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Eindhoven University of Technology (TU/e)

Analysis and Properties of Error Patterns in the Lee Channel

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University of Zurich

joint work with Hannes Bartz, Gianluigi Liva
and Joachim Rosenthal

Outline

1. Introduction
2. The Lee Channel
3. Error Pattern Construction
4. Scalar Multiplication in the Lee Metric

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2 The Lee Channel

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4 Scalar Multiplication in the Lee Metric

Channel Coding

Let $\mathcal{X}$ and $\mathcal{Y}$ the input and output alphabet of the channel, respectively.

With a given channel law

$$P_{Y|X}(y_i \mid x_i) := \mathbb{P}(Y = y_i \mid X = x_i)$$

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Example: q-ary Symmetric Channel (qSC)

- Alphabets:
  $\mathcal{X} = \mathcal{Y} = \{0, 1, \dots, q-1\}$

Input	Output
0	0
1	1
$\vdots$	$\vdots$
$q-1$	$q-1$

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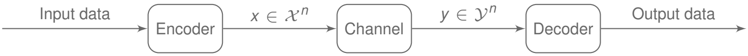

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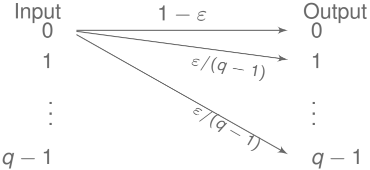

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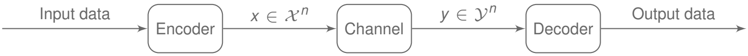

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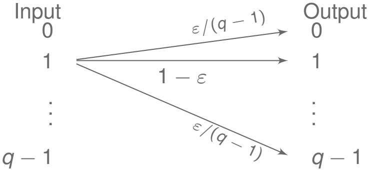

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Linear Block Codes

Let $\mathbb{F}_q$ be a finite field of order q and let n be a positive integer.

Definition [Linear Code]

An $[n, k]_q$-*linear code* $\mathcal{C} \subset \mathbb{F}_q^n$ is a k-dimensional subspace of $\mathbb{F}_q^n$. The elements of $\mathcal{C}$ are called *codewords*.

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Definition [Hamming Weight/Distance]

For any two codewords $x, y \in \mathcal{C}$ we define

- the *Hamming weight* of x, $\mathrm{wt}_H(x) = |\{i \in \{1, \ldots, n\} \mid x_i \neq 0\}|$
- the *Hamming distance* between x and y, $\mathrm{d}_H(x, y) := \mathrm{wt}_H(x - y)$

The Lee Metric

We will denote by $\mathbb{Z}_q$ the ring of integers modulo q.

Definition [Lee weight]

For any integer $a \in \mathbb{Z}_q$ its *Lee weight* is defined as

$$\mathrm{wt}_L(a) := \min(a, q - a) \tag{1}$$

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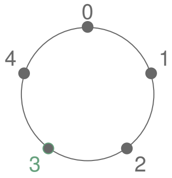

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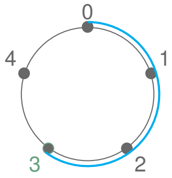

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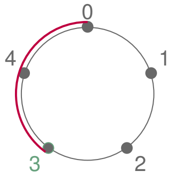

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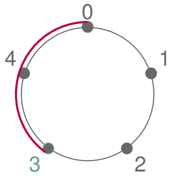

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$\Longrightarrow \mathrm{wt}_L(3) = 2$

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Properties

For every $a \in \mathbb{Z}_q$ it holds:

- $\mathrm{wt}_L(a) = \mathrm{wt}_L(q-a)$

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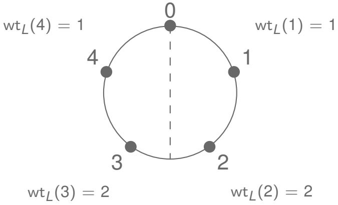

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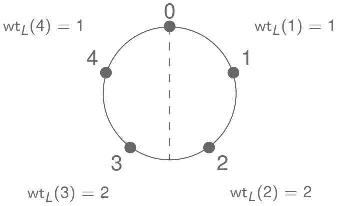

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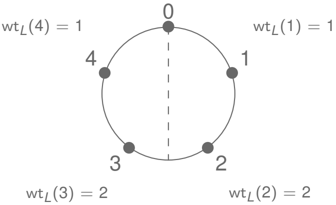
- $\mathrm{wt}_H(a) \leq \mathrm{wt}_L(a)$
  If $q \in \{2, 3\}$, the Lee weight is equivalent to the Hamming weight.

Example



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Let $x = (x_1, \ldots, x_n) \in \mathbb{Z}_q^n$ be a tuple of length n. The *Lee weight* of x is the sum of the Lee weight of its entries, i.e.,

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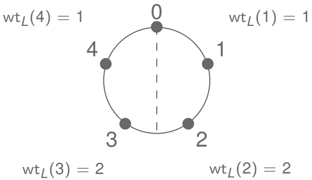
$$x = (0, 2, 4, 3, 0, 3)$$
$$\mathrm{wt}_L(x) =$$



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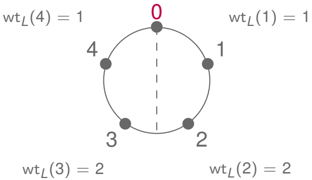

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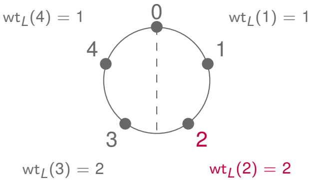

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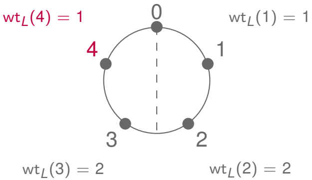

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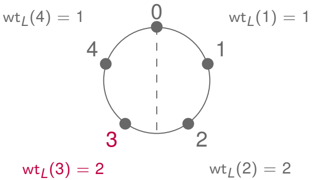

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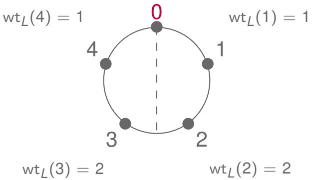

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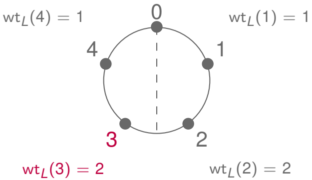

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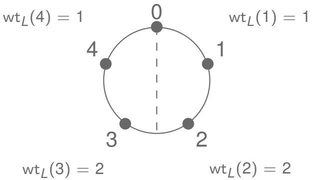
$$x = (0, 2, 4, 3, 0, 3)$$
$$\mathrm{wt}_L(x) = 0 + 2 + 1 + 2 + 0 + 2 = 7$$



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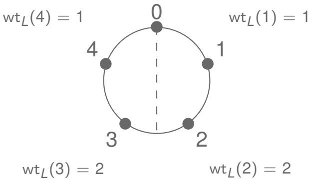
$$\begin{aligned} x &= (0, 2, 4, 3, 0, 3) \\ \mathrm{wt}_L(x) &= 0 + 2 + 1 + 2 + 0 + 2 = 7 \\ \mathrm{wt}_H(x) &= 4 \end{aligned}$$



Why Lee Metric?

- Transmitting symbols over a nonbinary noisy channel
  $\longrightarrow$ primarily those using phase-shift keying modulation

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  - Generic decoding is NP-hard in the Lee Metric[1]

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[2] Paolo Santini et al. "Low-Lee-Density Parity-Check Codes". In: *ICC 2020-2020 IEEE International Conference on Communications (ICC)*. IEEE. 2020, pp. 1–6.

Why Lee Metric?

- Transmitting symbols over a nonbinary noisy channel
  $\longrightarrow$ primarily those using phase-shift keying modulation
- Design code-based cryptosystems with reduced key sizes
- Used in magnetic and DNA storage systems.
- Recently: gained attention in cryptographic applications
  - Generic decoding is NP-hard in the Lee Metric[1]
  - Low-Lee-Density Parity-Check Codes were defined[2]

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Outline

1 Introduction

2 The Lee Channel

3 Error Pattern Construction

4 Scalar Multiplication in the Lee Metric

The Lee Channel

Originally introduced by Chiang and Wolf[3].

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Assume the alphabet is $\mathbb{Z}_q$.
Goal: Describe $\mathbb{P}(i \mid j) = \mathbb{P}(i - j \mid 0)$.

$$
\begin{array}{ccc}
 & & -\lfloor q/2 \rfloor \\
 & & \vdots \\
 & & -1 \\
0 & & 0 \\
 & & 1 \\
 & & \vdots \\
 & & \lfloor q/2 \rfloor
\end{array}
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$$p_i := \mathbb{P}(i \mid 0) = \mathbb{P}(-i \mid 0)$$

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$\vdots$

-1

$0 \xrightarrow{p_0} 0$

1

$\vdots$

$\lfloor q/2 \rfloor$

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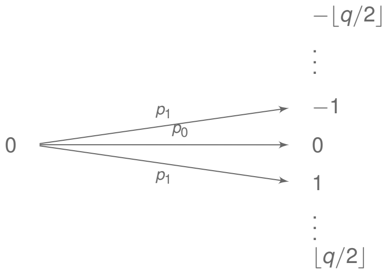

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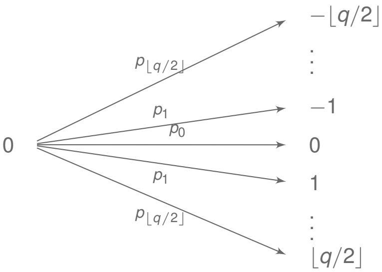

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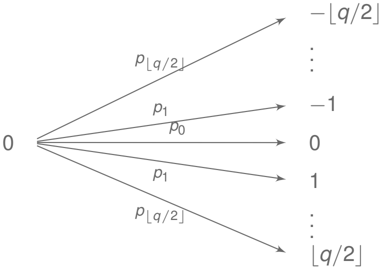
Question: How is p_i defined?



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The Lee Channel Law

For $y, x, e \in \mathbb{Z}_q$, consider a discrete memoryless channel (DMC)

$$\underset{\text{channel output}}{y} = \underset{\text{channel input}}{x} + \underset{\text{additive error term}}{e} \tag{3}$$

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Restrict to: e a realization of a random variable E with

$$\mathbb{P}(E = e) \propto \exp(-\lambda \, \mathrm{wt}_L(e)), \qquad \lambda > 0,$$

$$P_{Y|X}(y|x) = \frac{1}{Z} \exp\left(-\lambda \, \mathrm{d}_L(x, y)\right), \qquad Z := \sum_{e=0}^{q-1} \exp(-\lambda \, \mathrm{wt}_L(e))$$

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Note

- The expectation of $\mathrm{wt}_L(E)$ can be written as $\delta = -\frac{\mathrm{d} \log Z(\lambda)}{\mathrm{d}\,\lambda}$.
- Defining $p_i := \mathbb{P}(\mathrm{wt}_L(e) = i) = \frac{1}{Z} \exp\left(-\lambda i\right)$ for $i \in \{0, 1, \ldots, \lfloor q/2 \rfloor\}$, we easily see

$$p_0 > p_1 \quad \text{and} \quad p_i = \frac{p_1^i}{p_0^{i-1}} \quad \text{for all } i = 2, \ldots, \lfloor q/2 \rfloor.$$

The Constant Lee Weight Channel

Consider now $y, x, e \in \mathbb{Z}_q^n$ and $y = x + e$, where e has a fixed Lee weight $t \in \mathbb{Z}$ and is drawn uniformly at random from $\mathcal{S}_{t,q}^n := \left\{ x \in \mathbb{Z}_q^n \,\middle|\, \mathrm{wt}_L(x) = t \right\}$.

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Theorem

For every $j \in \{1, \ldots, n\}$ the marginal weight distribution of an entry e_j is given by

$$p_i := \mathbb{P}(\mathrm{wt}_L(e_j) = i) = \frac{1}{\sum_{j=0}^{q-1} \exp(-\beta\, \mathrm{wt}_L(j))} \exp\left(-\beta i\right), \forall i \in \{0, \ldots, \lfloor q/2 \rfloor\}$$

where $\beta > 0$ is the solution to $\frac{t}{n} = \frac{(r-1)\mathrm{e}^{(r+1)\beta} - r\mathrm{e}^{r\beta} + \mathrm{e}^{\beta}}{(\mathrm{e}^{\beta r} - 1)(\mathrm{e}^{\beta} - 1)}$ with $r = \lfloor q/2 \rfloor + 1$.

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Proof idea.
Solve an optimization problem to find a distribution $(p_0, p_1, \ldots, p_{\lfloor q/2 \rfloor})$ that is

... maximizing $H(p_0, \ldots, p_{\lfloor q/2 \rfloor}) := -\sum_{i=0}^{\lfloor q/2 \rfloor} p_i \cdot \log(p_i)$,

... subject to $\sum_{i=0}^{\lfloor q/2 \rfloor} p_i \cdot i = \frac{t}{n}$.

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Let t be a positive integer. An *(integer) partition* of t of length k is a k-tuple $\lambda = (\lambda_1, \dots, \lambda_k)$ satisfying

1. $\lambda_1 + \ldots + \lambda_k = t$,
2. $\lambda_1 \geq \ldots \geq \lambda_k > 0$.

The elements λ_i are called *parts* and their corresponding values are the *part sizes*.

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Notation:
$\mathcal{P}(t)$: Set of integer partitions of a positive integer $t \in \mathbb{Z}$.
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Example

Let us consider $t = 4$.

$$\begin{aligned}
\mathcal{P}(4) &= \{(3, 1), (2, 2), (2, 1, 1), (1, 1, 1, 1)\} \\
\mathcal{P}_2(4) &= \{(2, 2), (2, 1, 1), (1, 1, 1, 1)\} \\
\mathcal{P}_1(4) &= \{(1, 1, 1, 1)\}
\end{aligned}$$

Tuples of Weight Decomposition λ over $\mathbb{Z}_q$

Definition [Vectors of Weight Decomposition λ]

For a positive integer n and a given partition $\lambda \in tr$ of a positive integer t, we say that a length-n vector x has *weight decomposition* λ *over* $\mathbb{Z}_q$ if there is a one-to-one correspondence between the Lee weight of the nonzero entries of x and the parts of λ.

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Consider $\mathbb{Z}_5$, $t = n = 4$ and $\lambda = (2, 1, 1)$ a partition of t over $\mathbb{Z}_5$.
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DLR

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$$\mathcal{V}_{4,(2,1,1)}^{(4)} = \{(2, 1, 1, 0), (2, 1, 0, 1), \ldots, (1, 2, 1, 0), \ldots, (3, 4, 1, 0), \ldots\}$$

Number of Tuples of weight decomposition λ over $\mathbb{Z}_q$

Lemma

Let n, q and t be positive integers and consider the set of partitions $\mathcal{P}_{\lfloor q/2 \rfloor}(t)$ of t with part sizes not exceeding $\lfloor q/2 \rfloor$. For any $\lambda \in \mathcal{P}_{\lfloor q/2 \rfloor}(t)$ the number of vectors of length n over $\mathbb{Z}_q$ of type λ is given by

$$\left| \mathcal{V}_{q,t,\lambda}^{(n)} \right| = \begin{cases} 2^{\ell_\lambda} \left| \Pi_\lambda \right| \binom{n}{\ell_\lambda} & \text{if } q \text{ is odd,} \\ 2^{\ell_\lambda - c_{\lfloor q/2 \rfloor, \lambda}} \left| \Pi_\lambda \right| \binom{n}{\ell_\lambda} & \text{else} \end{cases} \tag{4}$$

where $c_{\lfloor q/2 \rfloor, \lambda} = |\{ i \in \{1, \ldots, \ell_\lambda\} \mid \lambda_i = \lfloor q/2 \rfloor \}|$.

Drawing Tuples of Fixed Lee Weight

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3. Choose randomly k positions of the tuple x and assign the values $a_1, \ldots, a_k$ to them.
4. The remaining entries are zero.

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Example

Consider $\mathbb{Z}_7 \implies \lfloor 7/2 \rfloor = 3$ is the maximal Lee weight for an entry. Say we want a tuple $x = (\_, \_, \_, \_, \_, \_)$ of length 6 with Lee weight $t = 4$.

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$(1,1,1,1)$	$(2,1,1)$	$(2,2)$	$(3,1)$
$\left\lvert \mathcal{V}^{(6)}_{4,(1,1,1,1)} \right\rvert = 240$	$\left\lvert \mathcal{V}^{(6)}_{4,(2,1,1)} \right\rvert = 480$	$\left\lvert \mathcal{V}^{(6)}_{4,(2,2)} \right\rvert = 60$	$\left\lvert \mathcal{V}^{(6)}_{4,(3,1)} \right\rvert = 120$

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2. Assign to each λ_i an element $a_i \in \mathbb{Z}_7$ with $\mathtt{wt}_L(a_i) = \lambda_i$:

$$\lambda_1 = 2 \longrightarrow 5, \quad \lambda_2 = 1 \longrightarrow 1, \quad \lambda_3 = 1 \longrightarrow 6$$

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Consider $\mathbb{Z}_7 \implies \lfloor 7/2 \rfloor = 3$ is the maximal Lee weight for an entry. Say we want a tuple $x = (\_, \_, \_, \_, \_, \_)$ of length 6 with Lee weight $t = 4$.

1. The partitions of $t = 4$ with no part exceeding 3 are:

$(1,1,1,1)$	$(2,1,1)$	$(2,2)$	$(3,1)$
$\left\lvert \mathcal{V}^{(6)}_{4,(1,1,1,1)} \right\rvert = 240$	$\left\lvert \mathcal{V}^{(6)}_{4,(2,1,1)} \right\rvert = 480$	$\left\lvert \mathcal{V}^{(6)}_{4,(2,2)} \right\rvert = 60$	$\left\lvert \mathcal{V}^{(6)}_{4,(3,1)} \right\rvert = 120$

   Say we pick $(\lambda_1, \lambda_2, \lambda_3) = (2, 1, 1)$.
2. Assign to each λ_i an element $a_i \in \mathbb{Z}_7$ with $\mathtt{wt}_L(a_i) = \lambda_i$:

$$\lambda_1 = 2 \longrightarrow 5, \quad \lambda_2 = 1 \longrightarrow 1, \quad \lambda_3 = 1 \longrightarrow 6$$

3. Choose randomly 3 positions of x and assign them to one of the above values

$$x = (\_, 6, \_, 5, 1, \_)$$

DLR

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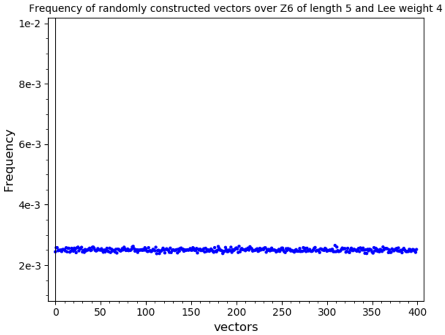
   $$x = (\_, 6, \_, 5, 1, \_)$$
4. $x = (0, 6, 0, 5, 1, 0)$

Distribution

Theorem

Let n, q and t be positive integers. The when sampling a sufficiently large number of n-tuples using the before shown algorithm, we obtain a uniform distribution on $\mathcal{S}_q^n(t)$.



Outline

1. Introduction
2. The Lee Channel
3. Error Pattern Construction
4. Scalar Multiplication in the Lee Metric

Generic Decoding

Assume we receive a vector $y = \underbrace{x}_{\text{original message}} + \underbrace{e}_{\text{error vector}}$.

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  Is NP-hard for the Hamming- and the Lee metric.

Introduction to the Problem

Example 1
Let $x = (0, 2, 3, 1, 0, 3) \in \mathbb{Z}_5^6$

Lee	Hamming
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DLR

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Risk: From a cryptographic point of view, an attacker could decrease the weight and retrieve the original message.

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Goal: We want this probability to be small!

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Applying the union bound, we have

$$
\begin{aligned}
\mathbb{P}(F) &= \mathbb{P}\left(\mathrm{wt}_L(a \cdot x) < \mathrm{wt}_L(x) \mid Q \text{ is "close" to } \mathcal{B}\right) \mathbb{P}\left(Q \text{ is "close" to } \mathcal{B}\right) \\
&\quad + \mathbb{P}\left(\mathrm{wt}_L(a \cdot x) < \mathrm{wt}_L(x) \mid Q \text{ is "not close" to } \mathcal{B}\right) \mathbb{P}\left(Q \text{ is "not close" to } \mathcal{B}\right) \\
&\leq \mathbb{P}\left(\mathrm{wt}_L(a \cdot x) < \mathrm{wt}_L(x) \mid Q \text{ is "close" to } \mathcal{B}\right) + \mathbb{P}\left(Q \text{ is "not close" to } \mathcal{B}\right)
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"Close" Distributions

Definition [Kullback-Leibler divergence]

Let X be a random variable over an alphabet $\mathcal{X}$ with probability distribution P, where $P(x) := \mathbb{P}(X = x)$. Furthermore, let us assume that X can approximated by another distribution $Q \neq P$. We define the *Kullback-Leibler divergence* of Q and P by

$$D(P \,||\, Q) := \sum_{x \in \mathcal{X}} P(x) \log\left(\frac{P(x)}{Q(x)}\right) \tag{6}$$

Note

- By convention: $0 \log(0) = 0$.
- The two distributions Q and P are *close* to each other if $D(Q \,||\, P) \leq \varepsilon$, for some $\varepsilon > 0$.

Conditional Limit Theorem

Theorem
Conditional Limit Theorem

Let E be a closed convex set of probability distributions over an alphabet $\mathcal{X}$ and let Q be a distribution over $\mathcal{X}$ but not in E. Let $X_1, \dots, X_n$ be discrete random variables drawn i.i.d. $\sim Q$. Define $X^n = (X_1, \dots, X_n)$ and let $P^\star = \arg\min_{P \in E} D(P \,||\, Q)$. Then

$$\mathbb{P}\left(X_1 = a \,|\, P_{X^n} \in E\right) \longrightarrow P^\star(a)$$

in probability as n grows large for any $a \in \mathcal{X}$.
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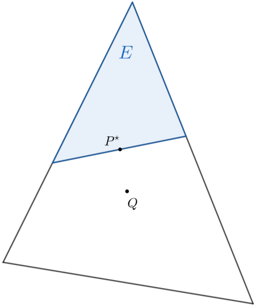

[4] Thomas M Cover. *Elements of information theory*. John Wiley & Sons, 1999

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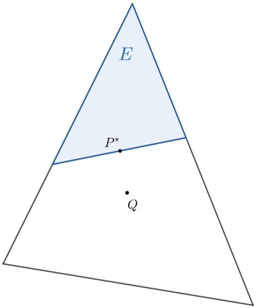

In our case:

$Q \sim \mathcal{U}(\mathbb{Z}_q)$; E set of distributions of tuples in $\mathcal{S}_q^n(t)$. Then $\mathcal{B} = \arg\min_{P \in E} D(P \,||\, Q)$.

[4] Cover, *Elements of information theory*

Asymptotic Regime

Recall, $F = \{\mathrm{wt}_L(a \cdot x) < \mathrm{wt}_L(x)\}$ and

$$\mathbb{P}(F) \leq \mathbb{P}\left(\mathrm{wt}_L(a \cdot x) < \mathrm{wt}_L(x) \,|\, Q \text{ is "close" to } \mathcal{B}\right) + \mathbb{P}\left(Q \text{ is "not close" to } \mathcal{B}\right)$$

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Let $x \in \mathbb{Z}_q^n$, for some positive integer $q > 3$, of average Lee weight $\delta = t/n$ be drawn randomly from $\mathcal{S}_q^n(t)$ with the shown algorithm. Let Q denote the empirical distribution of the entries of x. For any nonzero $a \in \mathbb{Z}_q$ it holds

$$\mathbb{P}\left(Q \text{ not close to } \mathcal{B}\right) \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

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By CLT $\leq \mathbb{P}\left(\mathrm{wt}_L(a \cdot x) < \mathrm{wt}_L(x) \mid Q \sim \mathcal{B}\right)$

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DLR

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q	5	7	8	9	11	31	33	53
$\lfloor q/2 \rfloor$	2	3	4	4	5	15	16	26
$\delta^\star$	1	1.5	1.534	1.703	2.5	7.5	7.03	13

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Thank you for your attention!