The Existence of MacWilliams-Type Identities for the Lee, Homogeneous and Subfield Metric

Combinatorics 2024, Carovigno

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Motivation

MacWilliams identity

Given the Hamming weight enumerator $W^{\mathsf{H}}(\mathcal{C})$ of a code $\mathcal{C}$, the Hamming weight enumerator $W^{\mathsf{H}}(\mathcal{C}^{\perp})$ of the dual $\mathcal{C}^{\perp}$ is in relation with $W^{\mathsf{H}}(\mathcal{C})$ as

$$W^{\mathsf{H}}_{\mathcal{C}^{\perp}}(j) = \frac{1}{|\,\mathcal{C}\,|} \sum_{i=0}^{n} K_j(i) W^{\mathsf{H}}_{\mathcal{C}}(i)$$

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- Various approaches to the identities
- Existence/Non-Existence Results
- Linear Programming bound

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Here: Focus on $\mathbb{Z}/p^s\mathbb{Z}$ and the Lee/Homogeneous metric

Codes over $\mathbb{Z}/p^s\mathbb{Z}$

Consider a finite integer residue ring $\mathbb{Z}/p^s\mathbb{Z}$ and a positive integer n.

An $\mathbb{Z}/p^s\mathbb{Z}$-submodule $\mathcal{C}$ of $(\mathbb{Z}/p^s\mathbb{Z})^n$ is called a *ring-linear code* of (block) length n. Its elements are called *codewords*. We define the *dual code* $\mathcal{C}^\perp$ of $\mathcal{C}$ as

$$\mathcal{C}^\perp = \{x \in (\mathbb{Z}/p^s\mathbb{Z})^n \mid \langle x, c \rangle = 0 \ \text{ for all } \ c \in \mathcal{C}\}.$$

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Given $\mathbb{Z}/p^s\mathbb{Z}$, a *weight* over $\mathbb{Z}/p^s\mathbb{Z}$ is a function $\mathrm{wt}\colon \mathbb{Z}/p^s\mathbb{Z} \to \mathbb{Q}$ satisfying

- $\mathrm{wt}(0) = 0$ and $\mathrm{wt}(a) > 0$ for all $a \neq 0$;
- $\mathrm{wt}(a) = \mathrm{wt}(-a)$;
- $\mathrm{wt}(a+b) \leq \mathrm{wt}(a) + \mathrm{wt}(b)$.

In particular, given $x \in (\mathbb{Z}/p^s\mathbb{Z})^n$, we define $\mathrm{wt}(x) = \sum_{i=1}^n \mathrm{wt}(x_i)$. We call this weight an *additive weight*.

Additive Weights over $\mathbb{Z}/p^s\mathbb{Z}$

Hamming weight

$$\mathrm{wt}_{\mathsf{H}}(a) := \begin{cases} 0 & \text{if } a = 0 \\ 1 & \text{else} \end{cases}.$$

Example

Consider $(0, 1, 6, 4) \in (\mathbb{Z}/8\mathbb{Z})^4$

$$\mathrm{wt}_{\mathsf{H}}((0, 1, 6, 4)) = 0 + 1 + 1 + 1 = 3$$

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Lee weight

$$\mathrm{wt}_{\mathsf{L}}(a) := \min\left\{a, p^s - a\right\}.$$

Note that $\mathrm{wt}_{\mathsf{H}}(a) \leq \mathrm{wt}_{\mathsf{L}}(a) \leq \lfloor p^s/2 \rfloor =: M$

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Homogenous weight

$$\mathrm{wt}_{\mathsf{Hom}}(a) := \begin{cases} 0 & \text{if } a = 0 \\ 1 & \text{if } a \notin \left\langle p^{s-1} \right\rangle \setminus \{0\} \\ \frac{p}{p-1} & \text{if } a \in \left\langle p^{s-1} \right\rangle \end{cases}$$

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MacWilliams Identities for the Hamming Weight Enumerator

Hamming weight enumerator: $W_{\mathcal{C}}^{\mathsf{H}}(i) = \left| \{ c \in \mathcal{C} \mid \mathrm{wt}_{\mathsf{H}}(c) = i \} \right|$ for every $i \in \{0, \ldots, n\}$

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MacWilliams Identity [MacWilliams '63]
Given a linear code $\mathcal{C} \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n$ and its dual $\mathcal{C}^\perp$. Then, for every $j \in \{0, \ldots, n\}$, it holds that

$$W_{\mathcal{C}^\perp}^{\mathsf{H}}(j) = \frac{1}{|\mathcal{C}|} \sum_{i=0}^{n} K_j(i) W_{\mathcal{C}}^{\mathsf{H}}(i),$$

where, given a p-th root of unity ξ, $K_j(i) := \sum_{\substack{x \in (\mathbb{Z}/p^s\mathbb{Z})^n \\ \mathrm{wt}_{\mathsf{H}}(x) = i}} \xi^{\langle a, x \rangle}$ for a $a \in \mathcal{C}^\perp$ with $\mathrm{wt}_{\mathsf{H}}(a) = j$.

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Note! The Krawtchouk coefficient $K_j(i)$ does only exist if it is **independent** on the choice of $a \in (\mathbb{Z}/p^s\mathbb{Z})^n$ with $\mathrm{wt}_{\mathsf{H}}(a) = j$.

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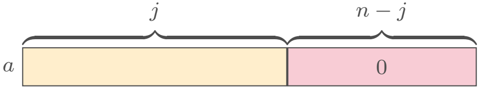

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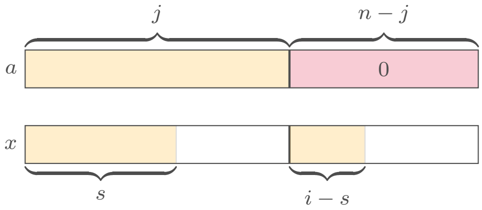

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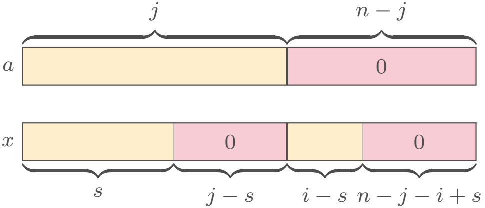

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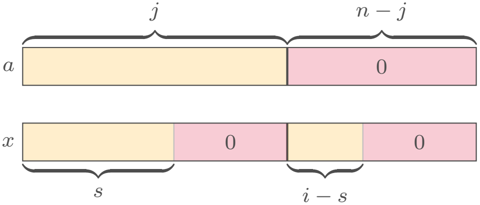

$$K_j(i) = \sum_{s=0}^{i} \binom{j}{s} \binom{n-j}{i-s} \prod_{k=0}^{n} \sum_{x_k} \xi^{x_k a_k}$$

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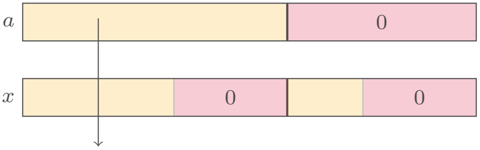

$\sum_{t\neq 0} \xi^t = (-1)$

$$K_j(i) = \sum_{s=0}^{i} \binom{j}{s}\binom{n-j}{i-s} \prod_{k=0}^{n} \sum_{x_k} \xi^{x_k a_k}$$

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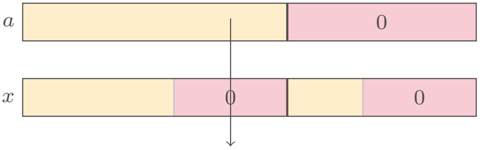

$$\sum_{t=0} \xi^t = 1$$

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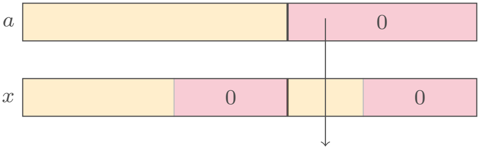

$$\textstyle\sum_{t \neq 0} \xi^0 = (p^s - 1)$$

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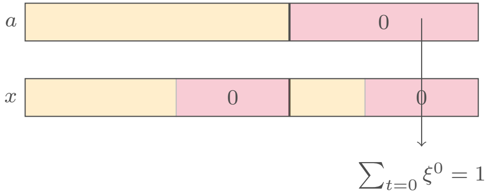

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Non-Existence for Lee/Homogeneous Weight Enumerator

Given any $a \in (\mathbb{Z}/p^s\mathbb{Z})^n \;:\; \mathrm{wt}_{\mathsf{L}}(a)/\,\mathrm{wt}_{\mathsf{Hom}}(a) = j$

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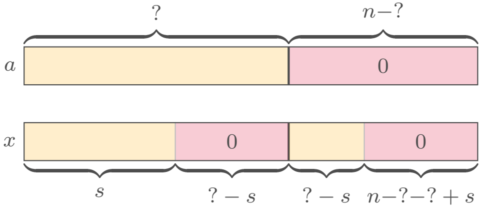

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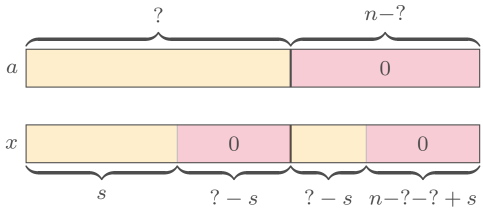

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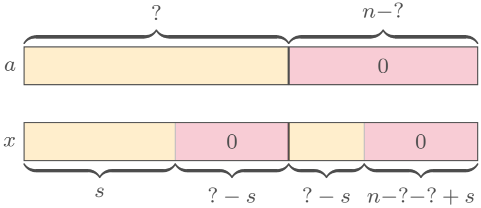

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Lee Weight Decomposition

For any $x \in (\mathbb{Z}/p^s\mathbb{Z})^n$ we define its *Lee weight decomposition* $\pi^{\mathsf{L}}(x) = (\pi^{\mathsf{L}}_0(x), \pi^{\mathsf{L}}_1(x), \ldots, \pi^{\mathsf{L}}_M(x))$ by

$$\pi^{\mathsf{L}}_i(x) = \left| \{k = 1, \ldots, n \mid \mathrm{wt}_{\mathsf{L}}(x_k) = i\} \right|.$$

Note! We can derive any additive weight $\mathrm{wt}(x)$ for $x \in \mathbb{Z}/p^s\mathbb{Z}$ from this decomposition, i.e.,

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Set of Lee decompositions $\mathbb{D}^{\mathsf{L}}_{p^s,n} := \left\{ \pi \in \{0, \dots, n\}^{M+1} \;\middle|\; \sum_{i=0}^{M+1} \pi_i = n \right\}$

Lee decomposition enumerator $\mathcal{D}^{\mathsf{L}}_{\pi}(\mathcal{C}) := \left| \left\{ c \in \mathcal{C} \mid \pi^{\mathsf{L}}(c) = \pi \right\} \right|$

Krawtchouk coefficient $K^{\mathsf{L}}_{\pi}(\rho) = \sum_{\substack{x \in (\mathbb{Z}/p^s\mathbb{Z})^n \\ \pi^{\mathsf{L}}(x) = \rho}} \xi^{\langle x, a \rangle}, \quad a \in (\mathbb{Z}/p^s\mathbb{Z})^n \;:\; \pi^{\mathsf{L}}(a) = \pi$

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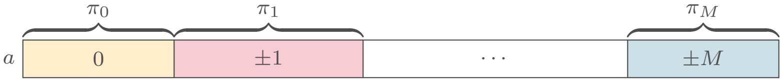
Question Is $K^{\mathsf{L}}_{\pi}(\rho)$ independent on the choice of $a \in (\mathbb{Z}/p^s\mathbb{Z})^n$?

Lee Metric - MacWilliams-like Identity



$\pi^{\mathsf{L}}(a) = \pi$

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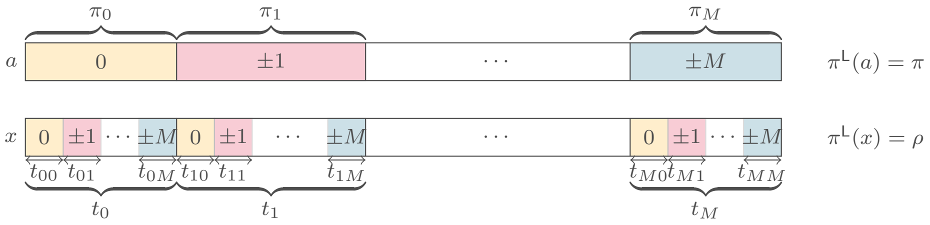

Requirements

- $\sum_{j=0}^{M} t_{ij} = \pi_i$ for every $i = 1, \dots, n$ (1)
- $\sum_{i=0}^{M} t_{ij} = \rho_j$ for every $j = 1, \dots, n$ (2)

$$\mathrm{Comp}^{\mathsf{L}}_{\rho<\pi} := \{t = (t_0, \dots, t_M) \mid (1) \text{ and } (2) \text{ are satisfied}\}$$

MacWilliams Identity for the Lee Decomposition Enumerator

$$\mathcal{D}^{\mathsf{L}}_{\rho}(\mathcal{C}^{\perp}) = \frac{1}{|\,\mathcal{C}\,|} \sum_{\pi \in \mathbb{D}^{\mathsf{L}}_{p^s,n}} K^{\mathsf{L}}_{\rho}(\pi) \mathcal{D}^{\mathsf{L}}_{\pi}(\mathcal{C}),$$

where the Krawtchouk coefficient exists and is given by

$$K^{\mathsf{L}}_{\rho}(\pi) = \begin{cases} \sum_{t \in \mathrm{Comp}^{\mathsf{L}}_{\rho<\pi}} \left(\prod_{i=0}^{M} \binom{\pi_i}{t_{i0},\dots,t_{iM}} \prod_{j=1}^{M-1} \left(\xi^{-ij} + \xi^{ij}\right)^{t_{ij}} \xi^{iM} \right) & \text{if } p = 2 \\ \sum_{t \in \mathrm{Comp}^{\mathsf{L}}_{\rho<\pi}} \left(\prod_{i=0}^{M} \binom{\pi_i}{t_{i0},\dots,t_{iM}} \prod_{j=1}^{M} \left(\xi^{-ij} + \xi^{ij}\right)^{t_{ij}} \right) & \text{otherwise} \end{cases}$$

MacWilliams Identity for the Lee Decomposition Enumerator

$$\mathcal{D}^{\mathsf{L}}_{\rho}(\mathcal{C}^{\perp}) = \frac{1}{|\,\mathcal{C}\,|} \sum_{\pi \in \mathbb{D}^{\mathsf{L}}_{p^s,n}} K^{\mathsf{L}}_{\rho}(\pi)\mathcal{D}^{\mathsf{L}}_{\pi}(\mathcal{C}),$$

where the Krawtchouk coefficient exists and is given by

$$K^{\mathsf{L}}_{\rho}(\pi) = \begin{cases} \sum_{t\in \mathrm{Comp}^{\mathsf{L}}_{\rho<\pi}} \left(\prod_{i=0}^{M} \binom{\pi_i}{t_{i0},\ldots,t_{iM}} \prod_{j=1}^{M-1} \left(\xi^{-ij}+\xi^{ij}\right)^{t_{ij}} \xi^{iM}\right) & \text{if } p=2 \\ \sum_{t\in \mathrm{Comp}^{\mathsf{L}}_{\rho<\pi}} \left(\prod_{i=0}^{M} \binom{\pi_i}{t_{i0},\ldots,t_{iM}} \prod_{j=1}^{M} \left(\xi^{-ij}+\xi^{ij}\right)^{t_{ij}}\right) & \text{otherwise} \end{cases}$$

Credits:

- MacWilliams in 1963: Over $\mathbb{F}_{p^s}$
- Astola in 1982: Association schemes
- Solé in 1986: Association schemes
- B., Cavicchioni, Weger in 2024: Identity is true over any finite chain ring $\mathcal{R}$ for all additive weights

Homogeneous Weight and Unit Decomposition

Recall the homogeneous weight over $\mathbb{Z}/p^s\mathbb{Z}$

$$\mathrm{wt}_{\mathsf{Hom}}(x) := \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \notin \left\langle p^{s-1} \right\rangle \setminus \{0\} \\ \frac{p}{p-1} & \text{if } x \in \left\langle p^{s-1} \right\rangle \end{cases}$$

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- Homogeneous weight decomposition
  - H. Gluesing-Luerssen, *Partitions of Frobenius Rings Induced by the Homogeneous Weight*, 2021.
- Consider the following partition of $\mathbb{Z}/p^s\mathbb{Z}$: $\mathcal{P}^{\mathsf{Hom}} = Z \mid U \mid S \mid R$, where

$$Z := \{0\}, \quad U := (\mathbb{Z}/p^s\mathbb{Z})^\times, \quad S := p^{s-1}(\mathbb{Z}/p^s\mathbb{Z}), \quad R := \{x \in \mathbb{Z}/p^s\mathbb{Z} \mid x \notin Z \cup U \cup S\}$$

Homogeneous Metric - MacWilliams-like Identity

Homogeneous weight and unit decomposition $\pi^{\mathsf{Hom}}(x) = (\pi_Z^{\mathsf{Hom}}(x), \pi_U^{\mathsf{Hom}}(x), \pi_S^{\mathsf{Hom}}(x), \pi_R^{\mathsf{Hom}}(x)),$

$$\pi_I^{\mathsf{Hom}}(x) = |\{k = 1, \ldots, n \mid x_k \in I\}|$$

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MacWilliams-like Identity for the Homogeneous Weight [B., Cavicchioni, Weger '24]

$$\mathcal{D}_\rho^{\mathsf{Hom}}(\mathcal{C}^\perp) = \frac{1}{|\mathcal{C}|} \sum_{\pi \in \mathbb{D}_{p^s,n}^{\mathsf{Hom}}} K_\rho^{\mathsf{Hom}}(\pi) \mathcal{D}_\pi^{\mathsf{Hom}}(\mathcal{C}),$$

where the Krawtchouk coefficient exists and is given by

$$K_\rho^{\mathsf{Hom}}(\pi) = \sum_{t \in \mathrm{Comp}_{\rho<\pi}^{\mathsf{Hom}}} \left(\prod_{i \in I_{\mathsf{Hom}}} \binom{\pi_i}{t_{i0}, \ldots, t_{iM}} \right) (-1)^{t_{US}} (-p^{s-1})^{t_{SU}} \left(p^{s-1}(p-1)\right)^{t_{ZU}}$$
$$\left(p^{s-1} - p\right)^{t_{ZR}} (p-1)^{t_{ZS}+t_{SS}+t_{SR}+t_{RS}} \mathbb{1}_{\{t_{UU}=t_{UR}=t_{RU}=t_{RR}=0\}}.$$

Other Results

- Coarsest partition for the Lee/Homogeneous metric
- Linear Programming bounds
- MacWilliams identity for λ-subfield metric over $\mathbb{F}_{p^s}$, $\lambda \geq 1$:

$$\mathrm{wt}_\lambda(a) = \begin{cases} 0 & \text{if } a = 0 \\ 1 & \text{if } a \in \mathbb{F}_p^\times \\ \lambda & \text{if } a \in \mathbb{F}_{p^s} \setminus \mathbb{F}_p \end{cases}$$

- $\mathcal{O}_{a_0} = \{0\}\,, \quad \mathcal{O}_{a_1} = \mathbb{F}_p^\times$
- iteratively for $i = 2, \dots, \frac{p^s-1}{p-1}$:
  $\mathcal{O}_{a_i} = a_i \mathbb{F}_p^\times$ for some $a_i \in \mathbb{F}_{p^s} \setminus \left(\bigsqcup_{j=0}^{i-1} \mathcal{O}_{a_j} \right)$

Further Results and Open Questions

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Future Research

- Is there a coarser partition for the subfield weight?
- Other metrics?

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Thank you for your attention!