TUM School of Computation, Information and Technology
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Partition-based MacWilliams Identities in the Lee, Homogeneous and Subfield Metric

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joint work with G. Cavicchioni and V. Weger

Seminar on Galois Geometries and Their Applications

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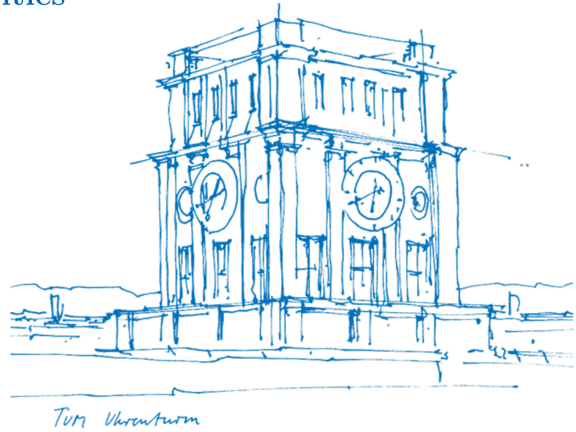

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- Non-Existence Results for the Lee and Homogeneous Metric

Can we derive a MacWilliams-like identity for a similar enumerator in the Lee/Homogeneous metric?

Outline

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Codes over $\mathbb{Z}/p^s\mathbb{Z}$

$$\begin{aligned} \text{integer residue ring}: &\quad \mathbb{Z}/p^s\mathbb{Z} = \{0, \ldots, p^s - 1\} \\ \text{standard inner product}: &\quad \langle x, y \rangle := \sum_{i=1}^{n} x_i y_i, \quad x, y \in (\mathbb{Z}/p^s\mathbb{Z})^n \end{aligned}$$

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Linear code over $(\mathbb{Z}/p^s\mathbb{Z})^n$ and its dual

Over $\mathbb{Z}/p^s\mathbb{Z}$, a *linear code* $\mathcal{C}$ *of length* n is a $\mathbb{Z}/p^s\mathbb{Z}$-submodule of $(\mathbb{Z}/p^s\mathbb{Z})^n$. The elements of $\mathcal{C}$ are called *codewords*. The *dual code* of $\mathcal{C}$ is

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Example over $\mathbb{Z}/3\mathbb{Z}$

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Hamming Weight

Hamming weight and weight enumerator

Let $x \in (\mathbb{Z}/p^s\mathbb{Z})^n$ and let $\mathcal{C} \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n$ be a linear code.

$$\begin{aligned} \textit{Hamming weight of } x: \quad & \mathrm{wt}_{\mathsf{H}}(x) = |\{i = 1, \ldots, n \mid x_i \neq 0\}| \\ \textit{Weight } i \textit{ enumerator of } \mathcal{C}: \quad & W_{\mathcal{C}}^{\mathsf{H}}(i) = |\{c \in \mathcal{C} \mid \mathrm{wt}_{\mathsf{H}}(c) = i\}| \qquad \text{for } i = 0, \ldots, n \end{aligned}$$

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$$\begin{aligned} W_{\mathcal{C}}^{\mathsf{H}}(0) &= 1 \\ W_{\mathcal{C}}^{\mathsf{H}}(1) &= 4 \\ W_{\mathcal{C}}^{\mathsf{H}}(2) &= 4 \\ W_{\mathcal{C}}^{\mathsf{H}}(3) &= 0 \end{aligned}$$

The MacWilliams Identity

- Relation between $W^{\mathsf{H}}_{\mathcal{C}}$ and $W^{\mathsf{H}}_{\mathcal{C}^\perp}$

MacWilliams Identity [MacWilliams '63]

Given a linear code $\mathcal{C} \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n$ and its dual $\mathcal{C}^\perp$. For every $j \in \{0, \ldots, n\}$, it holds that

$$W^{\mathsf{H}}_{\mathcal{C}^\perp}(j) = \frac{1}{|\,\mathcal{C}\,|} \sum_{i=0}^{n} K_j(i) W^{\mathsf{H}}_{\mathcal{C}}(i),$$

where, given a p-th root of unity ξ, $K_j(i) := \sum_{\substack{a \in (\mathbb{Z}/p^s\mathbb{Z})^n \\ \mathrm{wt}_{\mathsf{H}}(a) = j}} \xi^{\langle a, x \rangle}$ for any $x \in \mathcal{C}$ with $\mathrm{wt}_{\mathsf{H}}(x) = i$.

Krawtchouk Coefficient for the Hamming Weight Enumerator

$$\text{Given any} \quad x \in (\mathbb{Z}/p^s\mathbb{Z})^n \;:\; \mathrm{wt}_{\mathsf{H}}(x) = i$$

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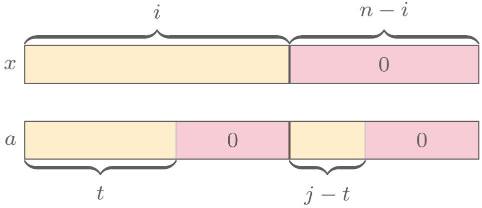

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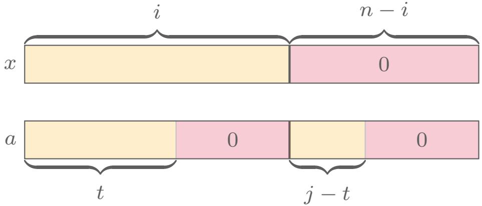

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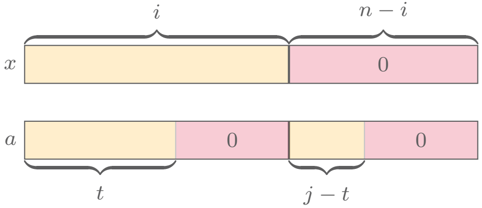

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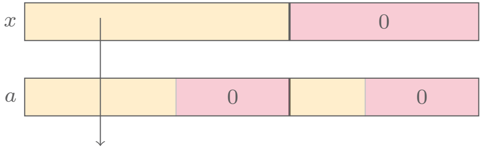

$\sum_{k\neq 0} \xi^k = (-1)$

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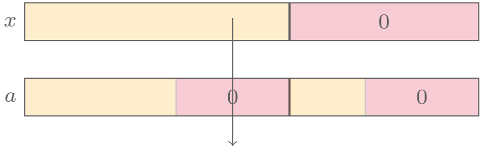

$$\sum_{k=0} \xi^k = 1$$

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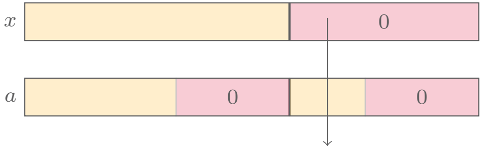

$\sum_{k \neq 0} \xi^0 = (p^s - 1)$

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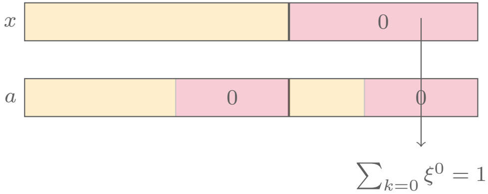

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Example over $\mathbb{Z}/3\mathbb{Z}$

MacWilliams Identity

$$W^{\mathsf{H}}_{\mathcal{C}^\perp}(j) = \frac{1}{|\,\mathcal{C}\,|}\sum_{i=0}^{n}\left(\sum_{t=0}^{j}\binom{i}{t}\binom{n-i}{j-t}(-1)^t(p^s-1)^{i-t}\right)W^{\mathsf{H}}_{\mathcal{C}}(i)$$

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$$\mathcal{C}^\perp = \Big\{(0,0,0),(0,1,0),(0,0,1),(0,2,0),(0,0,2),(0,1,1),(0,1,2),(0,2,1),(0,2,2)\Big\}$$

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$$W^{\mathsf{H}}_{\mathcal{C}^\perp}(1) = \frac{1}{3}\Big(W^{\mathsf{H}}_{\mathcal{C}}(0)K_1(0) + W^{\mathsf{H}}_{\mathcal{C}}(1)K_1(1) + W^{\mathsf{H}}_{\mathcal{C}}(2)K_1(2) + W^{\mathsf{H}}_{\mathcal{C}}(3)K_1(3)\Big)$$

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$$W^{\mathsf{H}}_{\mathcal{C}^\perp}(1) = \frac{1}{3} \left(1 \cdot K_1(0) + 2 \cdot K_1(1) \right)$$

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Outline

1. The MacWilliams Identity

2. Failure of the Classical Identities

3. Reflexive Partitions of Finite Abelian Groups

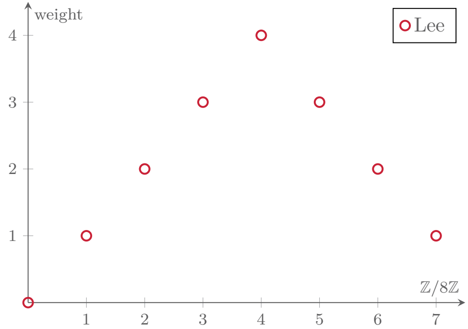
4. Application to some Additive Metrics over Finite Chain Rings

Lee and Homogeneous Weight over $\mathbb{Z}/p^s\mathbb{Z}$

Lee Weight

$$\mathrm{wt}_{\mathsf{L}}(a) := \min\{a, p^s - a\}$$

Example over $\mathbb{Z}/8\mathbb{Z}$



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Lee Weight

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Homogeneous Weight

$$\mathrm{wt}_{\mathsf{Hom}}(a) := \begin{cases} 0 & a = 0 \\ 1 & a \notin \langle p^{s-1} \rangle \\ p/(p-1) & a \in \langle p^{s-1} \rangle \setminus \{0\} \end{cases}$$

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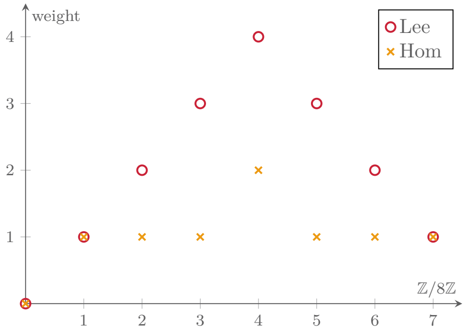

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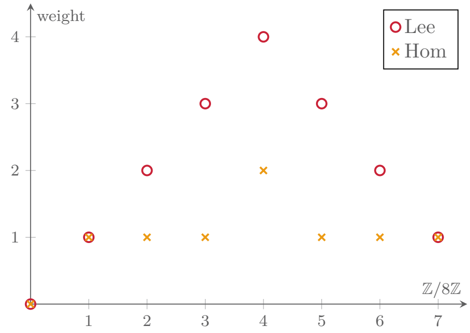

We extend both weights additively, i.e., for $x \in (\mathbb{Z}/p^s\mathbb{Z})^n$ we have $\mathrm{wt}_{\mathsf{L}/\mathsf{Hom}}(x) = \sum_{i=1}^{n} \mathrm{wt}_{\mathsf{L}/\mathsf{Hom}}(x_i)$

$$\text{Given any} \quad x \in (\mathbb{Z}/p^s\mathbb{Z})^n \; : \; \mathrm{wt}_{\mathsf{L}}(x)/\,\mathrm{wt}_{\mathsf{Hom}}(x) = i$$

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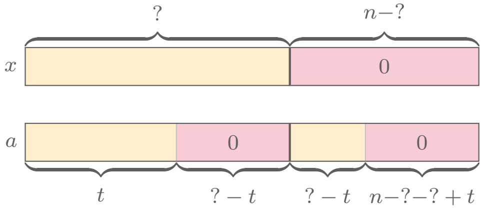

- $\mathrm{wt}_{\mathsf{L}}(x)/\,\mathrm{wt}_{\mathsf{Hom}}(x) = i$ does not imply $\mathrm{supp}(a) = i$.
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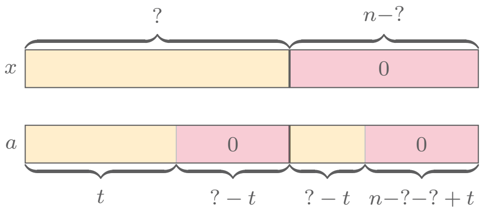

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- J. Wood, *Homogeneous weight enumerators over integer residue rings and failures of the MacWilliams identities*, 2023.

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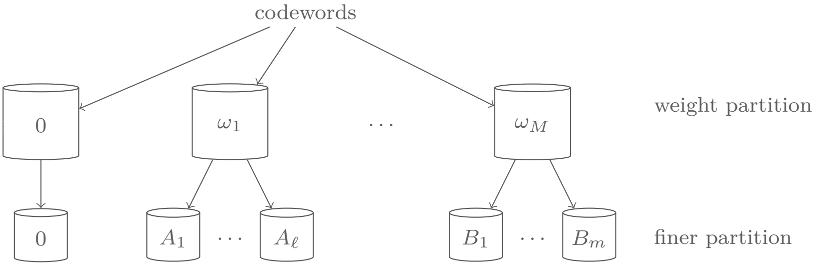

Outline

Group Characters

G finite abelian group

Definition - Character

A character χ of G is a complex-valued map $\chi : G \longrightarrow \mathbb{C}^*$, such that for every $\alpha, \beta \in G$ it holds

$$\chi(\alpha + \beta) = \chi(\alpha)\chi(\beta).$$

- $\widehat{G} = \mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{C}^*)$: set of all characters

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  $\longrightarrow$ Extends to G^n as $\chi_a(x) = \xi^{\mathrm{Tr}(\langle ax \rangle)}$

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A *partition* $\mathcal{P} = P_1 \mid \cdots \mid P_m$ of a finite abelian group G consists of nonempty, pairwise disjoint sets $P_i \subseteq G$ that cover G.

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Example of reflexive partition ([Gluessing-Luerssen, 2015])

- G finite additive abelian group, $H \leq \mathrm{Aut}(G)$
- $\mathcal{P}_H$ partition of G obtained by orbits of H

$\Longrightarrow$ $\mathcal{P}_H$ is reflexive

Induced Partitions

- Go from a partition $\mathcal{P} = P_1 \mid \ldots \mid P_m$ of G to a partition of G^n
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$$\mathcal{P}^2_{\mathsf{H},\mathrm{sym}} = \left\{g \in G^2 \mid \mathrm{wt}_{\mathsf{H}}(g) = 0\right\} \mid \left\{g \in G^2 \mid \mathrm{wt}_{\mathsf{H}}(g) = 1\right\} \mid \left\{g \in G^2 \mid \mathrm{wt}_{\mathsf{H}}(g) = 2\right\}$$

Reflexive Symmetrized Partitions

Theorem [Gluessing-Luerssen, 2015]
Let $\mathcal{P}$ be a partition of G such that $\{0\}$ is a block of $\mathcal{P}$. It holds that $\widehat{\mathcal{P}^n_{\mathrm{sym}}} = \widehat{\mathcal{P}}^n_{\mathrm{sym}}$. Consequently, if $\mathcal{P}$ is reflexive, then $\mathcal{P}^n_{\mathrm{sym}}$ is reflexive too.

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Let $\mathcal{P}$ be a partition of G such that $\{0\}$ is a block of $\mathcal{P}$. It holds that $\widehat{\mathcal{P}^n_{\text{sym}}} = \widehat{\mathcal{P}}^n_{\text{sym}}$. Consequently, if $\mathcal{P}$ is reflexive, then $\mathcal{P}^n_{\text{sym}}$ is reflexive too.

Examples

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- (symmetrized) Hamming weight partition $\mathcal{P}_{\mathsf{H}}$ and $\mathcal{P}^n_{\mathsf{H},\mathrm{sym}}$
- Singleton partition $\mathcal{P}_{\mathrm{Sing}} = \{0\} \mid \{g_1\} \mid \ldots \mid \{g_m\}$, for all $g_i \in G$, and $\mathcal{P}^n_{\mathrm{Sing},\mathrm{sym}}$

Outline

Set-Up and Goal

Consider a finite chain ring $\mathcal{R}$, a ring-linear code $\mathcal{C} \subseteq \mathcal{R}^n$, and an additive weight over $\mathcal{R}$.

Wanted

A partition-based enumerator of $\mathcal{R}^n$...

- giving rise to the weight enumerator (in the relative metric),
- with more information about the codeword's structure,
- allowing for a well-defined Krawtchouk coefficient.

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Question: How fine should the partition be?

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Question: What is the coarsest such partition?

Lee Weight Partition

Lee Partition

For any $x \in (\mathbb{Z}/p^s\mathbb{Z})^n$ we define its *Lee partition* $\pi^{\mathsf{L}}(x) = (\pi_0^{\mathsf{L}}(x), \pi_1^{\mathsf{L}}(x), \dots, \pi_M^{\mathsf{L}}(x))$ by

$$\pi_i^{\mathsf{L}}(x) = \left| \{k = 1, \dots, n \mid \mathrm{wt}_{\mathsf{L}}(x_k) = i\} \right| .$$

Example over $\mathbb{Z}/4\mathbb{Z}$

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Example over $\mathbb{Z}/4\mathbb{Z}$

$$\mathcal{C} = \Big\{(0,0,0),(0,1,0),(0,0,1),(0,3,0),(0,0,3),(0,2,0),(0,0,2),(0,1,1),$$
$$(0,1,3),(0,3,1),(0,3,3),(0,1,2),(0,2,1),(0,3,2),(0,2,3),(0,2,2)\Big\}$$

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$$(0,\,1,\,2),\,(0,\,2,\,1),(0,\,3,\,2),\,(0,\,2,\,3)\,, \; (0,\,2,\,2)\Big\}$$

$$\begin{array}{rl} \text{codewords of Lee weight } 2: & \underbrace{(0,2,0),(0,0,2)}_{(2,0,1)},\ \underbrace{(0,1,1),(0,1,3),(0,3,1),(0,3,3)}_{(1,2,0)} \\ \text{Lee partition } \pi^{\mathsf{L}}: & \end{array}$$

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Lee partition π^{L} :

Note! We can derive any additive weight $\mathrm{wt}(x)$ for $x \in \mathbb{Z}/p^s\mathbb{Z}$ from this decomposition, i.e.,

$$\mathrm{wt}(x) = \sum_{i=0}^{M} \pi^{\mathsf{L}}_i(x)\, \mathrm{wt}(i).$$

Symmetrized Lee Partition

Set of Lee partitions $\quad \mathbb{D}^{\mathsf{L}}_{p^s,n} := \left\{ \pi \in \{0,\dots,n\}^{M+1} \;\middle|\; \sum_{i=0}^{M+1} \pi_i = n \right\}$

symmetrized Lee partition $\quad \mathcal{P}^{n}_{\mathsf{L},\mathrm{sym}} = P^{\mathsf{L}}_{\pi^{(0)}} \mid P^{\mathsf{L}}_{\pi^{(1)}} \mid \cdots \mid P^{\mathsf{L}}_{\pi^{(D-1)}}, \quad \pi^{(i)} \in \mathbb{D}^{\mathsf{L}}_{p^s,n}$

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Proposition [B., Cavicchioni, Weger, ´24]
The symmetrized Lee partition $\mathcal{P}^n_{\mathsf{L},\mathrm{sym}}$ is reflexive.

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Idea.

- Underlying partition $\mathcal{P}_{\mathsf{L}}$ of $\mathbb{Z}/p^s\mathbb{Z}$ into Lee weights is based on orbits of $\{\pm 1\}$
- $\mathcal{P}_{\mathsf{L}}$ is reflexive
- $\mathcal{P}^n_{\mathsf{L},\mathrm{sym}}$ is reflexive

Krawtchouk Coefficient for the Lee Partition

Set of Lee partitions	$\mathbb{D}^{\mathsf{L}}_{p^s,n} := \left\{ \pi \in \{0,\ldots,n\}^{M+1} \;\middle|\; \sum_{i=0}^{M+1} \pi_i = n \right\}$
Lee partition enumerator	$\mathcal{D}^{\mathsf{L}}_{\pi}(\mathcal{C}) := \left| \left\{ c \in \mathcal{C} \mid \pi^{\mathsf{L}}(c) = \pi \right\} \right|$
Krawtchouk coefficient	$K^{\mathsf{L}}_{\pi}(\rho) = \sum_{\substack{a \in (\mathbb{Z}/p^s\mathbb{Z})^n \\ \pi^{\mathsf{L}}(x) = \rho}} \xi^{\langle x,a \rangle}, \quad x \in (\mathbb{Z}/p^s\mathbb{Z})^n \;:\; \pi^{\mathsf{L}}(x) = \pi$

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$$x \quad \overbrace{\boxed{\;0\;}}^{\pi_0}\overbrace{\boxed{\;\pm 1\;}}^{\pi_1}\boxed{\;\cdots\;}\overbrace{\boxed{\;\pm M\;}}^{\pi_M} \qquad \pi^{\mathsf{L}}(x) = \pi$$

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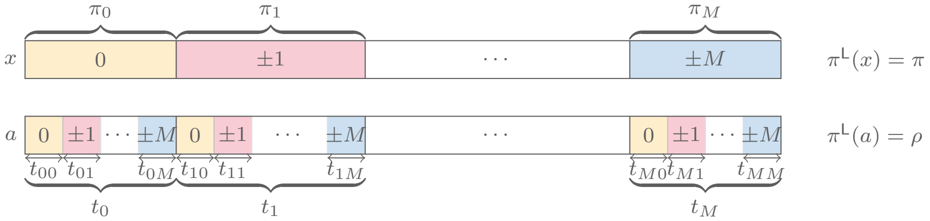

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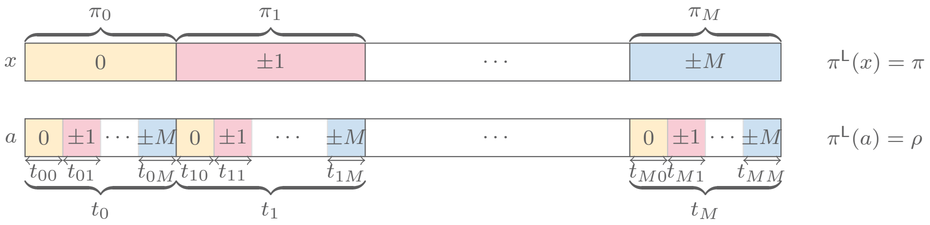

Note! $K^{\mathsf{L}}_{\pi}(\rho)$ is independent on the choice of $x \in (\mathbb{Z}/p^s\mathbb{Z})^n$

MacWilliams Identity for the Lee Partition Enumerator

$$\mathcal{D}^{\mathsf{L}}_{\pi}(\mathcal{C}^{\perp}) = \frac{1}{|\,\mathcal{C}\,|} \sum_{\pi \in \mathbb{D}^{\mathsf{L}}_{p^s,n}} K^{\mathsf{L}}_{\rho}(\pi)\mathcal{D}^{\mathsf{L}}_{\rho}(\mathcal{C}),$$

where the Krawtchouk coefficient exists and is given by

$$K^{\mathsf{L}}_{\rho}(\pi) = \begin{cases} \sum_{t \in \mathrm{Comp}^{\mathsf{L}}_{\rho<\pi}} \left(\prod_{i=0}^{M} \binom{\pi_i}{t_{i0},\dots,t_{iM}} \prod_{j=1}^{M-1} \left(\xi^{-ij} + \xi^{ij}\right)^{t_{ij}} \xi^{iM} \right) & \text{if } p = 2 \\ \sum_{t \in \mathrm{Comp}^{\mathsf{L}}_{\rho<\pi}} \left(\prod_{i=0}^{M} \binom{\pi_i}{t_{i0},\dots,t_{iM}} \prod_{j=1}^{M} \left(\xi^{-ij} + \xi^{ij}\right)^{t_{ij}} \right) & \text{otherwise} \end{cases}$$

Lee Metric - MacWilliams-like Identity

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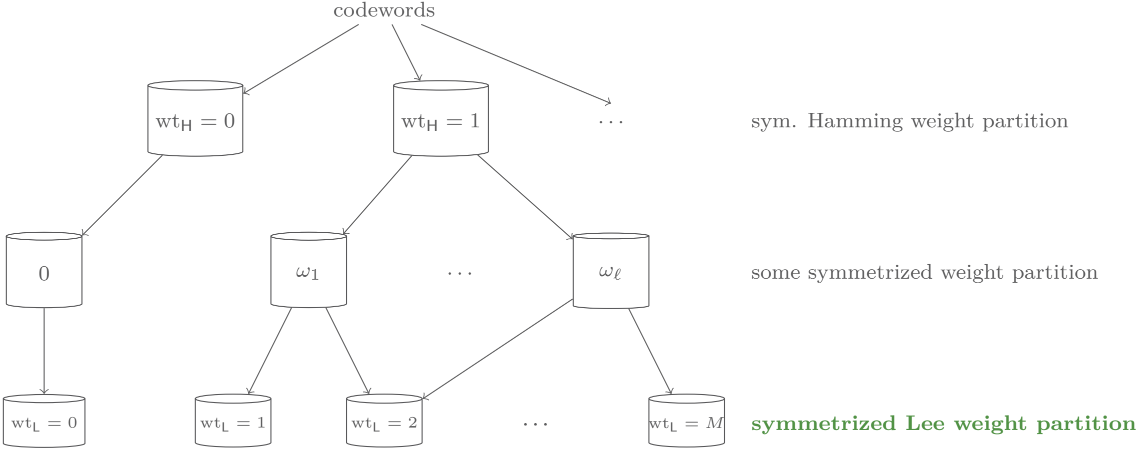
Credits:

- MacWilliams in 1963: Over $\mathbb{F}_{p^s}$
- Astola in 1982: Association schemes
- Solé in 1986: Association schemes
- Gluessing-Luerssen 2015: Fourier-reflexive partitions
- B., Cavicchioni, Weger in 2024: Identity is true over any finite chain ring $\mathcal{R}$ for all additive weights.

A Partition for any Additive Weight



Homogeneous Weight-Unit Partition

Recall the homogeneous weight over $\mathbb{Z}/p^s\mathbb{Z}$

$$\mathrm{wt}_{\mathsf{Hom}}(x) := \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \notin \left\langle p^{s-1} \right\rangle \setminus \{0\} \\ \frac{p}{p-1} & \text{if } x \in \left\langle p^{s-1} \right\rangle \end{cases}$$

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- Homogeneous weight partition
  - H. Gluesing-Luerssen, *Partitions of Frobenius Rings Induced by the Homogeneous Weight*, 2021.

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- Homogeneous weight partition
  - H. Gluesing-Luerssen, *Partitions of Frobenius Rings Induced by the Homogeneous Weight*, 2021.
- Consider the following partition of $\mathbb{Z}/p^s\mathbb{Z}$: $\mathcal{P}^{\mathsf{Hom}} = Z \mid U \mid S \mid R$, where

$$Z := \{0\}\,, \quad U := (\mathbb{Z}/p^s\mathbb{Z})^{\times}, \quad S := p^{s-1}(\mathbb{Z}/p^s\mathbb{Z}), \quad R := \{x \in \mathbb{Z}/p^s\mathbb{Z} \mid x \notin Z \cup U \cup S\}$$

Homogeneous Weight-Unit Partition

Given a finite chain ring $\mathcal{R}$ and $x \in \mathcal{R}^n$, define the *homogeneous weight-unit partition* of x as

$$\pi^{\mathsf{Hom}}(x) = (\pi_Z^{\mathsf{Hom}}(x), \pi_U^{\mathsf{Hom}}(x), \pi_S^{\mathsf{Hom}}(x), \pi_R^{\mathsf{Hom}}(x)), \quad \pi_I^{\mathsf{Hom}}(x) = |\,\{k = 1, \ldots, n \mid x_k \in I\}\,|\,.$$

Symmetrized Homogeneous Weight-Unit Partition

Set of hom. partitions $\mathbb{D}^{\mathsf{Hom}}_{p^s,n} := \left\{ \pi \in \{0,\ldots,n\}^4 \;\middle|\; \sum_{I\in\{Z,U,R,S\}} \pi_I = n \right\}$

symmetrized hom. partition $\mathcal{P}^n_{\mathsf{Hom},\mathrm{sym}} := P^{\mathsf{Hom}}_{\pi^{(0)}} \mid P^{\mathsf{Hom}}_{\pi^{(1)}} \mid \cdots \mid P^{\mathsf{Hom}}_{\pi^{(D-1)}}, \quad \pi^{(i)} \in \mathbb{D}^{\mathsf{Hom}}_{p^s,n}$

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Proposition [B., Cavicchioni, Weger, ´24]
The symmetrized homogeneous weight-unit partition $\mathcal{P}^n_{\mathsf{Hom},\mathrm{sym}}$ is reflexive.

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Idea.
For each $I, J \in \{Z, U, R, S\}$ compute the expression $\sum_{x\in J} \xi^{\mathrm{Tr}(a\cdot x)}$:

$I \setminus J$	Z	U	S	R
Z	1	$q^{s-1}(q-1)$	$q-1$	$q^{s-1}-q$
U	1	0	-1	0
S	1	$-q^{s-1}$	$q-1$	$q^{s-1}-q$
R	1	0	$q-1$	$-q$

Homogeneous Metric - MacWilliams-like Identity

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MacWilliams-like Identity for the Homogeneous Weight [B., Cavicchioni, Weger '24]

$$\mathcal{D}_\rho^{\mathsf{Hom}}(\mathcal{C}^\perp) = \frac{1}{|\mathcal{C}|} \sum_{\pi \in \mathbb{D}_{p^s,n}^{\mathsf{Hom}}} K_\rho^{\mathsf{Hom}}(\pi) \mathcal{D}_\pi^{\mathsf{Hom}}(\mathcal{C}),$$

where the Krawtchouk coefficient exists and is given by

$$K_\rho^{\mathsf{Hom}}(\pi) = \sum_{t \in \mathrm{Comp}_{\rho<\pi}^{\mathsf{Hom}}} \left(\prod_{i \in I_{\mathsf{Hom}}} \binom{\pi_i}{t_{i0}, \ldots, t_{iM}} \right) (-1)^{t_{US}} (-p^{s-1})^{t_{SU}} \left(p^{s-1}(p-1)\right)^{t_{ZU}}$$

$$\left(p^{s-1} - p\right)^{t_{ZR}} (p-1)^{t_{ZS}+t_{SS}+t_{SR}+t_{RS}} \mathbb{1}_{\{t_{UU}=t_{UR}=t_{RU}=t_{RR}=0\}}.$$

Subfield Metric

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Given a finite field $\mathbb{F}_{p^s}$ the *subfield weight* of an element $a \in \mathbb{F}_{p^s}$ is defined as

$$\mathrm{wt}_\lambda(a) = \begin{cases} 0 & \text{if } a = 0 \\ 1 & \text{if } a \in \mathbb{F}_p^\times \\ \lambda & \text{if } a \in \mathbb{F}_{p^s} \setminus \mathbb{F}_p \end{cases}$$

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- MacWilliams identities fail, in general, for the subfield weight partition
- Similar approach as for homogeneous metric:

$$\mathcal{O}_{a_0} = \{0\}\,, \quad \mathcal{O}_{a_1} = \mathbb{F}_p^\times$$

$$\mathcal{O}_{a_i} = a_i\mathbb{F}_p^\times\,, \quad \text{for } a_i \in \mathbb{F}_{p^s} \setminus \left(\sqcup_{j=0}^{i-1}\mathcal{O}_{a_j}\right) \text{ and } i = 2, \ldots \frac{p^s-1}{p-1}.$$

$\Longrightarrow$ Is reflexive (since based on orbits of $\mathbb{F}_p^\times$)

Subfield-Trace Partition

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Given a finite field $\mathbb{F}_{p^s}$. The *subfield-trace partition* of $\mathbb{F}_{p^s}$ is defined as $\mathcal{P}_\lambda = I_0^\lambda \mid I_1^\lambda \mid I_2^\lambda \mid I_3^\lambda$, where

$$I_0^\lambda := \{0\}\,,\ I_1^\lambda := \mathbb{F}_p^\times,\ I_2^\lambda := \{x \in \mathbb{F}_{p^s} \setminus \mathbb{F}_p \mid \mathrm{Tr}(x) = 0\} \text{ and } I_3^\lambda := \{x \in \mathbb{F}_{p^s} \setminus \mathbb{F}_p \mid \mathrm{Tr}(x) \neq 0\}\,.$$

- Denote the symmetrized subfield-trace partition of $\mathbb{F}_{p^s}^n$ by $\mathcal{P}_{\lambda,\mathrm{sym}}^n$.

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Proposition [B., Cavicchioni, Weger, ´24]

The subfield-trace partition $\mathcal{P}_\lambda$ of $\mathbb{F}_{p^s}$ is reflexive (and so is $\mathcal{P}_{\lambda,\mathrm{sym}}^n$).

Subfield Metric - MacWilliams-like Identity

Set of λ-trace partitions	$\mathbb{D}^{\lambda}_{p,r,1,n} = \left\{ \pi \in \{0,\ldots,n\}^4 \;\middle|\; \sum_{i=0}^{3} \pi_i = n \right\}$
Symmetrized λ-trace partition	$\mathcal{P}^{n}_{\lambda,\mathrm{sym}} := P^{\lambda}_{\pi^{(0)}} \mid \cdots \mid P^{\lambda}_{\pi^{(D-1)}}$
λ-trace enumerator	$\mathcal{D}^{\lambda}_{\pi}(\mathcal{C}) := \left\| P^{\lambda}_{\pi} \cap \mathcal{C} \right\| \quad \pi^{(i)} \in \mathbb{D}^{\lambda}_{p^r,n}$

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MacWilliams-like Identity for the Homogeneous Weight [B., Cavicchioni, Weger '24]

$$\mathcal{D}^{\lambda}_{\rho}(\mathcal{C}^{\perp}) = \frac{1}{|\,\mathcal{C}\,|} \sum_{\pi \in \mathbb{D}^{\lambda}_{p,r,1,n}} K^{\lambda}_{\pi}(\rho)\mathcal{D}^{\lambda}_{\pi}(\mathcal{C}),$$

where the Krawtchouk coefficient exists and is given by $K^{\lambda}_{\pi}(\rho) = \sum_{t \in \mathrm{Comp}^{\lambda}_{\pi}(\rho)} \left(\prod_{i=0}^{3} \binom{\pi_i}{t_{i0},\ldots,t_{i3}}\right) \cdot \kappa^{\lambda}$ and

$$\kappa^{\lambda} := \begin{cases} (p-1)^{t_{01}+t_{11}+t_{21}+t_{03}}(-1)^{t_{31}}(-p)^{t_{22}}(p^{r-1}-p)^{t_{02}+t_{12}}(-p^{r-1})^{t_{13}}(p^{r-1})^{t_{03}} & \text{if } r \mid p, \\ (p-1)^{t_{01}+t_{03}}(-1)^{t_{11}+t_{22}+t_{33}+t_{32}}(1-p)^{t_{23}}(p^{r-1}-1)^{t_{02}+t_{03}+t_{12}+t_{21}}(1-p^{r-1})^{t_{13}} & \text{otherwise.} \end{cases}$$

Wrap up

Found

A partition-based enumerator...

- giving rise to the weight enumerator (in the relative metric),
- with more information about the codeword's structure,
- allowing for a well-defined Krawtchouk coefficient,
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- Other metrics?

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Thank you for your attention!