(gh) = b(g) + \pi(g)b(h) \tag{4.1}$$

for all $g, h \in G$. We speak of **quasi-cocycles** if, in lieu of condition (4.1), a map $b: G \to \mathcal{H}$ has finite defect, i.e.

$$\sup_{g,h \in G} \|b(gh) - (b(g) + \pi(g)b(h))\| < +\infty. \tag{4.2}$$

A group $G$ has **property** (TTT) if every wq-cocycle on $G$ is bounded. Here, a wq-cocycle is a Borel locally bounded map $b: G \to \mathcal{H}$, together with a Borel locally bounded map $\pi: G \to \mathcal{U}(\mathcal{H})$, satisfying inequality (4.2).

Let us introduce the notion of relative property for a pair $(G, H)$, where $H$ is a closed subgroup of $G$. A pair $(G, H)$ is said to have **relative property** (TTT) (resp. rel. (TT), resp. rel. (T)) if every wq-cocycle (resp. quasi-cocycle, resp. cocycle) is bounded on the subgroup $H$. Of course, this relative definition coincides with the original definition whenever $H = G$. Various characterizations of relative property (T) for pairs $(G, H)$ can be found in [Jol2]. In particular, we will frequently use the following standard formulation : A pair of topological groups $(G, H)$ has relative property (T) if there exists a non-empty compact subset $Q \subset G$ and a real number $\delta > 0$, such that, for any unitary representation $(\pi, \mathcal{H})$ of $G$ with a $(Q, \delta)$-invariant vector $\xi \in \mathcal{H}$, i.e. satisfying

$$\sup_{x \in Q} \|\pi(x)\xi - \xi\| < \delta \|\xi\|,$$

there exists a non-zero $H$-invariant vector, that is

$$\{\xi \in \mathcal{H} : \pi(h)\xi = \xi \text{ for every } h \in H\} \neq \{0\}.$$

A unitary representation that has a $(Q, \delta)$-invariant vector for any compact subset $Q \subset G$ and any real number $\delta > 0$ is said to "almost have invariant vectors".

4.1. **Kernels of positive type.** Let $G$ be a locally compact group. We denote by $L^\infty(G)$ the space of (equivalence classes of) complex-valued measurable functions on $G$ that are essentially bounded with respect to the Haar measure $dg$.

A function $\theta \in L^\infty(G \times G)$ is called a **kernel of positive type** if

$$\int_G \int_G \theta(g,h)\xi(g)\overline{\xi(h)}\, dg dh \geq 0$$

holds for every $\xi \in L^1(G)$. If $\mathscr{H}$ is a separable Hilbert space and given a vector-valued map $P \in L^\infty(G, \mathscr{H})$, the assignment $(g,h) \mapsto \langle P(g), P(h)\rangle_{\mathscr{H}}$ defines a kernel of positive type. Refer to Appendix A for a review of vector-valued measure and integration theory. The content of the next proposition is that, conversely, every non-zero kernel of positive type has such a representation.

**Proposition 4.1.** [Fol] *If $\theta \in L^\infty(G \times G)$ is a non-zero kernel of positive type, then there exists a Hilbert space $\mathscr{H}_\theta$ and a vector-valued map $P \in L^\infty(G, \mathscr{H}_\theta)$ such that*

$$\theta(g,h) = \langle P(g), P(h)\rangle \quad a.e.$$

*Moreover, if $\theta$ is continuous, then $P$ can be taken continuous.*

*Proof.* Because $\theta$ is assumed to be of positive type, for any $\xi, \eta \in L^1(G)$,

$$\langle \xi, \eta \rangle_\theta := \int_G \int_G \theta(g,h)\xi(g)\overline{\eta(h)}\, dg dh$$

defines a positive semi-definite Hermitian form. In fact, observe that $\overline{\theta(g,h)} = \theta(h,g)$ a.e. follows from the basic identity $\Im(z) = \frac{z-\bar{z}}{2i}$ for $z := \int\int \theta(g,h)\xi(g)\overline{\xi(h)}\, dg dh$. In particular, $\langle \cdot, \cdot \rangle_\theta$ is an inner product on the quotient $L^1(G)/\mathscr{N}$, where

$$\mathscr{N} = \{\xi \in L^1(G) : \langle \xi, \xi \rangle_\theta = 0\},$$

and we denote by $\mathscr{H}_\theta$ the Hilbert space completion of $L^1(G)/\mathscr{N}$. Letting $T : L^1(G) \to \mathscr{H}_\theta$ denote the quotient map, we have

$$\langle T\xi, T\eta \rangle_\theta = \int_G \int_G \theta(g,h)\xi(g)\overline{\eta(h)}\, dg dh.$$

Since, by duality, there is a map $P \in L^\infty(G, \mathscr{H}_\theta)$ such that $T(\xi) = \int_G \xi(g)P(g)dg$, it follows that $\theta(g,h) = \langle P(g), P(h)\rangle_\theta$ a.e. and $\|P\|_\theta = \|T\|_\theta = \|\theta\|_\infty^{1/2}$.

$\square$

A kernel of positive type $\theta$ is said to be **normalized** if $\theta(g,g) = 1$ for every $g \in G$. Note that for every kernel of positive type $\theta$ that satisfies $\theta(g,g) = \langle P(g), P(g)\rangle \approx 1$ for all $g \in G$, there

is a normalized kernel of positive type given by $\widetilde{\theta}(g,h) := \left\langle \frac{P(g)}{\|P(g)\|}, \frac{P(h)}{\|P(h)\|} \right\rangle$. Finally, we endow $L^\infty(G \times G)$ with the completely bounded norm, or cb-norm,

$$\|\theta\|_{\mathrm{cb}} := \inf\{\|P\|\,\|Q\| : \ P, Q \in L^\infty(G, \mathscr{H}_\theta), \ \theta(g,h) = \langle P(g), Q(h) \rangle \ \text{a.e.}\}.$$

By convention, if the set over which the infimum is taken is empty, we set "$\|\theta\|_{\mathrm{cb}} = +\infty$". The choice of this atypical norm is motivated by the following characterization of completely bounded multipliers of the Fourier algebra of $G$ (cf. [Jol1] for definitions, background and proof). A continuous function $\varphi : G \to \mathbb{C}$ is a completely bounded multiplier if and only if $\|\theta\|_{\mathrm{cb}} < +\infty$, where $\theta(g,h) := \varphi(g^{-1}h)$ for every $g, h \in G$. Completely bounded multipliers are important in Ozawa's proof of property (TTT) for the algebraic group $\mathrm{SL}_n(\mathbb{K})$ ($n \geq 3$) over a local field $\mathbb{K}$. In this thesis, we however present an alternate argument based on Bochner's Theorem and will not discuss completely bounded multipliers any further.

Observe then that any kernel of positive type $\theta$ satisfying $\theta(g,g) \approx 1$ for each $g \in G$ is within bounded distance of its normalization $\widetilde{\theta}$. We may thus always consider normalized kernels of positive type.

## 4.2. **Relations between variations : Properties** $(\mathrm{T_P})$, $(\mathrm{T_Q})$. Let the group $G$ act on $L^\infty(G \times G)$ via

$$(g.\theta)(x,y) = \theta(g^{-1}x, g^{-1}y). \tag{4.3}$$

**Definition 4.2.** *The pair* $(G,H)$ *is said to have **relative property** $(\mathrm{T_P})$ if, for every* $\varepsilon > 0$, *there exists a pair* $(Q, \delta)$, *where $Q$ is a compact subset of $G$ and $\delta > 0$, such that, for any normalized Borel kernel of positive type* $\theta : G \times G \to \mathbb{C}$ *satisfying*

$(P_1)$ $\sup_{g \in G} \|g.\theta - \theta\|_{\mathrm{cb}} < \delta$ *and*
$(P_2)$ $\sup_{g^{-1}h \in Q} |\theta(g,h) - 1| < \delta$,

*we have*

$$\sup_{g,h \in H} |\theta(g,h) - 1| < \varepsilon.$$

*The pair* $(G,H)$ *is said to have **relative property** $(\mathrm{T_Q})$ if, for every* $\varepsilon > 0$, *there exists a pair* $(Q, \delta)$ *such that, for any triple* $(\pi, \mathscr{H}, \xi)$ *of a map* $\pi : G \to \mathscr{U}(\mathscr{H})$ *together with a unit vector* $\xi \in \mathscr{H}$ *that satisfy*

$(Q_1)$ $\sup_{g,h \in G} \|\pi(gh)\xi - \pi(g)\pi(h)\xi\|_{\mathscr{H}} < \delta$ *and*
$(Q_2)$ $\sup_{g \in Q} \|\pi(g)\xi - \xi\|_{\mathscr{H}} < \delta$,

*then*

$$\sup_{h \in H} \|\pi(h)\xi - \xi\|_{\mathscr{H}} < \varepsilon.$$

In the above definitions, the compact subset $Q$ can always be taken symmetric (i.e. $Q = Q^{-1}$) and we may assume it contains the unit element $e$. If $\theta$ is a kernel of positive type satisfying $(P_2)$ for some $(Q, \delta)$ where $Q$ is assumed to have positive measure, then

$$\widetilde{\theta}(g,h) = \frac{1}{|Q|^2} \int_{Q^2} \theta(gq, hq') dq dq',$$

defines a continuous kernel of positive type, uniformly close to $\theta$. We note the following inequality, implicit in several arguments in [Oza].

**Lemma 4.3.** *Let $\theta$ be a normalized kernel of positive type on a locally compact group G. If there exist a symmetric compact subset $Q$ and a $\delta > 0$ such that $\sup_{x^{-1}y \in Q} |\theta(x,y) - 1| < \delta$, then*

$$\sup_{x,y \in Q} \|\theta(\cdot x, \cdot y) - \theta(\cdot, \cdot)\| < 2\sqrt{2}\delta^{1/2}.$$

*Proof.* Observe that since $\theta$ is taken to be a normalized kernel of positive type, we have that

$$
\begin{aligned}
|\theta(gx, hy) - \theta(g,h)| &\leq |\theta(gx, hy) - \theta(g, hy)| + |\theta(g, hy) - \theta(g,h)| \\
&\leq \|P(gx) - P(g)\| + \|P(hy) - P(h)\|.
\end{aligned}
$$

If $x \in Q$, then by assumption :

$$\|P(gx) - P(g)\|^2 = \|P(gx)\|^2 + \|P(g)\|^2 - 2\,\Re\,\langle P(gx), P(g) \rangle = 2\,\Re(1 - \theta(gx, g)) < 2\delta.$$

The conclusion follows immediately.

$\square$

Let $\mathscr{H}^{\otimes n}$ denote the $n$-fold tensor product of a Hilbert space $\mathscr{H}$. In other words, it is the Hilbert space obtained by completion of the space of finite linear combinations of elements of the form $\xi_1 \otimes \cdots \otimes \xi_n$, where $\xi_i \in \mathscr{H}$ for all $i \in \{1, ..., n\}$, together with the inner product

$$\langle \xi_1 \otimes \cdots \otimes \xi_n, \eta_1 \otimes \cdots \otimes \eta_n \rangle = \langle \xi_1, \eta_1 \rangle \cdots \langle \xi_n, \eta_n \rangle.$$

Depending on whether the Hilbert space $\mathscr{H}$ is real or complex, we set $\mathscr{H}^0 = \mathbb{R}$ or $\mathscr{H}^0 = \mathbb{C}$. The **full Fock space** is the Hilbert space

$$\mathscr{F} = \bigoplus_{n \geq 0} \mathscr{H}^{\otimes n}$$

consisting of infinite series $f = \sum_{n\geq 0} f_n$ such that $\|f\|^2 = \sum_{n\geq 0} \|f_n\|^2 < +\infty$. There is a continuous map $\mathscr{H} \to \mathscr{F}$ which associates to each vector $\xi \in \mathscr{H}$ an exponential vector

$$\mathrm{EXP}(\xi) = \bigoplus_{n\geq 0} \frac{\xi^{\otimes n}}{\sqrt{n!}},$$

that satisfies $\langle \mathrm{EXP}(\xi), \mathrm{EXP}(\eta) \rangle = \exp \langle \xi, \eta \rangle$.

**Theorem 4.4.** *Let H be a subgroup of G. Then for the pair $(G,H)$ the following implications hold.*

$$\textit{rel. } (T_P) \implies \textit{rel. } (TTT) \implies \textit{rel. } (TT)$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\textit{rel. } (T_Q) \implies \textit{rel. } (T).$$

*Proof.* The implications rel. (TTT) $\Rightarrow$ rel. (TT) $\Rightarrow$ rel. (T) follow from the definitions.

$(T_P) \Rightarrow (TTT)$. For $\varepsilon = \frac{1}{2}$, there exists, by assumption, a pair $(Q, \delta)$. Let $b : G \to \mathscr{H}$ be a Borel locally bounded wq-cocycle. We denote by $\mathscr{F}$ the full Fock space generated by $\mathscr{H}$ and consider the map $E : \mathscr{H} \to \mathscr{F}$, defined by

$$E(\xi) = \exp(-\|\xi\|^2)\mathrm{EXP}(\sqrt{2}\xi).$$

This map is continuous and into the unit sphere of $\mathscr{F}$ since $\langle E(\xi), E(\eta) \rangle = e^{-\|\xi - \eta\|^2}$. Then,

$$\theta(g,h) := \langle E[b(g)], E[b(h)] \rangle = e^{-\|b(g) - b(h)\|^2}$$

defines a normalized kernel of positive type. Since $b$ is assumed to be locally bounded, we have

$$
\begin{aligned}
|\theta(g,h) - 1| &= 1 - \exp(-\|b(g) - b(h)\|^2) \\
&= 1 - \exp(-\|b(g)\|^2 - \|b(h)\|^2 - 2\|b(g)\| \|b(h)\|) \\
&\leq 1 - \exp(-4\delta_0^2)
\end{aligned}
$$

for all $g, h$ such that $g^{-1}h \in Q$, where $\delta_0$ is chosen large enough such that $\|b(g)\| < \delta_0$ for all $g \in Q$.

By definition, for all $x, y \in G$,

$$\langle E[b(x)], E[b(y)] \rangle = \left\langle E\left[b(g^{-1}) + \pi(g^{-1})b(x)\right], E\left[b(g^{-1}) + \pi(g^{-1})b(y)\right] \right\rangle$$

for any $g \in G$ and we can write $g.\theta(x,y) - \theta(x,y)$ as

$$g.\theta(x,y) - \theta(x,y) = \left( \begin{array}{c} \langle E[b(g^{-1}x)] - E[b(g^{-1}) + \pi(g^{-1})b(x)], E[b(g^{-1}y)] \rangle \\ + \\ \langle E[b(g^{-1}) + \pi(g^{-1})b(x)], E[b(g^{-1}y)] - E[b(g^{-1}) + \pi(g^{-1})b(y)] \rangle \end{array} \right).$$

Then, taking $\delta'_0 > 0$ to be large enough to bound the defect of $b$,

$$
\begin{aligned}
\|g.\theta - \theta\|_{\mathrm{cb}} &\leq 2 \sup_{x \in G} \left\| E[b(g^{-1}x)] - E\left[b(g^{-1}) + \pi(g^{-1})b(x)\right] \right\| \\
&\leq 2 \sup_{x \in G} \left( 2 - 2\, e^{-\left\| b(g^{-1}x) - b(g^{-1}) - \pi(g^{-1})b(x) \right\|^2} \right)^{1/2} \\
&< 2 \left( 2 - 2\, e^{-\delta'^2_0} \right)^{1/2}.
\end{aligned}
$$

If we choose $\delta_0$ and $\delta'_0$ small enough as to satisfy property $(\mathrm{T_P})$, it follows that, for all $h \in H$,

$$
|\theta(h,e) - 1| = 1 - \exp(-\|b(h) - b(e)\|^2) < \frac{1}{2}.
$$

That is, $\|b(h) - b(e)\|^2 < \log(2)$ for all $h \in H$.

$(\mathrm{TTT}) \Rightarrow (\mathrm{T_Q})$. The proof of rel. $(\mathrm{TTT}) \Rightarrow$ rel. $(\mathrm{T_Q})$ follows very closely the proof of Proposition 2.4.5 in [BHV]. For completeness, we include it here. Let us argument by contradiction that the pair $(G,H)$ does not have property $(\mathrm{T_Q})$. As a Hausdorff locally compact second countable topological group, $G$ is $\sigma$-compact, i.e., it can be written as a countable union of interiors of compact sets

$$
G = \bigcup_{n \geq 0} \mathring{Q}_n
$$

where we took care to index the sequence $(Q_n)_{n \geq 0}$ following the partial order given by inclusion, that is $Q_n \subseteq Q_{n+1}$ for every $n \geq 0$. Pick $\varepsilon$ such that, for any pair $(Q_n, \sqrt{2^{-n}})$, $n \in \mathbb{N}_0$, there is a triple $(\pi_n, \mathscr{H}_n, \xi_n)$ for which $(Q_1)$, $(Q_2)$ and $\sup_{h \in H} \|\pi_n(h)\xi_n - \xi_n\| \geq \varepsilon$ hold. We will show that the map $b : G \to \bigoplus_{n \geq 0} \mathscr{H}_n$, defined by

$$
b(g) = \sum_{n \geq 0} n\left( \pi_n(g)\xi_n - \xi_n \right).
$$

defines a continuous wq-cocycle which is unbounded on $H$.

In fact, for each $g \in G$, there exists $N \in \mathbb{N}_0$ such that $g \in Q_N$ and $\|\pi_n(g)\xi_n - \xi_n\|^2 < \frac{1}{2^n}$ for all $n \geq N$ so that, for some bounded constant $c$,

$$
\|b(g)\|^2 = \sum_{n \geq 0} n^2 \|\pi_n(g)\xi_n - \xi_n\|^2 < c + \sum_{n \geq N} \frac{n^2}{2^n} < +\infty.
$$

In other words, the map $b$ is uniformly convergent on compact subsets and in particular well defined and continuous. Moreover, setting $\pi = \oplus \pi_n$,

$$
\begin{aligned}
\|b(gh) - b(g) - \pi(g)b(h)\|^2 &= \left\| \sum_{n \geq 0} n \left( \pi_n(gh)\xi_n - \pi_n(g)\pi_n(h)\xi_n \right) \right\|^2 \\
&= \sum_{n \geq 0} n^2 \left\| \pi_n(gh)\xi_n - \pi_n(g)\pi_n(h)\xi_n \right\|^2 \\
&< \sum_{n \geq 0} \frac{n^2}{2^n} < +\infty,
\end{aligned}
$$

we verify that $b$ is indeed a wq-cocycle for $\pi : G \to \mathscr{U}(\oplus \mathscr{H}_n)$. Since, by assumption, $\sup_{h \in H} \|\pi_n(h)\xi_n - \xi_n\| \geq \varepsilon$, it follows that $b$ is unbounded on $H$.

$(\mathrm{T_Q}) \Rightarrow (\mathrm{T})$. Assuming that $(G, H)$ has relative property $(\mathrm{T_Q})$, pick $\varepsilon < 1$ and let $(\pi, \mathscr{H})$ be a unitary representation of $G$ for which

$$
\sup_{x \in Q} \|\pi(x)\xi - \xi\| < \delta
$$

holds for some unit vector $\xi \in \mathscr{H}$, where $(Q, \delta)$ is given by $(\mathrm{T_Q})$. We show that there exists a non-zero vector in $\mathscr{H}$ which is $H$-invariant.

In fact, since $\pi$ is a unitary representation, it follows from $(\mathrm{T_Q})$ that

$$
\sup_{h \in H} \|\pi(h)\xi - \xi\| < \varepsilon. \tag{4.4}
$$

We consider the non-empty bounded subset $\pi(H)\xi$ of $\mathscr{H}$ and let $C$ denote the closure of the convex hull

$$
\left\{ \sum_{i=1}^n \alpha_i \pi(h_i)\xi \ : \ n \geq 1, \ h_i \in H, \ \alpha_i \in [0,1], \ \sum_{i=1}^n \alpha_i = 1 \right\}.
$$

By the "Lemma of the center" (Lemma 2.2.7 [BHV]), there exists a unique vector $\eta \in C$ of minimal norm. By uniqueness and since $\|\pi(h)\eta\| = \|\eta\|$ for all $h \in H$, the vector $\eta$ is $H$-invariant. Moreover, it follows from (4.4) that any element $\zeta$ of $C$ verifies $\|\zeta - \xi\| < \varepsilon$, hence in particular $\eta \neq 0$.

$\square$

We will later see that if the pair $(G, A)$ is made up of a semi-direct product $G = G_0 \ltimes A$, where $G_0$ acts continuously by automorphisms on an abelian group $A$, all the above properties are equivalent (Coro. 6.4). We end this section with the following application of property $(\mathrm{T_Q})$ for unitary $\delta$-representations. Note that the theorem does not concern the unitary $\delta$-representations constructed in Proposition 3.5 since $\mathbb{F}_n$ is not a property (T) group.

**Theorem 4.5** (Thm. 5.5, [BOT]). *Let $G$ be a locally compact group with property* $(T_Q)$ *and let* $\delta > 0$. *Any $d$-dimensional unitary $\delta$-representation of $G$ is* $F_G^{(d)}(\delta)$*-close to an actual unitary representation with* $\lim_{\delta \to 0} F_G^{(d)}(\delta) = 0$, *where*

$$F_G^{(d)}(\delta) \ := \ \sup\{D(\pi) : \sup_{g,h \in G} \|\pi(gh) - \pi(g)\pi(h)\| \leq \delta\}$$

*with the supremum taken over all $d$-dimensional Hilbert spaces and maps* $\pi : G \to \mathscr{U}(\mathscr{H})$ *with* $\pi(e) = 1$ *and where*

$$D(\pi) \ := \ \inf\{\|\pi - \rho\| : \rho \in \mathrm{Hom}(G, \mathscr{U}(\mathscr{H}))\}.$$

*Proof.* Fix $d \geq 1$. Let $\varepsilon > 0$ and assume by contradiction that there exists a sequence $(\pi_n)_{n \geq 1}$ of unitary $\frac{1}{n}$-representations $\pi_n : G \to \mathscr{U}(d)$ that are $\varepsilon$-away from any actual unitary representation converging *pointwise* to a unitary representation $\pi$. The space of all $d \times d$ matrices endowed with the Hilbert-Schmidt inner product (cf. Section 3.2) is a Hilbert space, which we denote $\mathrm{HS}(d)$. Define a companion sequence $(\sigma_n)_{n \geq 1}$ of maps $\sigma_n : G \to \mathscr{U}(\mathrm{HS}(d))$, given by

$$\sigma_n(g)A \ := \ \pi_n(g)A\pi(g)^*$$

for all $g \in G$. Recall the normalized Hilbert-Schmidt bi-invariant metric of 3.1. Then, for every $n \geq 1$, $\mathrm{d}_{\mathrm{HS}}(\sigma_n(g)I, I) = \mathrm{d}_{\mathrm{HS}}(\pi_n(g), \pi(g))$ and

$$\begin{aligned}
\mathrm{d}_{\mathrm{HS}}(\sigma_n(gh)I, \sigma_n(g)\sigma(h)I) &= \mathrm{d}_{\mathrm{HS}}(\pi_n(gh)\pi(gh)^*, \pi_n(g)\pi_n(h)\pi(h)^*\pi(g)^*) \\
&= \mathrm{d}_{\mathrm{HS}}(\pi_n(gh), \pi_n(g)\pi_n(h)) \qquad \leq \quad \frac{1}{n}
\end{aligned}$$

Since $G$ has property $(T_Q)$, it follows that for $n$ large enough $\mathrm{d}_{\mathrm{HS}}(\sigma_n(g)I, I) \ < \ \varepsilon$ and

$$\|\pi_n(g) - \pi(g)\| \ \leq \ \frac{1}{\sqrt{d}}\|\sigma_n(g)I - I\|_{\mathrm{HS}} \ \to \ \infty$$

uniformly on $G$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

## 5. PROPERTY (TTT) FOR $SL_n(\mathbb{K})$ ($n \geq 3$)

We will follow the line of proof given in Section 1.4 [BHV] for property (T). The pivotal point of the argument for (T) is a theorem of Y. Shalom that links rel. (T) for a semi-direct product with an abelian group $(G \ltimes A, A)$ to the (in)existence of $G$-invariant, finitely additive probability measures on the unitary dual $\widehat{A}$ (Theorem 5.5, [Sh1]).

### 5.1. Bochner's Theorem.
Let $A$ be a locally compact abelian group. As a consequence of Schur's lemma, every irreducible representation of $A$ is one dimensional, thus of the form $\pi(a)\xi := \chi(a)\xi$, where $\xi \in \mathbb{C}$, and $\chi : A \to S^1$ with $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ is a continuous homomorphism called a **(unitary) character** of $A$. The set of all characters, that is, of all equivalence classes of irreducible representations of $A$, is called the **(unitary) dual group** of $A$, and is denoted by $\widehat{A}$. Its unit is the character that associates value 1 to every $a \in A$, and will be denoted 1. The dual group, when endowed with the topology of uniform convergence on compacta of $A$, is a locally compact abelian group (see Ch. 4.1 in [Fol]).

A **function of positive type** on a locally compact group $G$ is a map $\varphi \in L^\infty(G)$ such that the kernel defined by $G \times G \to \mathbb{C}$, $(g, h) \mapsto \varphi(h^{-1}g)$, is of positive type, that is, if

$$\int_G \int_G \varphi(h^{-1}g)\xi(g)\overline{\xi(h)}dgdh \;\geq\; 0$$

holds for all $\xi \in L^1(G)$. If $G = A$ is a locally compact abelian group, then **Bochner's Theorem** states that for every continuous function of positive type $\varphi$ on $A$, there exists a unique finite non-negative complex Radon measure $\mu$ on the Borel $\sigma$-algebra $\mathscr{B}(\widehat{A})$ of the unitary dual $\widehat{A}$ such that, for every $a \in A$,

$$\varphi(a) \;=\; \int \chi(a)d\mu(\chi).$$

### 5.2. Strengthening Shalom's Theorem.
Let $G$ be a locally compact group acting continuously on a locally compact abelian group $A$ by conjugation. Endowed with the product topology, the semi-direct product group $G \ltimes A$ is locally compact, with $G$ and $A$ as closed resp. closed normal subgroup. The induced action of $G$ on $\widehat{A}$ dual to the conjugation action is then given by $g.\chi(a) = \chi(gag^{-1})$, for all $g \in G$, $\chi \in \widehat{A}$. Clearly, the unit character is invariant under the dual action of $G$.

A **mean $m$ on $\widehat{A}$** is a finitely additive probability measure on the Borel $\sigma$-algebra $\mathscr{B}(\widehat{A})$. The dual action of $G$ on $\widehat{A}$ induces an isometric action of $G$ on $m$, given by $g_*m(B) = m(g^{-1}B)$ for all $B \in \mathscr{B}(\widehat{A})$ and all $g \in G$. A mean $m$ that is invariant under this action is called a **$G$-invariant mean**.

The Dirac mass at the unit character 1, i.e.

$$\delta_1(B) = \begin{cases} 1 & 1 \in B \\ 0 & 1 \notin B \end{cases} \qquad \text{for every } B \in \mathscr{B}(\widehat{A}),$$

is the trivial example of a $G$-invariant mean. Theorem 5.5 in [Sh1] states that if there is no $G$-invariant mean for the dual action of $G$ on $\widehat{A}$ other than the Dirac mass at the unit character, then the pair $(G \ltimes A, A)$ has relative property (T). With arguments of [Oza] and Bochner's Theorem, we will show that this condition even yields relative property $(T_P)$.

Abelian groups are amenable (Thm. G.2.1, [BHV]). The relation between the two characterizations of amenability given in Definition 2.5 allows to construct asymptotic invariants for an abelian group. In fact, let $(A_n)_{n \geq 0}$ be a Følner sequence in a locally compact abelian group $A$. A free ultrafilter on $\mathbb{N}_0$ may be expressed as a finitely additive measure $\omega : \mathscr{P}(\mathbb{N}_0) \to \{0,1\}$, where $\mathscr{P}(\mathbb{N}_0)$ denotes the power set of $\mathbb{N}_0$, such that $\omega(F) = 0$ for every finite subset $F \in \mathscr{P}(\mathbb{N}_0)$. By Zorn's Lemma, a free ultrafilter exists on $\mathbb{N}_0$. Let $f \in L^\infty(A)$ be a complex-valued function and consider the sequence $(a_n)_{n \geq 0}$ given by

$$a_n := \frac{1}{|A_n|} \int_{A_n} f(a) \, da$$

where $da$ denotes the Haar measure on $A$. The ultralimit $a := \underset{\omega}{\mathrm{Lim}} \, a_n$ associated to $\omega$ exists (and is unique) if, for any $\varepsilon > 0$, $\omega(\{n \in \mathbb{N}_0 : |a_n - a| < \varepsilon\}) = 1$. In particular, since $(a_n)_{n \geq 0}$ is a bounded sequence in $\mathbb{C}$, the ultralimit $a$ exists (cf. Section 3.1, [KL]). The associated ultralimit is a generalized limit in the sense that it may be seen as a positive unital linear functional on $\ell^\infty(\mathbb{N}_0)$ that coincides with the usual limit of convergent sequences. The sequence $(a_n)_{n \geq 0}$ defined above depends on the choice of a map $f \in L^\infty(A)$ In particular, thanks to the asymptotic invariance of the Følner sequence,

$$a(f) := \underset{\omega}{\mathrm{Lim}} \frac{1}{|A_n|} \int_{A_n} f(a) \, da \qquad \text{for all } f \in L^\infty(A)$$

is an invariant mean on $L^\infty(A)$ where each $a(f)$ is an asymptotic invariant that inherits (to some extent) properties of $f$. This is of course not the case if we simply pick randomly an invariant mean for $A$.

**Theorem 5.1.** *Let $G$ be a locally compact group and let $G \ltimes A$ be a semi-direct product with a locally compact abelian group $A$. If there is no invariant mean under the dual action of $G$ on $\widehat{A} - \{1\}$, then the pair $(G \ltimes A, A)$ has rel. $(T_P)$.*

*Proof.* By contradiction, assume that there exists an $\varepsilon > 0$ such that, for any compact subset $Q$ of $G \times A$ and for any $\delta > 0$, we can find a normalized kernel of positive type $\theta$ on $G \ltimes A$ that satisfies

$$\sup_{g \in G \ltimes A} \|g.\theta - \theta\|_{\mathrm{cb}} < \delta, \qquad \sup_{x^{-1}y \in Q} |\theta(x,y) - 1| < \delta \qquad \text{and} \qquad \sup_{x,y \in A} |\theta(x,y) - 1| \geq \varepsilon.$$

We will moreover assume that the compact set $Q$ is symmetric and contains a neighborhood of the unit element. Restrict $\theta$ to $A \times A$. The resulting function, which we will still denote by $\theta$ is a normalized kernel of positive type with $\|\theta|_{A \times A}\|_{\mathrm{cb}} \leq \|\theta\|_{\mathrm{cb}}$ (cf. last paragraph of Section 2 in [Oza]). For each $a \in A$ and $g \in Q$, set

$$\theta_a^g(x,y) \ = \ \frac{1}{|Q|^2} \int_Q \int_Q \theta(agxg^{-1}q, agyg^{-1}q') \, dq dq'.$$

With our assumptions, $\{\theta_a^g\}_{g \in Q, a \in A}$ defines a family of equicontinuous kernels of positive type with

$$|\theta_a^g(x,y) - \theta_a^g(x_0, y_0)| \ \leq \ 2\sqrt{2}\delta^{1/2}$$

if $x \in x_0 Q$, $y \in y_0 Q$, and, by Lemma 4.3,

$$\|\theta_a^g - \theta\|_{\mathrm{cb}} \leq \|(ag)^{-1}.\theta - \theta\|_{\mathrm{cb}} + \sup_{g \in Q} \|\theta(\cdot g^{-1}, \cdot g^{-1}) - \theta(\cdot, \cdot)\| < \delta + 2\sqrt{2}\delta^{1/2} =: \delta_1$$

for every $a \in A$ and $g \in Q$. Fix a free ultrafilter $\omega$ on $\mathbb{N}$ with associated ultralimit $\underset{\omega}{\mathrm{Lim}}$ and introduce for every $g \in Q$

$$\bar{\theta}^g(x,y) \ := \ \underset{\omega}{\mathrm{Lim}} \, \frac{1}{|A_n|} \int_{A_n} \theta_a^g(x,y) da.$$

Each $\bar{\theta}^g$ is a continuous $A$-invariant kernel of positive type with $\|\bar{\theta}^g - \theta\|_{\mathrm{cb}} \leq \|\theta_a^g - \theta\|_{\mathrm{cb}} < \delta_1$.

We define a continuous function of positive type on $A$ with $\varphi(x^{-1}y) = \bar{\theta}^e(x,y)$. Since $\bar{\theta}^e$ is $A$-invariant, the map $\varphi : A \to \mathbb{C}$ is well-defined, with $\varphi(e) = \bar{\theta}^e(e,e)$. Let $\mu_{\bar{\theta}^e}$ denote the non-negative complex measure on $\mathscr{B}(\widehat{A})$ associated to $\varphi$ by Bochner's Theorem such that

$$\varphi(a) = \int_{\widehat{A}} \chi(a) d\mu_{\bar{\theta}^e}(\chi)$$

for all $a \in A$. Since characters of $A$ are homomorphisms, $\mu_{\bar{\theta}^e}(\widehat{A}) = \varphi(e)$ and we may assume that $\mu_{\bar{\theta}^e}$ is a probability measure on $\mathscr{B}(\widehat{A})$. The dual action of $G$ on $\widehat{A}$ induces an isometric action of $G$ on $\mu_{\bar{\theta}^e}$ so that, for every $g \in Q$, we have

$$\varphi(g^{-1}ag) = \bar{\theta}^g(e,a) = \int \chi(a) d(g_* \mu_{\bar{\theta}^e})(\chi)$$

and finally, for every $g \in Q$,

$$\|g_* \mu_{\bar{\theta}^e} - \mu_{\bar{\theta}^e}\| \leq \|\bar{\theta}^g - \bar{\theta}^e\|_{\mathrm{cb}} < 2\delta_1.$$

By assumption, there exist $x, y \in A$ such that

$$0 < \varepsilon \leq |\theta(x,y) - 1| \leq |\theta(x,y) - \bar{\theta}^e(x,y)| + |\bar{\theta}^e(x,y) - 1| < \delta_1 + \int_{\widehat{A}} |\chi(x^{-1}y) - 1| d\mu_{\bar{\theta}^e}(\chi).$$

If we set $C := \{ \chi \in \widehat{A} : |\chi(x^{-1}y) - 1| \neq 0 \}$ and pick $\delta$ small enough so that $\varepsilon - \delta_1 > 0$, then

$$0 < \varepsilon - \delta_1 \leq \int_C |\chi(x^{-1}y) - 1| d\mu_{\bar{\theta}^e}(\chi) \leq 2 \mu_{\bar{\theta}^e}(C) \leq 2 \mu_{\bar{\theta}^e}(\widehat{A} - \{1\}).$$

Choose $\delta$ such that $\delta_1 < \varepsilon/2$. Then $\mu_{\bar{\theta}^e}(\widehat{A} - \{1\}) \geq \varepsilon/4$. Since the semi-direct product $G \ltimes A$ is $\sigma$-compact, there exists an increasing sequence of compact subsets $(Q_n)_{n \geq 1}$ that cover $G \ltimes A$. We define a sequence $(\delta_n)_{n \geq 1}$ by $\delta_n := \delta/n$, where $\delta$ is chosen such that $\delta_1 < \varepsilon/2$. Then, for every $n \geq 1$, there exists a finite non-negative complex Radon measure $\mu_n$ on the Borel $\sigma$-algebra $\mathscr{B}(\widehat{A})$ such that

(a) $\mu_n(\widehat{A} - \{1\}) \geq \frac{\varepsilon - (\delta_n + 2\sqrt{2}\delta_n^{1/2})}{2} \geq \frac{\varepsilon - \frac{1}{\sqrt{n}}\delta_1}{2} \geq \varepsilon/4$ and

(b) $\|g_* \mu_n - \mu_n\| < 2\frac{1}{\sqrt{n}}\delta_1 < \frac{\varepsilon}{\sqrt{n}}$ for every $g \in Q_n$.

Then, for every $n \geq 1$, the normalization

$$\nu_n := \frac{1}{\mu_n(\widehat{A} - \{1\})} \mu_n|_{\widehat{A} - \{1\}}$$

defines a probability measure on $\mathscr{B}(\widehat{A} - \{1\})$ such that $\|g_* \nu_n - \nu_n\| < \frac{4}{\sqrt{n}}$ for all $g \in Q_n$. With Banach-Alaoglu, the sequence $(\nu_n)_{n \geq 1}$ converges along a subsequence to a $G$-invariant probability measure on $\mathscr{B}(\widehat{A} - \{1\})$.

$\square$

## 5.3. Property (TTT) for $\mathrm{SL}_n(\mathbb{K})$.

A **local field** is a field $\mathbb{K}$ for which there exists an absolute value $|\cdot|$ inducing a topology on the additive group $(\mathbb{K}, +)$ such that it is locally compact and non-discrete. Here, an **absolute value** on a field $\mathbb{K}$ is a map $|\cdot| : \mathbb{K} \to \mathbb{R}$ that satisfies, for all $x, y \in \mathbb{K}$,

(i) $|xy| = |x||y|$      (ii) $|x| \geq 0$ and $|x| = 0$ if and only if $x = 0$      (iii) $|x + y| \leq |x| + |y|$.

The metric $d(x,y) = |x - y|$ then induces a topology on the field $\mathbb{K}$. Let $n \geq 1$ and fix a unitary character $1 \neq \chi \in \widehat{\mathbb{K}}$. Each $t \in \mathbb{K}^n$ defines a continuous homomorphism $\chi_t \in \widehat{\mathbb{K}^n}$ by $\chi_t(x) = \chi(\sum_{i=1}^n t_i x_i)$. In particular, the assignment $\mathbb{K}^n \to \widehat{\mathbb{K}^n}$, $t \mapsto \chi_t$ is a topological group isomorphism (Coro. D.4.6 [BHV]).

**Corollary 5.2.** *Let* $\mathrm{SL}_2(\mathbb{K})$ *be the group of* $2 \times 2$ *matrices over a local field* $\mathbb{K}$ *with determinant 1. The pair* $(\mathrm{SL}_2(\mathbb{K}) \ltimes \mathbb{K}^2, \mathbb{K}^2)$ *has rel. property* $(\mathrm{T}_P)$.

*Proof.* The dual group $\widehat{\mathbb{K}^n}$ of $\mathbb{K}^n$ can be identified with $\mathbb{K}^n$, for each $n \in \mathbb{N}$. It is the content of Proposition 1.4.12 in [BHV] that the only $\mathrm{SL}_2(\mathbb{K})$-invariant mean on the Borel $\sigma$-algebra of $\mathbb{K}^2$ is the Dirac mass at 0. For completeness, we transcript the proof here. Let $m$ be a $\mathrm{SL}_2(\mathbb{K})$-invariant mean on $\mathbb{K}^2$ and consider the subset

$$\Omega = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{K}^2 - \{0\} : |x| \leq |y| \right\}$$

where $|\cdot|$ is the absolute value responsible for the topology on $\mathbb{K}$. Let a sequence $(\lambda_n)_{n \in \mathbb{N}} \subset \mathbb{K}$ with $|\lambda_{n+1}| > |\lambda_n| + 2$ for $n \in \mathbb{N}$ define

$$\Omega_n = \begin{pmatrix} 1 & \lambda_n \\ & 1 \end{pmatrix} \Omega = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{K}^2 - \{0\} : |y|(|\lambda_n| - 1) \leq |x| \leq |y|(|\lambda_n| + 1) \right\}.$$

Because

$$\cdots \leq |y|(|\lambda_{n-1}| + 1) < |y|(|\lambda_n| - 1) \leq |y|(|\lambda_n| + 1) < |y|(|\lambda_{n+1}| - 1) \leq \ldots$$

for all $y \in \mathbb{K}$, the sets $\Omega_n$ are pairwise disjoint. Because the mean $m$ is $\mathrm{SL}_2(\mathbb{K})$-invariant and

$$\sum_{n \in \mathbb{N}} m(\Omega_n) \leq m(\mathbb{K}^2 - \{0\}) \leq 1,$$

we have that $m(\Omega) = 0$. Consider also the subset

$$\Omega' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Omega = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{K}^2 - \{0\} : |x| \geq |y| \right\}.$$

Again by $\mathrm{SL}_2(\mathbb{K})$-invariance,

$$m(\mathbb{K}^2 - \{0\}) = m(\Omega \cup \Omega') \leq m(\Omega) + m(\Omega') = 2m(\Omega) = 0.$$

The mean $m$ is the Dirac mass at 0.

$\square$

An **elementary matrix** $E_{ij}(t) \in \mathrm{SL}_n(\mathbb{K})$, where $t \in \mathbb{K}$ and $i \neq j$ is a matrix

$$\left( E_{ij}(t) \right)_{m,n} = \begin{cases} 1 & m = n \\ t & m = i \text{ and } n = j \\ 0 & \text{elsewhere.} \end{cases}$$

It is a standard result of linear algebra that $\mathrm{SL}_n(\mathbb{K})$ is generated by elementary matrices, that is, each matrix $A$ in $\mathrm{SL}_n(\mathbb{K})$ can be written as a finite product of elementary matrices.[4] In particular, there exists a finite integer $N$ for $\mathrm{SL}_n(\mathbb{K})$, that depends only on $n$, such that every matrix in $\mathrm{SL}_n(\mathbb{K})$ may be written as a product of at most $N$ elementary matrices. In a more general setting where the

---

[4] Each elementary row operation to reduce $A$ to a diagonal matrix produces an elementary matrix. If we take care to make each successive entry on the diagonal 1, the very last entry will need to be $\det(A)$. Since $\det(A) = 1$, this is the identity matrix, and $A$ is the finite product of the elementary matrices.

local field $\mathbb{K}$ may be replaced by any commutative ring with unit, the following lemma provides the necessary step to pass from $(\mathrm{SL}_2(\mathbb{K}) \ltimes \mathbb{K}^2, \mathbb{K}^2)$ to $\mathrm{SL}_n(\mathbb{K})$.

**Lemma 5.3.** [Sh2] *Fix an integer $n \geq 3$ and let* R *be a commutative ring with unit. For every elementary matrix $E_{ij}(t) \in \mathrm{SL}_n(R)$, there is a natural embedding of $\mathrm{SL}_2(R) \ltimes R^2$ that contains $E_{ij}(t)$ in some subgroup isomorphic to $R^2$.*

*Proof.* When $n = 3$, there are four natural embeddings of $\mathrm{SL}_2(R) \ltimes R^2$ in $\mathrm{SL}_3(R)$, namely

$$\left( \begin{array}{c|cc} 1 & x & y \\ \hline 0 & & \\ 0 & & A \end{array} \right), \left( \begin{array}{c|cc} 1 & 0 & 0 \\ \hline x & & \\ y & & A \end{array} \right), \left( \begin{array}{cc|c} & & x \\ \multicolumn{2}{c|}{A} & y \\ \hline 0 & 0 & 1 \end{array} \right), \left( \begin{array}{cc|c} & & 0 \\ \multicolumn{2}{c|}{A} & 0 \\ \hline x & y & 1 \end{array} \right)$$

with $A \in \mathrm{SL}_2(R)$ and $\begin{pmatrix} x \\ y \end{pmatrix} \in R^2$. It is obvious that every elementary matrix belongs to a copy of the subgroup $\{I_2\} \ltimes R^2$ under one of the above natural embeddings in $\mathrm{SL}_3(R)$.

For the general case, we proceed by induction. Pick an elementary matrix $E_{ij}(t)$ in $\mathrm{SL}_n(R)$. If neither $(i, j) = (1, n)$ nor $(i, j) = (n, 1)$, then $E_{ij}(t)$ is contained in a copy of either

$$\left( \begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & \mathrm{SL}_{n-1}(R) & \\ 0 & & & \end{array} \right) \quad \text{or} \quad \left( \begin{array}{ccc|c} & & & 0 \\ & \mathrm{SL}_{n-1}(R) & & \vdots \\ & & & 0 \\ \hline 0 & \cdots & 0 & 1 \end{array} \right).$$

In the cases where $(i, j) = (1, n)$ or $(i, j) = (n, 1)$, take the embeddings of $\mathrm{SL}_2(R) \ltimes R^2$ with respective images the subgroups

$$\left( \begin{array}{ccc} \mathrm{SL}_2(R) & & R^2 \\ & I_{n-4} & \\ & & I_2 \end{array} \right) \quad \text{resp.} \quad \left( \begin{array}{ccc} \mathrm{SL}_2(R) & & \\ & I_{n-4} & \\ R^2 & & I_2 \end{array} \right).$$

$\square$

**Theorem 5.4.** *Fix $n \geq 3$ and let $\mathbb{K}$ be a local field. The group $\mathrm{SL}_n(\mathbb{K})$ has property (TTT).*

*Proof.* Take $b$ a wq-cocycle on $\mathrm{SL}_n(\mathbb{K})$ with finite defect

$$D := \sup_{g,h \in \mathrm{SL}_n(\mathbb{K})} \|b(gh) - (b(g) + \pi(g)b(h))\| < +\infty.$$

By Corollary 5.2, the pair $(\mathrm{SL}_2(\mathbb{K}) \ltimes \mathbb{K}^2, \mathbb{K}^2)$ has rel. property (TTT), hence $b$ is bounded on each elementary matrix in $\mathrm{SL}_n(\mathbb{K})$ on account of Lemma 5.3. Let $N$ be a finite integer such that any matrix in $\mathrm{SL}_n(\mathbb{K})$ may be written as a product of at most $N$ elementary matrices. Pick $g \in \mathrm{SL}_n(\mathbb{K})$ arbitrarily and let $g = g_1 \cdots g_m$ be its decomposition in a finite product of $m \leq N$

elementary matrices. Then

$$
\begin{aligned}
\|b(g)\| &\leq \|b(g_1 \cdots g_{m-1}) + \pi(g_1 \cdots g_{m-1})b(g_m)\| + D \\
&\leq \|b(g_1 \cdots g_{m-1})\| + \|b(g_m)\| + D \\
&\leq \sum_{i=1}^{m} \|b(g_i)\| + (m-1)D \\
&\leq N \sup_{g} \|b(g)\| + (N-1)\,D,
\end{aligned}
$$

where the supremum is taken over all elementary matrices. By generality, every wq-cocycle on $SL_n(\mathbb{K})$ is bounded.

□

Actually, we can even deduce from Lemma 5.3 and Corollary 5.2 that $SL_n(\mathbb{K})$ has property $(T_P)$. See Theorem 5 in [Oza].

## 6. CHARACTERIZATION OF REL. (TTT) FOR SEMI-DIRECT PRODUCTS WITH ABELIAN GROUPS

Let a locally compact group $G$ act continuously on a locally compact abelian group $A$ by automorphisms and let $G \ltimes A$ denote their semi-direct product.

In [CT], Y. de Cornulier and R. Tessera establish that if we consider only countably approximable means[5], Shalom's condition that there may *not* exist a $G$-invariant mean on $\widehat{A}$ other that the Dirac mass at the unit character, is not only sufficient but also necessary for the pair $(G \ltimes A, A)$ to have rel. (T). Ozawa extends this characterization of rel. (T) to one of rel. $(T_P)$.

The building blocks in the characterization of rel. (T) presented in [CT] are previous results of A. Ioana, M. Burger and Y. Shalom. If $G$ and $A$ are discrete countable groups, A. Ioana showed that rel. (T) for $(G \ltimes A, A)$ infers that there exists no sequence of probability measures $(\mu_n)$ on the unitary dual $\widehat{A}$, which would be asymptotically $G$-invariant with mass concentrated in a neighborhood of the unit character, yet distinct from the Dirac mass at the unit character. This result (Thm. 6.1, [Ioa]), reformulated below in Theorem 6.1 to fit the terminology of [Oza], provided the converse implication to a result of M. Burger (Prop. 7, [Bur]). The latter was later generalized by Shalom (Thm. 5.5, [Sh1], which we extended to rel. $(T_P)$ in Theorem 5.1) in two directions : replacing probability measures by means, and discrete groups by locally compact groups.

**Theorem 6.1.** *Let $G$ be a discrete countable group acting by automorphisms on a discrete countable abelian group $A$. If the pair $(G \ltimes A, A)$ has rel. property* (T)*, then the property* (M) *below holds.*

(M) *For any $\varepsilon > 0$, there is a pair $(Q, \delta)$, where $Q$ is a compact subset of $G$ and $\delta > 0$, such that, for any Borel probability measure $\mu$ on the Borel $\sigma$-algebra $\mathscr{B}(\widehat{A})$ that satisfies*

$(M_1)$ $\|g_*\mu - \mu\| \leq \delta$ *for every $g \in Q$ and*

$(M_2)$ $\left| \int_{\widehat{A}} \chi(a) d\mu(\chi) - 1 \right| < \delta$ *for every $a \in Q \cap A$,*

*we have, for every $a \in A$,*

$$\left| \int_{\widehat{A}} \chi(a) d\mu(\chi) - 1 \right| < \varepsilon.$$

---

[5] By *countably approximable*, de Cornulier and Tessera mean the following. For a Borel measure space $(X, \mathscr{B}, \mu)$, there is a one-to-one correspondence between Borel means $m$ on $\mathscr{B}$ and appropriate positive linear functionals $M$ on $L^\infty(X) := L^\infty(X, \mathscr{B}, \mu)$, given by

$$M(\chi_B) = m(B) \qquad B \in \mathscr{B},$$

where $\chi_B$ denotes the characteristic function of the Borel set $B$, i.e. the function that takes value 1 on elements contained in $B$ and value 0 otherwise. A Borel mean is said to be countably approximable if it lies in the weak$^*$ closure in $L^\infty(X)^*$ of a countable set of Borel probability measures (seen as means) on $\widehat{A}$.

Let $e_A$ and $e_G$ denote the unit elements of $A$ and $G$ respectively. Since $(a,g) = (a,e_G) \cdot (e_A,g)$ in $G \ltimes A$, any unitary representation in its restrictions to $A$ and $G$, i.e. $\pi = \pi|_A \otimes \pi|_G$. Conversely, the next lemma characterizes such semi-direct product of unitary representations.

**Lemma 6.2.** *Let $G$ and $A$ be as before, and let $(\sigma,\mathcal{H})$ and $(\rho,\mathcal{H})$ be unitary representations of $G$ resp. $A$. Then $\pi = \rho \otimes \sigma$ is a unitary representation of $G \ltimes A$ if and only if*

$$\sigma(g)\rho(a)\sigma(g)^{-1} = \rho(g.a),$$

*for all $a \in A$, $g \in G$.*

*Proof.* Let $a,a_0 \in A$ and $g,g_0 \in G$ be arbitrary group elements. Then

$$
\begin{aligned}
\sigma(g)\rho(a)\sigma(g)^{-1} = \rho(g.a) \;&\Leftrightarrow\; \sigma(g)\rho(a) = \rho(g.a)\sigma(g) \\
&\Leftrightarrow\; \rho(a_0)\sigma(g)\rho(a)\sigma(g_0) = \rho(a_0 g.a)\sigma(gg_0) \\
&\Leftrightarrow\; \pi(a_0,g)\pi(a,g_0) = \pi(a_0 g.a, gg_0).
\end{aligned}
$$

$\square$

*Proof of Theorem 6.1.* Let $\varepsilon > 0$ and take $(Q,\delta_0)$ such that any unitary representation of $G \ltimes A$ with a $(Q,\delta_0)$-invariant vector admits a non-zero $A$-invariant vector. We assume that $Q$ contains the unit element of $G \ltimes A$. Let $\{g_n : n \in \mathbb{N}\}$ be an enumeration of the elements in $G$ and let $\mu$ be a Borel probability measure satisfying $(M_1)$ and $(M_2)$ for $(Q,\delta)$ ($\delta$ is to be determined later on). Observe that the probability measure defined by $\sum_{n \in \mathbb{N}} 2^{-n}(g_n)_*\mu$ also satisfies $(M_1)$ and $(M_2)$ for $(Q,\delta)$ :

$$\sum_{n \geq 1} \frac{1}{2^n} \|(g_n)_* g_* \mu - (g_n)_* \mu\| < \left(\sum_{n \geq 1} \frac{1}{2^n}\right) \delta = \delta$$

for any $g \in Q$, and

$$\left| \sum_{n \geq 1} \frac{1}{2^n} \left( \int_{\widehat{A}} \chi(a) d(g_n)_* \mu(\chi) - 1 \right) \right| \leq \sum_{n \geq 1} \frac{1}{2^n} \left| \int_{g_n^{-1}\widehat{A}} g_n \chi(a) d\mu(\chi) - 1 \right| < \delta$$

for any $a \in Q \cap A$. We may thus replace $\mu$ by $\sum_{n \in \mathbb{N}} 2^{-n}(g_n)_*\mu$, which is $G$-quasi-invariant, i.e. $g_*\mu(B) = 0$ if and only if $\mu(B) = 0$ for any Borel subset $B \in \mathscr{B}(\widehat{A})$ and all $g \in G$. In particular, the Radon-Nikodym derivative

$$\frac{dg_*\mu}{d\mu}(\chi) \in [0,+\infty]$$

is well-defined on $\widehat{A}$ for all $g \in G$. Define unitary representations $(\sigma, L^2(\widehat{A}, \mu))$ of $G$ and $(\rho, L^2(\widehat{A}, \mu))$ of $A$ by

$$(\sigma(g)f)(\chi) = \sqrt{\frac{dg_*\mu}{d\mu}}(\chi)f(g^{-1}\chi),$$

$$(\rho(a)f)(\chi) = \chi(a)f(\chi).$$

By Lemma 6.2, $\pi = \rho \otimes \sigma$ defines a unitary representation of the semi-direct product $G \ltimes A$ in $L^2(\widehat{A}, \mu)$. We claim that the unit vector $1_{\widehat{A}}$ is $(Q, \delta)$-invariant. In fact, for every $g \in Q \cap (G \setminus A)$, we have

$$
\begin{aligned}
\left\| \pi(g)1_{\widehat{A}} - 1_{\widehat{A}} \right\|^2 &= \int_{\widehat{A}} \left| \pi(g)1_{\widehat{A}}(\chi) - 1_{\widehat{A}}(\chi) \right|^2 d\mu(\chi) \\
&= \int_{\widehat{A}} \left| \sqrt{\frac{dg_*\mu}{d\mu}}(\chi) - 1 \right|^2 d\mu(\chi) \\
&\overset{(*)}{\leq} \int_{\widehat{A}} \left[ \frac{dg_*\mu}{d\mu}(\chi) - 1 \right] d\mu(\chi) \\
&\leq \|g_*\mu - \mu\| \\
&< \delta,
\end{aligned}
$$

where $(*)$ comes from the inequality $|1 - \sqrt{x}| \leq \sqrt{1 - |x|}$ for all $x \geq 0$, and, for every $a \in Q \cap A$,

$$\left\| \pi(a)1_{\widehat{A}} - 1_{\widehat{A}} \right\|^2 \leq 2\Re\left[ 1 - \int_{\widehat{A}} \chi(a)d\mu(\chi) \right] < 2\delta.$$

Thus, picking $\delta < \delta_0^2/2$, rel. property (T) yields a non-zero vector $f \in L^2(\widehat{A}, \mu)$ satisfying

$$[\chi(a) - 1]f(\chi) = 0$$

for all $a \in A$ and $\mu$-a.e. $\chi \in \widehat{A}$, and, since $f \neq 0$, it follows that

$$\left| \int_{\widehat{A}} \chi(a)d\mu(\chi) - 1 \right| \leq \int_{\widehat{A}} |\chi(a) - 1| d\mu(\chi) < \varepsilon$$

holds for all $a \in A$. □

A key point in the paper of de Cornulier and Tessera is that, in the previous theorem, every probability measure $\mu$ satisfying $(M_1)$ and $(M_2)$ may be assumed $G$-quasi-invariant (cf. $(P) \Rightarrow (PQ)$, [CT]). Observe that with this additional assumption, the proof of the previous theorem generalizes to locally compact groups.

**Proposition 6.3** (Prop. 3, [Oza])**.** *The following statements are equivalent.*

$(T_P)$ *The pair $(G \ltimes A, A)$ has relative property $(T_P)$;*

$(T)$ *The pair $(G \ltimes A, A)$ has relative property $(T)$;*

(M) *For any $\varepsilon > 0$, there is a pair $(Q, \delta)$, where $Q$ is a compact subset of $G$ and $\delta > 0$, such that, for any Borel probability measure $\mu$ on the Borel $\sigma$-algebra $\mathscr{B}(\widehat{A})$ that satisfies*

    $(M_1)$ *$\|g_*\mu - \mu\| \leq \delta$ for every $g \in Q$ and*

    $(M_2)$ *$\left|\int_{\widehat{A}} \chi(a) d\mu(\chi) - 1\right| < \delta$ for every $a \in Q \cap A$,*

    *then we have, for every $a \in A$,*

$$\left|\int_{\widehat{A}} \chi(a) d\mu(\chi) - 1\right| < \varepsilon.$$

*Proof.* By Theorem 4.4, rel. $(T_P)$ yields rel. $(T)$.

$(T) \Rightarrow (M)$. Following [CT], we may assume $\mu$ to be $G$-quasi-invariant. The conclusion then follows by the proof of Theorem 6.1.

$(M) \Rightarrow (T_P)$. Let $\varepsilon > 0$ and $(Q, \delta_0)$ such that for a Borel probability measure $\mu$ satisfying $(M_1)$ and $(M_2)$,

$$\left|\int_{\widehat{A}} \chi(a) d\mu(\chi) - 1\right| < \varepsilon. \tag{6.1}$$

As in the proof of Theorem 5.1, we associate to any normalized continuous kernel of positive type $\theta$ satisfying $(P_1)$ and $(P_2)$ for $(Q, \delta)$, a probability measure $\mu_{\bar{\theta}^e}$ on $\mathscr{B}(\widehat{A})$ that satisfies

$$\sup_{g \in Q} \|g_*\mu_{\bar{\theta}^e} - \mu_{\bar{\theta}^e}\| < 2\delta_1 \qquad \text{and} \qquad \sup_{a \in Q \cap A} \left|\int_{\widehat{A}} \chi(a) d\mu_{\bar{\theta}^e}(\chi) - 1\right| < 2\delta_1$$

where $\delta_1$ is a positive real number depending only on $\delta$. Picking $\delta_1 < \delta_0/2$, it follows by 6.1 that for any $x, y \in A$,

$$|\theta(x,y) - 1| \leq \|\bar{\theta}^e - \theta\|_{\mathrm{cb}} + |\bar{\theta}^e(x,y) - 1| < \delta_1 + \left|\int_{\widehat{A}} \chi(x^{-1}y) d\mu(\chi) - 1\right| < \delta_1 + \varepsilon.$$

$\square$

**Corollary 6.4.** *All relative properties in Theorem 4.4 are equivalent for pairs $(G \ltimes A, A)$.*

Let R be a commutative ring with unit, let $n \geq 3$, and denote by $\mathrm{EL}_n(\mathrm{R})$ the subgroup of $\mathrm{SL}_n(\mathrm{R})$ generated by elementary matrices. The group $\mathrm{EL}_n(\mathrm{R})$ is called **boundedly elementary generated** if there exists some finite $N$ such that any matrix in $\mathrm{EL}_n(\mathrm{R})$ may be written as a product of at most $N$ elementary matrices. We assume that R is moreover finitely generated, and with a topology. Y. Shalom proved in [Sh2] that the pair $(\mathrm{EL}_2(\mathrm{R}) \ltimes \mathrm{R}^2, \mathrm{R}^2)$ has rel. property (T). Then it admits rel. (TTT) so that $\mathrm{EL}_n(\mathrm{R})$ has property (TTT) (compare with Lemma 5.3 and Thm. 5.4).

# 7. INHERITANCE OF PROPERTY $(\mathrm{T_P})$ TO LATTICES

The goal of this section is to present the proof of

**Theorem 7.1** (Thm. 6, [Oza]). *Let $G$ be a locally compact group and let $\Gamma \leqslant G$ be a lattice. Then, $G$ has property $(\mathrm{T_P})$ if and only if $\Gamma$ has property $(\mathrm{T_P})$.*

## 7.1. Canonical cohomology classes of lattices.

Take $(X, \mu)$ to be a standard probability space[6] and let $G$ and $H$ be locally compact groups with a group action $G \curvearrowright X$ given by $g.x = g^{-1}x$. An **$H$-valued cocycle over the action $G \curvearrowright X$** is a Borel map $\beta : X \times G \to H$ that satisfies, for all $g$, $h \in G$ and a.e. $x \in X$, the **cocycle identity**

$$\beta(x, gh) = \beta(x, g)\beta(g^{-1}x, h). \tag{7.1}$$

Trivially, any homomorphism $\varphi : G \to H$ defines a "constant" cocycle via $\beta(\cdot, g) := \varphi(g)$. Two $H$-valued cocycles $\alpha$ and $\beta$ over $G \curvearrowright X$ are **cohomologous** if the functional equation

$$\beta(x, g) = \sigma(x)^{-1}\alpha(x, g)\sigma(g^{-1}x)$$

is satisfied by some Borel map $\sigma : X \to H$. Conversely, given an $H$-valued cocyle $\alpha$ over $G \curvearrowright X$, any Borel map $\sigma : X \to H$ defines via the above equation a cocycle $\beta$ cohomologous to $\alpha$.

A **lattice** $\Gamma$ of a group $G$ is a discrete subgroup $\Gamma \leqslant G$ such that the quotient $G/\Gamma$ carries finite invariant measure. By rescaling, we may assume that $|G/\Gamma| = 1$ and, for convenience, we adopt Ozawa's notation $X := G/\Gamma$. Then $X$ is a standard probability space and the invariance of the probability measure implies that $G \curvearrowright X$ is measure preserving, that is, the measure of any Borel set is left invariant by the action of $G$.

Let $\sigma : X \to G$ be a Borel cross-section for the canonical projection $\pi : G \to X$, that is, a Borel map such that the following diagram commutes :

$$\begin{array}{ccc} & & G \\ & \sigma \nearrow & \downarrow \pi \\ X & \xrightarrow{\mathrm{id}} & G/\Gamma \end{array} \tag{7.2}$$

or, in other words, $\sigma(x)\Gamma = x$ for every $x \in X$. Consider the induced Borel cocycle $\beta : X \times G \to \Gamma$ given by $\beta(x, g) = \sigma(x)^{-1}g\sigma(g^{-1}x)$. Note that $\beta$..

(a) ..is independent of the choice of the cross-section (in the sense that different choices of the cross-section $\sigma$ lead to cohomologous cocycles) ;

---

[6] Refer to Appendix B for the definition of a standard probability space.

(b) ..is $\Gamma$-valued *(since $\sigma(x)^{-1}x$ is trivial in X, $\beta(x,g)\Gamma = \sigma(x)^{-1}x\Gamma = \Gamma$ for all $g \in G$ and a.e. $x \in X$)* ;

(c) ..has in fact range $\Gamma$ : Any $\gamma \in \Gamma$ may be written $\gamma = \beta(x, \sigma(x)\gamma\sigma(y)^{-1})$ for a.e. $(x,y) \in X \times X$.

The proof of the next theorem is the object of the second part of this section. It is the main technical result needed to establish Theorem 7.1. In that sense, it plays the rôle of the cohomological induction argument used for the inheritance of property (TT) to uniform lattices (compare with subsection 2.2). Notice how quasi-cocycles have been replaced by some non-additive type of length function on the measured groupoid $X \times G$. The theorem gives a sufficient condition for these "cocycles" to be bounded in some sense.

**Theorem 7.2** (Thm. C, [Oza]). *Let $G \curvearrowright X$ be a measure preserving action of a locally compact group G on a standard probability space X. Let $C \geq 1$ and let $\ell : X \times G \to \mathbb{R}_+$ be a measurable map such that*

$$\ell(x, gh) \leq C\left(\ell(x,g) + \ell(g^{-1}x, h)\right) \quad a.e.$$

*If*

$$D := \limsup_{n \to \infty} \int_G \int_X \ell(x,g) dx d\mu^{*n}(g) < +\infty,$$

*then there exists a function $h \in L^1(X)$ with $\|h\|_1 \leq 4C^2D$ and such that*

$$\ell(x,g) \leq h(x) + h(g^{-1}x) \quad a.e.$$

A **semi-length function** on G is a Borel map $\ell : G \to \mathbb{R}_+ := [0, +\infty]$ such that

(i) $\ell(\mathrm{id}_G) = 0$     (ii) $\ell(g) = \ell(g^{-1})$ for all $g \in G$     (iii) $\ell(gh) \leq \ell(g) + \ell(h)$.

We will speak of **semi-length-like function** whenever $\ell$ satisfies only (iii).

**Corollary 7.3** (Coro. 9, [Oza]). *Let $\Gamma$ be a lattice of a locally compact group G and let $\ell$ be a semi-length-like function on $\Gamma$. Take a Borel cocycle $\beta : X \times G \to \Gamma$ induced by some cross-section $\sigma : X \to G$ as above. If the induced semi-length-like function*

$$L(g) = \int_X \ell(\beta(x,g)) dx$$

*on G is essentially bounded, then $\ell$ is bounded on $\Gamma$.*

*Proof.* The composition $\ell \circ \beta$ satisfies the conditions of Theorem 7.2. Consequently, there exists a map $h \in L^1(X)$ such that

$$\ell(\gamma) = \ell(\beta(x, \sigma(x)\gamma\sigma(y)^{-1})) \leq h(x) + h(y)$$

for all $\gamma \in \Gamma$ and a.e. $x, y \in X$, and such that $\|h\|_1 \leq 4 \|L\|_\infty$. By restricting our attention to the subset of non-negligible measure $X_0 := \{x \in X : h(x) < 5 \|L\|_\infty\}$, we obtain that $\ell$ is bounded on $\Gamma$ by $10 \|L\|_\infty$. $\qquad\square$

## 7.2. **Inheritance of** (TTT) **to cocompact lattices.**

The arguments to prove that a lattice $\Gamma \leqslant G$ inherits property $(T_P)$ or property (TTT) (the latter under the additional requirement that $\Gamma$ is uniform, i.e. $G/\Gamma$ is compact) are in essence the same. However, as on account of Corollary 7.3, the setting appears quite intuitive in the case of property (TTT), we present here a full account of this special case by way of an introduction to the proof of Theorem 7.1.

**Proposition 7.4.** *Let $G$ be a locally compact group with property* (TTT) *and let $\Gamma \leqslant G$ be a uniform lattice. Then $\Gamma$ has property* (TTT).

*Proof.* Let $b$ be a wq-cocycle $\Gamma$ with finite defect $D := \sup_{\gamma, \gamma' \in \Gamma} \|b(\gamma\gamma') - (b(\gamma) + \pi(\gamma)b(\gamma'))\|_{\mathscr{H}}$. Our aim is to show that this wq-cocycle is bounded on $\Gamma$. Observe that for all $\gamma, \gamma' \in \Gamma$,

$$\|b(\gamma\gamma')\| \leq \|b(\gamma)\| + \|b(\gamma')\| + D,$$

so that $\ell(\gamma) = \|b(\gamma)\| + D$ defines a semi-length-like function on $\Gamma$. It suffices then to show that $\ell$ is bounded.

Recall the construction of the class of cocycles with range $\Gamma$ introduced at the beginning of the section ((c)). We take care to choose a regular cross-section[7] and define

$$\widetilde{b} : G \to L^2(X, \mathscr{H}), \quad \widetilde{b}(g)(x) = b(\beta(x,g))$$

together with

$$\widetilde{\pi} : G \to \mathscr{U}(L^2(X, \mathscr{H})), \quad (\widetilde{\pi}(g)\xi)(x) = \pi(\beta(x,g))\xi(g^{-1}x).$$

Since we suppose that $\Gamma$ is a uniform lattice, that $\beta$ is defined by a regular cross-section and that $b$ is locally bounded, hence bounded on the closure of $\{\beta(x,g) : x \in X\}$, $\widetilde{b}$ defines a square-integrable wq-cocycle with defect $\leq D$. Since $G$ has property (TTT), the wq-cocycle $\widetilde{b}$ is bounded and

$$L(g) = \int_X \ell(\beta(x,g))dx = \int_X \left\|\widetilde{b}(g)(x)\right\| dx + D$$

is essentially bounded. We conclude with Corollary 7.3. $\qquad\square$

**Corollary 7.5.** *Fix $n \geq 3$ and let $\mathbb{K}$ be a local field. The group $\mathrm{SL}_n(\mathbb{K})$ and its cocompact lattices have property* (TTT).

---

[7] Regular in the sense that $\sigma(K)$ is relatively compact in $G$ for any compact subset $K \subset X$. We may assume that we can pick a regular cross-section, cf. [Mac].

### 7.3. **Proof of Theorem 7.1.**

$$G \text{ with } (\mathrm{T_P}) \Rightarrow \Gamma \text{ with } (\mathrm{T_P})$$

Let $\varepsilon > 0$, let $(Q, \delta)$ be a pair for which any normalized kernel of positive type on $G$ satisfies the conditions of property $(\mathrm{T_P})$ and take a normalized kernel of positive type $\theta$ on $\Gamma$ to satisfy

$$\sup_{\gamma \in \Gamma} \|\gamma.\theta - \theta\| < \delta/2 \qquad \text{and} \qquad \sup_{\gamma^{-1}\gamma' \in Q'} |\theta(\gamma, \gamma') - 1| < \delta/2, \tag{7.3}$$

where we leave the compact subset $Q' \subset \Gamma$ to be determined later. We aim to show that

$$\sup_{\gamma, \gamma' \in \Gamma} |\theta(\gamma, \gamma') - 1|$$

is bounded by a positive constant depending on $\varepsilon$. Recall that by Proposition 4.1, there exists a map of unit norm $P \in L^{\infty}(G, \mathscr{H}_\theta)$ so that we observe the rough estimate

$$\sup_{\gamma, \gamma' \in \Gamma} |\theta(\gamma, \gamma') - 1| = \sup_{\gamma, \gamma' \in \Gamma} |\langle P(\gamma) - P(\gamma'), P(\gamma') \rangle| \leq 2 \sup_{\gamma \in \Gamma} \|P(\gamma) - P(e)\|.$$

Take a cocycle $\beta : X \times G \to \Gamma$ defined by a regular cross-section $\sigma$. Since any lattice element $\gamma$ can be written $\gamma = \beta(x, \sigma(x)\gamma\sigma(y)^{-1})$ for a.e. $(x, y) \in X^2$, the term $\|P(\cdot) - P(e)\|$ defines a map $\ell : X \times G \to \mathbb{R}_+$ given by

$$\ell(x, g) = \|P(\beta(x, g)) - P(e)\| + \sqrt{\delta}$$

so it is sufficient to prove that $\ell$ is bounded by a positive constant depending on $\varepsilon$. We verify that for all $g, h \in G$ and for a.e. $x \in X$,

$$
\begin{aligned}
\ell(x, gh) &= \|P(\beta(x, g)\beta(g^{-1}x, h)) - P(e)\| + \sqrt{\delta} \\
&\leq \|P(\beta(x, g)\beta(g^{-1}x, h)) - P(\beta(x, g))\| + \|P(\beta(x, g)) - P(e)\| + \sqrt{\delta} \\
&< \|P(\beta(g^{-1}x, h)) - P(e)\| + \sqrt{\delta} + \|P(\beta(x, g)) - P(e)\| + \sqrt{\delta} \\
&= \ell(g^{-1}x, h) + \ell(x, g),
\end{aligned}
$$

where the strict inequality is the result of the general inequality

$$\|P(\gamma\gamma') - P(\gamma)\|^2 < \|P(\gamma') - P(e)\|^2 + \delta$$

for all $\gamma, \gamma' \in \Gamma$. The semi-length-like function $\ell$ is connected to the induced normalized kernel of positive type on $G$

$$\widetilde{\theta}(g, h) = \int_X \theta(\beta(x, g), \beta(x, h)) dx \qquad (g, h \in G)$$

by the inequality

$$\int_X \ell(x,g)dx \leq \left[\int_X \|P(\beta(x,g)) - P(e)\|^2 dx\right]^{1/2} + \sqrt{\delta}$$

$$\leq \sqrt{2} \left|1 - \int_X \theta(\beta(x,g),e)dx\right|^{1/2} + \sqrt{\delta}$$

that holds for every $g \in G$. As we assume $G$ to have property $(T_P)$, the latter term is bounded if $\widetilde{\theta}$ satisfies $(P_1)$ and $(P_2)$. For the time being, admit this is the case, i.e.

$$\int_X \ell(x,g) \, dx < \sqrt{2\varepsilon} + \sqrt{\delta} < 3\sqrt{\varepsilon}.$$

Then, we conclude with Theorem 7.2 that there exists a function $h \in L^1(X)$ satisfying $\|h\|_1 \leq 12\sqrt{\varepsilon}$ and such that $\ell(x,g) \leq h(x) + h(g^{-1}x)$ for a.e. $(x,g) \in X \times G$. Because any $\gamma \in \Gamma$ can be written $\gamma = \beta(x, \sigma(x)\gamma\sigma(y)^{-1})$ for a.e. $(x,y) \in X^2$, restricting our attention to the subset of non-negligible measure $X_0 := \{x \in X : h(x) < 13\sqrt{\varepsilon}\} \subset X$, we obtain

$$\sup_{\gamma \in \Gamma} \|P(\gamma) - P(e)\| \leq \sup_{\gamma \in \Gamma} \left[\ell(x, \sigma(x)\gamma\sigma(y)^{-1})\right] \leq h(x) + h(y) < 26\sqrt{\varepsilon}.$$

Thus

$$\sup_{\gamma,\gamma' \in \Gamma} \left|\theta(\gamma,\gamma') - 1\right| < 52\sqrt{\varepsilon}.$$

We still have to show that $\widetilde{\theta}$ satisfies $(P_1)$ and $(P_2)$. Set $Q' = \{\beta(x,g) : x \in X_1, g \in Q\}$ where we choose $X_1$ a compact subset of $X$ to satisfy $|X_1| > 1 - \frac{\delta}{4}$. Since $\beta$ is defined by a regular cross-section, the set $Q' \subset \Gamma$ is finite. The conditions imposed on $\theta$ in 7.3 imply that for all $g_0, g, h \in G$,

$$\left|g_0.\widetilde{\theta}(g,h) - \widetilde{\theta}(g,h)\right| = \int_X \left[\theta(\beta(x,g_0^{-1}g), \beta(x,g_0^{-1}h)) - \theta(\beta(x,g), \beta(x,h))\right] dx$$

$$= \int_X \left[\beta(g_0 x, g_0).\theta(\beta(g_0 x, g), \beta(g_0 x, h)) - \theta(\beta(g_0 x, g), \beta(g_0 x, h))\right] dx$$

$$\leq \int_X \|\beta(g_0 x, g_0).\theta - \theta\|_{cb} \, dx$$

$$< \frac{\delta}{2}$$

and, since $\beta(gx,g)^{-1}\beta(gx,h) = \beta(x,g^{-1}h)$, we have, for all $g, h \in G$ such that $g^{-1}h \in Q$,

$$\left|\widetilde{\theta}(g,h) - 1\right| \leq \int_X |\theta(\beta(gx,g), \beta(gx,h)) - 1| \, dx$$

$$\leq \int_{X_1} |\theta(\beta(gx,g), \beta(gx,h)) - 1| \, dx + 2|X - X_1|$$

$$< \frac{\delta}{2} + \frac{\delta}{2}.$$

$$\Gamma \text{ with } (T_P) \Rightarrow G \text{ with } (T_P)$$

Let $\varepsilon > 0$ and assume there exists a pair $(Q, \delta)$ for which $\Gamma$ has property $(T_P)$. Let $\beta$ be a cocycle induced by some cross-section $\sigma$ and assume $Q'$ to be a symmetric compact subset of $X$ such that $|\sigma(X) \cap Q'| > 1 - \varepsilon$ and such that $Q \subset Q'$. If we take $\theta$ be a normalized kernel of positive type on $G$ that satisfies $(P_1)$ and $(P_2)$ for $(Q', \delta)$, then, by Lemma 4.3,

$$\sup_{g,h \in \sigma(X) \cap Q'} \|\theta(\cdot g, \cdot h) - \theta(\cdot, \cdot)\| < 2\sqrt{2}\delta^{1/2}$$

and consequently, for every $g, h \in G$,

$$
\begin{aligned}
\left| \theta(g,h) - \int_X \theta(\beta(x,g), \beta(x,h)) dx \right| &\leq \int_X \left[ \|\sigma(x).\theta - \theta\|_{\mathrm{cb}} + \left| \theta(g\sigma(g^{-1}x), h\sigma(h^{-1}x)) - \theta(g,h) \right| \right] dx \\
&\leq \delta + 2\sqrt{2}\delta^{1/2} + 2\|\theta\|_{\mathrm{cb}} \left| X \setminus (\sigma(X) \cap Q') \right| \\
&< \delta + 2\sqrt{2}\delta^{1/2} + 2\varepsilon.
\end{aligned}
$$

Because $Q \subset Q'$ and we assumed that $\Gamma$ has property $(T_P)$, it follows that $|\theta(\gamma, \gamma') - 1| < \varepsilon$ for all $\gamma, \gamma' \in \Gamma$ and

$$
\begin{aligned}
\sup_{g,h \in G} |\theta(g,h) - 1| &\leq \sup_{g,h \in G} \left| \theta(g,h) - \int_X \theta(\beta(x,g), \beta(x,h)) dx \right| + \int_X |\theta(\beta(x,g), \beta(x,h)) - 1| dx \\
&< \delta + 2\sqrt{2}\delta^{1/2} + 3\varepsilon.
\end{aligned}
$$

Hence a lattice $\Gamma \leqslant G$ induces property $(T_P)$ on the ambient group $G$. $\qquad\square$

7.4. **Bi-$\mu$-harmonic functions.** Let $(G, \mu)$ be a standard probability space, where $G$ is a locally compact group, and let $E$ be a separable Banach space on which $G$ acts continuously by linear isometries. There is an induced $G$-action on the Lebesgue-Bochner[8] space $L^\infty(G, E)$ given by $(g.f)(x) = g\left( f(g^{-1}x) \right)$ for every $f \in L^\infty(G, E)$ and $\mu$-a.e. $g \in G$. More generally, a **coefficient $G$-module** is the dual of a separable Banach space on which $G$ acts continuously by linear isometries. We underline that the separable predual in question is part of the data, i.e. should be specified in the definition. A coefficient $G$-module then comes with a specific weak* topology. In particular, the $G$-action induced on the coefficient $G$-module acts by weakly* continuous linear isometries. For example, $L^\infty(G, E) = L^1(G, E)^*$ is a coefficient $G$-module.

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[8] Refer to Appendix A for a review of Bochner integration.

Following Ozawa, we define the left and right convolution equations

$$(\mu * f)(g) \;=\; \int_G sf(s^{-1}g)\,d\mu(s) \quad \text{and} \quad (f * \mu)(g) \;=\; \int_G f(gt^{-1})\,d\mu(t). \qquad (7.4)$$

Both $\mu * f$, $f * \mu \in L^\infty(G,E)$. If we assume $\mu$ to be a symmetric probability measure, then $\mu$-Bochner integrable functions satisfying $(f * \mu)(g) = f(g)$ for all $g \in G$ are exactly the $\mu$-harmonic functions in $L^\infty(G,E)$.

Recall that a $\mu$-Bochner integrable function $f : G \to E$ is said to be $\mu$**-harmonic** if it satisfies the mean-value property

$$f(g) \;=\; \int_G f(gt)\,d\mu(t)$$

for all $g \in G$. The vector space of all bounded $\mu$-harmonic functions will be denoted by $\mathscr{H}_\mu^\infty(G,E)$. We sketch a construction due to Furstenberg (as in Prop. 2.9, [Fur]) of a class of bounded $\mu$-harmonic functions. Let $(X,\nu)$ be a standard probability space on which $G$ acts measurably and such that the measure $\nu$ is $\mu$-stationary, i.e. $\mu * \nu = \nu$.[9] Then there is an assignment $F_{(X,\nu)} : L^\infty(X,E) \to \mathscr{H}_\mu^\infty(G,E)$, $\phi \mapsto f_\phi$, where the bounded $\mu$-harmonic function $f_\phi$ is given by

$$f_\phi(g) \;=\; \int_X \phi(gx)\,d\nu(x).$$

Under some conditions on $\mu$, there exists a particular standard probability space $(X,\nu)$ such that the assignment above is an isometric isomorphism of Banach spaces. This (uniquely defined) measure space is called the **Poisson boundary** of $(G,\mu)$. More precisely, if the probability measure $\mu$ is..

(i) ..non-degenerate, i.e. the semi-group generated by the support of $\mu$ is $G$, and

(ii) ..spread out, i.e. for some $n$, the convolution power $\mu^{*n}$ is non-singular with respect to a Haar measure on $G$,

then $(G,\mu)$ admits a unique Poisson boundary $(B,\nu)$ such that $L^\infty(B,E) \cong \mathscr{H}_\mu^\infty(G,E)$.

We shortly discuss ergodic behavior with respect to the Poisson boundary. Let $G$ act measurably on a standard probability space $(X,\nu)$. The action is called **measure preserving** if $\nu(gA) = \nu(A)$ for all $g \in G$ and any Borel subset $A \subseteq X$. A measure preserving $G$-action is **ergodic** if any $G$-invariant Borel subset $A \subseteq X$ satisfies $\nu(A)\nu(X \setminus A) = 0$. That is, the space can not be broken down

---

[9] The convolution of $\mu$ with another probability measure $\nu$ on $G$ is the image of the product $\mu \otimes \nu$ under the multiplication $(g,h) \mapsto gh$, that is, more explicitely, for any Borel subset $A \subseteq G$,

$$\mu * \nu\,(A) = \int_{G \times G} 1_A(gh)\,d\mu \otimes \nu(g,h) = \int_G \mu(Ah^{-1})\,d\nu(h) = \int_G \nu(g^{-1}A)\,d\mu(g).$$

In particular, the convolution product is associative. Convolutions can be generalized to measurable $G$-spaces, i.e. locally compact spaces on which $G$ acts measurably, with probability measures. Let $(X,\nu)$ be a probability space endowed with a measurable $G$-action, then $\mu * \nu$ denotes the image of $\mu \otimes \nu$ under the $G$-action, i.e. $\mu * \nu\,(A) = \int 1_A(g.x)d\mu \otimes \nu(g,x)$. Note that $G$ acting by continuous left multiplication on itself is trivially a $G$-space and, in this case, $\mu * \nu$ denotes the usual convolution of probability measures on $G$, as defined above.

into smaller subsystems on each of which the group would act separately. Equivalently, a measure preserving $G$-action is ergodic if any $G$-invariant measurable function on $B$ is constant a.e. Let $E$ be a coefficient $G$-module. A measure preserving $G$-action is called **doubly $E$-ergodic** if any weak$^*$ measurable $G$-equivariant[10] function $f : X \times X \to E$ is constant a.e. Observe that if $E \neq 0$ is a separable trivial coefficient $G$-module, then double $E$-ergodicity is nothing more than ergodicity of the diagonal action on $X \times X$. For more general coefficient $G$-module, double ergodicity is however a stronger requirement (cf. [Mon]).

**Theorem 7.6** (Thm. 3, [Kai]). *Let $G$ be a locally compact group, let $E$ be a separable coefficient $G$-module and let $\mu$ be a non-degenerate, spread out, symmetric probability measure on $G$. Denote by $(B, \nu)$ the Poisson boundary associated to $(G, \mu)$. Then the measure preserving action $G \curvearrowright B$ is doubly $E$-ergodic.*

We call a $\mu$-Bochner integrable function $f : G \to E$ satisfying $\mu * f = f = f * \mu$ for the convolution equations given in 7.4 a **bi-$\mu$-harmonic function** and we denote by $\mathcal{H}^\infty_{\mathrm{bi}-\mu}(G, E)$ the vector space of bounded bi-$\mu$-harmonic $E$-valued functions on $G$.

**Lemma 7.7.** *Let $(G, \mu)$ be a standard probability space where $\mu$ is taken non-degenerate, spread-out and symmetric. Let $E$ be a separable coefficient G-module. Then $\mathcal{H}^\infty_{\mathrm{bi}-\mu}(G, E) = E$.*

*Proof.* We associate to $(G, \mu)$ its Poisson boundary $(B, \nu)$. Then $(B \times B, \nu \otimes \nu)$ is the Poisson boundary of $(G \times G, \mu \otimes \mu)$ (Coro. 3.2, [BS]). Let $f \in \mathcal{H}^\infty_{\mathrm{bi}-\mu}(G, E)$ and define $\widetilde{f} : G \times G \to E$, $(g, h) \mapsto (g.f)(h) = g\left(f(g^{-1}h)\right)$. For any $g, h \in G$,

$$
\begin{aligned}
\int_G \int_G \widetilde{f}(gs^{-1}, ht^{-1}) \, d\mu(s) d\mu(t) &= \int_G \int_G gs^{-1} f(sg^{-1}ht^{-1}) \, d\mu(s) d\mu(t) \\
&\overset{A.1}{=} \int_G g \int_G s^{-1} f(sg^{-1}ht^{-1}) d\mu(s) d\mu(t) \\
&= \int_G g f(g^{-1}ht^{-1}) d\mu(t) \\
&= \widetilde{f}(g, h).
\end{aligned}
$$

Therefore, $\widetilde{f}$ is a $G$-equivariant function in $\mathcal{H}^\infty_{\mu \otimes \mu}(G \times G, E) \cong L^\infty(B \times B, E)$. By the double ergodicity of the Poisson boundary (Theorem 7.6), $\widetilde{f}$ is constant a.e. and its value is a $G$-invariant vector in $E$. $\qquad\square$

We end this subsection with the construction, after Ozawa, of a class of bounded bi-$\mu$-harmonic functions. Let $(X, \mu)$ be a standard probability space and let $(f_n)_{n \in \mathbb{N}_0}$ be a bounded sequence in

---

[10] The action $G \curvearrowright X \times X$ is taken to be the diagonal action $g(x, y) = (gx, gy)$ for all $x, y \in X$ and all $g \in G$.

the dual Banach space $L^\infty(X,\mu) = L^1(X,\mu)^*$. Since $\mathbb{Z}$ is an amenable group (cf. subsection 2.3), there exists an invariant mean on $\ell^\infty(\mathbb{Z})$. As $\ell^\infty(\mathbb{N}_0)$ embedds naturally in $\ell^\infty(\mathbb{Z})$, we may fix an invariant mean $\mathrm{Lim} : \ell^\infty(\mathbb{N}_0) \to \mathbb{R}$. Define $\underset{n}{\mathrm{Lim}}\, f_n$ to be the unique element in $L^\infty(X,\mu)$ such that

$$\left\langle \underset{n}{\mathrm{Lim}} f_n, \xi \right\rangle = \underset{n}{\mathrm{Lim}} \ \langle f_n, \xi \rangle$$

for all $\xi \in L^1(X,\mu)$. By duality, $\left\| \underset{n}{\mathrm{Lim}}\, f_n \right\| \leq \underset{n}{\mathrm{Lim}} \|f_n\| < +\infty$.

**Lemma 7.8.** [Oza] *Let $(f_n)_{n\in\mathbb{N}_0}$ be a bounded sequence in $L^\infty(X,\mu)$. Then the limit along an invariant mean on $\mathbb{N}_0$ with respect to the weak$^*$ topology on $L^\infty(X,\mu)$ commutes with weak$^*$ continuous operators $\Psi : L^\infty(X) \to W = (W_*)^*$ into dual Banach spaces. In other words,*

$$\Psi(\underset{n}{\mathrm{Lim}}\, f_n) = \underset{n}{\mathrm{Lim}}\, \Psi f_n.$$

*Proof.* For every $\xi \in W_*$,

$$\left\langle \Psi\left(\underset{n}{\mathrm{Lim}}\, f_n\right), \xi \right\rangle = \left\langle \underset{n}{\mathrm{Lim}}\, f_n, \Psi^*\xi \right\rangle = \underset{n}{\mathrm{Lim}} \langle f_n, \Psi^*\xi \rangle = \left\langle \underset{n}{\mathrm{Lim}}\, \Psi(f_n), \xi \right\rangle.$$

$\square$

Let $G$ be a locally compact group and let $\mu$ be a probability measure on the Borel $\sigma$-algebra of $G$. Denote the $n$-fold convolution product of a probability measure by $\mu^{*n} = \underbrace{\mu * \mu \cdots * \mu}_{n\text{-times}}$, for $n \geq 1$. Fix a bounded function $f$ in $L^\infty(G,\mu)$ around which we define a bounded double sequence $(\mu^{*m} * f * \mu^{*n})_{m,n\in\mathbb{N}_0}$ in $L^\infty(G,\mu)$. Explicitly, for each $m,n \in \mathbb{N}_0$ and any $g \in G$, this spells out

$$(\mu^{*m} * f * \mu^{*n})\,(g) = \int_{G\times G} s f(s^{-1}gt^{-1})\, d\mu^{*m}(s)\, d\mu^{*n}(t).$$

Since $\mathbb{Z}^2$ is amenable, we may fix an invariant mean on $\ell^\infty(\mathbb{N}_0^2)$ and set $F := \underset{m,n}{\mathrm{Lim}}(\mu^{*m} * f * \mu^{*n})$ to be the limit taken along the invariant mean with respect to the weak$^*$ topology. Left and right convolutions by $\mu$ define operators $\Psi_\mu(f) = \mu * f$ and $\Psi^\mu(f) = f * \mu$ that are weak$^*$ continuous so that, by the previous lemma, we have, for instance,

$$\mu * F = \underset{m,n}{\mathrm{Lim}}\left(\mu^{*m+1} * f * \mu^{*n}\right).$$

As the limit is taken along a mean that is invariant, it follows that $\mu * F = F = F * \mu$. Hence $F \in \mathscr{H}^\infty_{\mathrm{bi}-\mu}(G)$.

7.5. **Proof of Theorem 7.2.** Fix some constant $R > 0$ and set $\ell_R(\cdot,\cdot) := \min\{\ell(\cdot,\cdot),R\}$. By the construction of the previous subsection,

$$f_R = \underset{m,n}{\mathrm{Lim}}\left(\mu^{*m} * \ell_R * \mu^{*n}\right)$$

is a bounded bi-$\mu$-harmonic function in $L^\infty(X \times G)$. By Fubini, there is a canonical isometric isomorphism $L^\infty(X \times G) \cong L^\infty(G, L^\infty(X))$. Moreover $L^\infty(X)$ embedds equivariantly in $L^2(X)$, which is a separable coefficient $G$-module. Then, by Lemma 7.7, $f_R$ is a.e. constant on $G$.

Observe that we have

$$
\begin{aligned}
\ell_R(x,g) &= \liminf_{m,n} \int_{G \times G} \ell_R(x,g) d\mu^{*m}(s) d\mu^{*n}(t) \\
&= \liminf_{m,n} \int_{G \times G} \ell_R(x, ss^{-1}g) d\mu^{*m}(s) d\mu^{*n}(t) \\
&\leq C \liminf_{m,n} \int_{G \times G} \left[ \ell_R(x,s) + \ell_R(s^{-1}x, s^{-1}g) \right] d\mu^{*m}(s) d\mu^{*n}(t) \\
&\leq C^2 \liminf_{m,n} \int_{G \times G} \left[ \ell_R(x,s) + \ell_R(s^{-1}x, s^{-1}gt^{-1}) + \ell_R(tg^{-1}x,t) \right] d\mu^{*m}(s) d\mu^{*n}(t) \\
&\leq C^2 \liminf_{n \to \infty} \int_G \left[ \ell_R(x,t) + \ell_R(tg^{-1}x,t) \right] d\mu^{*n}(t) + C^2 f_R(x)
\end{aligned}
$$

for any $(x,g) \in X \times G$. Then the function $h : X \to \mathbb{R}$ defined by

$$
h(x) = C^2 \liminf_{n \to \infty} \int_G \left[ \ell(x,t) + \ell(t^{-1}x,t) \right] d\mu^{*n}(t) + \frac{C^2}{2} \lim_{R \to \infty} f_R(x)
$$

satisfies $\ell_R(x,g) \leq h(x) + h(g^{-1}x)$ for a.e. $(x,g) \in X \times G$ and for any $R > 0$. We finally show that $h \in L^1(X)$. First, since the $G$-action is measure preserving, we see that

$$
\begin{aligned}
\int_X \liminf_{n \to \infty} \int_G \left[ \ell(x,t) + \ell(t^{-1}x,t) \right] d\mu^{*n}(t) dx &\leq \liminf_{n \to \infty} \int_G \int_X \left[ \ell(x,t) + \ell(t^{-1}x,t) \right] dx d\mu^{*n}(t) \\
&\leq 2D,
\end{aligned}
$$

where we used Fatou and Fubini in the first inequality. Secondly, as $f_R$ is increasing in $R$, we use the monotone convergence Theorem to isolate the integrand from the limit

$$
\int_X \lim_{R \to \infty} f_R(x) dx \leq \lim_{R \to \infty} \int_X f_R(x) dx
$$

and, picking a subset $A \subset \{ g \in G : \int_X \ell(x,g) fx \leq 2D \}$ of unit measure, this is

$$
\begin{aligned}
\int_{X \times A} f_R(x,g) \, d(x,g) &= \operatorname{Lim}_{m,n} \int_{X \times A} \int_{G^2} \ell_R(s^{-1}x, s^{-1}gt^{-1}) \, d\mu^{*m}(s) d\mu^{*n}(t) d(x,g) \\
&\leq C^2 \operatorname{Lim}_{m,n} \int_{X \times A} \int_G \ell(s^{-1}x, s^{-1}) + \ell(g^{-1}x, s^{-1}) d\mu^{*m}(s) d(x,g) + \\
&\qquad\qquad + C^2 \int_{X \times A} \ell(x,g) d(x,g) \\
&\leq C^2 \operatorname{Lim}_{m,n} \int_G \int_{X \times A} 2\ell(x,s) \, dx d\mu^{*m}(s) + 2C^2 D \\
&\leq 4C^2 D.
\end{aligned}
$$

## Part 3. Foundations

### APPENDIX A. VECTOR-VALUED INTEGRATION

Let $E$ be a Banach space and let $(G, \mu)$ be a probability measure space. We will want to work with $E$-valued functions defined on $G$. To this purpose, we gather here some notions of the theory of Bochner integration, which appears as a natural generalization of Lebesgue integration. In fact, results from the latter carry over into this more general framework of the Bochner integral as long as they are not based on some non-negativity requirement (e.g. Fatou's Lemma, the monotone convergence Theorem).

A function $f : G \to E$ is called $\mu$-simple if it can be written as a finite sum $f = \sum v_n 1_{A_n}$, where $1_{A_n}$ denotes the indicator function

$$1_{A_n}(g) = \begin{cases} 1 & \text{if } g \in A_n \\ 0 & \text{if } g \notin A_n, \end{cases}$$

where each $A_n$ is a measurable subset $A_n \subseteq G$ of finite measure and $v_n \in E$ for all $n$. A function $f : G \to E$ is **strongly $\mu$-measurable** if there exists a sequence $(f_n)$ of $\mu$-simple functions such that $\lim f_n = f$ $\mu$-a.e. In the case of a separable Banach space $E$, all measurable functions on $G$ are strongly $\mu$-measurable. A more extensive characterization of this property, the Pettis Theorem, has as consequence that the $\mu$-a.e. limit of a sequence of strongly $\mu$-measurable functions $f_n : G \to E$ is also strongly $\mu$-measurable.

A $\mu$-**Bochner integrable** function $f : G \to E$ is a strongly $\mu$-measurable function satisfying

$$\lim_{n \to \infty} \int_G \|f_n - f\|_E \, d\mu = 0,$$

where $(f_n)$ denotes the sequence of $\mu$-simple functions converging to $f$ $\mu$-a.e. The **Bochner integral** of $f$ is then given by

$$\int_G f \, d\mu = \lim_{n \to \infty} \int_G f_n \, d\mu.$$

A more practical characterization is that a strongly $\mu$-measurable function $f : G \to E$ is $\mu$-Bochner integrable if and only if $\int \|f\|_E \, d\mu < +\infty$.

We spell out some facts, which can be directly drawn from the definitions. In all that follows, we take $f : G \to E$ to be $\mu$-Bochner integrable. Denote by $E^*$ the dual space of $E$, that is, the Banach space of all continuous linear functionals on $E$, and denote by $\langle x^*, x \rangle := x^*(x)$ the duality coupling for any $x^* \in E^*$, $x \in E$. Then

$$\left\langle x^*, \int_G f \, d\mu \right\rangle = \int_G \langle x^*, f \rangle \, d\mu$$

for each $x^* \in E^*$. It also follows that the truncated function $f \cdot 1_A$ is $\mu$-Bochner integrable for any $A \in \mathscr{B}$ and that the restriction $f|_A$ is $\mu|_A$-Bochner integrable. Finally, if $T : E \rightarrow F$ is a bounded linear operator into a Banach space $F$, then the composition $Tf$ is $\mu$-Bochner integrable and

$$T \int_G f \, d\mu \; = \; \int_G Tf \, d\mu. \tag{A.1}$$

There are adequate definitions of Lebesgue-Bochner spaces $L^p(G,E)$, for $1 \leq p \leq +\infty$, generalizing the usual Lebesgue spaces. We will concentrate only on $L^\infty(G,E)$, the vector space of (equivalence classes of) strongly $\mu$-measurable functions $f : G \rightarrow E$ which are essentially bounded. Endowed with the usual ess-sup norm, $L^\infty(G,E)$ is a Banach space. We can also see $L^\infty(G,E)$ as the dual Banach space $L^\infty(G,E) = L^1(G,E)^*$, where the duality coupling is given by the canonical isometric isomorphism assigning to each $f \in L^\infty(G,E)$ the continuous linear functional $\xi \mapsto \langle f, \xi \rangle$ on $L^1(G,E)$, where

$$\langle f, \xi \rangle := \int_G f\xi \, d\mu.$$

## APPENDIX B. STANDARD PROBABILITY SPACES

A measure space $(X, \mathscr{A}, \mu)$, i.e. the data consisting of a set $X$ and a $\sigma$-algebra $\mathscr{A}$ in $X$ on which is defined a measure $\mu$, is said to be **standard** if it is isomorphic to the union of a closed interval $([0,a], \lambda)$, for some $a > 0$, with Lebesgue measure together with at most countably many points of positive measure. In this setting, two sets $A$ and $B$ that coincide are understood to do so modulo null sets, i.e. whenever the symmetric difference $A \triangle B$ is a set of null measure. If a locally compact group $G$ is endowed with a probability measure defined on its Borel $\sigma$-algebra and together with a natural action on itself, we may take advantage of another intrinsic characterization of standard probability spaces after which $(X, \mu)$ is a standard probability space if $X$ is polish (i.e. a complete and separable metric space) and $\mu$ is a probability measure defined on the Borel $\sigma$-algebra of $X$. It is known that every locally compact Hausdorff second countable space is polish (for reference, see e.g. IX [Bou]).

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