Coding Theory and Cryptography:
A Conference in Honor of Joachim Rosenthal's 60th Birthday

The Marginal Distribution of the Lee Channel and its Applications

Jessica Bariffi

joint work with Hannes Bartz and Gianluigi Liva
and with Karan Khathuria (UT) and Violetta Weger (TUM)

Institute of Communications and Navigation
German Aerospace Center, DLR

Outline

1. Preliminaries and Motivation
2. The Lee Channel and its Properties
3. Information Set Decoding
4. Information Set Decoding using Restricted Spheres
   - Bounded Minimum Distance Decoding
   - Decoding Beyond the Minimum Distance

DLR

Outline

DLR

Syndrome Decoding Problem

Assume we send a codeword $x \in C$ and receive a vector $y = x + e \in (\mathbb{Z}/p^s\mathbb{Z})^n$.

Syndrome Decoding Problem

Given an (n, k) n parity-check matrix H of C and a syndrome $s = yH^T$, find the length-n vector e such that

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The security of the McEliece cryptosystem relies on the hardness of the syndrome decoding problem

- Is an NP-hard problem (in the Hamming metric, Lee metric, . . .)
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- Is an NP-hard problem (in the Hamming metric, Lee metric, . . .)
- generic decoding has a large cost in the Lee metric

Has a unique solution for a relatively small weight (w.r.t. the GV bound)

Ring-Linear Codes

Let p a prime number and s and n two positive integers.

Definition

A *linear code* C $(\mathbb{Z}/p^s\mathbb{Z})^n$ is a $\mathbb{Z}/p^s\mathbb{Z}$-submodule of $(\mathbb{Z}/p^s\mathbb{Z})^n$.

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Parameters:

n is called the *length* of C

$k := \log_{p^s} |C|$ is the $\mathbb{Z}/p^s\mathbb{Z}$-*dimension* of C

$R := k/n$ denotes the *rate* of C.

The Lee Metric

Definition

For $a \in \mathbb{Z}/p^s\mathbb{Z}$ and $e = (e_1, \ldots, e_n) \in (\mathbb{Z}/p^s\mathbb{Z})^n$ we define their *Lee weight*, respectively, by

$$\mathrm{wt}_L(a) := \min(a, p^s - a),$$

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Example over $\mathbb{Z}/5\mathbb{Z}$

$0: \quad \mathrm{wt}_L(0) = 0$

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Properties:

For every $a \in \mathbb{Z}/p^s\mathbb{Z}$ and $e \in (\mathbb{Z}/p^s\mathbb{Z})^n$

$$\mathrm{wt}_L(a) = \mathrm{wt}_L(j p^s - a j)$$

$$\mathrm{wt}_H(a) \quad \mathrm{wt}_L(a) \quad bp^s/2c =: M$$

$$\mathrm{wt}_H(e) \quad \mathrm{wt}_L(e) \quad nM$$

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The Constant-Weight Lee Channel

Take a linear code $C \quad (\mathbb{Z}/p^s\mathbb{Z})^n$.

Introduce an error, i.e., we add to x

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Lemma

Let $a \in \mathbb{Z}/p^s\mathbb{Z}$ be chosen uniformly at random. Then

$$\mu_{p^s} := \mathrm{E}(\mathrm{wt}_L(a)) = \begin{cases} \frac{(p^s)^2 - 1}{4p^s} & \text{if } p^s \text{ is odd;} \\ \frac{p^s}{4} & \text{if } p^s \text{ is even:} \end{cases}$$

The Marginal Distribution

Let E be the random variable corresponding to the realization of a random entry of e.

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Theorem [1]

Assume that the asymptotic relative Lee weight is $T := \lim_{n \to \infty} \frac{t(n)}{n}$. For every $i \in \mathbb{Z}/p^s\mathbb{Z}$ the marginal distribution of E is given by

$$p_i := \mathrm{P}(E = i) = \frac{1}{\sum_{j=0}^{p^s-1} \exp(\quad \mathrm{wt_L}(j))} \exp(\quad i)$$

where is the solution to $T = \sum_{i=0}^{M} \mathrm{wt_L}(i) p_i$.

1

[1] "On the Properties of Error Patterns in the Constant Lee Weight Channel". In: *International Zurich Seminar on Information and Communication (IZS)*. 2022, pp. 44–48.

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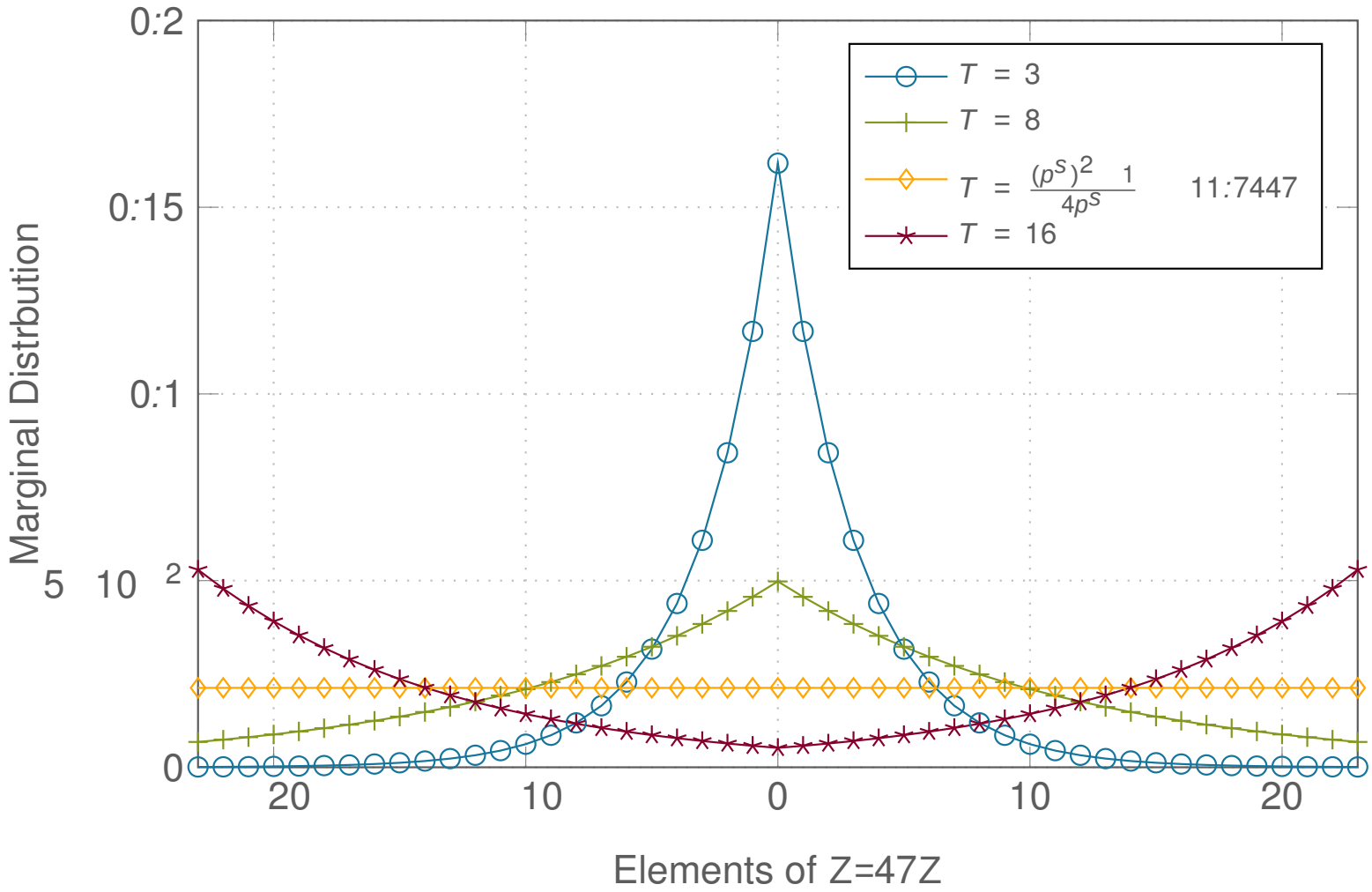
Note $T < \quad _{p^s} \quad (\quad) \quad > 0$

The Marginal Distribution - Example over Z=47Z



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Consider an instance of the Lee Syndrome Decoding Problem (LSDP):

$$\text{Given } H \in (\mathbb{Z}/p^s\mathbb{Z})^{(n-k)\times n};\ s \in (\mathbb{Z}/p^s\mathbb{Z})^{n-k} \text{ and } t \in \mathbb{N};$$
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- Recent improvements: using partial Gaussian elimination[1]

[1] Matthieu Finiasz and Nicolas Sendrier. "Security bounds for the design of code-based cryptosystems". In: *International Conference on the Theory and Application of Cryptology and Information Security*. Springer. 2009, pp. 88–105.

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[1] Anja Becker et al. "Decoding random binary linear codes in $2^{n/20}$: How 1+ 1= 0 improves information set decoding". In: *Annual international conference on the theory and applications of cryptographic techniques*. Springer. 2012, pp. 520–536.

[2] Alexander May, Alexander Meurer, and Enrico Thomae. "Decoding Random Linear Codes in $\mathcal{O}(2^{0.054n})$". In: *International Conference on the Theory and Application of Cryptology and Information Security*. Springer. 2011.

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  - ::: BJMM on 2 Levels is fastest in the Lee metric (non-amortized)[1]
  - ::: Wagner's approach is fastest in the Lee metric (amortized)[2]

[1] Violetta Weger et al. "On the hardness of the Lee syndrome decoding problem". In: *Advances in Mathematics of Communications* (2019). DOI: 10.3934/amc.2022029.

[2] André Chailloux, Thomas Debris-Alazard, and Simona Etinski. "Classical and Quantum algorithms for generic Syndrome Decoding problems and applications to the Lee metric". In: *International Conference on Post-Quantum Cryptography*. Springer, 2021, pp. 44–62.

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The cost of an ISD algorithm is given by

$$\underbrace{\text{nr. of iterations}}_{\frac{1}{\text{success probability per iter.}}} \quad \text{cost per iteration}$$

General Framework

We use the idea of partial Gaussian elimination to solve the problem:

1. Find $U \in \mathrm{GL}_{n-k}(\mathbb{Z}/p^s\mathbb{Z})$ such that

$$UH^\top = \begin{pmatrix} \mathrm{I}_{n-k} & 0 \\ A^\top & B^\top \end{pmatrix}$$

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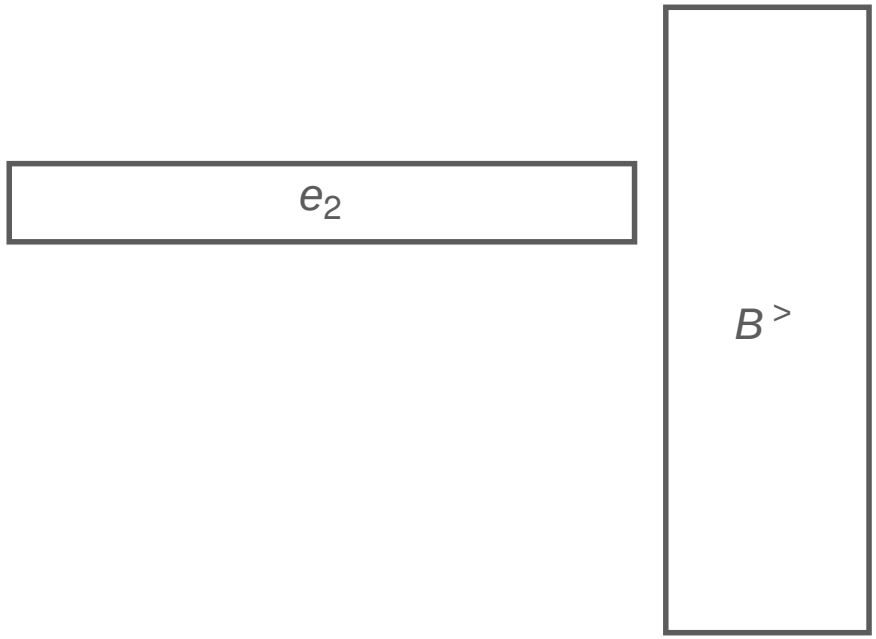
4. Solve the smaller instance of the LSDP. Immediately check whether $e_1 - s_1 - e_2 A^\top$ has Lee weight $t - v$.

Solving the Smaller Instance - Finding e_2

Focus on $e_2 B^{>} = s_2$, with $\mathrm{wt}_L(e_2) = v$



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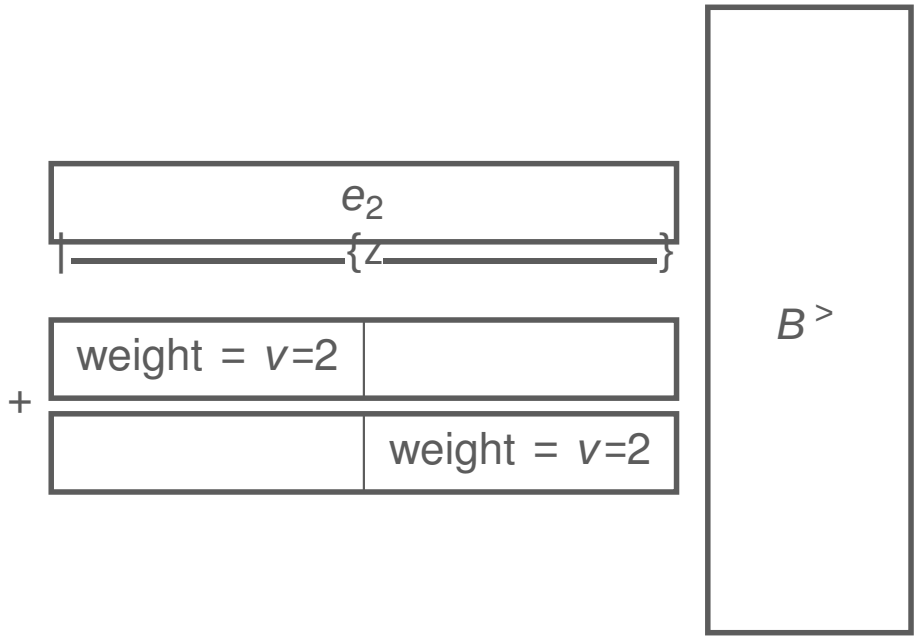
Stern/Dumer

Represent e_2 as

$$e_2 = y_1 + y_2;$$

where
$\mathrm{wt}_L(y_1) = \mathrm{wt}_L(y_2) = v=2$.



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Enumerate the following sets

$$L_1 := \{ y_1 B_1^{>} \mid \mathrm{wt}(y_1) = v{=}2 \}$$
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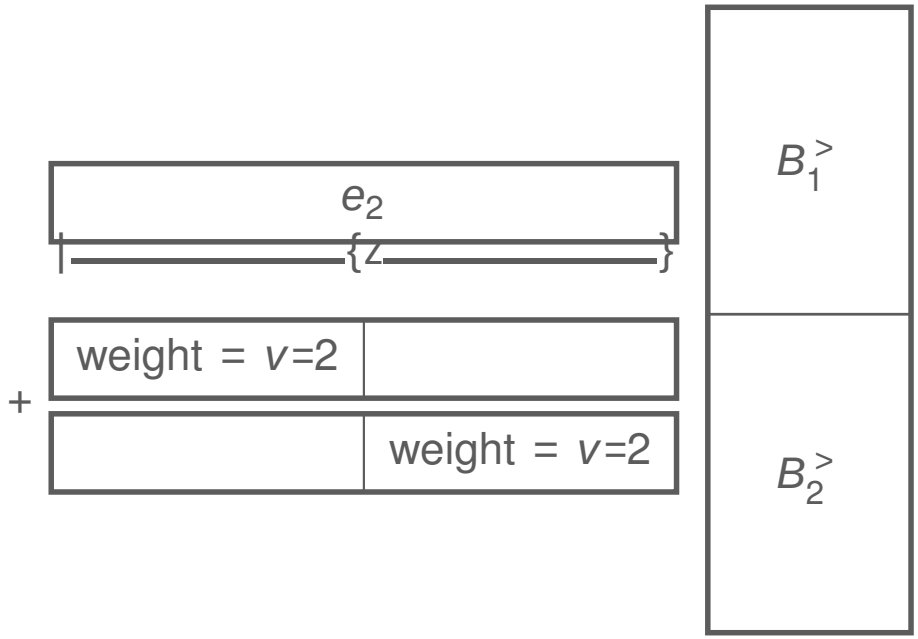

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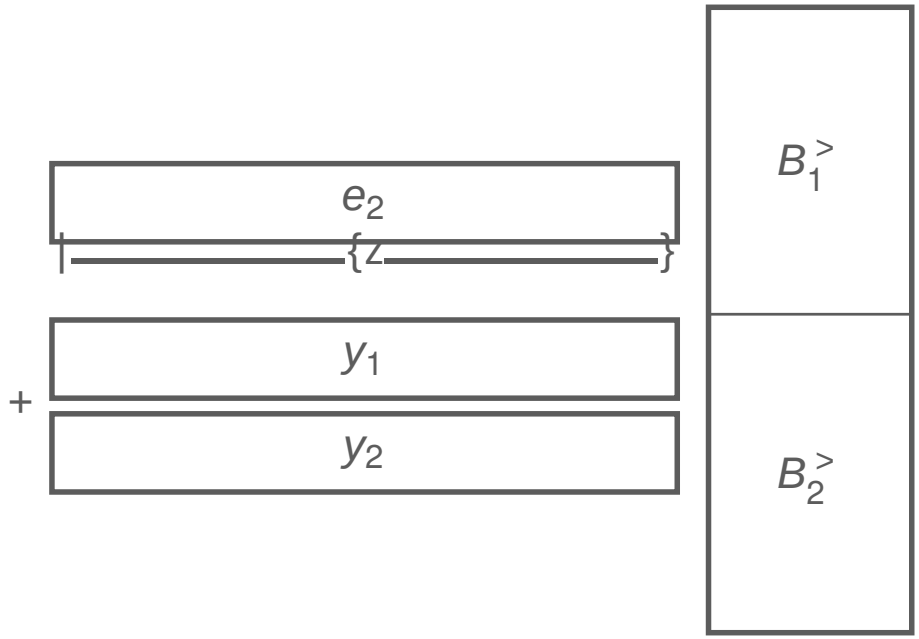

Note: The two vectors $y_1 \in L_1$ and $y_2 \in L_2$ share ε nonzero positions. The expected weight of $y_1 + y_2$ is still v.

Outline

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New Idea: Using Restricted Spheres

Focus on the small instance of the Lee syndrome decoding problem.

Given $B \in (\mathbb{Z}/p^s\mathbb{Z})^{k+\ell}$, $s_2 \in (\mathbb{Z}/p^s\mathbb{Z})^{\ell}$ and $v, t \in \mathbb{N}$

find $e_2 \in (\mathbb{Z}/p^s\mathbb{Z})^{k+\ell}$ s.t. $\mathrm{wt}_L(e_2) = v$ and $s_2 = e_2 B^T$:

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Given $B \in (\mathbb{Z}/p^s\mathbb{Z})^{k\times n}$, $s_2 \in (\mathbb{Z}/p^s\mathbb{Z})^{n-k}$ and $v, t \in \mathbb{N}$
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Use the marginal distribution, i.e.,

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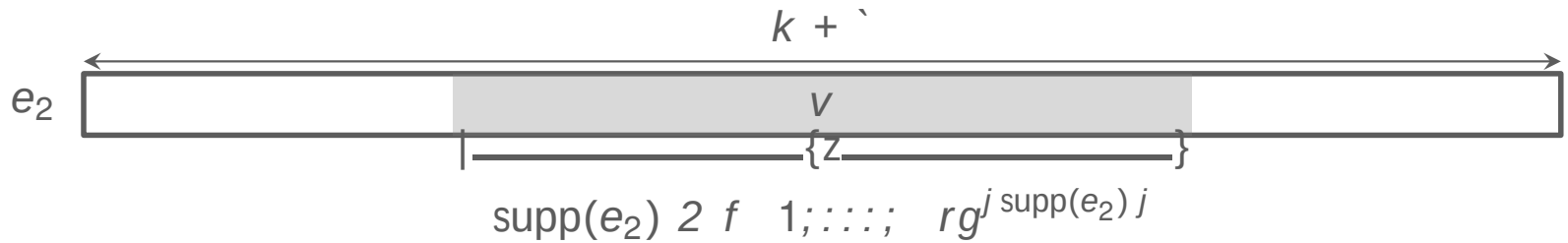
With high probability the least probable entries of e lie **outside** the information set, hence are not in e_2.

We will restrict e_2 to live either in $\{0, 1, \ldots, r\}^{k+\ell}$ or in $\{r, \ldots, M\}^{k+\ell}$, respectively.

Bounded Minimum Distance Decoding - Representation of e_2



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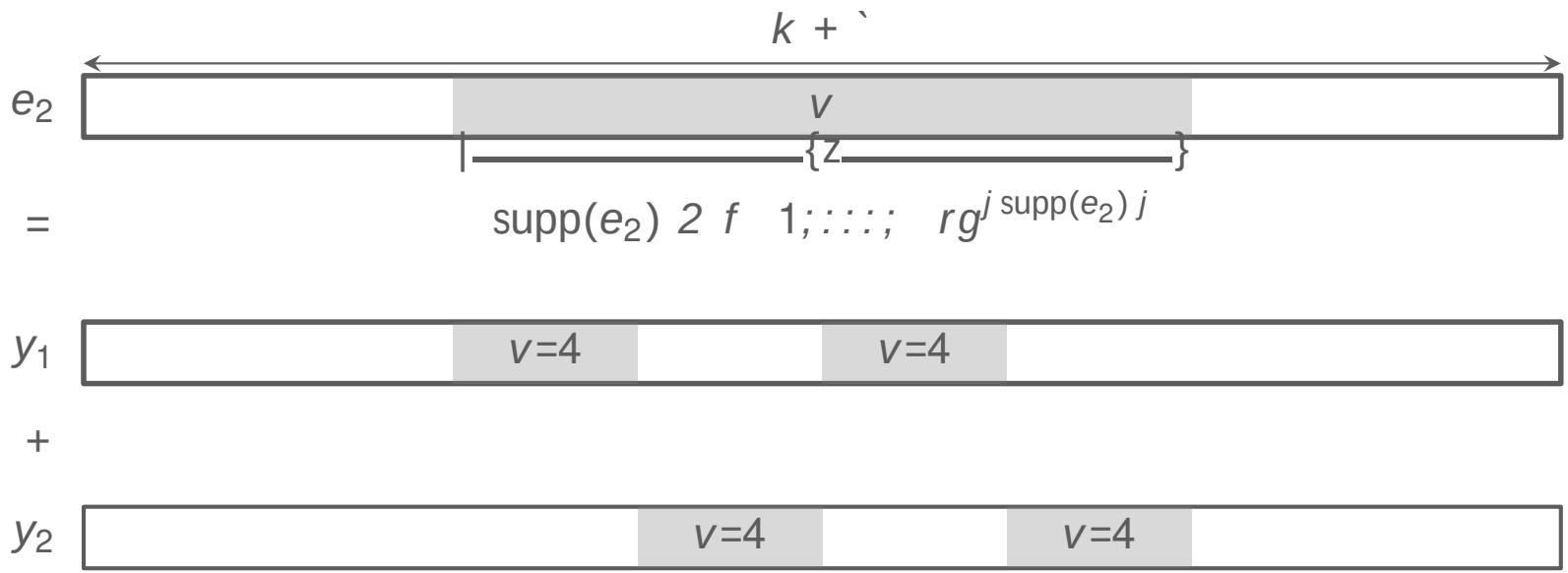

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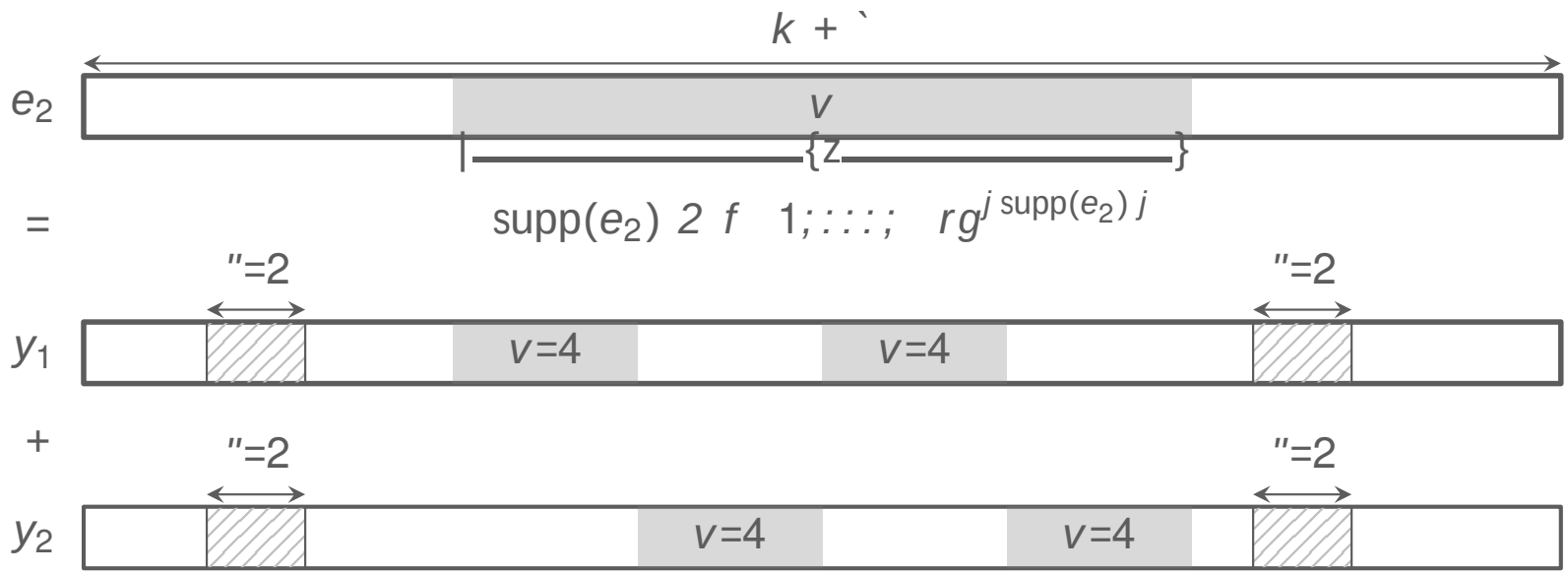
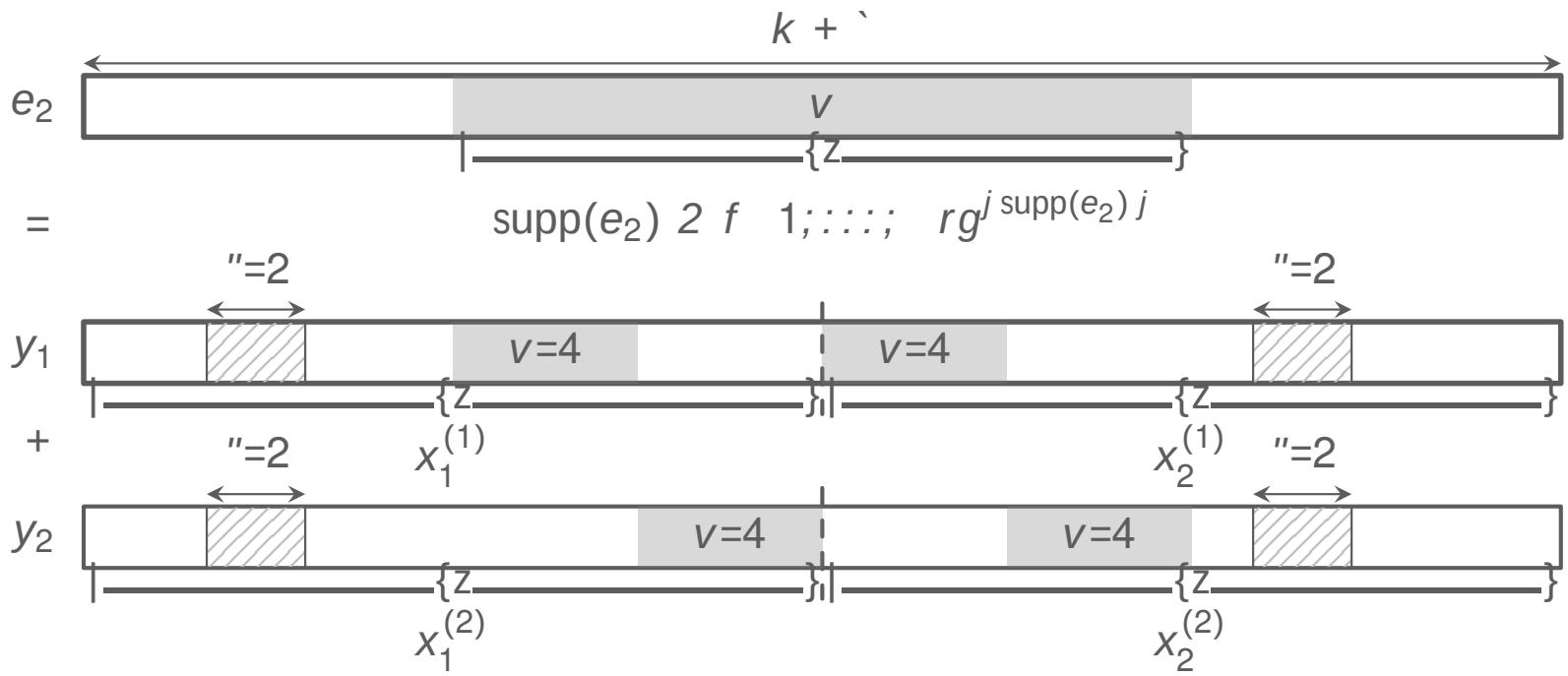
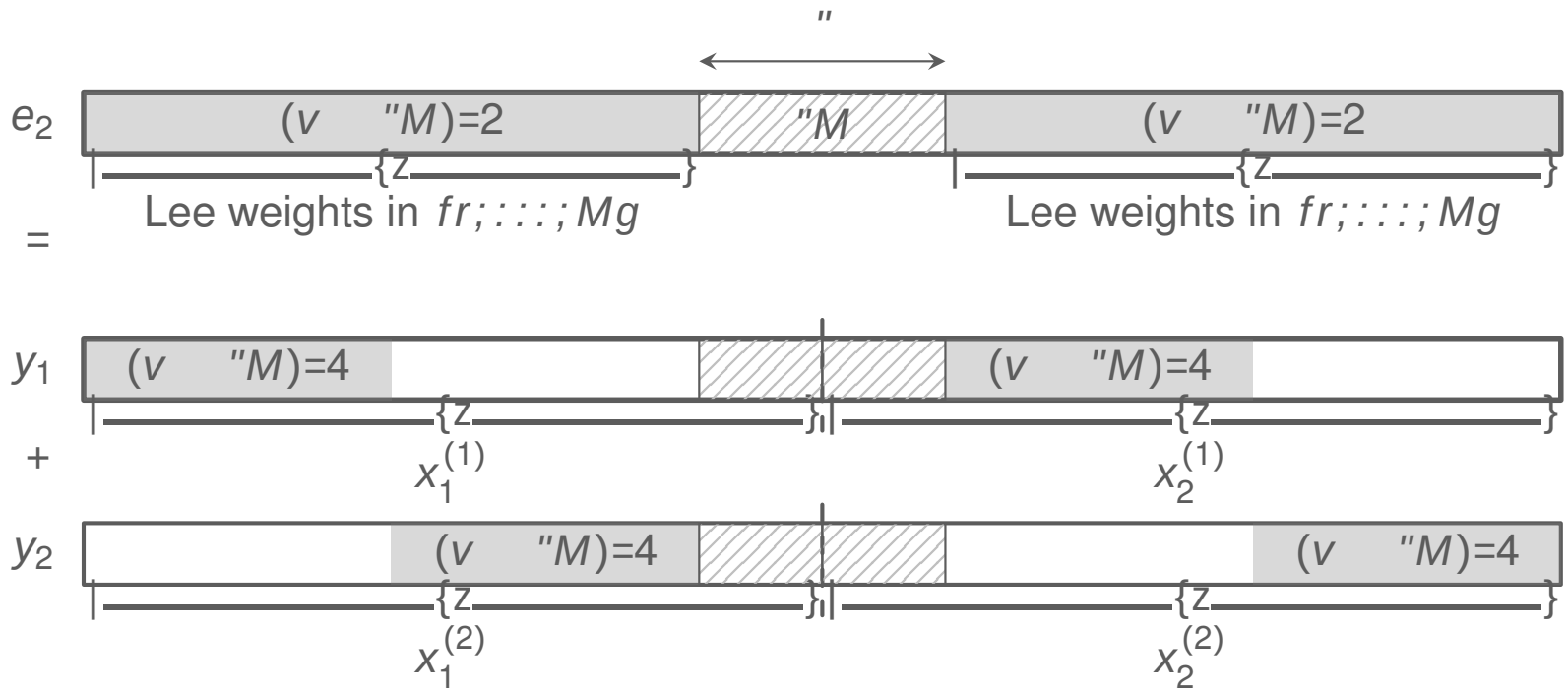


Bounded Minimum Distance Decoding - Representation of e_2



$$B_i = \{x) \ j \ x_{E_i^c} \ 2 \ f0; : : : ; \ rg^{(k+`\ \ "')=2}; \mathrm{wt_L}(x_{E_i^c}) = v=4; x_{E_i} \ 2 \ \ Z=p^s Z \ \ ^{"=2} ; \ \ 2 \ S_{(k+`)=2}$$

Decoding Beyond the Minimum Distance



Bounded Minimum Distance Decoding - BJMM Approach

Recall, $s_2 = e_2 B^\top$, where $e_2 = y_1 + y_2 = (x_1^{(1)}; x_2^{(1)}) + (x_1^{(2)}; x_2^{(2)})$.

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1. Splitting $B = (B_1\ B_2)$, for $i = 1; 2$ concatenate all $x_1^{(i)}; x_2^{(i)} \in B_i$ satisfying

$$x_1^{(1)} B_1^{>} =_u x_2^{(1)} B_2^{>};$$
$$x_1^{(2)} B_1^{>} =_u s_2 - x_2^{(2)} B_2^{>}:$$

They imply the syndrome equations for y_1 and y_2, respectively.

$$y_1 B^{>} = 0 \text{ and } y_2 B^{>} = s_2$$

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$$x_1^{(1)} B_1^\top = u \quad x_2^{(1)} B_2^\top;$$
$$x_1^{(2)} B_1^\top = u\ s_2 \quad x_2^{(2)} B_2^\top:$$

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$$s_2 = (y_1 + y_2) B^\top \text{ and } \mathrm{wt}_L(y_1 + y_2) = v;$$

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   $$s_2 = (y_1 + y_2) B^{>} \text{ and } \mathrm{wt}_L(y_1 + y_2) = v;$$
   b) the original LSDP is fulfilled as well
   $$\mathrm{wt}_L(s_1 - (y_1 + y_2) A^{>}) = t - v$$

Comparison - Bounded Minimum Distance Decoding in Z=47Z

[1]

Algorithm	$e(R\ ; p^S)$	R
Lee-BJMM	0.1618	0.451
Restricted Lee-BJMM for $r = 5$	0.1539	0.408
Amortized Lee-BJMM	0.1205	0.396
Amortized Restricted Lee-BJMM	0.1189	0.406
Amortized Lee-Wagner	0.1441	0.445
Amortized Restricted Lee-Wagner	0.1441	0.445

[1] André Chailloux, Thomas Debris-Alazard, and Simona Etinski. "Classical and Quantum algorithms for generic Syndrome Decoding problems and applications to the Lee metric". In: *International Conference on Post-Quantum Cryptography*. Springer. 2021, pp. 44–62.

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Thank you for your attention!

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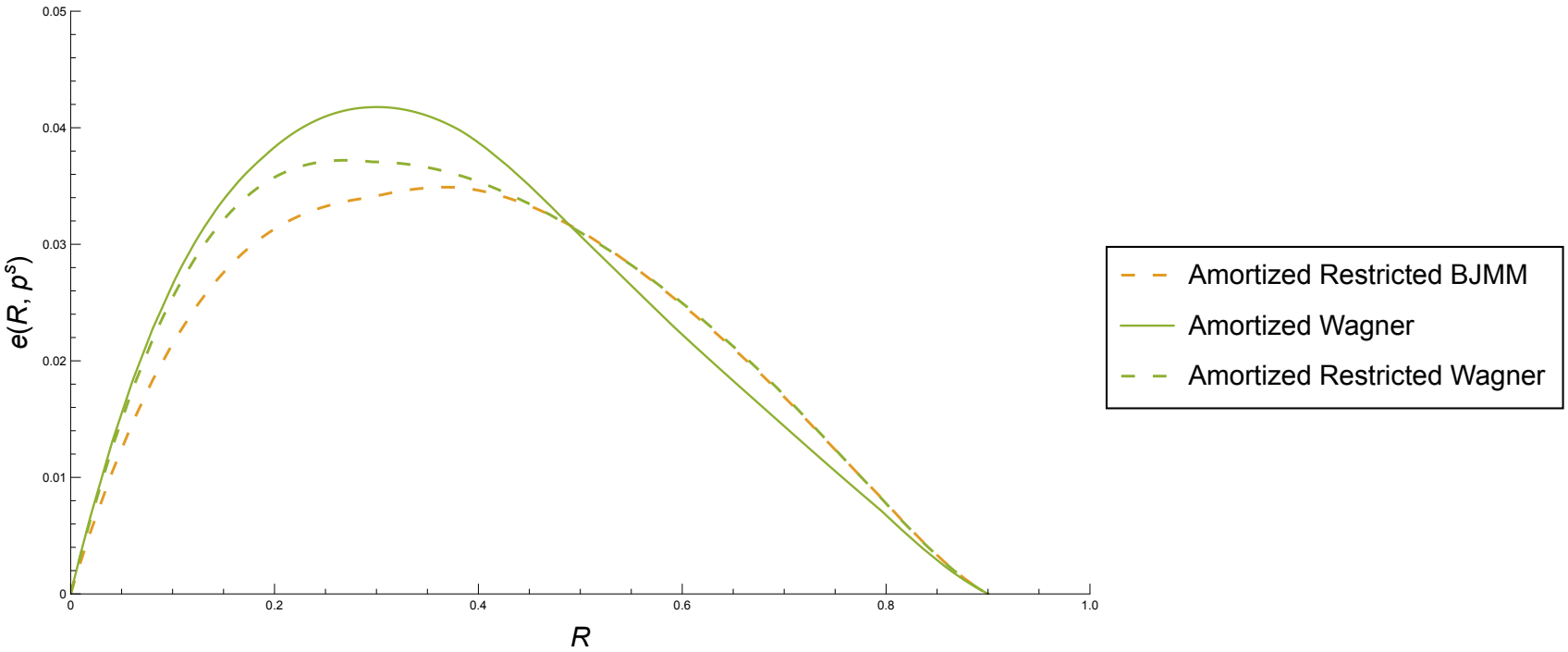
Frame Title


0.05
0.04
0.03
0.02
0.01
0
$e(R, p^s)$
0
0.2
0.4
0.6
0.8
1.0
R
Amortized Restricted BJMM
Amortized Wagner
Amortized Restricted Wagner