A Finite Geometry Construction for MDPC-Codes

Master Thesis

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Introduction

Motivation

- Code-based cryptography $\longrightarrow$ quantum-secure cryptosystems.

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    - Low-density parity-check (LDPC) codes
    - Moderate-density parity-check (MDPC) codes

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- Error-correction performance for random codes is asymptotic
- Give a construction of MDPC codes optimizing the error-correction performance after one round of the bit-flipping decoding algorithm

Coding Theory

Linear Codes

Let $GF(q)$ denote the finite field of q elements, q is a prime power.

Definition

A q*-ary linear code* C of length n and dimension k is a k-dimensional linear subspace of $GF(q)^n$.

Remarks

- We denote C as $[n, k]_q$—linear code.

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- We denote C as $[n,k]_q-$linear code.
- A codeword $c \in C$ is a vector of length n over a finite field $GF(q)$.

Dual Code

Definition

Let C be an $[n,k]_q-$linear code. Its *dual code* $C^\perp$ is given by

$$C^\perp = \{x \in GF(q)^n \,|\, x \cdot c^\top = 0, \forall c \in C\}.$$

Representation of linear codes

Definition

Let C be an $[n,k]_q-$ linear code. A *generator matrix* for C is a $(k \times n)$ matrix whose rows are formed from any k linearly independent vectors of C. Similarly we define a matrix $H \in GF(q)^{(n-k)\times n}$, the *parity check matrix* of C, to be the generator matrix of the dual code $C^{\perp}$.

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- It holds that $H \cdot G^\top = 0$.
- $C = \ker H = \{c \in GF(q)^n \mid Hc^\top = 0\}$.

Minimum Distance

Definition

Let x and y be two vectors of $GF(q)^n$. The *Hamming distance* $d(x, y)$ is the number of positions in which x and y differ, i.e.

$$d(x, y) = |\{i \in \{1, \ldots, n\} \mid x_i \neq y_i\}|.$$

The *minimum distance* of a code C, denoted $d(C)$, is the smallest possible Hamming distance two codewords c and $\tilde{c}$ of C,

$$d(C) := \min\{d(c, \tilde{c}) \mid c, \tilde{c} \in C, c \neq \tilde{c}\}.$$

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$C = \{(0, 0, 0, 0), (0, 0, 1, 1), (1, 1, 0, 0), (1, 1, 1, 1)\}$

- possible distances between two words: 2 or 4.
- $d(C) = 2$

Weight

Definition

The weight of a vector $x = (x_1, \ldots, x_n) \in GF(q)^n$ is the the number of non-zero positions of x, i.e. $wt(x) = |\{i = 1, \ldots, n | x_i \neq 0\}|$.

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Remark

If every row x of a matrix H has a constant weight $wt(x) = w$ then we say that H has *row-weight* w.

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- Advantage: high error-correction performance.
- Problem: due to the low-weight of the dual codewords, some variants of the McEliece cryptosystem can be attacked.
- Extension of LDPC-codes by increasing the row-weight $\longrightarrow$ MDPC-codes.

MDPC-Codes

Definition

A *moderate density parity-check code*, or simply MDPC-code, is a binary linear code of length n with a parity-check matrix whose row weight is $\mathcal{O}(\sqrt{n})$. If the weight of every column is v and the weight of every row is w we say the MDPC-code is *of type* (v, w).

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- Stops if syndrome is zero or if the maximal number of iterations $b_{\max}$ is reached.
- Complexity: $\mathcal{O}(nwb_{\max})$

Maximum Column Intersection

Definition

Let $H = (h_{ij})_{\substack{1\leq i\leq r\\ 1\leq j\leq n}}$ be a binary matrix. The *intersection number* of two different columns j and j' of H is equal to the number of rows i for which $h_{ij} = h_{ij'} = 1$. The *maximum column intersection*, denoted s_H, of H is equal to the maximum intersection number of two distinct columns of H.

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Then $s_H = 2$.

Error-Correction Capacity

Proposition, [5]

Let C be an MDPC-Code of type (v, w) with parity-check matrix $H = (h_{ij})_{\substack{1 \leq i \leq r \\ 1 \leq j \leq n}}$. Let s_H be the maximum column intersection with respect to the parity check matrix H. Performing one round of the bit-flipping decoding algorithm based on the matrix H one can correct all errors of weight at most $\lfloor \frac{v}{2s_H} \rfloor$.

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Remarks:

- The smaller s_H, the bigger the amount of errors that can be corrected.
- **Goal**: Give a construction of MDPC-codes with a small maximum column intersection.

Finite Geometry

Incidence Structure

Definition

An *incidence structure* is a triple $S = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, consisting of a set of points $\mathcal{P}$, a set of lines $\mathcal{L}$ that is distinct from the set of points, and an incidence relation $\mathcal{I} \subset \mathcal{P} \times \mathcal{L}$ between the points and the lines.

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- The point a is incident to line l and o.

Incidence Matrix

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$$a_{ij} = \begin{cases} 0, & \text{if point } p_i \text{ does not lie on line } l_j \\ 1, & \text{if point } p_i \text{ lies on line } l_j. \end{cases}$$

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- $$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} l & m & n & o \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

Projective plane

Definition

A *projective plane* is an incidence structure Π of points and lines satisfying the following properties

(P1) Any two distinct points are incident with exactly one line,

(P2) Any two distinct lines are incident with exactly one point,

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Remark:

- A *finite projective plane* is a projective plane of finitely many points and lines.

Projective Plane

Number of points incident to a line

Every point in a finite projective plane is incident to a constant $q + 1$ lines. Dually every line passes through a constant $q + 1$ points.

The number q is called the order of a projective plane.

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Total number of points and lines

In a finite projective plane of order q there are q^2+q+1 points and q^2+q+1 lines.

Desarguesian Plane

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- Identified with the equivalence classes of $GF(q)^3\backslash\{0\}/_\sim$, where:

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- For a given point $P = [x, y, z]$ the set of lines passing through P is given by

$$\langle a, b, c \rangle := \{[a, b, c] \in PG(2, q) \mid ax + by + cz = 0\}.$$

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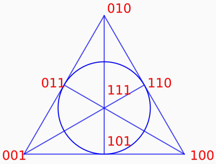

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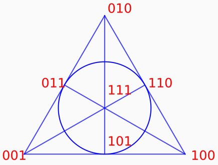
- An incidence matrix is given by

$$H = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$



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- The columns of H correspond to the lines and the rows of H correspond to the points.
- H is an $(q^2 + q + 1) \times (q^2 + q + 1)$ binary matrix of column-weight $v = q + 1 = \mathcal{O}(\sqrt{q^2 + q + 1})$.

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- $C_2(\Pi)^\perp = \ker(H)$ is a $[q^2 + q + 1, 1]_2-$linear code.

Projective Bundles

Conic

Definition

A *conic* $\mathcal{C}$ of a projective plane Π of order q is a set of points $[x, y, z]$ of Π satisfying a quadratic equation.

- A general quadratic equation is $ax^2 + by^2 + cz^2 + dyz + exz + fxy = 0$, where $a, b, c, d, e, f \in GF(q)$ not all zero. Its associated matrix is $A = \begin{pmatrix} 2a & f & e \\ f & 2b & d \\ e & d & 2c \end{pmatrix}$.

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$$A = \begin{pmatrix} 2a & f & e \\ f & 2b & d \\ e & d & 2c \end{pmatrix}.$$

Definition

A conic $\mathcal{C}$ is *non-degenerate* if $\det(A) \neq 0$ when q is odd, or if (d, e, f) is non-zero and not contained in $\mathcal{C}$ when q is even.
A *degenerate* conic is a conic which is not non-degenerate.

Conic

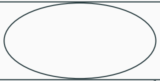
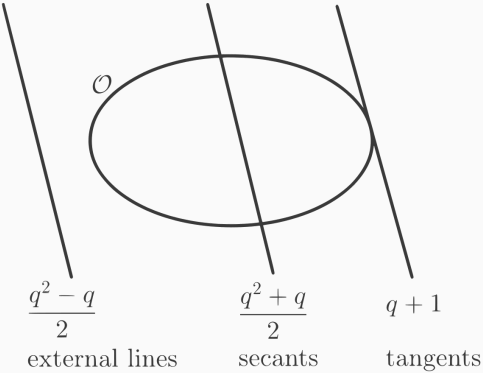
Type of conic	points in $\mathcal{C}$	Diagram
non-degenerate	$q+1$	
repeated line	$q+1$	
two real lines	$2q+1$	
two imaginary lines	1	

Table 1: The four distinct types of conics in $PG(2,q)$.

Non-Degenerate Conic and Lines



The relative positions between lines and a non-degenerate conic.

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- The rows of an incidence matrix are represented by the points and the columns are represented by the non-degenerate conics.

Existence in $PG(2, q)$

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- He proved the existence of projective bundles in $PG(2,q)$.
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- Each of the three types exists for an odd prime power q.
- Only one of these is a projective bundle also if q is an even prime power.

Algebraic Classification

Due to Singer ([4]).

Identify the points of $PG(2, q)$ with the integers mod $q^2 + q + 1$.

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If a set D of $q+1$ distinct integers $d_0,\dots,d_q$ has the property that $(d_i-d_j)_{0\le i\ne j\le q}$ are distinct mod q^2+q+1, then D is called a *perfect difference set*.

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Example

For $q=2$, the set $\{0,1,3\}$ of 3 integers is a perfect difference set mod 7, because all the possible differences

$$0-1\equiv\mathbf{6},\ 0-3\equiv\mathbf{4},\ 1-3\equiv\mathbf{5},\ 1-0\equiv\mathbf{1},\ 3-0\equiv\mathbf{3},\ 3-1\equiv\mathbf{2}$$

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Construction of a perfect difference set of $q+1$ integers:

- Write d_0 for the point in $PG(2,q)$ identified with 0 and d_1 for the point in $PG(2,q)$ identified with 1.

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- $D = \{d_0, d_1, \ldots, d_q\}$ is a perfect difference set.
- The following array with integers reduced mod q^2+q+1 represents the points and lines of $PG(2,q)$.

$$\begin{array}{cccc} d_0 & d_0+1 & \cdots & d_0+q^2+q \\ d_1 & d_1+1 & \cdots & d_1+q^2+q \\ \vdots & \vdots & \cdots & \vdots \\ d_q & d_q+1 & \cdots & d_q+q^2+q \end{array}$$

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Consider the projective plane $PG(2, q)$ of order q.

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Example

Fano plane $PG(2, 2)$:
$\mathcal{P} = \{0, 1, 2, 3, 4, 5, 6\}$,
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$\mathcal{L} = \{\{0, 1, 3\}, \{1, 2, 4\}, \{2, 3, 5\},$
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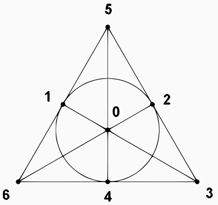

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Theorem [1]

For $N = q^2 + q + 1$, if $r \in \mathbb{Z}/_{\langle N \rangle}$ is relatively prime to N and $D = \{d_0, \ldots, d_q\}$ a perfect difference set for $PG(2, q)$, then the set $D/r = \{d_0/r, \ldots, d_q/r\}$ is the point set of some curve of degree r.

Notes:

- The order q is an odd prime power and $N = q^2 + q + 1$ is odd too.

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  3. *Self-polar bundle:* Image of $D/2$ under the cycle $S(i) = i + 1$, for $i \in \mathbb{Z}/_{\langle q^2+q+1 \rangle}$.

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- Hence an inscribed bundle is given by $\mathcal{B}_I=\{\{0+i,2+i,5+i,6+i\}\,|\,i\in\mathbb{Z}/_{\langle 13\rangle}\}$.

Construction

The Parity-Check Matrix

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Define $H = [H_1|H_2]$ of size $(q^2+q+1) \times 2(q^2+q+1)$.

$$H = \begin{array}{c} \\ p_1 \\ p_2 \\ \vdots \\ p_{q^2+q+1} \end{array} \begin{array}{c} \begin{array}{cccc|cccc} l_1 & l_2 & \dots & l_{q^2+q+1} & c_1 & c_2 & \dots & c_{q^2+q+1} \end{array} \\ \left(\begin{array}{cccc|cccc} & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \end{array} \right) \end{array}$$

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$$\left(\begin{array}{ccccccccccccc|ccccccccccccc}
1&0&0&0&1&0&0&0&0&0&1&0&1&1&0&0&0&0&0&0&1&1&0&0&1&0\\
1&1&0&0&0&1&0&0&0&0&0&1&0&0&1&0&0&0&0&0&0&1&1&0&0&1\\
0&1&1&0&0&0&1&0&0&0&0&0&1&1&0&1&0&0&0&0&0&0&1&1&0&0\\
1&0&1&1&0&0&0&1&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&1&1&0\\
0&1&0&1&1&0&0&0&1&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&1&1\\
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Constructed code:

- $C_2(\Pi \sqcup \Gamma)^\perp := \ker(H)$ is binary linear code of length $n = 2(q^2 + q + 1)$ and type $(v, w) = (q + 1, 2(q + 1))$.
- Indeed $v = \mathcal{O}(\sqrt{n})$. Thus, $C_2(\Pi \sqcup \Gamma)^\perp$ is an MDPC-code.

Proposition

The dimension of the MDPC-code $C_2(\Pi \sqcup \Gamma)^\perp$ is given by $\dim(C_2(\Pi \sqcup \Gamma)^\perp) = q^2 + q + 2$.

Theorem

Let d denote the minimum distance of the MDPC-code $C_2(\Pi \sqcup \Gamma)^\perp$. Then the following estimate holds

$$\left\lfloor \frac{2q+4}{3} \right\rfloor + 1 \leq d.$$

Error-correction

Recall:
MDPC-code of length n, column-weight v, parity-check matrix H and max. column intersection $s \implies$ after performing one round of bit-flipping algorithm one can correct errors of weight at most $\lfloor \frac{v}{2s} \rfloor$.

- For $C_2(\Pi \sqcup \Gamma)^{\perp}$ of length $n = 2(q^2 + q + 1)$ and parity-check matrix H we have:

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- For $C_2(\Pi \sqcup \Gamma)^\perp$ of length $n = 2(q^2 + q + 1)$ and parity-check matrix H we have:
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Theorem

After performing one round of bit-flipping decoding algorithm on a parity-check matrix H of $C_2(\Pi \sqcup \Gamma)^\perp$ we can correct errors of weight up to $\lfloor \frac{q+1}{4} \rfloor$, which is roughly $\sqrt{\frac{n}{32}}$.

Error-correction

q	inscribed bundle	circumscribed bundle	self-polar bundle
5	53.5%	53.5%	53.5%
7	4.2 %	3.9%	3.9%
9	75.9%	75.4%	76.0%
11	43.8%	42.8%	42.1%
13	91.9%	91.3%	90.5%
17	96.0%	96.6%	96.0%
19	91.5%	91.6%	91.3%
23	97.4%	98.0%	97.8%
25	98.9%	98.9%	100%

Table 2: Probability to decode a received word of $\lfloor \frac{q+1}{4} \rfloor + 1$ errors correctly after one round of the bit-flipping decoding algorithm.

Error-correction

q	inscribed bundle	circumscribed bundle	self-polar bundle
5	2.9%	2.9%	2.9%
9	6.1%	5.0%	5.9%
11	4.3%	4.8%	4.8%
13	16.9%	17.5%	17.0%
17	59.4%	58.6%	57.7%
19	45.1%	45.6%	46.6%
23	78.0%	80.1%	77.7%
25	95.8%	95.0%	94.5%

Table 3: Probability to decode a received word of $\lfloor \frac{q+1}{4} \rfloor + 2$ errors correctly after one round of the bit-flipping decoding algorithm.

Thank you for your attention!

Questions?

Algebraic Classification

Example

q	perfect difference set D
2	$\{0, 1, 3\}$
3	$\{0, 1, 3, 9\}$
5	$\{0, 1, 3, 8, 12, 18\}$
7	$\{0, 1, 3, 13, 32, 36, 43, 52\}$
9	$\{0, 1, 3, 9, 27, 49, 56, 61, 77, 81\}$

Table 4: Perfect difference sets for some initial values of q.

Minimum Distance

It is rather difficult to compute the minimum distance of $C_2(\Pi \sqcup \Gamma)^{\perp}$.

Estimation:

- Let $S = \{l_1, ..., l_r, c_1, ..., c_s\}$ be an arbitrary but minimal set of linearly dependent columns of H, where l_i are some columns corresponding to lines and c_i some corresponding non-degenerate conics of $PG(2, q)$, then:
  $(\bigcup_{i=1}^{r} l_i) \cup (\bigcup_{i=1}^{s} c_i) = \left(\bigcup_{i<j} l_i \cap l_j\right) \cup \left(\bigcup_{i<j} c_i \cap c_j\right) \cup \left(\bigcup_{i,j} l_i \cap c_j\right)$

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- $\lceil \frac{2(q+2)}{3} \rceil \leq d(C_2(\Pi \sqcup \Gamma)^\perp)$.

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