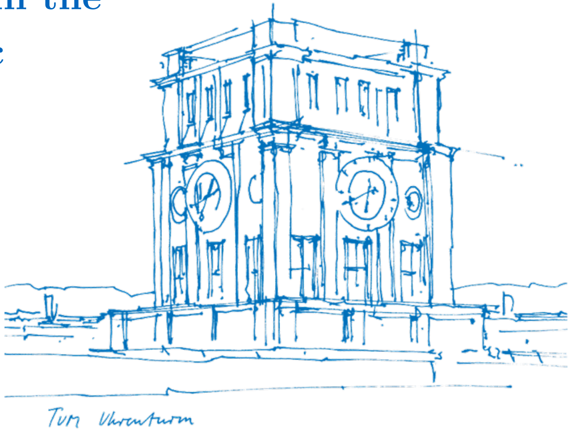
TUM School of Computation, Information and Technology
Department of Electrical and Computer Engineering
Technical University of Munich

MacWilliams-type Identities in the Lee and Homogeneous Metric



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joint work with G. Cavicchioni and V. Weger

Doctoral Seminar

July 25th and 26th, 2024

Motivation

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Count the number of codewords of a certain weight

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- Non-Existence Results for the Lee and Homogeneous Metric

Can we derive a MacWilliams-like identity for a similar enumerator in the Lee/Homogeneous metric?

Outline

1. The MacWilliams Identity

2. The Lee and Homogeneous Metric Case

3. MacWilliams-type Identities for Suitable Code Partitions

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Codes over $\mathbb{Z}/p^s\mathbb{Z}$

$$\begin{aligned} \text{integer residue ring}: &\quad \mathbb{Z}/p^s\mathbb{Z} = \{0, \dots, p^s - 1\} \\ \text{standard inner product}: &\quad \langle x, y \rangle := \sum_{i=1}^{n} x_i y_i, \quad x, y \in (\mathbb{Z}/p^s\mathbb{Z})^n \end{aligned}$$

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Linear code over $(\mathbb{Z}/p^s\mathbb{Z})^n$ and its dual

Over $\mathbb{Z}/p^s\mathbb{Z}$, a *linear code* $\mathcal{C}$ *of length* n is a $\mathbb{Z}/p^s\mathbb{Z}$-submodule of $(\mathbb{Z}/p^s\mathbb{Z})^n$. The elements of $\mathcal{C}$ are called *codewords*. The *dual code* of $\mathcal{C}$ is

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Hamming Weight

Hamming weight and weight enumerator

Let $x \in (\mathbb{Z}/p^s\mathbb{Z})^n$ and let $\mathcal{C} \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n$ be a linear code.

$$\textit{Hamming weight of } x: \quad \mathrm{wt}_{\mathsf{H}}(x) = |\{i = 1, \ldots, n \mid x_i \neq 0\}|$$

$$\textit{Weight } i \textit{ enumerator of } \mathcal{C}: \quad W_{\mathcal{C}}^{\mathsf{H}}(i) = |\{c \in \mathcal{C} \mid \mathrm{wt}_{\mathsf{H}}(c) = i\}| \qquad \text{for } i = 0, \ldots, n$$

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$$W_{\mathcal{C}}^{\mathsf{H}}(0) = 1$$
$$W_{\mathcal{C}}^{\mathsf{H}}(1) = 4$$
$$W_{\mathcal{C}}^{\mathsf{H}}(2) = 4$$
$$W_{\mathcal{C}}^{\mathsf{H}}(3) = 0$$

The MacWilliams Identity

- Relation between $W_{\mathcal{C}}^{\mathsf{H}}$ and $W_{\mathcal{C}^\perp}^{\mathsf{H}}$

MacWilliams Identity [MacWilliams '63]

Given a linear code $\mathcal{C} \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n$ and its dual $\mathcal{C}^\perp$. For every $j \in \{0, \ldots, n\}$, it holds that

$$W_{\mathcal{C}^\perp}^{\mathsf{H}}(j) = \frac{1}{|\mathcal{C}|} \sum_{i=0}^{n} K_j(i) W_{\mathcal{C}}^{\mathsf{H}}(i),$$

where, given a p-th root of unity ξ, $K_j(i) := \sum_{\substack{a \in (\mathbb{Z}/p^s\mathbb{Z})^n \\ \mathrm{wt}_{\mathsf{H}}(a) = j}} \xi^{\langle a, x \rangle}$ for any $x \in \mathcal{C}$ with $\mathrm{wt}_{\mathsf{H}}(x) = i$.

Krawtchouk Coefficient for the Hamming Weight Enumerator

$$\text{Given any} \quad x \in (\mathbb{Z}/p^s\mathbb{Z})^n \ : \ \mathrm{wt}_{\mathsf{H}}(x) = i$$

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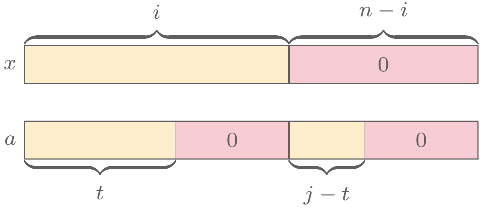

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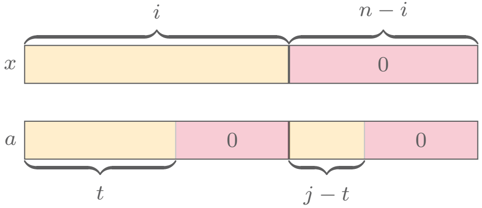

$$K_j(i) = \sum_{t=0}^{j} \binom{i}{t} \binom{n-i}{j-t} \prod_{k=0}^{n} \sum_{a_k} \xi^{x_k a_k}$$

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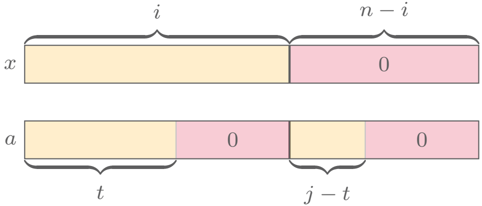

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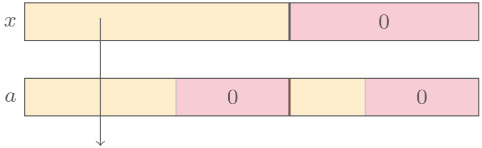

$\sum_{k \neq 0} \xi^k = (-1)$

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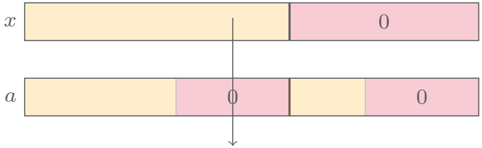

$\sum_{k=0} \xi^k = 1$

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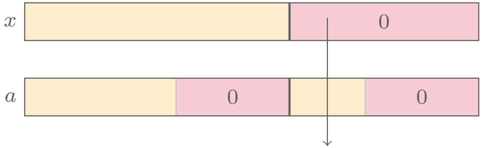

$\sum_{k \neq 0} \xi^0 = (p^s - 1)$

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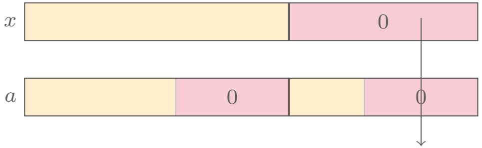

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Example over $\mathbb{Z}/3\mathbb{Z}$

MacWilliams Identity

$$W^{\mathsf{H}}_{\mathcal{C}^\perp}(j) = \frac{1}{|\,\mathcal{C}\,|}\sum_{i=0}^{n}\left(\sum_{t=0}^{j}\binom{i}{t}\binom{n-i}{j-t}(-1)^t(p^s-1)^{i-t}\right)W^{\mathsf{H}}_{\mathcal{C}}(i)$$

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$$W^{\mathsf{H}}_{\mathcal{C}^{\perp}}(1) = \frac{1}{3}\left(W^{\mathsf{H}}_{\mathcal{C}}(0)K_1(0) + W^{\mathsf{H}}_{\mathcal{C}}(1)K_1(1) + W^{\mathsf{H}}_{\mathcal{C}}(2)K_1(2) + W^{\mathsf{H}}_{\mathcal{C}}(3)K_1(3)\right)$$

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$$\begin{aligned} W^{\mathsf{H}}_{\mathcal{C}^\perp}(1) &= \frac{1}{3}\left(1\cdot K_1(0) + 2\cdot K_1(1)\right) \\ &= \frac{1}{3}\left(1\left[\binom{0}{0}\binom{3}{1}(-1)^0 2^1\right] + 2\left[\binom{1}{0}\binom{2}{1}(-1)^0 2^1 + \binom{1}{1}\binom{3}{0}(-1)^1 2^0\right]\right) \end{aligned}$$

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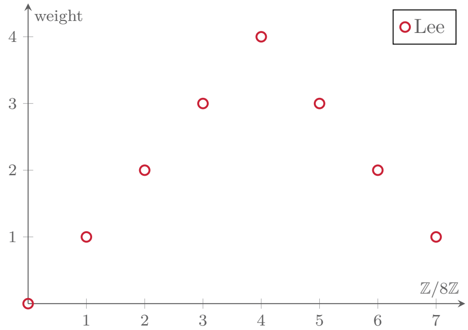
Outline

Lee and Homogeneous Weight over $\mathbb{Z}/p^s\mathbb{Z}$

Lee Weight

$$\mathrm{wt}_{\mathsf{L}}(a) := \min\{a, p^s - a\}$$

Example over $\mathbb{Z}/8\mathbb{Z}$



Lee and Homogeneous Weight over $\mathbb{Z}/p^s\mathbb{Z}$

Lee Weight

$$\mathrm{wt}_{\mathsf{L}}(a) := \min\{a, p^s - a\}$$

Homogeneous Weight

$$\mathrm{wt}_{\mathsf{Hom}}(a) := \begin{cases} 0 & a = 0 \\ 1 & a \notin \langle p^{s-1} \rangle \\ p/(p-1) & a \in \langle p^{s-1} \rangle \setminus \{0\} \end{cases}$$

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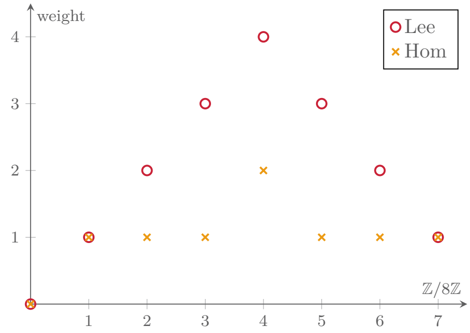

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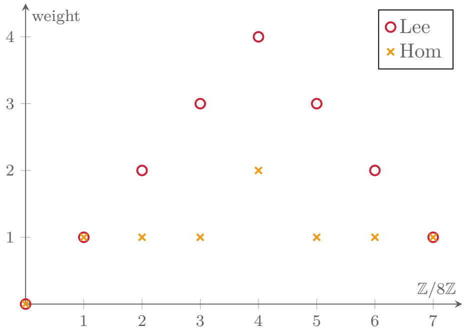

We extend both weights additively, i.e., for $x \in (\mathbb{Z}/p^s\mathbb{Z})^n$ we have $\mathrm{wt_{L/Hom}}(x) = \sum_{i=1}^{n} \mathrm{wt_{L/Hom}}(x_i)$

Non-Existence for Lee/Homogeneous Weight Enumerator

Given any $x \in (\mathbb{Z}/p^s\mathbb{Z})^n \ : \ \mathrm{wt}_{\mathsf{L}}(x)/\,\mathrm{wt}_{\mathsf{Hom}}(x) = i$

Krawtchouk coefficient $K_j(i) := \sum_{\substack{a \in (\mathbb{Z}/p^s\mathbb{Z})^n \\ \mathrm{wt}_{\mathsf{L}}(x)/\,\mathrm{wt}_{\mathsf{Hom}}(a) = j}} \xi^{\langle a, x \rangle}$

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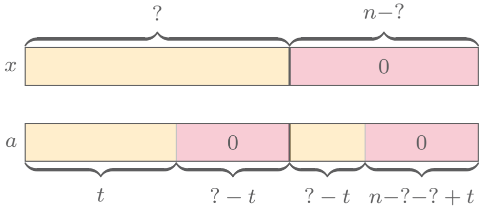

- $\mathrm{wt}_{\mathsf{L}}(x)/\,\mathrm{wt}_{\mathsf{Hom}}(x) = i$ does not imply $\mathrm{supp}(a) = i$.
- $\sum_{\substack{a\in(\mathbb{Z}/p^s\mathbb{Z})^n \\ \mathrm{wt}_{\mathsf{L}}(a)/\,\mathrm{wt}_{\mathsf{Hom}}(a)=j}} \xi^{\langle a,x\rangle}$ highly depends on the choice of x

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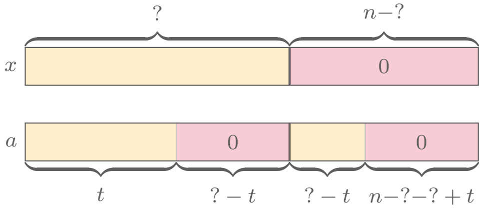

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- K. Shiromoto, *A note on a basic exact sequence for the Lee and Euclidean weights of linear codes over $\mathbb{Z}_\ell$*, 2015.
- N. Abdelghany, J. Wood, *Failure of the MacWilliams identities for the Lee weight enumerator over $\mathbb{Z}_m$, $m \geq 5$*, 2020.
- J. Wood, *Homogeneous weight enumerators over integer residue rings and failures of the MacWilliams identities*, 2023.

Outline

1. The MacWilliams Identity

2. The Lee and Homogeneous Metric Case

3. MacWilliams-type Identities for Suitable Code Partitions

Changing the Approach

Wanted

An enumerator ...

- with more information about the codeword's structure

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Idea: Refine the weight partition!

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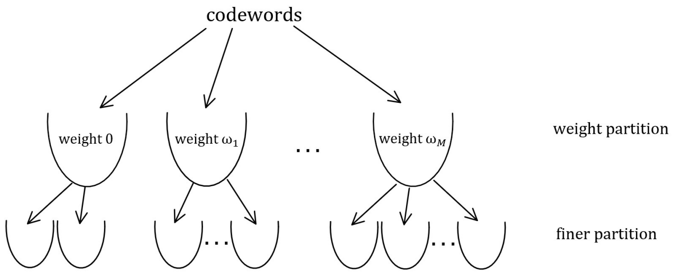

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Example over $\mathbb{Z}/4\mathbb{Z}$

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$$\mathcal{C} = \big\{ (0,0,0), (0,1,0), (0,0,1), (0,3,0), (0,0,3), (0,2,0), (0,0,2), (0,1,1), (0,1,3), (0,3,1), (0,3,3), (0,1,2), (0,2,1), (0,3,2), (0,2,3), (0,2,2) \big\}$$

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Note! We can derive any additive weight $\mathrm{wt}(x)$ for $x \in \mathbb{Z}/p^s\mathbb{Z}$ from this decomposition, i.e.,

$$\mathrm{wt}(x) = \sum_{i=0}^{M} \pi^{\mathsf{L}}_i(x)\,\mathrm{wt}(i).$$

Krawtchouk Coefficient for the Lee Decomposition

Set of Lee decompositions $\mathbb{D}^{\mathsf{L}}_{p^s,n} := \left\{ \pi \in \{0,\dots,n\}^{M+1} \;\middle|\; \sum_{i=0}^{M+1} \pi_i = n \right\}$

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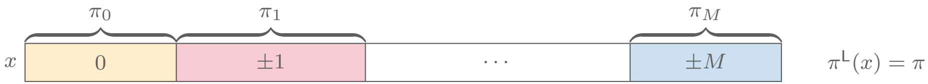

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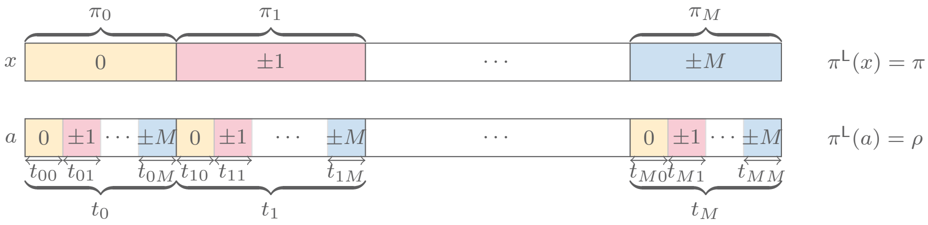

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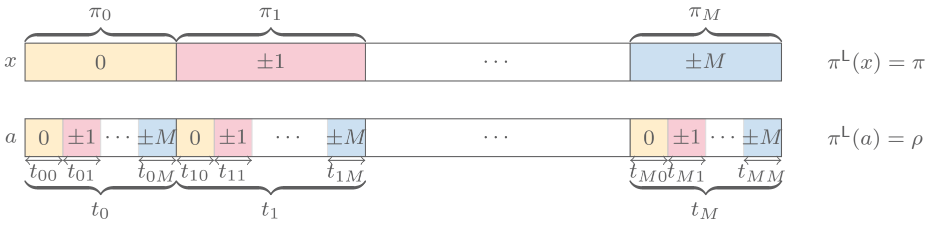

Note! $K^{\mathsf{L}}_{\pi}(\rho)$ is independent on the choice of $x \in (\mathbb{Z}/p^s\mathbb{Z})^n$

Lee Metric - MacWilliams-like Identity

MacWilliams Identity for the Lee Decomposition Enumerator

$$\mathcal{D}^{\mathsf{L}}_{\pi}(\mathcal{C}^{\perp}) = \frac{1}{|\,\mathcal{C}\,|} \sum_{\pi \in \mathbb{D}^{\mathsf{L}}_{p^s,n}} K^{\mathsf{L}}_{\rho}(\pi)\mathcal{D}^{\mathsf{L}}_{\rho}(\mathcal{C}),$$

where the Krawtchouk coefficient exists and is given by

$$K^{\mathsf{L}}_{\rho}(\pi) = \begin{cases} \sum_{t \in \mathrm{Comp}^{\mathsf{L}}_{\rho<\pi}} \left(\prod_{i=0}^{M} \binom{\pi_i}{t_{i0},\dots,t_{iM}} \prod_{j=1}^{M-1} \left(\xi^{-ij} + \xi^{ij}\right)^{t_{ij}} \xi^{iM} \right) & \text{if } p = 2 \\ \sum_{t \in \mathrm{Comp}^{\mathsf{L}}_{\rho<\pi}} \left(\prod_{i=0}^{M} \binom{\pi_i}{t_{i0},\dots,t_{iM}} \prod_{j=1}^{M} \left(\xi^{-ij} + \xi^{ij}\right)^{t_{ij}} \right) & \text{otherwise} \end{cases}$$

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Credits:

- MacWilliams in 1963: Over $\mathbb{F}_{p^s}$
- Astola in 1982: Association schemes
- Solé in 1986: Association schemes
- B., Cavicchioni, Weger in 2024: Identity is true over any finite chain ring $\mathcal{R}$ for all additive weights

Homogeneous Weight and Unit Decomposition

Recall the homogeneous weight over $\mathbb{Z}/p^s\mathbb{Z}$

$$\mathrm{wt}_{\mathsf{Hom}}(x) := \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \notin \left\langle p^{s-1} \right\rangle \setminus \{0\} \\ \frac{p}{p-1} & \text{if } x \in \left\langle p^{s-1} \right\rangle \end{cases}$$

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- Consider the following partition of $\mathbb{Z}/p^s\mathbb{Z}$: $\mathcal{P}^{\mathsf{Hom}} = Z \mid U \mid S \mid R$, where

$$Z := \{0\}\,, \quad U := (\mathbb{Z}/p^s\mathbb{Z})^\times, \quad S := p^{s-1}(\mathbb{Z}/p^s\mathbb{Z}), \quad R := \{x \in \mathbb{Z}/p^s\mathbb{Z} \mid x \notin Z \cup U \cup S\}$$

Homogeneous Metric - MacWilliams-like Identity

Homogeneous weight and unit decomposition $\pi^{\mathsf{Hom}}(x) = (\pi_Z^{\mathsf{Hom}}(x), \pi_U^{\mathsf{Hom}}(x), \pi_S^{\mathsf{Hom}}(x), \pi_R^{\mathsf{Hom}}(x)),$

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MacWilliams-like Identity for the Homogeneous Weight [B., Cavicchioni, Weger '24]

$$\mathcal{D}_\rho^{\mathsf{Hom}}(\mathcal{C}^\perp) = \frac{1}{|\,\mathcal{C}\,|} \sum_{\pi \in \mathbb{D}_{p^s,n}^{\mathsf{Hom}}} K_\rho^{\mathsf{Hom}}(\pi) \mathcal{D}_\pi^{\mathsf{Hom}}(\mathcal{C}),$$

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$$\left(p^{s-1} - p\right)^{t_{ZR}} (p-1)^{t_{ZS} + t_{SS} + t_{SR} + t_{RS}} \mathbb{1}_{\{t_{UU} = t_{UR} = t_{RU} = t_{RR} = 0\}}.$$

What More?

Further Results

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- Linear Programming bounds
- MacWilliams identity for λ-subfield metric over $\mathbb{F}_{p^s}$, $\lambda \geq 1$

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Thank you for your attention!