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PICSeminar

Channel Coding in the Lee Metric

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German Aerospace Center (DLR) &
University of Zurich

joint work with Hannes Bartz, Gianluigi Liva
and Joachim Rosenthal

Outline

DLR

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1. Introduction
2. The Lee Channel
3. Error Pattern Construction
4. Scalar Multiplication in the Lee Metric

Channel Coding

Let $\mathcal{X}$ and $\mathcal{Y}$ the input and output alphabet of the channel, respectively.

With a given channel law

$$P_{Y|X}(y_i \mid x_i) := \mathbb{P}(Y = y_i \mid X = x_i)$$

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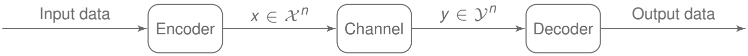

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Example: q-ary Symmetric Channel (qSC)

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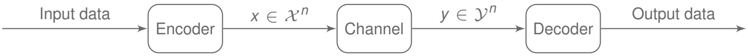

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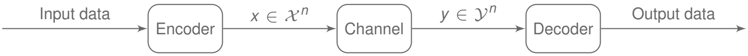

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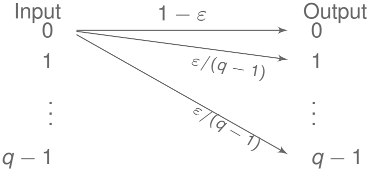

DLR

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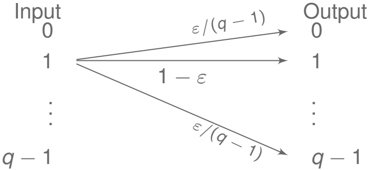

Linear Block Codes

Let $\mathbb{F}_q$ be a finite field of order q and let n be a positive integer.

Definition [Linear Code]

An $[n, k]_q$-*linear code* $\mathcal{C} \subset \mathbb{F}_q^n$ is a k-dimensional subspace of $\mathbb{F}_q^n$. The elements of $\mathcal{C}$ are called *codewords*.

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$\mathcal{C} = \{(0,0,0,0), (1,1,0,0), (0,0,1,1), (1,1,1,1)\}$ is a $[4,2]_2$-linear code.

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Definition [Hamming Weight/Distance]

For any two codewords $x, y \in \mathcal{C}$ we define

- the *Hamming weight* of x, $\mathrm{wt}_H(x) = |\{i \in \{1, \ldots, n\} \mid x_i \neq 0\}|$
- the *Hamming distance* between x and y, $\mathrm{d}_H(x, y) := \mathrm{wt}_H(x - y)$

The Lee Metric

We will denote by $\mathbb{Z}_q$ the ring of integers modulo q.

Definition [Lee weight]

For any integer $a \in \mathbb{Z}_q$ its *Lee weight* is defined as

$$\mathrm{wt}_L(a) := \min(a, q - a) \tag{1}$$

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Example: Consider $\mathbb{Z}_5$. The Lee weight of $a = 3$ is

$$\mathrm{wt}_L(3) = \min(3, 5 - 3) = 2$$

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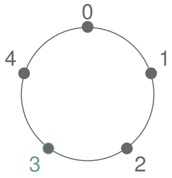

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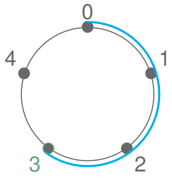

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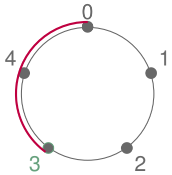

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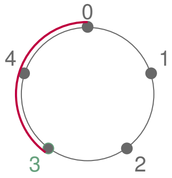

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$\Longrightarrow \mathrm{wt}_L(3) = 2$

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For every $a \in \mathbb{Z}_q$ it holds:

- $\mathrm{wt}_L(a) = \mathrm{wt}_L(q-a)$

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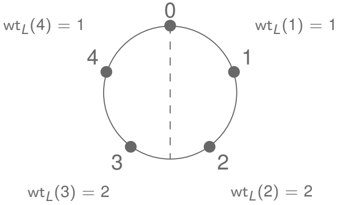

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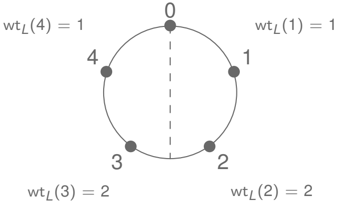

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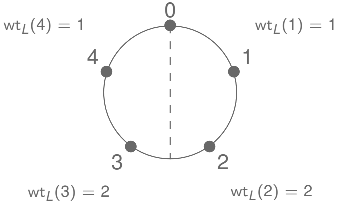
- $\mathrm{wt}_H(a) \leq \mathrm{wt}_L(a)$
  If $q \in \{2, 3\}$, the Lee weight is equivalent to the Hamming weight.

Example



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Definition [Lee weight]

Let $x = (x_1, \ldots, x_n) \in \mathbb{Z}_q^n$ be a tuple of length n. The *Lee weight* of x is the sum of the Lee weight of its entries, i.e.,

$$\mathrm{wt}_L(x) := \sum_{i=1}^{n} \mathrm{wt}_L(x_i) \tag{2}$$

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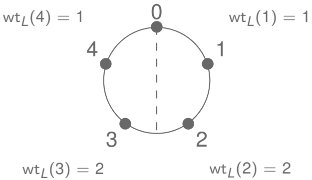

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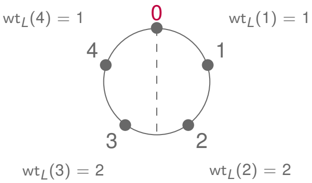

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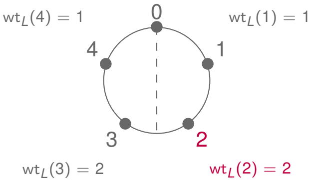

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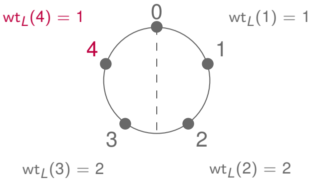

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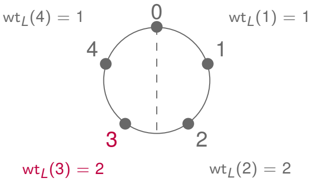

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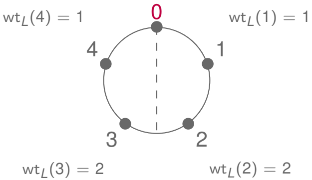

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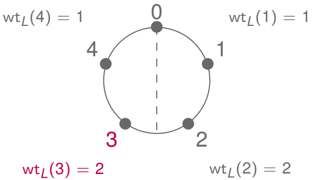

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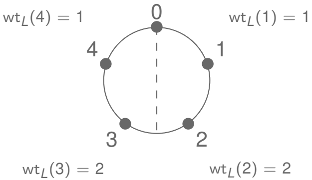

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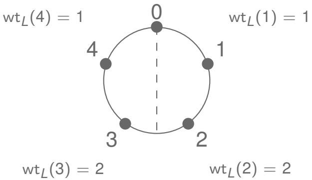
$$\mathrm{wt}_H(x) = 4$$



Why Lee Metric?

- Transmitting symbols over a nonbinary noisy channel
  $\longrightarrow$ primarily those using phase-shift keying modulation

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  - Low-Lee-Density Parity-Check Codes were defined[2]

[1] Anna-Lena Horlemann-Trautmann and Violetta Weger. “Information set decoding in the Lee metric with applications to cryptography”. In: *Applied and Computational Mathematics* (2019).

[2] Paolo Santini et al. “Low-Lee-Density Parity-Check Codes”. In: *ICC 2020-2020 IEEE International Conference on Communications (ICC)*. IEEE. 2020, pp. 1–6.

Outline

The Lee Channel

Originally introduced by Chiang and Wolf[3].

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Assume the alphabet is $\mathbb{Z}_q$.
Goal: Describe $\mathbb{P}(i \mid j) = \mathbb{P}(i - j \mid 0)$.

$-\lfloor q/2 \rfloor$
$\vdots$
-1
0 $\quad$ 0
1
$\vdots$
$\lfloor q/2 \rfloor$

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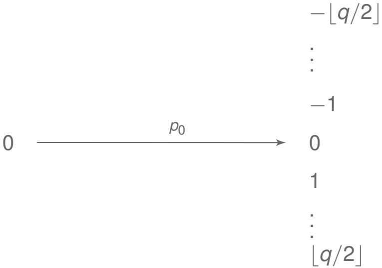

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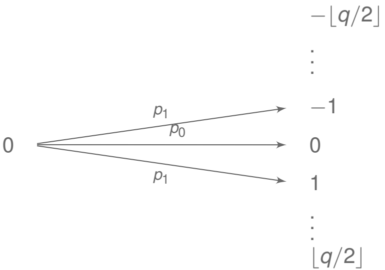

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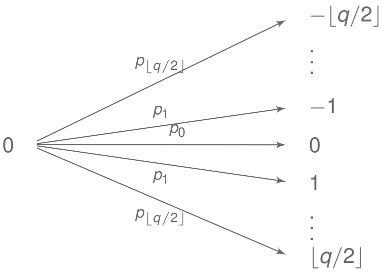

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DLR

The Lee Channel Law

For $y, x, e \in \mathbb{Z}_q$, consider a discrete memoryless channel (DMC)

$$\underset{\text{channel output}}{y} = \underset{\text{channel input}}{x} + \underset{\text{additive error term}}{e} \tag{3}$$

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Restrict to: e a realization of a random variable E with

$$\mathbb{P}(E = e) \propto \exp(-\lambda \, \mathrm{wt}_L(e)), \qquad \lambda > 0,$$

$$P_{Y|X}(y|x) = \frac{1}{Z} \exp\left(-\lambda \, \mathrm{d}_L(x, y)\right), \qquad Z := \sum_{e=0}^{q-1} \exp(-\lambda \, \mathrm{wt}_L(e))$$

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Note

- The expectation of $\mathrm{wt}_L(E)$, δ, can be written as $\delta = \frac{\mathrm{d} \log Z(\lambda)}{\mathrm{d} \, \lambda}$.
- Defining $p_i := \mathbb{P}(\mathrm{wt}_L(e) = i) = \frac{1}{Z} \exp\left(-\lambda i\right)$ for $i \in \{0, 1, \dots, \lfloor q/2 \rfloor\}$, we easily see

$$p_0 > p_1 \quad \text{and} \quad p_i = \frac{p_1^i}{p_0^{i-1}} \quad \text{for all } i = 2, \dots, \lfloor q/2 \rfloor.$$

The Constant Lee Weight Channel

Consider now $y, x, e \in \mathbb{Z}_q^n$ and $y = x + e$, where e has a fixed Lee weight $t \in \mathbb{Z}$ and is drawn uniformly at random from $\mathcal{S}_{t,q}^n := \left\{ x \in \mathbb{Z}_q^n \;\middle|\; \mathrm{wt}_L(x) = t \right\}$.

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Theorem

For every $j \in \{1, \ldots, n\}$ the marginal weight distribution of an entry e_j is given by

$$p_i := \mathbb{P}(\mathrm{wt}_L(e_j) = i) = \frac{1}{\sum_{j=0}^{q-1} \exp(-\beta \, \mathrm{wt}_L(j))} \exp\left(-\beta i\right), \forall i \in \{0, \ldots, \lfloor q/2 \rfloor\}$$

where $\beta > 0$ is the solution to $\frac{t}{n} = \frac{(r-1)\mathrm{e}^{(r+1)\beta} - r\mathrm{e}^{r\beta} + \mathrm{e}^{\beta}}{(\mathrm{e}^{\beta r} - 1)(\mathrm{e}^{\beta} - 1)}$ with $r = \lfloor q/2 \rfloor + 1$.

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Proof idea.
Solve an optimization problem to find a distribution $(p_0, p_1, \ldots, p_{\lfloor q/2 \rfloor})$ that is

... maximizing $H(p_0, \ldots, p_{\lfloor q/2 \rfloor}) := -\sum_{i=0}^{\lfloor q/2 \rfloor} p_i \cdot \log(p_i)$,

... subject to $\sum_{i=0}^{\lfloor q/2 \rfloor} p_i \cdot i = \frac{t}{n}$.

Outline

Integer Partitions

Definition [Integer Partition]

Let $n \in \mathbb{Z}$. An *(integer) partition* of n of length k is a k-tuple $\lambda = (\lambda_1, \ldots, \lambda_k)$ satisfying

1. $\lambda_1 + \ldots + \lambda_k = n$,
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For a partition λ of t, we will denote the set of all n-tuples of type λ by $\mathcal{V}_{t,\lambda}^{(n)}$.

Tuples of type λ over $\mathbb{Z}_q$

Note: Integer partitions of some type λ over $\mathbb{Z}_q$ have part sizes not exceeding $\lfloor q/2 \rfloor$.

Example

Consider $\mathbb{Z}_5$, $t = n = 4$ and $\lambda = (2, 1, 1)$ a partition of t over $\mathbb{Z}_5$. Then:
$\mathcal{V}^{(4)}_{4,(2,1,1)} = \{(2, 1, 1, 0), (2, 1, 0, 1), \ldots, (1, 2, 1, 0), \ldots, (3, 4, 1, 0), \ldots\}$

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Lemma

Let n, q and t be positive integers and consider the set of partitions $\mathcal{P}_{\lfloor q/2 \rfloor}(t)$ of t with part sizes not exceeding $\lfloor q/2 \rfloor$. For any $\lambda \in \mathcal{P}_{\lfloor q/2 \rfloor}(t)$ the number of vectors of length n over $\mathbb{Z}_q$ of type λ is given by

$$\left|\mathcal{V}^{(n)}_{t,\lambda}\right| = \begin{cases} 2^{\ell_\lambda} \left|\Pi_\lambda\right| \binom{n}{\ell_\lambda} & \text{if } q \text{ is odd,} \\ 2^{\ell_\lambda - c_{\lfloor q/2 \rfloor,\lambda}} \left|\Pi_\lambda\right| \binom{n}{\ell_\lambda} & \text{else} \end{cases} \tag{4}$$

where $c_{\lfloor q/2 \rfloor,\lambda} = |\{i \in \{1, \ldots, \ell_\lambda\} \mid \lambda_i = \lfloor q/2 \rfloor\}|$.

Drawing Tuples of Fixed Lee Weight

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4. The remaining entries are zero.

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Example

Consider $\mathbb{Z}_7 \Longrightarrow \lfloor 7/2 \rfloor = 3$ is the maximal Lee weight for an entry. Say we want a tuple $x = (\_, \_, \_, \_, \_, \_)$ of length 6 with Lee weight $t = 4$.

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$\left\lvert \mathcal{V}^{(6)}_{4,(1,1,1,1)} \right\rvert = 240$	$\left\lvert \mathcal{V}^{(6)}_{4,(2,1,1)} \right\rvert = 480$	$\left\lvert \mathcal{V}^{(6)}_{4,(2,2)} \right\rvert = 60$	$\left\lvert \mathcal{V}^{(6)}_{4,(3,1)} \right\rvert = 120$

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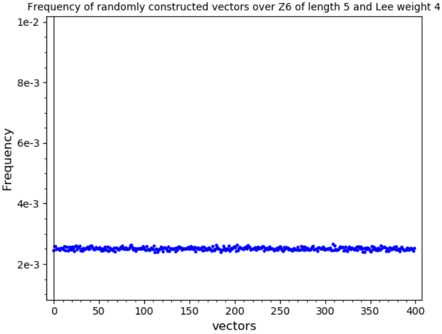
4. $x = (0, 6, 0, 5, 1, 0)$

Distribution

Theorem

Let n, q and t be positive integers. The when sampling a sufficiently large number of n-tuples using the before shown algorithm, we obtain a uniform distribution on $\mathcal{S}_q^n(t)$.



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Assume we receive a vector $y = \underbrace{x}_{\text{original message}} + \underbrace{e}_{\text{error vector}}$.

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  Is NP-hard for the Hamming- and the Lee metric.

Introduction to the Problem

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Let $x = (0, 2, 3, 1, 0, 3) \in \mathbb{Z}_5^6$

Lee $\mathrm{wt}_L(x) = 7$, Hamming $\mathrm{wt}_H(x) = 4$

Introduction to the Problem

Example 1	Lee	Hamming
Let $x = (0, 2, 3, 1, 0, 3) \in \mathbb{Z}_5^6$	$\mathrm{wt}_L(x) = 7,$	$\mathrm{wt}_H(x) = 4$
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DLR

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Example 2		
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DLR

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DLR

Introduction to the Problem

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Why can decreasing the Lee weight be a problem?

Generic (or syndrome) decoding is based on the weight of the error term.

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Risk: From a cryptographic point of view, an attacker could decrease the weight and retrieve the original message.

Problem Statement

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Goal: We want this probability to be small!

Preparation

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Applying the union bound, we have

$$\begin{aligned}\mathbb{P}(F) &= \mathbb{P}\left(\mathrm{wt}_L(a \cdot x) < \mathrm{wt}_L(x) \mid Q \text{ is "close" to } \mathcal{B}\right)\mathbb{P}\left(Q \text{ is "close" to } \mathcal{B}\right) \\ &\quad + \mathbb{P}\left(\mathrm{wt}_L(a \cdot x) < \mathrm{wt}_L(x) \mid Q \text{ is "not close" to } \mathcal{B}\right)\mathbb{P}\left(Q \text{ is "not close" to } \mathcal{B}\right) \\ &\leq \mathbb{P}\left(\mathrm{wt}_L(a \cdot x) < \mathrm{wt}_L(x) \mid Q \text{ is "close" to } \mathcal{B}\right) + \mathbb{P}\left(Q \text{ is "not close" to } \mathcal{B}\right)\end{aligned}$$

"Close" Distributions

Definition [Kullback-Leibler divergence]

Let X be a random variable over an alphabet $\mathcal{X}$ with probability distribution P, where $P(x) := \mathbb{P}(X = x)$. Furthermore, let us assume that X can approximated by another distribution $Q \neq P$. We define the *Kullback-Leibler divergence* of Q and P by

$$D(Q \,||\, P) := \sum_{x \in \mathcal{X}} Q(x) \log \left(\frac{Q(x)}{P(x)} \right) \tag{6}$$

Note

- By convention: $0 \log(0) = 0$.
- An approximated distribution Q is *close* to the exact distribution P, if $D(Q \,||\, P) \leq \varepsilon$, for some $\varepsilon > 0$.

DLR

Conditional Limit Theorem

Theorem
Conditional Limit Theorem

Let E be a closed convex set of probability distributions over an alphabet $\mathcal{X}$ and let Q be a distribution over $\mathcal{X}$ but not in E. Let $X_1, \ldots, X_n$ be discrete random variables drawn i.i.d. $\sim Q$. Define $X^n = (X_1, \ldots, X_n)$ and let $P^\star = \arg\min_{P \in E} D(P \,||\, Q)$. Then

$$\mathbb{P}\left(X_1 = a \,|\, P_{X^n} \in E\right) \longrightarrow P^\star(a)$$

in probability as n grows large for any $a \in \mathcal{X}$.
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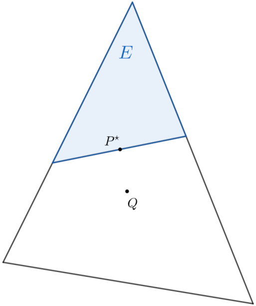

[4] Thomas M Cover. *Elements of information theory*. John Wiley & Sons, 1999

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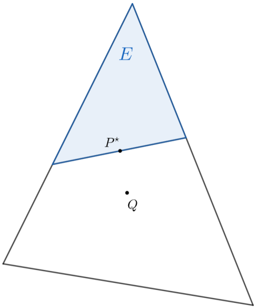

In our case:

$Q \sim \mathcal{U}(\mathbb{Z}_q)$; E set of distributions of tuples in $\mathcal{S}_q^n(t)$. Then $\mathcal{B} = \arg\min_{P \in E} D(P \,||\, Q)$.

[4] Cover, *Elements of information theory*

Asymptotic Regime

Recall, $F = \{\mathrm{wt}_L(a \cdot x) < \mathrm{wt}_L(x)\}$ and

$$\mathbb{P}(F) \leq \mathbb{P}\left(\mathrm{wt}_L(a \cdot x) < \mathrm{wt}_L(x) \mid Q \text{ is "close" to } \mathcal{B}\right) + \mathbb{P}\left(Q \text{ is "not close" to } \mathcal{B}\right)$$

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Let $x \in \mathbb{Z}_q^n$, for some positive integer $q > 3$, of average Lee weight $\delta = t/n$ be drawn randomly from $\mathcal{S}_q^n(t)$ with the shown algorithm. Let Q denote the empirical distribution of the entries of x. For any nonzero $a \in \mathbb{Z}_q$ it holds

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By CLT $\leq \mathbb{P}\left(\mathrm{wt}_L(a \cdot x) < \mathrm{wt}_L(x) \mid Q \sim \mathcal{B}\right)$

DLR

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