July 12, 2021
Institutskolloquium, KN-IColl

Decoding Performance of LDPC Codes over the Lee Channel

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joint work with Hannes Bartz, Gianluigi Liva
and Joachim Rosenthal

Outline

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Motivation

- Transmission of data over a noisy channel

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- Error correction/detection

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- Fast encoding and decoding performance

Channel Coding

Let $\mathcal{X}$ and $\mathcal{Y}$ the input and output alphabet of the channel, respectively.

With a given channel law

$$P_{Y|X}(y \mid x) := \mathbb{P}(Y = y \mid X = x)$$

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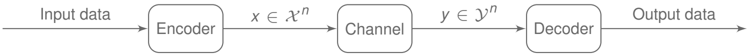

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Example: q-ary Symmetric Channel (qSC)

- Alphabets: $\mathcal{X} = \mathcal{Y} = \{0, 1, \ldots, q-1\}$

Input	Output
0	0
1	1
$\vdots$	$\vdots$
$q-1$	$q-1$

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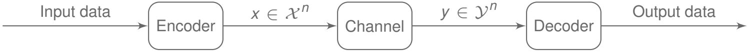

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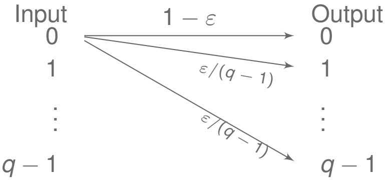

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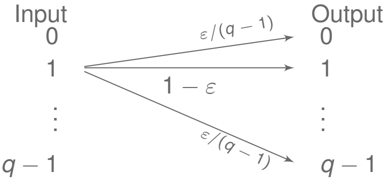

The Hamming Weight

Let $\mathbb{F}_q$ be a finite field of order q and let n be a positive integer. We will denote by $\mathbb{Z}_q$ the ring of integers modulo q.

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For any two vectors $x, y \in \mathbb{F}_q^n$ we define

- the *Hamming weight* of x, $\mathrm{wt}_H(x) = |\{i \in \{1, \dots, n\} \mid x_i \neq 0\}|$
- the *Hamming distance* between x and y, $\mathrm{d}_H(x, y) := \mathrm{wt}_H(x - y)$

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An $[n, k]_q$-linear code $\mathcal{C}$ can be represented by an $(n-k) \times n$ matrix H satisfying

$$\mathcal{C} = \ker(H) = \{x \in \mathbb{F}_q^n \mid Hx^\top = 0\}.$$

We call H a *parity-check matrix* of $\mathcal{C}$.

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Assume we receive a vector $y = \underbrace{x}_{\text{original message}} + \underbrace{e}_{\text{error vector}}$.

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  Is NP-hard for the Hamming- and the Lee metric.

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Definition [Lee weight]

For any integer $a \in \mathbb{Z}_q$ its *Lee weight* is defined as

$$\mathrm{wt}_L(a) := \min(a, q - a) \quad (1)$$

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$$\mathrm{wt}_L(3) = \min(3, 5 - 3) = 2$$

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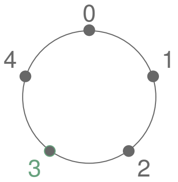

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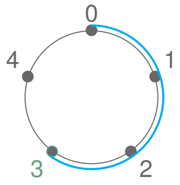

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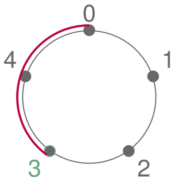

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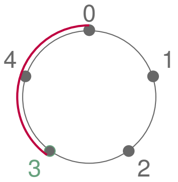

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$\Longrightarrow \mathrm{wt}_L(3) = 2$

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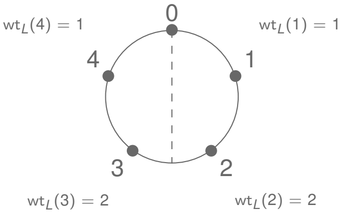
For every $a \in \mathbb{Z}_q$ it holds:

- $\mathrm{wt}_L(a) = \mathrm{wt}_L(m - a)$

Example



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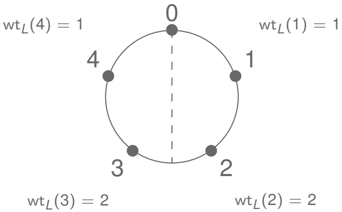

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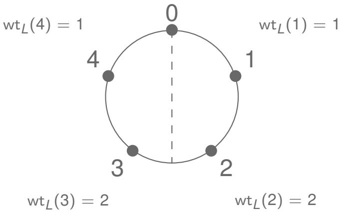
- $\mathrm{wt}_H(a) \leq \mathrm{wt}_L(a)$
  If $q \in \{2, 3\}$, the Lee weight is equivalent to the Hamming weight.

Example



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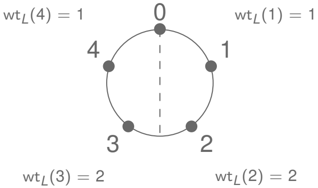

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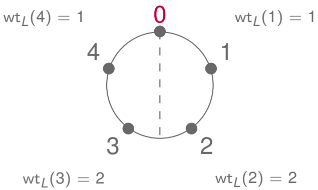

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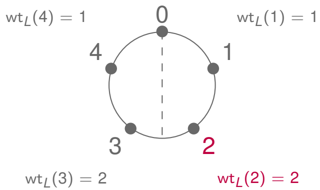

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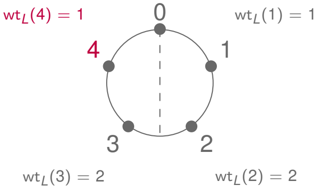

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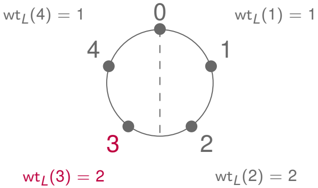

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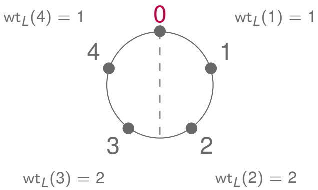

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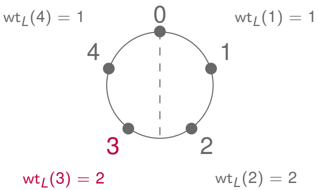

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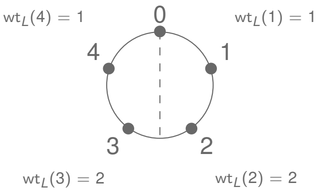

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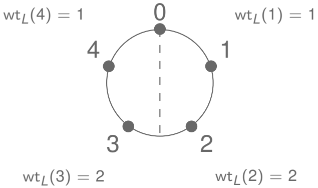

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DLR

Interesting Problem

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Complexity of generic (or information-set) decoding depends on the weight of the error vector.

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Asymptotically: The probability of decreasing the weight is negligible as the length grows large.

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- Transmitting symbols over a nonbinary noisy channel
  $\longrightarrow$ primarily those using phase-shift keying modulation

[1] Anna-Lena Horlemann-Trautmann and Violetta Weger. "Information set decoding in the Lee metric with applications to cryptography". In: *arXiv preprint arXiv:1903.07692* (2019).

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  $\longrightarrow$ primarily those using phase-shift keying modulation
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Outline

The Lee Channel

Originally introduced by Chiang and Wolf[3].

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Assume the alphabet is $\mathbb{Z}_q$.
Goal: Describe $\mathbb{P}(i \mid j) = \mathbb{P}(i - j \mid 0)$.

0

$-\lfloor q/2 \rfloor$
$\vdots$
-1
0
1
$\vdots$
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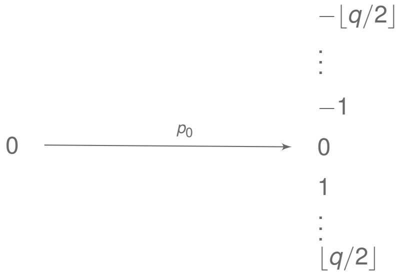

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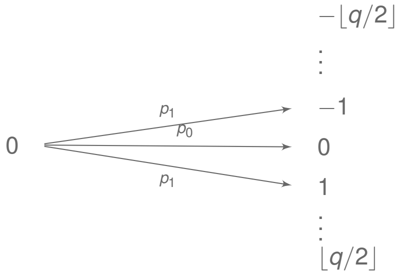

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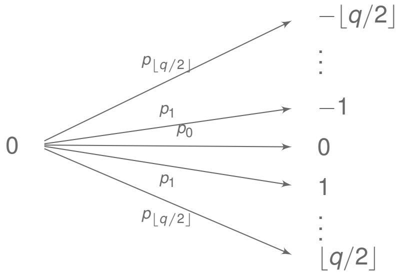

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The Lee Channel

Theorem [Chiang and Wolf]

The channel described before is strictly matched to the Lee metric for maximum likelihood decoding if and only if the following two properties hold.

$$p_0 > p_1 \quad \text{and} \quad p_i = \frac{p_1^i}{p_0^{i-1}} \text{ for all } i = 2, \ldots, \lfloor q/2 \rfloor .$$

DLR

The Lee Channel

For $y, x, e \in \mathbb{Z}_q$, consider a discrete memoryless channel (DMC)

$$\underbrace{y}_{\text{channel output}} = \underbrace{x}_{\text{channel input}} + \underbrace{e}_{\text{additive error term}} \tag{3}$$

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$$\mathbb{P}(Y = y \,|\, X = x) =: P_{Y|X}(y|x) = \frac{1}{Z} \exp\left(-\lambda \, \mathrm{d}_L(x, y)\right), \tag{4}$$

where $Z := \sum_{e=0}^{q-1} \exp(-\lambda \, \mathrm{wt}_L(e))$ and $\lambda > 0$.

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Note

- The channel defined in (4) is the DMC matched to the Lee metric.
- The conditional distribution (4) arises (in the limit of large n) as the marginal distribution of a constant-weight Lee channel.

The Constant-Weight Lee Channel

Let $y, x, e \in \mathbb{Z}_q^n$, where $\mathrm{wt}_L(e) = t$ for some fixed positive integer t. Consider again

$$y = x + e.$$

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Note

The error vector e is chosen uniformly at random from the set of all length-n vectors of Lee weight t:

$$\mathcal{S}_t^n := \left\{x \,\middle|\, x \in \mathbb{Z}_q^n, \mathrm{wt}_L(x) = t\right\}.$$

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Question

What would $P_{Y \mid X}(y \mid x)$ look like?

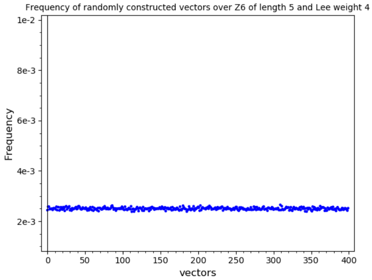
DLR

Channel Distribution

Theorem

Let $e \in \mathbb{Z}_q^n$ with $\mathrm{wt}_L(x) = t$ be the error term from before. Then it holds

i. With our algorithm e is drawn uniformly from the set of all length-n vectors of Lee weight t over $\mathbb{Z}_q$.



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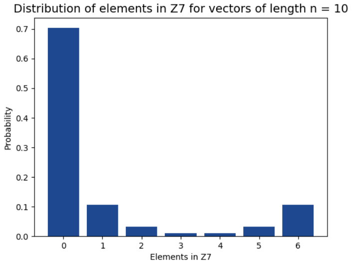
i. With our algorithm e is drawn uniformly from the set of all length-n vectors of Lee weight t over $\mathbb{Z}_q$.

ii. Every entry e_i has the following probability

$$\mathbb{P}(e_i = j) = \kappa \exp(-\lambda \,\mathrm{wt}_L(j)),$$

where $\kappa = \sum_{k=0}^{m-1} \exp(-\lambda \,\mathrm{wt}_L(k))$ and $j \in \mathbb{Z}_q$.



Outline

LDPC Codes over Finite Integer Rings

According to Sridhara and Fuja

Definition [LDPC Code]

An $[n, k]_q$ LDPC code over $\mathbb{Z}_q$ is defined by a sparse parity-check matrix H, whose nonzero entries lie in the set of units $\mathbb{Z}_q^\times$.

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Can be described by a bipartite graph $\mathcal{G}$ consisting of

- variable nodes (VN) $\{v_1, \ldots, v_n\} \longrightarrow$ columns of H.
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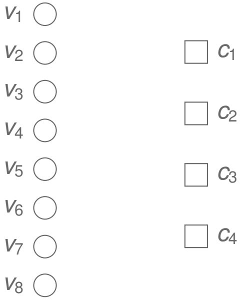
VN v_j is connected to CN c_i if and only if $h_{ij} \neq 0$.

LDPC Codes over Finite Integer Rings

Example

$$H = \begin{bmatrix} 0 & 1 & 0 & 2 & 4 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 3 & 1 \\ 1 & 0 & 0 & 1 & 4 & 0 & 1 & 0 \end{bmatrix} \in \mathbb{Z}_5^{4\times 8}$$



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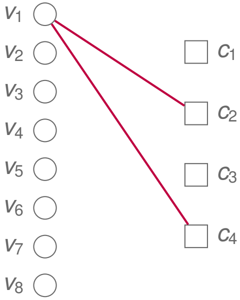

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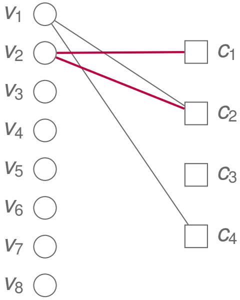

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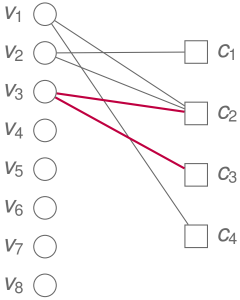

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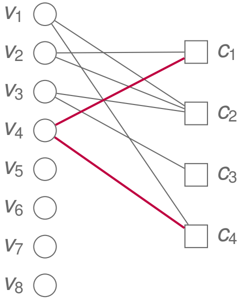

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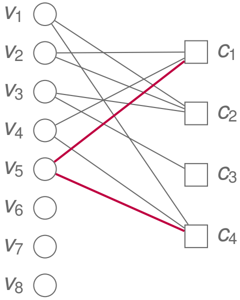

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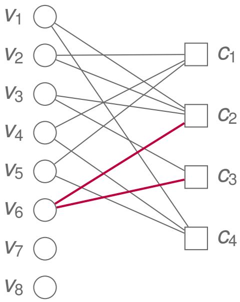

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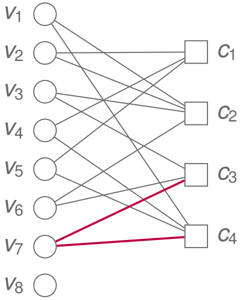

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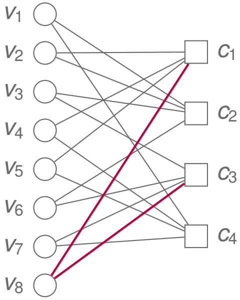

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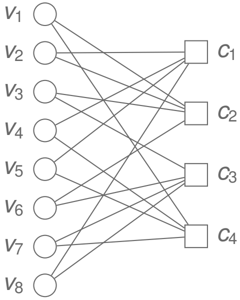

An LDPC code is (k, ℓ)-*regular*, if every VN connects to k CNs and every CN connects to ℓ VNs, for some fixed positive integer k and ℓ.

Symbol Message Passing

Consider a nonbinary LDPC code $\mathcal{C}$ with VNs $\{v_1, \dots, v_n\}$ and CNs $\{c_1, \dots, c_m\}$ and parity-check matrix H. Denote by $\mathcal{N}(v_j)$ and $\mathcal{N}(c_i)$ the set of all connecting elements to VN v_j and CN c_i, respectively.

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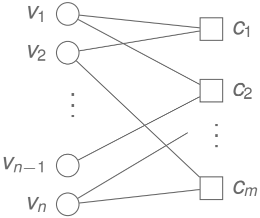

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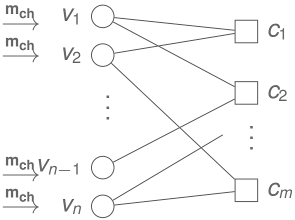
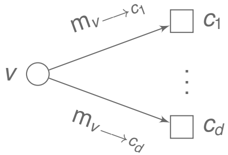
Every VN v receives the channel observation
$\mathbf{m_{ch}} := (P_{Y|X}(y \mid 0), \ldots, P_{Y|X}(y \mid q-1))$



Symbol Message Passing

Initialization.
Each VN v sends channel observation to the neighboring CNs $c \in \mathcal{N}(v)$

$$m_{v \longrightarrow c} = \mathbf{m_{ch}}.$$

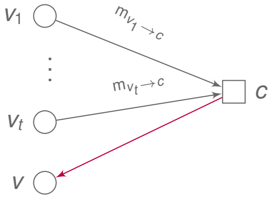


Symbol Message Passing

CN-to-VN step.
Each CN computes for every $v \in \mathcal{N}(c)$

$$m_{c\to v} = h_{c,v}^{-1} \sum_{v' \in \mathcal{N}(c)\setminus\{v\}} h_{c,v'} m_{v'\to c}.$$

Note: $h_{c,v}^{-1}$ exists, since we said the nonzero entries of H are units.



Symbol Message Passing

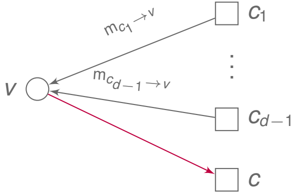
VN-to-CN step.

Define the aggregated extrinsic L-vector

$$E = L(y) + \sum_{c' \in \mathcal{N}(v) \setminus \{c\}} L\left(m_{c' \to v}\right),$$

where y is the channel output and
$L(y) = (L_0(y), \ldots, L_{q-1}(y))$ with
$L_x(y) = \log\left(P_{Y|X}(y \mid x)\right)$.
Note: We assume the CN-to-VN messages are modelled as a qSC.
Then the VN-to-CN messages are

$$m_{v \to c} = \arg\max_{x \in \mathbb{Z}_q} E_x.$$



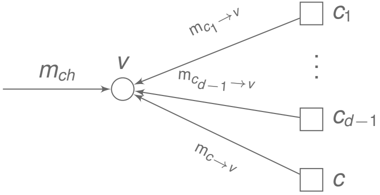
Symbol Message Passing

Final decision.
The final decision at each VN v is

$$\hat{x} = \arg\max_{x \in \mathbb{Z}_q} L_x^{\mathrm{FIN}}$$

where

$$L^{\mathrm{FIN}} = L(m_{\mathrm{ch}}) + \sum_{\mathrm{c} \in \mathcal{N}(\mathrm{v})} L\left(m_{\mathrm{c} \to \mathrm{v}}\right).$$



The qSC-Assumption for SMP

Motivation for the qSC assumption in the extrinsic channel

- Assumption is true for finite fields (i.e. $\mathbb{Z}_q$ with q a prime)

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If q is not a prime:

- The approximation is especially accurate when $\mathbb{Z}_q$ consists of many units.
- Decoding becomes particularly simple.

Simulations

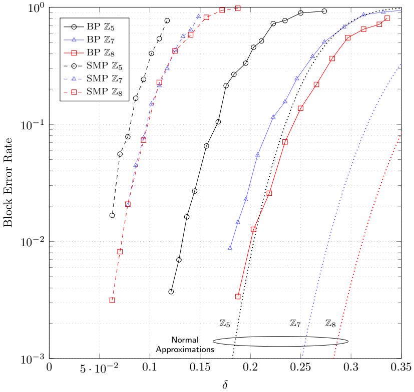
Decoding performance for both BP and SMP over both the Lee channel and the constant-weight Lee channel using

- $(3, 6)$ regular nonbinary LDPC codes of length 256 and 2048,
- For the constant-weight Lee channel, the error vectors are drawn uniformly at random from the set of vectors with a given weight.

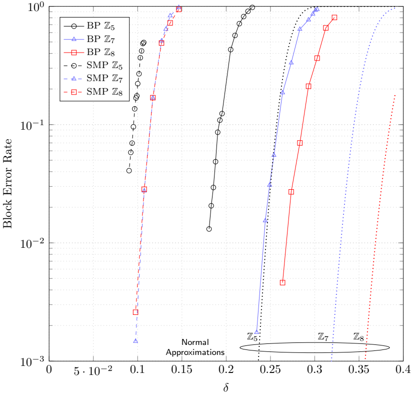
Simulations

Block error rate vs. average Lee weight δ for regular $(3, 6)$ nonbinary LDPC codes in the Lee channel for BP and SMP decoding.

Code length n = 256

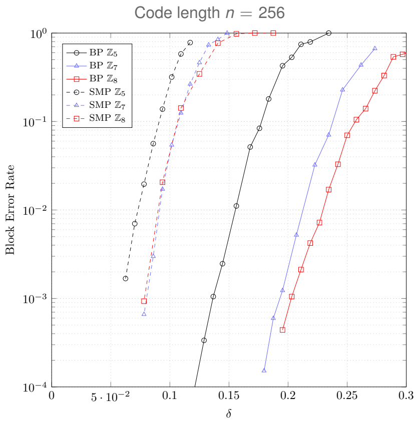


Code length n = 2048



Simulations

Block error rate vs. average Lee weight δ for regular $(3, 6)$ nonbinary LDPC codes in the constant-weight Lee channel for BP and SMP decoding.



Simulations

Block error rate vs. average Lee weight δ for regular $(3, 6)$ nonbinary LDPC codes in the constant-weight Lee channel for BP and SMP decoding.

Thank you very much for your attention!