THE $1/4$ THRESHOLD IN THE SPECTRAL THEORY OF HYPERBOLIC SURFACES

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Abstract. We explain the origin and significance of the threshold $1/4$ in the spectral theory of hyperbolic surfaces. Recent developments in the study of random hyperbolic surfaces highlight the universal role of this threshold in a probabilistic setting. This note is a write-up of the author's introductory Bourbaki lecture to Bram Petri's survey of these developments [Pet26].

1. Introduction

The purpose of this note is to explain why the number $1/4$ appears so persistently in the spectral theory of hyperbolic surfaces. It is not a normalization accident. It is the bottom of the spectrum of the hyperbolic plane $\mathbf{H}^2$, the point at which the Selberg spectral parameter changes nature, and the asymptotic upper threshold for the first positive eigenvalue of closed hyperbolic surfaces in large genus. Recent work on random hyperbolic surfaces shows that the same value also governs the typical high-genus picture.

The note is structured as follows. We first recall how the Laplace–Beltrami operator packages the geometry of a Riemannian manifold into an analytic object. We then specialize to closed hyperbolic surfaces, written as quotients $\Gamma\backslash\mathbf{H}^2$, and introduce the basic geometric objects that will enter the discussion. The central bridge between spectrum and geometry is the Selberg trace formula. We use it to explain how the eigenvalues of the Laplacian interact with the length spectrum of closed geodesics, and to highlight several places where the threshold $1/4$ already appears: Huber's theorem, the prime geodesic theorem, and Huber's large-genus bound for λ_1.

We then turn to the source of the threshold. On the universal cover $\mathbf{H}^2$, radial eigenfunctions of the Laplacian satisfy an elementary differential equation whose asymptotic behaviour changes at $\lambda = 1/4$. It turns out that $1/4$ is exactly the bottom of the spectrum of the Laplacian,

$$\mathrm{Spec}(-\Delta_{\mathbf{H}^2}) = [1/4, \infty). \tag{1}$$

Just as in the Euclidean setting, this can be deduced from the fact that the (Helgason) Fourier transform diagonalizes the Laplacian. We instead include a proof in appendix that

$$\inf \mathrm{Spec}(-\Delta_{\mathbf{H}^2}) = \frac{1}{4}$$

based on Cheeger's inequality that highlights more geometrically the influence of the hyperbolic metric.

In the last section, we present a "soft" proof of Huber's large-genus bound, following an argument communicated by A. Boulanger and relayed to the author by V. Giri, which connects explicitly the spectral gap λ_1 to the bottom of the spectrum of the universal cover via the heat kernel. One advantage of this approach is that is that it avoids the use of the Selberg trace formula and therefore extends beyond the locally symmetric setting, as in Boulanger's treatment of general Riemannian manifolds [Bou25].

2. Background

2.1. The Laplacian and its spectrum. We begin in a slightly more general setting than that of hyperbolic surfaces. Let (M, g) be an n-dimensional Riemannian manifold. The metric g intrinsically defines a Laplacian,

$$\Delta_g = \mathrm{div}_g \circ \nabla_g.$$

This is the geometric analogue of the Euclidean Laplacian

$$\Delta_{\mathbf{R}^n} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}. \tag{2}$$

Let us briefly recall its geometric interpretation. The Taylor expansion of a test function $f \in C^\infty(\mathbf{R}^n)$ gives

$$\frac{1}{\mathrm{vol}(B(x,\varepsilon))} \int_{B(x,\varepsilon)} f = f(x) + \frac{\varepsilon^2}{2(n+2)} \Delta f(x) + o(\varepsilon^2),$$

with $B(x, \epsilon)$ denoting a Euclidean ball of radius ϵ centered at x, and where linear terms vanish because the domain of integration is symmetric about x. This tells us that the Laplacian measures the extent to which the value of a function at a point differs from its average over an infinitesimal neighbourhood. The most vivid physical interpretation is given by the heat equation

$$\partial_t u(x,t) = \Delta_x u(x,t).$$

If the temperature at x increases, i.e., if $\partial_t u(x, t) > 0$, it means that the average temperature around x is higher than the temperature at x, and so heat flows toward x.

From the formula

$$\Delta f(x) := \lim_{\varepsilon \to 0} \frac{2(n+2)}{\varepsilon^2} \left(\frac{1}{\mathrm{vol}(B(x,\varepsilon))} \int_{B(x,\varepsilon)} f - f(x) \right),$$

we also see that Δ is invariant under Euclidean isometries $T(x) = Ax + y$, with $A \in O(n)$ and $y \in \mathbf{R}^n$. Alternatively, one may verify directly that (2) is unchanged under orthogonal changes of coordinates and translations. This observation is in fact much more general: for a diffeomorphism $T : M \to M$, one has

$$\Delta_g(f \circ T) = (\Delta_g f) \circ T \iff T^* g = g.$$

Thus, in studying geometric questions about M, one can use tools from functional analysis and spectral theory rather than working directly with the metric tensor, which is local and coordinate dependent. If M is compact, the spectral theorem applies because Δ_g is an elliptic, essentially self-adjoint operator on $L^2(M, g)$.

Theorem 2.1 (Spectral theorem). *If (M, g) is a compact Riemannian manifold, then the spectrum*[1] *of $-\Delta$ on $L^2(M, g)$ is*

$$0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \to \infty,$$

where each eigenvalue is counted with multiplicity. In particular, there exists an orthonormal basis of eigenfunctions $\{\varphi_j\}_{j\geq 0}$ in $L^2(M, g)$ such that $-\Delta\varphi_j = \lambda_j\varphi_j$, hence $\lambda \geq 0$.

Note that choosing to work with $-\Delta$ rather than Δ is a simple convention designed to have non-negative rather than negative eigenvalues: if $(\Delta + \lambda)\varphi = 0$ then $\lambda\|\varphi\|^2 = \langle -\Delta\varphi, \varphi\rangle = +\|\nabla\varphi\|^2$ and hence $\lambda \geq 0$.

2.2. **Hyperbolic surfaces.** We now specialize this discussion to the particularly rich setting of hyperbolic surfaces. For the rest of the note, M will be a closed hyperbolic surface, meaning compact and without boundary.

Definition 2.2. *A hyperbolic surface is a connected, complete Riemannian surface with constant Gaussian curvature $K = -1$. Equivalently, a hyperbolic surface is a connected Riemannian surface whose universal cover is isometric to the hyperbolic plane.*

As a model of the hyperbolic plane, we take the Poincaré upper half-plane

$$\mathbf{H}^2 = \{x + iy \in \mathbf{C} \mid y > 0\},$$

equipped with the metric $ds^2 = y^{-2}(dx^2 + dy^2)$. Its geodesics are vertical half-lines or semicircles orthogonal to the real axis, the Gaussian curvature is $K = -1$, and the Laplacian is

$$\Delta_{\mathbf{H}^2} = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

The orientation-preserving isometry group of $\mathbf{H}^2$ is given by the Möbius transformations preserving the real axis:

$$\mathrm{Isom}^+(\mathbf{H}^2) = \mathrm{PSL}(2, \mathbf{R}) = \mathrm{SL}(2, \mathbf{R})/\{\pm I\}.$$

This has the important advantage of allowing us to work with 2×2 matrices. By the Poincaré uniformization theorem, every closed hyperbolic surface M can be realized as a quotient $M = \Gamma\backslash\mathbf{H}^2$, where $\Gamma < \mathrm{Isom}^+(\mathbf{H}^2)$ is a discrete, cocompact, and torsion-free subgroup.

Topologically, closed hyperbolic surfaces are classified by their genus g. By Gauss–Bonnet, $g \geq 2$. The moduli space $\mathcal{M}_g$ of closed hyperbolic surfaces of genus g, up to isometry, is a real orbifold of dimension $6g - 6$.

[1] That is, the set of λ for which $(\Delta + \lambda)^{-1}$ is not defined.

3. The Selberg trace formula

The preceding discussion fixes the geometric setting. The next question is how the spectrum of $-\Delta_M$ sees the hyperbolic geometry of M. For hyperbolic surfaces, the answer is supplied by the Selberg trace formula, which relates eigenvalues of the Laplacian to lengths of closed geodesics. For closed hyperbolic surfaces, the trace formula takes on a relatively simple form. See [Sel56, Hej76, Bus92] for standard treatments.

Theorem 3.1 (Selberg trace formula)**.** *Let M be a closed hyperbolic surface, let $f \in C_c^\infty(\mathbf{R})$, and let $\widehat{f}$ denote its Fourier transform. Then*[2]

$$\sum_{j\geq 0} \widehat{f}(t_j) = \frac{\mathrm{vol}(M)}{4\pi} \int_{\mathbf{R}} \widehat{f}(t) \tanh(\pi t) t \, dt + \sum_{c \text{ closed geod. on } M} f(\ell_c) \frac{\ell_c^*}{2\sinh(\ell_c/2)},$$

where $\lambda_j = 1/4 + t_j^2$, with $t_j \geq 0$ if $\lambda_j \geq 1/4$ and $\mathrm{Im}(t_j) \geq 0$ if $0 \leq \lambda_j \leq 1/4$, and where ℓ_c^* is the primitive length of the geodesic c.

The formula should be read as a dictionary. The left-hand side is spectral: it is built out of the eigenvalues of $-\Delta_M$. The right-hand side is geometric: the first term is the contribution of the identity element, and the second term is a sum over the closed geodesics of M. Thus the trace formula says that the closed orbits of the geodesic flow are encoded by the Laplace spectrum.

Let us point out a few gems that can be deduced from the trace formula. A first striking consequence is Huber's theorem. Let $\mathrm{LSpec}(M)$ denote the length spectrum of M, namely the multiset of lengths of primitive closed geodesics, counted with multiplicity.

Theorem 3.2 ([Hub59])**.** *The spectrum of a closed hyperbolic surface is completely determined by the length spectrum. In other words, if M_1 and M_2 are two closed hyperbolic surfaces, then*

$$\mathrm{Spec}(M_1) = \mathrm{Spec}(M_2) \iff \mathrm{LSpec}(M_1) = \mathrm{LSpec}(M_2).$$

The classical isospectrality problem asks for something stronger: whether isospectral surfaces are necessarily isometric. The answer is no, as was shown by sophisticated constructions of Vignéras and Sunada [Vig80, Sun85]. Via the trace formula, Huber's theorem says instead that isospectral closed hyperbolic surfaces must have the same unmarked length spectrum, and that conversely, the full length spectrum determines the Laplace spectrum. Wolpert later proved that the length-spectrum map on $\mathcal{M}_g$ is finite-to-one [Wol79]; thus, for fixed genus, the length spectrum of a closed hyperbolic surface determines its isometry class up to finitely many possibilities.

The trace formula also gives quantitative information about how closed geodesics are distributed. Below Li refers to the logarithmic integral $\mathrm{Li}(T) = \int_2^T \frac{dt}{\ln t}$.

[2] Here one uses the analytic continuation of $\widehat{f}$ to $\mathbf{C}$; for instance, the eigenvalue $\lambda_0 = 0$ corresponds to the spectral parameter $t_0 = i/2$.

Theorem 3.3 (Prime geodesic theorem)**.** *One has an asymptotic expansion of the form*

$$\#\{c \text{ primitive closed geod.} \mid \ell_c \le L\} = \mathrm{Li}(e^L) + \sum_{0<\lambda_j<1/4} \mathrm{Li}(e^{L(1/2+|t_j|)}) + O\left(\mathrm{Li}(e^{3L/4})\right).$$

The name of this theorem reflects its evident analogy with the prime number theorem, especially Riemann's explicit formula. Selberg developed this analogy through the Selberg zeta function, which remains a model for many problems in spectral geometry.

The secondary terms in the expansion come exactly from the small eigenvalues $0 < \lambda_j < 1/4$. In particular, the smaller the spectral gap λ_1, the larger its contribution to the geodesic count. This phenomenon also has a dynamical meaning. The geodesic flow on T^1M is one of the simplest models of chaotic dynamics: it is an Anosov flow, and it is ergodic and mixing with respect to Liouville measure. The larger the spectral gap, the faster the mixing of the geodesic flow; see, for instance, [Rat87].

Since $\lambda = 1/4$ corresponds to $t = 0$, one might expect the error term in the prime geodesic theorem to be of order $O(\mathrm{Li}(e^{L(1/2+\varepsilon)}))$. No general error term of that strength is known; the obstruction comes from a technical difficulty inherent in harmonic analysis, namely the uncertainty principle: one cannot localize both a test function f and its Fourier transform. Much of the difficulty in extracting sharp counting results from the trace formula lies in managing this trade-off.

These consequences already show that the number $1/4$ is not merely a parameter in the trace formula: it separates the exceptional eigenvalues $0 < \lambda < 1/4$ from the rest of the spectrum. The next theorem, also due to Huber, shows a third role of the same threshold, now in the large-genus geometry of moduli space.

Theorem 3.4 ([Hub74])**.**

$$\limsup_{g\to\infty} \sup\{\lambda_1(X) \mid X \in \mathcal{M}_g\} \le 1/4.$$

It is necessary to restrict to maximal spectral gaps: pinching a short separating geodesic on a fixed surface $M \in \mathcal{M}_g$ degenerates $\lambda_1 \to 0$ but does not change the topology of the surface. It was conjectured by Buser (1984) that this upper bound is sharp: there should exist a sequence (M_g) of closed hyperbolic surfaces with $g \to \infty$ and $\lambda_1(M_g) \to 1/4$. This was proved by Hide and Magee using random covers of finite-area hyperbolic surfaces together with a compactification procedure of Buser–Burger–Dodziuk [BBD88, HM23].

Even more strikingly, the same behaviour is typical in the Weil–Petersson model. Recent work of Anantharaman and Monk proves that, for every $\varepsilon > 0$,

$$\mathbb{P}_g^{\mathrm{WP}}\left(\lambda_1(X) \ge \frac{1}{4} - \varepsilon\right) \longrightarrow 1 \qquad (g \to \infty).$$

Together with Huber's upper bound, this says that a typical high-genus hyperbolic surface has asymptotically optimal spectral gap. The proof is again a trace-method argument, but now averaged over moduli space: one uses the Selberg trace formula together with deep information about the statistics of closed geodesics on Weil–Petersson random surfaces [AM25, Mon26].

At present, the strongest proven mechanisms reaching the $1/4$ threshold in high genus are therefore probabilistic, based on different random models of hyperbolic surfaces. There is, however, a parallel arithmetic dream: Selberg's eigenvalue conjecture predicts that congruence surfaces have no exceptional eigenvalues below $1/4$.[3] In that arithmetic setting, the best known general lower bound remains the Kim–Sarnak bound [KS03].

So far $1/4$ has appeared in three different forms: in the spectral parametrization $\lambda = 1/4 + t^2$, in the contribution of small eigenvalues to the prime geodesic theorem, and in Huber's large-genus upper bound for λ_1. We now explain why this is the natural number by returning to the universal cover $\mathbf{H}^2$.

4. Why 1/4?

To understand why the threshold $1/4$ appears in these results, it is useful to turn first to harmonic analysis on $\mathbf{H}^2$. The starting point of the Selberg trace formula is the study of invariant kernels on $\mathbf{H}^2$, and therefore the study of radial functions on $\mathbf{H}^2$. Let us look for a radial eigenfunction on $\mathbf{H}^2$. Since φ is radial, we write $\varphi(z) = F(d_{\mathbf{H}^2}(z, w))$ for a fixed point $w \in \mathbf{H}^2$. Expressing

$$-\Delta\varphi = \lambda\varphi$$

in hyperbolic geodesic polar coordinates gives

$$F''(r) + \coth(r)F'(r) = -\lambda F.$$

To read off the behaviour relevant for square-integrability, we study the limiting linear ODE as $r \to \infty$:

$$F'' + F' + \lambda F = 0.$$

The general solution of this homogeneous second-order linear ODE depends on the sign of its discriminant

$$D = 1 - 4\lambda.$$

Thus the asymptotic behaviour is governed by the threshold $1/4$. Moreover, the *Ansatz* $F(r) = e^{-sr}$ leads to $\lambda = s(1 - s)$, which explains the spectral parametrization

$$s = 1/2 + it, \qquad \lambda = 1/4 + t^2$$

in Selberg's trace formula. The solutions of the ODE at infinity are oscillatory when $\lambda > 1/4$ and exponential when $\lambda < 1/4$. For comparison, in the Euclidean setup, working with polar coordinates and letting $r \to \infty$ leads to the linear ODE $F'' + \lambda F = 0$.

[3] Although the classical congruence surfaces in Selberg's conjecture are noncompact, the Jacquet–Langlands correspondence transfers the analogous automorphic spectral-gap question to certain compact arithmetic surfaces [Sel65, JL70].

This radial calculation explains the threshold at the level of asymptotics. To turn it into a spectral statement, one can use the harmonic analysis of $\mathbf{H}^2$, where the Helgason Fourier transform plays the role of the ordinary Fourier transform on $\mathbf{R}^n$ and diagonalizes the Laplacian. This is used to prove the following general fact.

$$\mathrm{Spec}(-\Delta_{\mathbf{H}^2}) = [\tfrac{1}{4}, +\infty).$$

In the appendix, we include a purely geometric proof of the fact that the bottom of the Laplacian spectrum on $\mathbf{H}^2$ is $1/4$. This fact also clarifies the meaning of Huber's theorem: the large-genus upper bound is the bottom of the spectrum of the common universal cover. One might expect that, as $\mathrm{vol}(M) \to \infty$, the spectral geometry of such surfaces begins to resemble that of their universal cover $\mathbf{H}^2$. Since $\inf \mathrm{Spec}(-\Delta_{\mathbf{H}^2}) = 1/4$, one would then expect an upper bound of the following type: for every $\varepsilon > 0$ and all sufficiently large g, there exists $M \in \mathcal{M}_g$ such that $\lambda_1(M) \leq 1/4 + \varepsilon$. The following proof of Huber's Theorem 3.4 makes this principle transparent. It is adapted from the proof of the Alon–Boppana bound in graph theory.

Proof of Theorem 3.4. Let $k_M(t; x, y)$ denote the heat kernel on M, and let $k_{\mathbf{H}}(t; \widetilde{x}, \widetilde{y})$ denote the heat kernel on $\mathbf{H}^2$. The heat kernel on M is given by the periodization formula

$$k_M(t; x, y) = \sum_{\gamma \in \Gamma} k_{\mathbf{H}^2}(t; \widetilde{x}, \gamma\widetilde{y}).$$

Since every term is nonnegative, the identity term gives

$$k_M(t; x, y) \geq k_{\mathbf{H}^2}(t; \widetilde{x}, \widetilde{y}).$$

Choose a radius $r > 0$ such that every closed hyperbolic surface contains an embedded hyperbolic ball of radius r. Let $B = B(x_0, r) \subset M$ be such a ball, and let $\widetilde{B} \subset \mathbf{H}^2$ be a lift. Hence

$$\int_B \int_B k_M(t; x, y)\, dx\, dy \geq \int_{\widetilde{B}} \int_{\widetilde{B}} k_{\mathbf{H}^2}(t; \widetilde{x}, \widetilde{y})\, d\widetilde{x}\, d\widetilde{y}.$$

The standard explicit formula for the heat kernel on $\mathbf{H}^2$ is

$$k_{\mathbf{H}^2}(t; \widetilde{x}, \widetilde{y}) = \frac{\sqrt{2}}{(4\pi t)^{3/2}} e^{-t/4} \int_{d_{\mathbf{H}^2}(\widetilde{x}, \widetilde{y})} \frac{s e^{-s^2/(4t)}}{\sqrt{\cosh s - \cosh d_{\mathbf{H}^2}(\widetilde{x}, \widetilde{y})}} ds.$$

For $\widetilde{x}, \widetilde{y} \in \widetilde{B}$, we have $d_{\mathbf{H}^2}(\widetilde{x}, \widetilde{y}) \leq 2r$. In particular, $k_{\mathbf{H}^2}(t; \widetilde{x}, \widetilde{y}) \geq c_r t^{-3/2} e^{-t/4}$ for all sufficiently large t. This implies that, for every $\delta > 0$, there exist constants $C_\delta > 0$ and $T_\delta > 0$ such that for all $t \geq T_\delta$,

$$\int_{\widetilde{B}} \int_{\widetilde{B}} k_{\mathbf{H}^2}(t; \widetilde{x}, \widetilde{y})\, d\widetilde{x}\, d\widetilde{y} \geq C_\delta e^{-(1/4+\delta)t}.$$

Now set

$$f := \mathbf{1}_B - \frac{\mathrm{area}(B)}{\mathrm{area}(M)},$$

so that $\int_M f = 0$, i.e. f is orthogonal to constants. Then

$$\left\langle e^{-t\Delta_M} f, \mathbf{1}_B \right\rangle = \int_B \int_B k_M(t;x,y)\,dx\,dy - \frac{\text{area}(B)^2}{|M|} \geq C_\delta e^{-(1/4+\delta)t} - \frac{\text{area}(B)^2}{\text{area}(M)}$$

for $t \geq T_\delta$.

On the other hand, since f is orthogonal to constants,

$$\begin{aligned}\left\langle e^{-t\Delta_M} f, \mathbf{1}_B \right\rangle = \left\langle e^{-t\Delta_M} f, f \right\rangle &= \sum_{j\geq 1} e^{-\lambda_j t} \left|\langle f, \varphi_j\rangle\right|^2 \\ &\leq e^{-\lambda_1 t}\|f\|_2^2 = e^{-\lambda_1 t}\left(\text{area}(B) - \frac{\text{area}(B)^2}{\text{area}(M)}\right) \leq e^{-\lambda_1 t}\text{area}(B).\end{aligned}$$

Combining the two estimates gives

$$e^{-\lambda_1 t}\text{area}(B) \geq C_\delta e^{-(1/4+\delta)t} - \frac{\text{area}(B)^2}{\text{area}(M)}.$$

Hence choosing t large enough implies that $\lambda_1(M) \leq 1/4 + \varepsilon$ for all sufficiently large genus, since $\text{area}(M) = 4\pi(g-1)$. □

This argument is also the natural bridge to the probabilistic picture: if large hyperbolic surfaces often look locally like $\mathbf{H}^2$, then one should expect their low spectrum to be constrained by the bottom of the spectrum of $\mathbf{H}^2$.

Appendix A.

We conclude this note with a geometric, measure-theoretic proof of the fact that

$$\inf \text{Spec}(-\Delta_{\mathbf{H}^2}) = \frac{1}{4}.$$

The lower bound will follow from the Cheeger inequality

$$\inf \text{Spec}(-\Delta_{\mathbf{H}^2}) \geq \frac{h(\mathbf{H}^2)^2}{4}, \qquad h(\mathbf{H}^2) = \inf_{\Omega\subset\mathbf{H}^2} \frac{\text{length}(\partial\Omega)}{\text{area}(\Omega)},$$

where Ω ranges over relatively compact domains with smooth boundary.

The proof of Cheeger's inequality is a simple, beautiful application of the coarea formula. We start from the fact that the bottom of the spectrum of the Laplacian on $\mathbf{H}^2$ is realized by the Rayleigh quotient

$$\inf \text{Spec}(-\Delta_{\mathbf{H}^2}) = \inf_{f\in C_c^\infty(\mathbf{H}^2)} \frac{\int_{\mathbf{H}^2} |\nabla f|^2}{\int_{\mathbf{H}^2} |f|^2}.$$

For $f \in C_c^\infty(\mathbf{H}^2)$, Cauchy–Schwarz gives

$$\frac{\int_{\mathbf{H}^2} |\nabla f|^2}{\int_{\mathbf{H}^2} |f|^2} \geq \left(\frac{\int_{\mathbf{H}^2} |\nabla f|\,|f|}{\int_{\mathbf{H}^2} |f|^2}\right)^2 = \frac{1}{4}\left(\frac{\int_{\mathbf{H}^2} |\nabla(f^2)|}{\int_{\mathbf{H}^2} |f|^2}\right)^2.$$

The numerator is now in a form to which we can apply the coarea formula:

$$\int_{\mathbf{H}^2} |\nabla f^2| = \int_0^\infty \operatorname{length}(\partial\{f^2 \geq t\})\, dt$$
$$\geq h(\mathbf{H}^2) \int_0^\infty \operatorname{area}(\{f^2 \geq t\})\, dt = h(\mathbf{H}^2) \int_{\mathbf{H}^2} |f|^2.$$

Therefore

$$\frac{\int_{\mathbf{H}^2} |\nabla f|^2}{\int_{\mathbf{H}^2} |f|^2} \geq \frac{h(\mathbf{H}^2)^2}{4} = \frac{1}{4}.$$

The isoperimetric inequality in the hyperbolic plane implies $h(\mathbf{H}^2) \geq 1$, and the bound is sharp for hyperbolic disks: their perimeter-to-area ratio tends to 1 as the radius grows. This surprising feature of the hyperbolic metric is responsible for the fact that $h(\mathbf{H}^2) = 1$.

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