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Analysis of Low-Density Parity-Check Codes over Finite Integer Rings for the Lee Channel

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joint work with Hannes Bartz, Gianluigi Liva and Joachim Rosenthal (UZH)

Outline

1. Ring-linear Codes and the Lee Metric

2. Channel Coding in the Lee Metric

3. Performance Analysis of LDPC Codes

Outline

Ring Linear Codes

Notation:

$\mathbb{Z}/q\mathbb{Z} := \{0, 1, 2, \dots, q-1\}$	integer residue ring
$(\mathbb{Z}/q\mathbb{Z})^\times$	set of units (i.e. integers coprime to q)

Note: If q is prime, then $\mathbb{Z}/q\mathbb{Z} \cong \mathbb{F}_q$ is a finite field of q elements.

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A *linear code* $\mathcal{C} \subseteq (\mathbb{Z}/q\mathbb{Z})^n$ is a $\mathbb{Z}/q\mathbb{Z}$-submodule of $(\mathbb{Z}/q\mathbb{Z})^n$. The elements of $\mathcal{C}$ are called *codewords* of length n.

Parameters:

- n is called the *length* of $\mathcal{C}$
- $k := \log_q |\mathcal{C}|$ is the $\mathbb{Z}/q\mathbb{Z}$*-dimension* of $\mathcal{C}$
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The *Hamming weight* of a codeword $c \in \mathcal{C}$ is the number of nonzero entries of c, i.e.,

$$\mathrm{wt_H}(c) := |\{i \in \{1, \ldots, n\} \mid c_i \neq 0\}|$$

The Lee Metric

We will denote by $\mathbb{Z}/q\mathbb{Z} = \{0, 1, \ldots, q-1\}$ the ring of integers modulo q.

For any integer $a \in \mathbb{Z}/q\mathbb{Z}$ and any vector $x, y \in (\mathbb{Z}/q\mathbb{Z})^n$ we define their *Lee weight* as

$$\mathrm{wt_L}(a) := \min(a, |\,q-a\,|) \quad \text{and} \quad \mathrm{wt_L}(x) := \sum_{i=1}^{n} \mathrm{wt_L}(x_i)$$

The *Lee distance* between x and y is given by $\mathrm{d_L}(x, y) := \mathrm{wt_L}(x-y)$.

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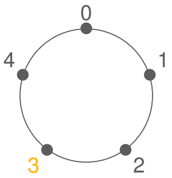

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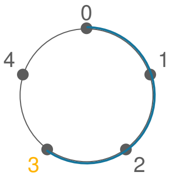

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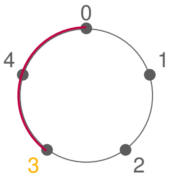

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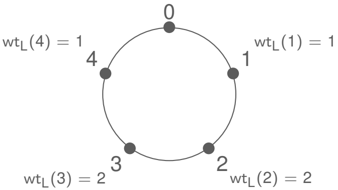

Properties

For every $a \in \mathbb{Z}/q\mathbb{Z}$ and $x \in (\mathbb{Z}/q\mathbb{Z})^n$

- $\mathrm{wt_L}(a) = \mathrm{wt_L}(q-a)$
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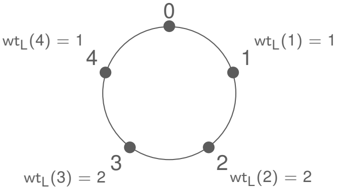

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- $\mathrm{wt_H}(x) \leq \mathrm{wt_L}(x) \leq nM$

The Expected Lee Weight

Let $a \in \mathbb{Z}/q\mathbb{Z}$ be chosen uniformly at random.

Lemma

The expected Lee weight of a is then given by

$$\delta_q := \mathbb{E}(\mathrm{wt_L}(a)) = \begin{cases} \frac{q^2-1}{4q} & \text{if } q \text{ is odd,} \\ \frac{q}{4} & \text{if } q \text{ is even.} \end{cases}$$

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Let $e \in \mathcal{S}_{t,q}^n := \left\{ x \in (\mathbb{Z}/q\mathbb{Z})^n \mid \mathrm{wt_L}(x) = t \right\}$ be chosen uniformly at random.

How does the distribution for each entry e_i look like?

DLR

The Marginal Distribution

Let $T := \lim_{n \longrightarrow \infty} t(n)/n$ be the asymptotic relative Lee weight of e.
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Theorem [1]

Assume that the asymptotic relative Lee weight is $T := \lim_{n\rightarrow\infty} \frac{t(n)}{n}$. For every $i \in \mathbb{Z}/q\mathbb{Z}$ the marginal distribution of E is given by

$$p_i := \mathbb{P}(E = i) = \frac{1}{\sum_{j=0}^{q-1} \exp(-\beta \, \mathrm{wt_L}(j))} \exp\left(-\beta i\right)$$

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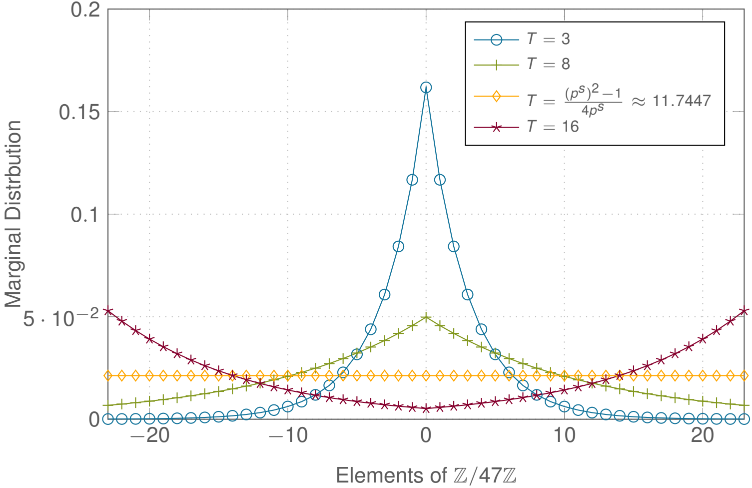
Note $T < \delta_q \iff \beta > 0$

The Marginal Distribution - Example over $\mathbb{Z}/47\mathbb{Z}$



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Channel Coding

Take a linear code $\mathcal{C} \subset (\mathbb{Z}/q\mathbb{Z})^n$.

$$\xrightarrow{\text{Message}} \boxed{\text{Encoder}} \xrightarrow{x \in \mathcal{C}} \boxed{\text{Channel}} \xrightarrow{y \in (\mathbb{Z}/q\mathbb{Z})^n} \boxed{\text{Decoder}} \xrightarrow{\text{decoded message}}$$

With given channel law

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We consider here an additive channel, i.e., $\quad y = x + e.$

Memoryless Lee Channel

Restrict, for every $i = 1, \dots, n$, to e_i as a realization of a random variable E_i with

$$\mathbb{P}(E_i = e_i) \propto \exp(-\lambda \, \mathrm{wt_L}(e_i)), \qquad \lambda > 0,$$

$$P_{Y_i|X_i}(y_i|x_i) = \frac{1}{Z(\lambda)} \exp\left(-\lambda \, \mathrm{d_L}(x_i, y_i)\right), \qquad Z(\lambda) := \sum_{e_i=0}^{q-1} \exp(-\lambda \, \mathrm{wt_L}(e_i))$$

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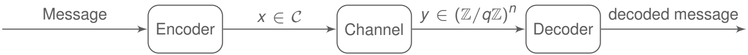

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Constant Lee Weight Channel

The error e has fixed Lee weight t and is chosen uniformly at random from $\{z \in (\mathbb{Z}/q\mathbb{Z})^n \mid \mathrm{wt_L}(z) = t\}$.

Random Coding Union Bounds

$\mathcal{C}$: best random (n, nR) code over $\mathbb{Z}/q\mathbb{Z}$

δ : normalized weight of error

$P_B(\mathcal{C})$: Error probability

$$\mathrm{H}_\delta^+ := \begin{cases} \mathrm{H}_\delta & \text{if } \delta \leq \delta_q \\ \log q & \text{otherwise.} \end{cases}$$

RCU bound

Constant Lee Weight Channel

$$\mathbb{E}\left[P_B(\mathcal{C})\right] < \exp\left(-n\left[(1-R)\log q - \mathrm{H}_\delta^+\right]^+\right)$$

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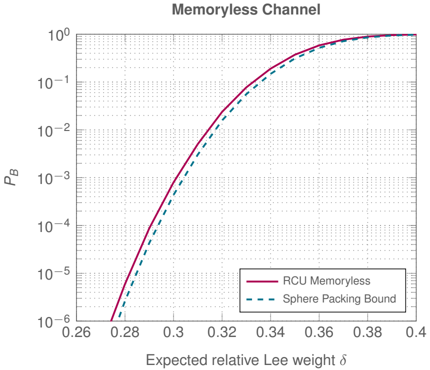
$$\mathbb{E}\left[P_B(\mathcal{C})\right] < \mathbb{E}\left[\exp\left(-n\left[(1-R)\log q - \mathrm{H}_{D/n}^+\right]^+\right)\right]$$



DLR

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LDPC Codes over $\mathbb{Z}/q\mathbb{Z}$

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Can be described by a bipartite graph $\mathcal{G}$ consisting of

- variable nodes (VN) $\{v_1, \ldots, v_n\}$
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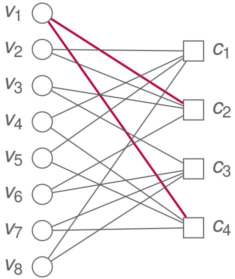
$$H = \begin{bmatrix} 0 & 1 & 0 & 2 & 4 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 3 & 1 \\ 1 & 0 & 0 & 3 & 4 & 0 & 1 & 0 \end{bmatrix} \in (\mathbb{Z}/5\mathbb{Z})^{4\times 8}$$



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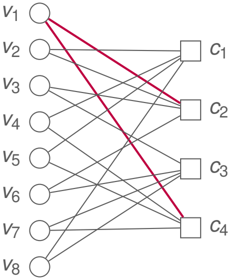
$$H = \begin{bmatrix} 0 & 1 & 0 & 1 & 5 & 0 & 0 & 1 \\ 1 & 5 & 1 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 5 & 1 \\ 1 & 0 & 0 & 5 & 5 & 0 & 1 & 0 \end{bmatrix} \in (\mathbb{Z}/6\mathbb{Z})^{4\times 8}$$



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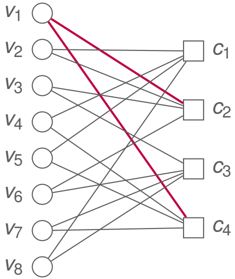

An LDPC code is (k, ℓ)-**regular**, if every VN connects to k CNs and every CN connects to ℓ VNs, for some fixed positive integer k and ℓ.

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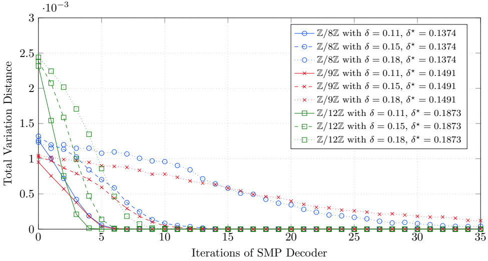
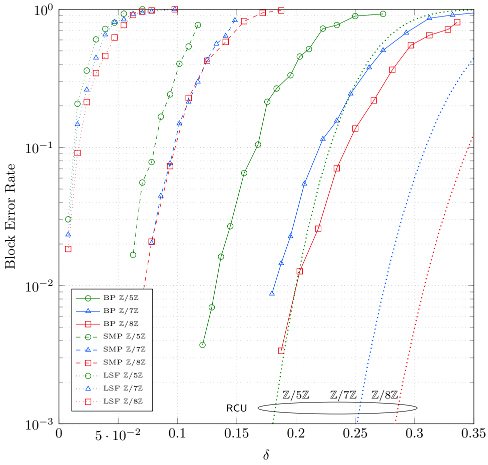
    $\longrightarrow$ valid for q nonprime as the total variation distance tends to zero



Simulation - Memoryless Lee Channel



Parameters:

- length $n = 256$
- regular $(3, 6)$ LDPC Codes
- Considered residue rings: $\mathbb{Z}/5\mathbb{Z}, \mathbb{Z}/7\mathbb{Z}$ and $\mathbb{Z}/8\mathbb{Z}$
- Decoders:
  - Lee Symbol Flipping
  - Message Passing
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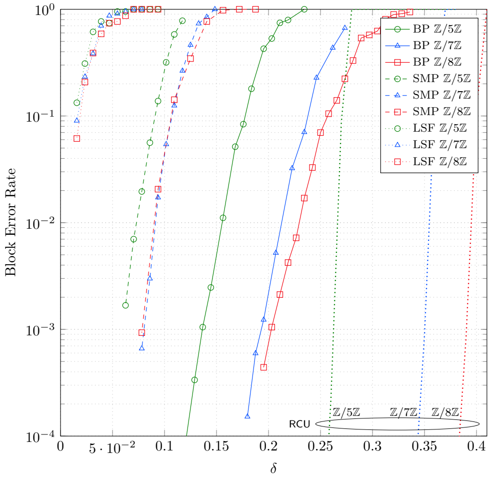
Simulation - Constant Lee Weight Channel



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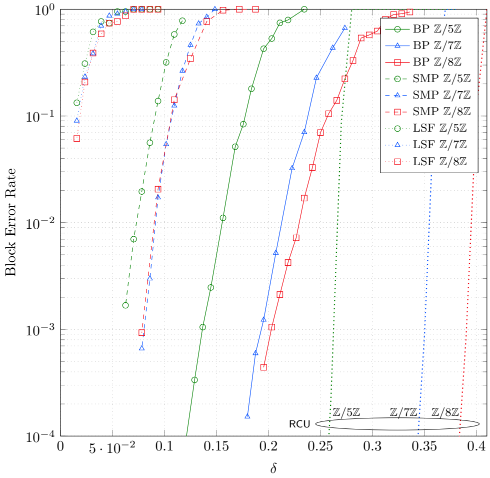

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Thank you for your attention!