Sphere Packings and Magical Functions

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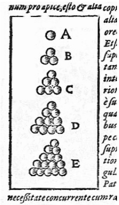
The sphere packing problem (in dimension 3)

What is the most space-efficient way to stack oranges?

How do you prove it...?



Johannes Kepler (1611): No packing of balls of the same radius in three dimensions has density greater than the cannonball packing.

Thereafter called the Kepler conjecture.

Kepler's conjecture

1611: Kepler's conjecture is stated
1831: Partial progress (Gauss)
1900: One of Hilbert's problems for the 20th century
1953: Fejes Tóth shows that proof can be reduced to a finite (but very large) number of calculations
1992: Hales and PhD student Ferguson reduce proof to solving about 100'000 linear programming problems
1998: The proof is completed in around 300 pages, over 3 GB of data, 40k lines of code. A jury of 12 experts is assigned to verify the validity of the proof
2003: Jury announces to be "99% certain" of the correctness
2005: Hales publishes a 100-pages article in the *Annals* on the non-computer part of the proof
2017: Formal (computer-verifiable) proof of Kepler's conjecture

Beyond 3 dimensions

A sphere in $\mathbb{R}^n$, with radius r and center $x_0 \in \mathbb{R}^n$, is the set of points

$$S_r(x_0) = \{x \in \mathbb{R}^n : |x - x_0| = r\}$$

Our physical world is three dimensional but our information world is very much higher dimensional.

- Google matrix has billions of dimensions (a googol is 10^{100});
- Megapixel as points in $\mathbb{R}^{3000000}$, etc.
- Sphere packings in communication theory/coding theory:

If the agreed vocabulary of signals are too close together, the error balls will intersect, leading to ambiguity.

However if there is no overlap, then there is no ambiguity.

Figure: From: Tsukerman, Communication and ball packing, Plus Mag.

Higher dimensions are highly mysterious

- Stacking optimal layers of packing in dimension $n-1$ need not lead to optimal packing in dimension n (Conway, Sloane 1995)
- We mostly have no idea what the densest sphere packings look like in high dimensions...
- ...but in dimensions 8 and 24, we have full solutions

Annals of Mathematics **185** (2017), 991–1015
https://doi.org/10.4007/annals.2017.185.3.7

The sphere packing problem in dimension 8

By Maryna S. Viazovska

Abstract

In this paper we prove that no packing of unit balls in Euclidean space $\mathbb{R}^8$ has density greater than that of the E_8-lattice packing.

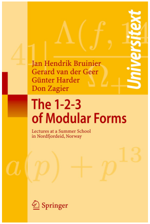
Annals of Mathematics **185** (2017), 1017–1033
https://doi.org/10.4007/annals.2017.185.3.8

The sphere packing problem in dimension 24

By Henry Cohn, Abhinav Kumar, Stephen D. Miller, Danylo Radchenko, and Maryna Viazovska

Maryna Viazovska

1984: Birth in Kyiv, Ukraine
2005: BSc in Mathematics,
Taras Shevchenko National University
2007: MSc in Mathematics, Kaiserlauten
2013: PhD under Don Zagier, Bonn



2013-2017: Postdoc, Berlin
Since 2018: Professor at EPFL
2022: Recipient of the Fields Medal

In Times of Scarcity, War and Peace, a Ukrainian Finds the Magic in Math

5

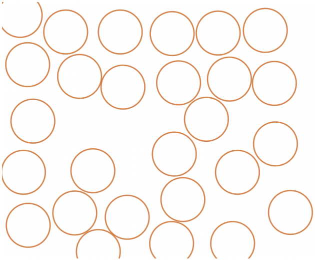
With her homeland mired in war, the sphere-packing number theorist Maryna Viazovska has become the second woman to win a Fields Medal in the award's 86-year history.

What is... a sphere packing?

- Choose a point-configuration $\{x_i\}$ in $\mathbb{R}^n$
- At each point, center a sphere $S_r(x_i)$
- All spheres should have fixed radius r and not overlap
- The corresponding sphere packing is

$$\mathcal{P} = \bigcup_{x_i} B_r(x_i)$$

- Problem: Find $\{x_i\}$ such that density of $\mathcal{P}$ is maximal



Let's consider a special type of point-configuration.
Definition: A lattice in $\mathbb{R}^n$ is the integer span

$$\Lambda = \{m_1 v_1 + \cdots + m_n v_n | m_1, \ldots, m_n \in \mathbb{Z}\}$$

of a choice of basis $\{v_1, \ldots, v_n\}$ for $\mathbb{R}^n$.

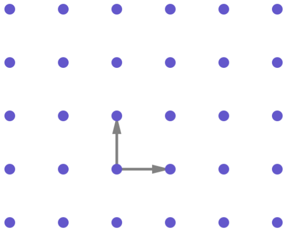


Figure: Lattice $\mathbb{Z}^2$ generated by $\{e_1, e_2\}$

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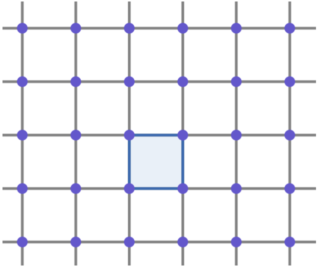


Figure: Tessellation of the plane $\mathbb{R}^2$ by the lattice $\mathbb{Z}^2$

Linear algebra: Volume $|\Lambda|$ of fundamental parallelepiped for lattice Λ is $|\Lambda| := |\det(v_1|\cdots|v_n)|$

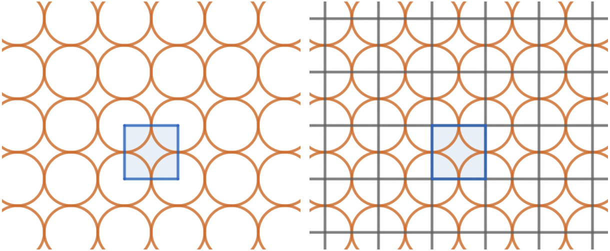


Figure: Lattice packing associated to $\mathbb{Z}^2$

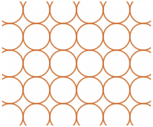
The density of a (Λ-)lattice packing $\mathcal{P}$ is given by

$$\Delta_{\mathcal{P}} = \frac{\mathrm{vol}(B_{\ell/2})}{|\Lambda|} = \frac{\ell^n \,\mathrm{vol}(B_1)}{2^n|\Lambda|}$$

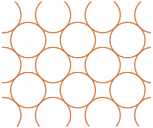
where ℓ is the length of the shortest vector v_i of Λ.

Some examples of lattice packings

Cubic lattice: $\Delta_{\mathbb{Z}^n} = \frac{\mathrm{vol}(B_1)}{2^n}$



Checkerboard lattice: $D_n = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n : \sum x_i \text{ is even}\}$,
$\ell = \sqrt{2}$, $|D_n| = 2$; $\Delta_{D_n} = \frac{\mathrm{vol}(B_1)}{2 \cdot 2^{n/2}}$



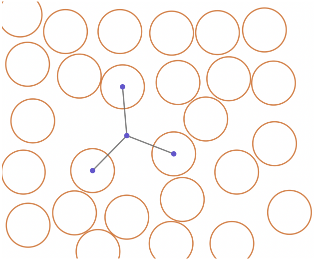
In $\mathbb{R}^8$: $E_8 = D_8 \cup (D_8 + (\frac{1}{2}, \ldots, \frac{1}{2}))$, $\ell = \sqrt{2}$, $|E_8| = 1$; $\Delta_{E_8} = \frac{\mathrm{vol}(B_1)}{2^4}$

The density of a sphere packing $\mathcal{P} = \bigcup B_r(x_i)$ is

$$\Delta_{\mathcal{P}} := \limsup_{R\to\infty} \frac{\mathrm{vol}(\mathcal{P} \cap B_R(0))}{\mathrm{vol}(B_R)}$$

The sphere packing constant in $\mathbb{R}^n$ is $\Delta_n = \sup_{\mathcal{P}\subset\mathbb{R}^n} \Delta_{\mathcal{P}}$

- "Greedy" lower bound: $\Delta_n \geq 2^{-n}$



- When n small, best known packings often lattice packings
- When n large, 2^{-n} much greater than density of known lattice packings
- Folklore: lattice packings not optimal in most high dimensions

Starting point of Viazovska's proof

Annals of Mathematics, **157** (2003), 689–714

New upper bounds on sphere packings I

By Henry Cohn and Noam Elkies*

Abstract

We develop an analogue for sphere packing of the linear programming bounds for error-correcting codes, and use it to prove upper bounds for the density of sphere packings, which are the best bounds known at least for dimensions 4 through 36. We conjecture that our approach can be used to solve the sphere packing problem in dimensions 8 and 24.

Dimension	Best Packing Known	Rogers' Bound	New Upper Bound
1	0.5	0.5	0.5
2	0.28868	0.28868	0.28868
3	0.17678	0.1847	0.18616
4	0.125	0.13127	0.13126
5	0.08839	0.09987	0.09975
6	0.07217	0.08112	0.08084
7	0.0625	0.06981	0.06933
8	0.0625	0.06326	0.06251
9	0.04419	0.06007	0.05900
10	0.03906	0.05953	0.05804
11	0.03516	0.06136	0.05932
12	0.03704	0.06559	0.06279
13	0.03516	0.07253	0.06870
14	0.03608	0.08278	0.07750
15	0.04419	0.09735	0.08999
16	0.0625	0.11774	0.10738
17	0.0625	0.14624	0.13150
18	0.07508	0.18629	0.16503
19	0.08839	0.24308	0.21202
20	0.13154	0.32454	0.27855
21	0.17678	0.44289	0.37389
22	0.33254	0.61722	0.51231
23	0.5	0.87767	0.71601
24	1.0	1.27241	1.01998
25	0.70711	1.8798	1.48001

The main theoretical result behind these new upper bounds is

THEOREM 3.2. *Suppose* $f : \mathbb{R}^n \to \mathbb{R}$ *is an admissible function satisfying the following three conditions*:

(1) $f(0) = \widehat{f}(0) > 0$,

(2) $f(x) \leq 0$ *for* $|x| \geq r$, *and*

(3) $\widehat{f}(t) \geq 0$ *for all* t.

Then the center density of sphere packings in $\mathbb{R}^n$ *is bounded above by* $(r/2)^n$.

The Fourier transform $\widehat{f}$ of an integrable function $f : \mathbb{R}^n \to \mathbb{R}$ is

$$\widehat{f}(y) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, y \rangle} dx$$

Theorem 3.2. *Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is an admissible function satisfying the following three conditions:*

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Then the center density of sphere packings in $\mathbb{R}^n$ *is bounded above by* $(r/2)^n$.

Proof for lattice packings with shortest vector length r:

- The center density is

$$\delta_n := \frac{\Delta_n}{\mathrm{vol}(B_1)} = \frac{\mathrm{vol}(B_{r/2})}{\mathrm{vol}(B_1)|\Lambda|} = \frac{r^n}{2^n|\Lambda|}.$$

- Admissible here means that we can apply the Poisson summation formula

$$f(0) \overset{(2)}{\ge} \sum_{\lambda \in \Lambda} f(\lambda) = \frac{1}{|\Lambda|} \sum_{\lambda \in \Lambda^*} \widehat{f}(\lambda) \overset{(3)}{\ge} \frac{\widehat{f}(0)}{|\Lambda|}$$

Searching for a magic function

Key: If we find f as in Thm 3.2 with $r = \sqrt{2}$, we prove that E_8 is optimal sphere packing in dimension 8

Theorem 3.2. *Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is an admissible function satisfying the following three conditions:*

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Then the center density of sphere packings in $\mathbb{R}^n$ is bounded above by $(r/2)^n$.

- Wlog, we can assume f is radial, $f(x) = f(|x|)$
- For $\Lambda = E_8$, $|E_8| = 1$, $E_8^* = E_8$, $r = \sqrt{2}$, we get

$$f(0) \overset{(2)}{\geq} \sum_{\lambda \in E_8} f(\lambda) = \sum_{\lambda \in E_8} \widehat{f}(\lambda) \overset{(3)}{\geq} \widehat{f}(0)$$

(1)-(3) implies $f(\lambda) = \widehat{f}(\lambda) = 0$ for all $\lambda \in E_8$, $\lambda \neq 0$

Magic functions and the uncertainty principle

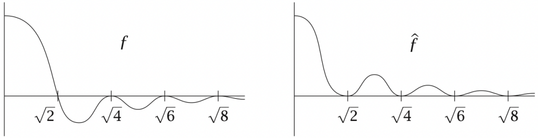

Figure 5. A schematic diagram showing the roots of the magic function f and its Fourier transform $\hat{f}$ in eight dimensions. The figure is not to scale, because the actual functions decrease too rapidly for an accurate plot to be illuminating.

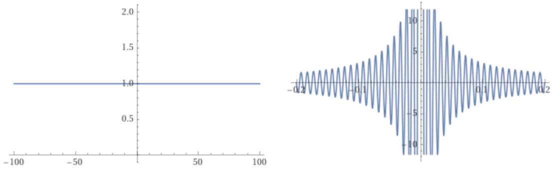

Figure: The characteristic function and its Fourier transform

Radial eigenfunctions of Fourier transform

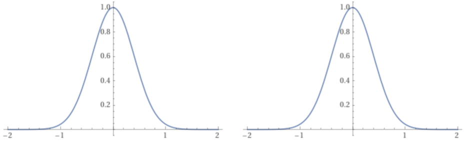

Figure: The Gaussian function $f(x) = e^{-\pi x^2}$ and its Fourier transform

Space of radial eigenfunctions for Fourier transform spanned by (infinite countable) basis of

$$f(x) = ((\text{Laguerre) polynomial})(|x|)e^{-\pi|x|^2}$$

- Cohn–Elkies: Computer search to look for good finite linear combinations of such functions (finite "root forcing")
- Viazovska: Magic function is

$$f(x) = \sin^2\left(\frac{\pi|x|^2}{2}\right)\int_0^\infty \phi_0(it)e^{-\pi t|x|^2}\,dt$$

for some linear combination ϕ_0 of (quasi-)modular forms

What is... a modular form?

Definition: A modular form f of weight k and level 1 must

1. be holomorphic on extended upper half-plane $\mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$
2. satisfy $f(z+1) = f(z)$ and $f(-1/z) = z^k f(z)$ for all $z \in \mathbb{H}$.

- The space M_k of all modular forms of weight k is a finite dimensional vector space
- The ring of modular forms $M_* = \oplus_k M_k$ is freely generated by Eisenstein series G_4, G_6, i.e., $M_* \cong \mathbb{C}[G_4, G_6]$
- A quasimodular form is a linear combination of derivatives of modular forms and G_2

"There are five fundamental operations of arithmetic: addition, subtraction, multiplication, division, and modular forms" (Eichler)

Modular magic

- Magic function has form

$$f(x) = \sin^2(\tfrac{\pi|x|^2}{2}) \int_0^\infty \phi_0(it) e^{-\pi t|x|^2} dt$$

- By complex analysis (Euler's identity, contour integration)

$$\widehat{f} = f \iff \begin{cases} \phi(z+1) = \phi(z) \\ 2\phi(z) = \phi(\tfrac{-1}{z-1})(z-1)^2 + \phi(\tfrac{-1}{z+1})(z+1)^2 - 2\phi(\tfrac{-1}{z})z^2 \end{cases}$$

for $\phi(z) := \phi_0(\tfrac{-1}{z})z^2$

And so starts an arduous process of trial and error (informed by many smart insights) that eventually leads Viazovska to the correct rational function

$$\phi_0 = \frac{\text{poly}(G_2, G_4, G_6)}{\text{poly}(G_4, G_6)}$$

Further reading

For more on the mathematics behind the proofs, there are two very nice articles

- Thomas Hales, Cannonballs and honeycombs (2000);
- Henry Cohn, A conceptual breakthrough in sphere packing (2017);

that you can easily find available online.

For an introduction to the beautiful theory of modular forms:

- Part 1 (Zagier) of The 1-2-3 of Modular Forms
- Lecture notes "L-functions and modular forms" (Chapters 4 and 5) on my website