Constructing Moderate-Density Parity-Check Codes from Projective Bundles

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Eindhoven University of Technology

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Outline

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1. MDPC Codes
2. Projective Bundles
3. New MDPC Code Family

Linear Codes

Consider a finite field $\mathbb{F}_q$ of q elements.

Linear code

A k-dimensional subspace $\mathcal{C} \subset \mathbb{F}_q^n$ is called a *q-ary linear code* of length n and dimension k. Its elements are called *codewords*.

Notation $\mathcal{C}$ is an $[n, k]_q$-linear code.

Example of a 2-dimensional subspace of $\mathbb{F}_2^4$

$$\mathcal{C} := \{(0,0,0,0), (0,0,1,1), (1,1,0,0), (1,1,1,1)\} = \langle (0,0,1,1), (1,1,0,0) \rangle$$

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$$\mathcal{C} := \{(0,0,0,0),(0,0,1,1),(1,1,0,0),(1,1,1,1)\} = \langle (0,0,1,1),(1,1,0,0) \rangle$$

Dual code

The *dual code* of an $[n,k]_q$-linear code is given by

$$\mathcal{C}^\perp = \left\{ x \in \mathbb{F}_q^n \,|\, x \cdot c^\top = 0 \text{ for all } c \in \mathcal{C} \right\}.$$

Example

$$\mathcal{C} = \langle (0,0,1,1),(1,1,0,0) \rangle = \mathcal{C}^\perp$$

Hamming Metric

Given two vectors $x, y \in \mathbb{F}_q^n$.

$$\begin{aligned} &\text{Hamming weight:} \quad && \mathrm{wt}_{\mathsf{H}}(x) && := \left| \{ i = 1, \ldots, n \mid x_i \neq 0 \} \right| \\ &\text{Hamming distance:} \quad && \mathrm{d}_{\mathsf{H}}(x, y) && := \mathrm{wt}_{\mathsf{H}}(x - y) \end{aligned}$$

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Minimum distance

The *minimum Hamming distance* $\mathrm{d}_{\mathsf{H}}(\mathcal{C})$ of an $[n, k]_q$-linear code $\mathcal{C}$ is the minimal Hamming weight of a nonzero codeword, i.e.,

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Properties of the minimum distance

- $\mathrm{d}_{\mathsf{H}}(\mathcal{C}) = d$ means that $\lfloor (d-1)/2 \rfloor$ errors can be corrected
- Singleton bound: $d \leq n - k + 1$

Code Representation

Parity-Check Matrix

A *parity-check matrix* of the code $\mathcal{C} \subset \mathbb{F}_q^n$ is a matrix $H \in \mathbb{F}_q^{(n-k)\times n}$ satisfying

$$\mathcal{C} = \ker(H) = \left\{ x \in \mathbb{F}_q^n \mid xH^\top = 0 \right\}.$$

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Example over $\mathbb{F}_2^4$

The code $\mathcal{C} = \langle (0,0,1,1),(1,1,0,0) \rangle$ has a parity-check matrix of the form

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Given a code $\mathcal{C} \subseteq \mathbb{F}_q^n$ with parity-check matrix H. We say it has *type* (v,w), if

- every column of H has a constant weight v,
- every row of H has a constant weight w.

MDPC Codes

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- Introduced as an extension to low-density parity-check codes [Gal62]
  $\longrightarrow$ especially interesting for code-based cryptography [MTSB13]
- Different constructions exist for MDPC codes
  $\longrightarrow$ random, cyclic, quasi-cyclic, ...
- MDPC codes can be decoded with low complexity [MTSB13]
- Several decoding algorithms analysed for MDPC codes [BL18]
  $\longrightarrow$ Bit-Flipping Decoding [Gal63]

Decoding performance result MDPC codes - Bit-flipping [Til18]

Given an MDPC code $\mathcal{C} \subseteq \mathbb{F}_q^n$ with

- a parity-check matrix H of type (v, w), and
- a *maximum column intersection number* $s_H := \max \left| \left\{ i = 1, \ldots, n \,|\, h_{ij} = h_{ij'} = 1 \text{ and } j \neq j' \right\} \right|$.

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  Example:

$$H = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \implies s_H = 1$$

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Then performing one round of bit-flipping allows to correct errors of weight at most

$$\left\lfloor \frac{v}{2 s_H} \right\rfloor .$$

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Construct codes with good performance.
Construct codes with small maximum column intersection s.

Outline

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2. Projective Bundles

3. New MDPC Code Family

Projective Planes

$\mathsf{PG}(2, q)$: projective plane in $\mathbb{F}_q$ consisting of

- $q^2 + q + 1$ points
- $q^2 + q + 1$ lines

Properties

1. Two points lie on exactly one common line and vice versa.
2. Each point lies on $q + 1$ lines & each line contains $q + 1$ points.
3. No three points among four are collinear.

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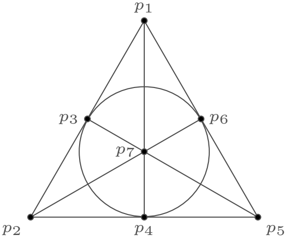
Fano Plane $\mathsf{PG}(2,2)$



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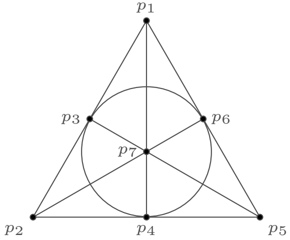

Representation through incidence matrix

$\mathsf{PG}(2,q)$ can be represented by a matrix $A \in \mathbb{F}_2^{(q^2+q+1)\times(q^2+q+1)}$ defined as

$$A_{p\ell} = \begin{cases} 1 & \text{point } p \text{ is incident to line } \ell \\ 0 & \text{otherwise} \end{cases}$$

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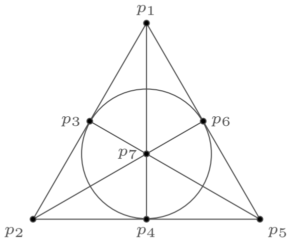

Representation through incidence matrix

$$A_{\mathsf{Fano}} = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} \ell_1 & \ell_2 & \ell_3 & \ell_4 & \ell_5 & \ell_6 & \ell_7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Codes from Planes

Given an incidence matrix H of $\Pi = \mathsf{PG}(2,q)$ over $\mathbb{F}_2$. Then we define the binary code $\mathcal{C}_2(\Pi)^\perp \subset (\mathbb{F}_2)^{q^2+q+1}$ by

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Parameters of $\mathcal{C}_2(\Pi)^\perp$ - [GM66, AJMJ70]

If q is odd:

- $n = q^2 + q + 1$
- $k = 1$
- $\mathrm{d}_{\mathsf{H}}(\mathcal{C}) = q^2 + q + 1$

If $q = 2^h$:

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Bit-Flipping Error Correction

After performing one round of the bit-flipping algorithm on a parity-check matrix H of $\mathcal{C}_2(\Pi)^\perp$, errors of weight up to $\left\lfloor \frac{\mathrm{d_H}(\mathcal{C})-1}{2} \right\rfloor$ can be corrected.

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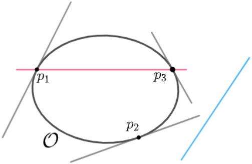
- Codes from planes are powerful and optimal w.r.t. bit-flipping decoding.
- Drawback: Only codes from planes of even order are interesting.

Projective Bundles

Ovals in $\mathsf{PG}(2, q)$

- set $\mathcal{O}$ of $q+1$ points of $\mathsf{PG}(2, q)$
- every line of $\mathsf{PG}(2, q)$ intersects $\mathcal{O}$ in at most 2 points
- exactly $q+1$ tangents - one in every point



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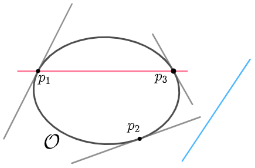

Projective Bundle - [Gly78]

A *projective bundle* is a collection of q^2+q+1 ovals of $\mathsf{PG}(2,q)$ mutually intersecting in a unique point.

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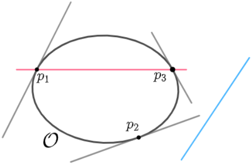

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- Projective bundles exist for any q.
- For q odd, there are three distinct types of projective bundles.

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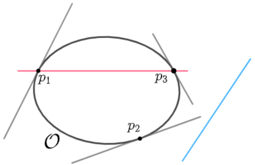

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Question: Why are they interesting to us?

Projective Bundles vs. Projective Planes

Projective Plane $\mathsf{PG}(2,q)$

With incidence matrix A

- q^2+q+1 points mutually intersecting in one line
- q^2+q+1 lines mutually intersecting in one point
- each point lies on $q+1$ lines
- each line contains $q+1$ points

Projective Bundle $\mathcal{B}$ in $\mathsf{PG}(2,q)$

With incidence matrix B

- q^2+q+1 ovals mutually intersecting in one tangent line
- q^2+q+1 lines in $\mathsf{PG}(2,q)$ mutually tangent to one oval
- each oval has $q+1$ tangent lines
- each line has $q+1$ tangent ovals

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Projective Plane from Ovals

Given a projective bundle $\mathcal{B}$ over $\mathsf{PG}(2, q)$. Identify the ovals of $\mathcal{B}$ and the lines of $\mathsf{PG}(2, q)$ as the points and lines, respectively, of $\mathsf{PG}(2, q)$ with incidence defined by tangency. Then this point-line geometry is a projective plane of order q.

In other words: $A^\top B$ is an incidence matrix of a projective plane again.

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MDPC Codes from Projective Bundles

- A: incidence matrix of $\Pi = \mathsf{PG}(2, q)$
- B: incidence matrix of the projective plane Γ induced by ovals of a projective bundle and lines.
- Let $H = (A \mid B) \in \mathbb{F}_2^{(q^2+q+1)\times 2(q^2+q+1)}$.

Code from projective bundles

A binary linear code with parity-check matrix H is called a *projective bundle code* and we write

$$\mathcal{C}_2(\Pi \sqcup \Gamma)^{\perp} := \ker(H).$$

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- dimension: $k = \begin{cases} q^2 + q + 2 & \text{if } q \text{ if odd,} \\ 2^{2h+1} + 2^{h+1} - 2(3^h) + 1 & \text{if } q = 2^h \end{cases}$

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- type: $(v,w) = (q+1, 2(q+1))$

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- minimum distance: $\mathrm{d}_{\mathsf{H}}(\mathcal{C}_2(\Pi \sqcup \Gamma)^{\perp}) = q + 2$
- type: $(v, w) = (q + 1, 2(q + 1))$

MDPC Codes from Projective Bundles

- A: incidence matrix of $\Pi = \mathsf{PG}(2, q)$
- B: incidence matrix of the projective plane Γ induced by ovals of a projective bundle and lines.
- Let $H = (A \mid B) \in \mathbb{F}_2^{(q^2+q+1)\times 2(q^2+q+1)}$.

Code from projective bundles

A binary linear code with parity-check matrix H is called a *projective bundle code* and we write

$$\mathcal{C}_2(\Pi \sqcup \Gamma)^{\perp} := \ker(H).$$

Parameters

- block length: $n = 2(q^2 + q + 1)$
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$C_2(\Pi \sqcup \Gamma)^{\perp}$ **is an MDPC code!**

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Alternative way to represent $\mathsf{PG}(2, q)$:

- Points are identified with the integers modulo $q^2 + q + 1$.
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Perfect Difference Set - [Hir98]
A set $D = \{d_0, \ldots, d_q\} \subset \mathbb{Z}/(q^2 + q + 1)\mathbb{Z}$ is a *perfect difference set* if all differences $(d_i - d_j)$ with $i \neq j$ are distinct modulo $q^2 + q + 1$.

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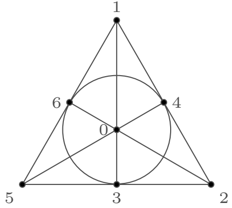
Examle Fano Plane $\mathsf{PG}(2, 2)$

- Set of points:
  $\mathcal{P} = \{0, 1, \ldots, 6\}$
- Perfect difference set modulo $q^2 + q + 1 = 7$:
  $D = \{0, 1, 3\}$
- Set of lines:
  $\mathcal{L} = \{\{0 + i, 1 + i, 3 + i\} \mid i \in \mathbb{Z}/7\mathbb{Z}\}$



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For q odd, the three types of projective bundles can be described via cyclic shifts of $-D, 2D$ and $D/2$.

Example of a Projective Bundle Code for $q = 3$

- Points: $\mathcal{P} = \{0, 1, \ldots, 13\}$
- Perfect difference set modulo 13: $D = \{0, 1, 3, 9\}$
- Lines: $\mathcal{L} = \{\{0 + i, 1 + i, 3 + i, 9 + i\} \mid i \in \mathbb{Z}/13\mathbb{Z}\}$
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$$
H = \left(\begin{array}{ccccccccccccc|ccccccccccccc}
1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 & \cdot \\
1 & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 \\
\cdot & 1 & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\
1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\
\cdot & 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 \\
\cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
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The maximum column intersection is $s_H = 2$. Hence, performing one round of the bit-flipping decoder on H corrects errors of weight up to $\left\lfloor \frac{q+1}{4} \right\rfloor$.

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! $s_H = 2$ for a matrix H of size $(q^2+q+1) \times c$ with $c > q^2+q+1$!

Generalization

Consider the projective plane $\Pi = \mathsf{PG}(2, q)$

- Take $t > 1$ many disjoint projective bundles in Π.
  (Existence proven in [BBEF94])
- Denote the resulting projective planes by $\Gamma_1, \ldots, \Gamma_t$ and the incidence matrices by $B_1, \ldots, B_t$
- Define the binary linear code

$$\mathcal{C}_2(\Pi \sqcup \Gamma_1 \sqcup \ldots, \sqcup \Gamma_t)^\perp := \ker \Big((A \,|\, B_1 \,|\, \ldots \,|\, B_t) \Big)$$

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- minimum distance: $\mathrm{d}_{\mathsf{H}}(\mathcal{C}_2(\Pi \sqcup \Gamma_1 \sqcup \ldots, \sqcup \Gamma_t)^\perp) \geq \left\lceil \frac{q+2}{2} \right\rceil$
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Thank you for your attention!

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