International Workshop on Code-Based Cryptography (CBCrypto) 2022

Information Set Decoding for Lee-Metric Codes using Restricted Spheres

Jessica Bariffi

joint work with Karan Khathuria (UT) and Violetta Weger (TUM)

Institute for Communications and Navigation
German Aerospace Center, DLR

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Code-based cryptography for quantum secure cryptosystems

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- ! Alternative metrics are considered

The security relies on the hardness of the syndrome decoding problem

- ! Generic decoding in the Lee metric has a large cost
- ! NP-hard in the Lee metric

Outline

1. Preliminaries

2. Information Set Decoding using Restricted Spheres
   - Up to Minimum Distance Decoding
   - Decoding Beyond the Minimum Distance

3. Comparison

Outline

DLR

Ring-Linear Codes

Let p a prime number and s and n two positive integers.

Definition

A *linear code* C $(\mathbb{Z}/p^s\mathbb{Z})^n$ is a $\mathbb{Z}/p^s\mathbb{Z}$-submodule of $(\mathbb{Z}/p^s\mathbb{Z})^n$.

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Parameters:

n is called the *length* of C

$k := \log_{p^s} |C|$ is the $\mathbb{Z}/p^s\mathbb{Z}$-*dimension* of C

$R := k/n$ denotes the *rate* of C.

The Lee Metric

Definition

For $a \in \mathbb{Z}/p^s\mathbb{Z}$ and $e = (e_1, \ldots, e_n) \in (\mathbb{Z}/p^s\mathbb{Z})^n$ we define their *Lee weight*, respectively, by

$$\mathrm{wt}_L(a) := \min(a, p^s - a),$$

$$\mathrm{wt}_L(e) := \sum_{i=1}^{n} \mathrm{wt}_L(e_i).$$

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Example over $\mathbb{Z}/5\mathbb{Z}$

$0: \quad \mathrm{wt}_L(0) = 0$

$1: \quad \mathrm{wt}_L(1) = 1$

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Properties:

For every $a \in \mathbb{Z}/p^s\mathbb{Z}$ and $x \in (\mathbb{Z}/p^s\mathbb{Z})^n$

$$\mathrm{wt}_L(a) = \mathrm{wt}_L(p^s - a)$$

$$\mathrm{wt}_H(a) \quad \mathrm{wt}_L(a) \quad bp^s/2c =: M$$

$$\mathrm{wt}_H(e) \quad \mathrm{wt}_L(e) \quad nM$$

The Expected Lee Weight

Let $a \in \mathbb{Z}/p^s\mathbb{Z}$ be chosen uniformly at random.

Lemma

The expected Lee weight of a is then given by

$$\mu_{p^s} := \mathrm{E}(\mathrm{wt}_{\mathrm{L}}(a)) = \begin{cases} \frac{(p^s)^2 - 1}{4p^s} & \text{if } p^s \text{ is odd;} \\ \frac{p^s}{4} & \text{if } p^s \text{ is even:} \end{cases}$$

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Let $T := \lim_{n \to \infty} t(n)/n$ be the asymptotic relative Lee weight of e.

The Marginal Distribution

Let E be the random variable corresponding to the realization of a random entry of e.

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Theorem [1]

Assume that the asymptotic relative Lee weight is $T := \lim_{n \to \infty} \frac{t(n)}{n}$. For every $i \in \mathbb{Z}/p^s\mathbb{Z}$ the marginal distribution of E is given by

$$p_i := \mathrm{P}(E = i) = \frac{1}{\sum_{j=0}^{p^s-1} \exp(\lambda \mathrm{wt_L}(j))} \exp(\lambda i)$$

where λ is the solution to $T = \sum_{i=0}^{M} \mathrm{wt_L}(i) p_i$.

[1] "On the Properties of Error Patterns in the Constant Lee Weight Channel". In: *International Zurich Seminar on Information and Communication (IZS)*. 2022, pp. 44–48.

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$$\mathrm{P}(\mathrm{wt_L}(E) = j) = \begin{cases} \mathrm{P}(E = j) & \text{if } (j = 0) \text{ or } (j = M \text{ and } p \text{ is even}); \\ 2\mathrm{P}(E = j) & \text{if } (1 \quad j \quad M \quad 1) \text{ or } (j = M \text{ and } p \text{ is odd}): \end{cases}$$

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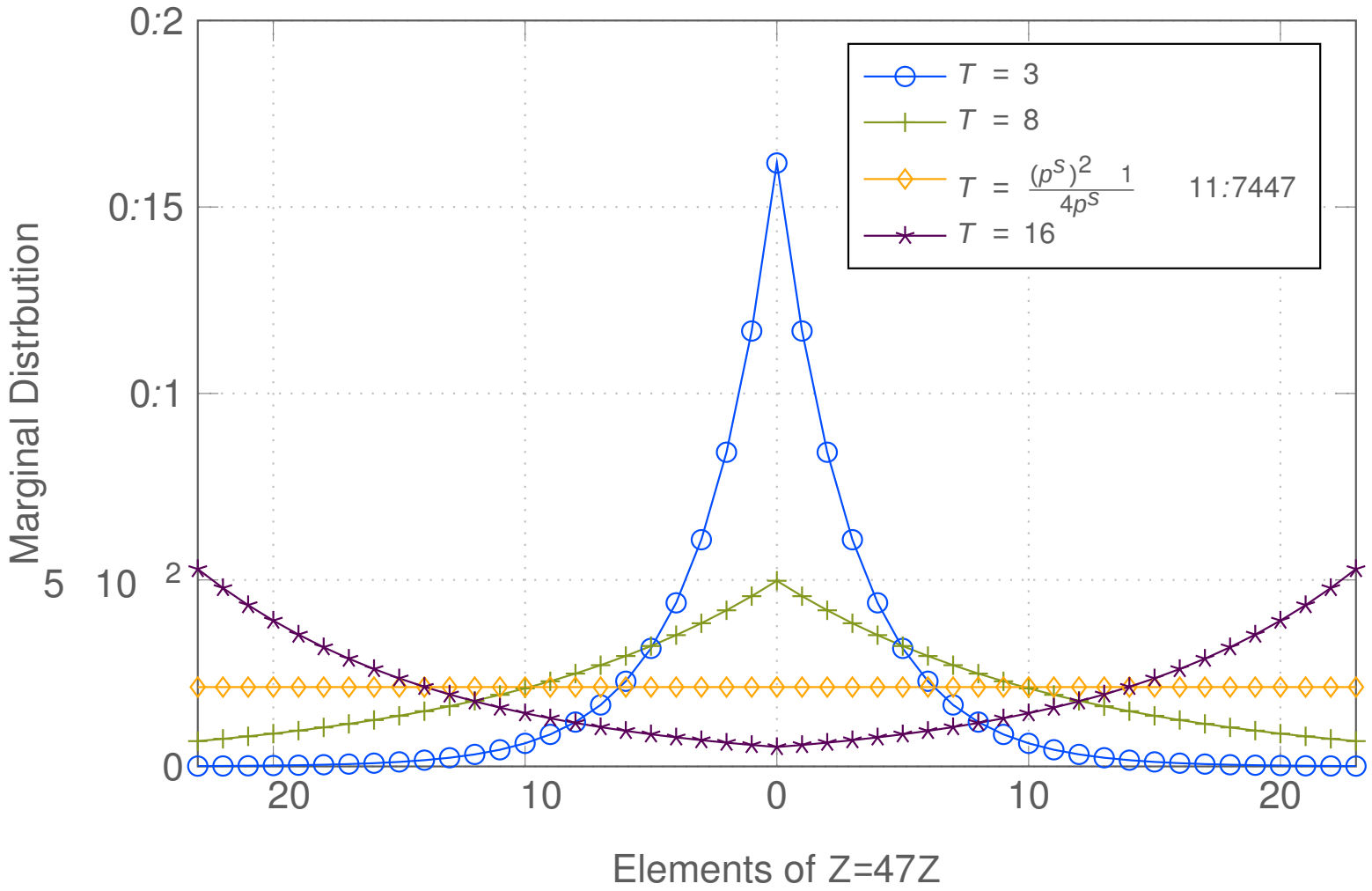
Note: $T < \quad _{p^s} \quad (\quad) \quad > 0$

The Marginal Distribution - Example over Z=47Z



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Information Set Decoding Algorithms

Consider an instance of the Lee Syndrome Decoding Problem (LSDP):

Given $H \in (\mathbb{Z}/p^s\mathbb{Z})^{(n-k)\times n}$, $s \in (\mathbb{Z}/p^s\mathbb{Z})^{n-k}$ and $t \in \mathbb{N}$,
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- ! Recent improvements: using partial Gaussian elimination[1]

[1] Matthieu Finiasz and Nicolas Sendrier. "Security bounds for the design of code-based cryptosystems". In: *International Conference on the Theory and Application of Cryptology and Information Security*. Springer. 2009, pp. 88–105.

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[1] Anja Becker et al. "Decoding random binary linear codes in $2^{n/20}$: How 1+ 1= 0 improves information set decoding". In: *Annual international conference on the theory and applications of cryptographic techniques*. Springer. 2012, pp. 520–536.

[2] Alexander May, Alexander Meurer, and Enrico Thomae. "Decoding Random Linear Codes in $\mathcal{O}(2^{0.054n})$". In: *International Conference on the Theory and Application of Cryptology and Information Security*. Springer. 2011, pp. 107–124.

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  - ::: Representation technique or Wagner's approach
  - ::: BJMM on 2 Levels is fastest in the Lee metric (non-amortized)[1]
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[1] Violetta Weger et al. "On the hardness of the Lee syndrome decoding problem". In: *Advances in Mathematics of Communications* (2019). DOI: 10.3934/amc.2022029.

[2] André Chailloux, Thomas Debris-Alazard, and Simona Etinski. "Classical and Quantum algorithms for generic Syndrome Decoding problems and applications to the Lee metric". In: *International Conference on Post-Quantum Cryptography*. Springer. 2021, pp. 44–62.

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We use the idea of partial Gaussian elimination to solve the problem:

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4. Solve the smaller instance of the LSDP. Immediately check whether $e_1 = s_1 - e_2A^\top$ has Lee weight $t - v$.

New Framework: using Restricted Spheres

Focus on the small instance of the Lee syndrome decoding problem

Given $B \in (\mathbb{Z}/p^s\mathbb{Z})^{(k+\ell)}$, $s_2 \in (\mathbb{Z}/p^s\mathbb{Z})^{\ell}$ and $v, t \in \mathbb{N}$

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Use the marginal distribution, i.e.,

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$$\text{Given } B \in (\mathbb{Z}/p^s\mathbb{Z})^{k\times n},\ s_2 \in (\mathbb{Z}/p^s\mathbb{Z})^{n} \text{ and } v, t \in \mathbb{N}$$
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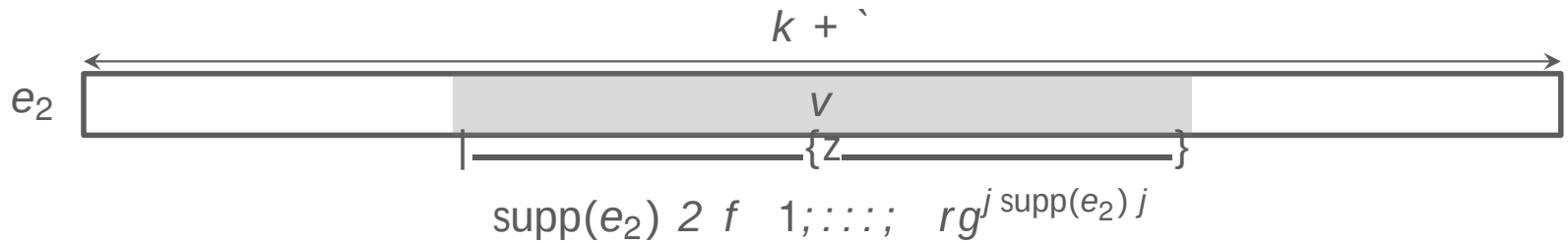
We will restrict e_2 to live either in $\{0, 1, \dots, r\}^{k+\ell}$ or in $\{r, \dots, M\}^{k+\ell}$, respectively.

Up to Minimum Distance Decoding - The BJMM Approach



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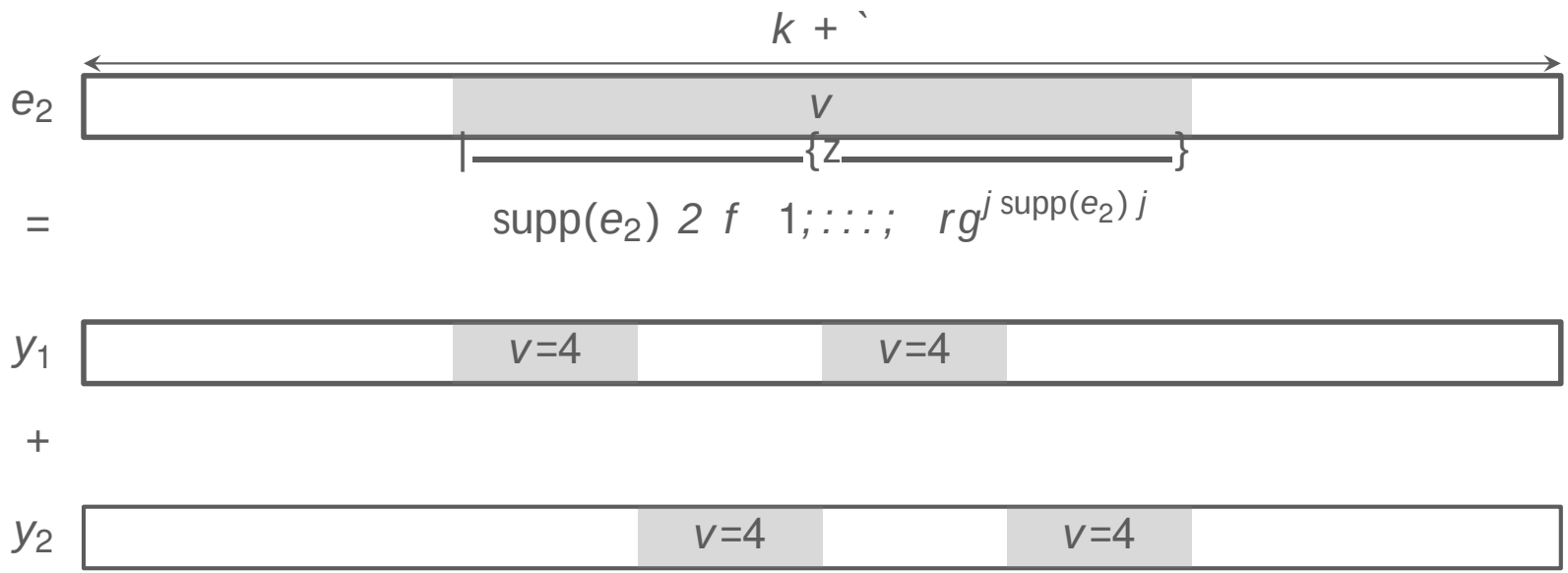

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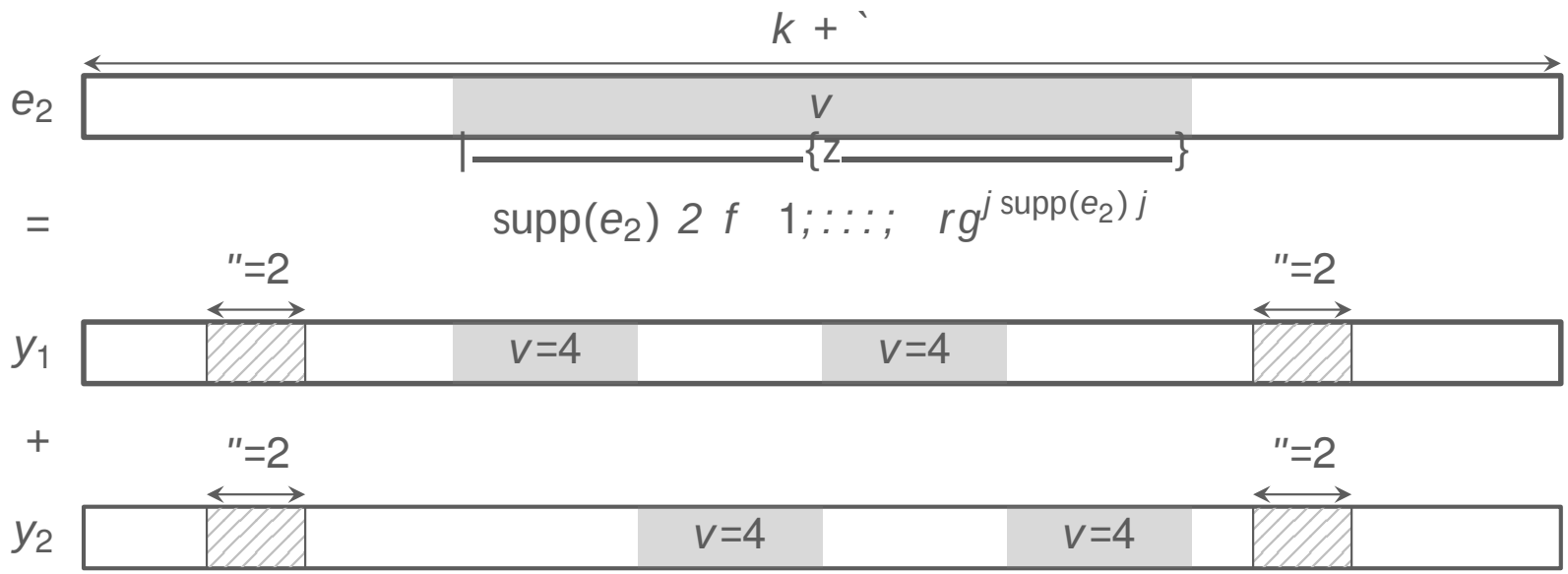

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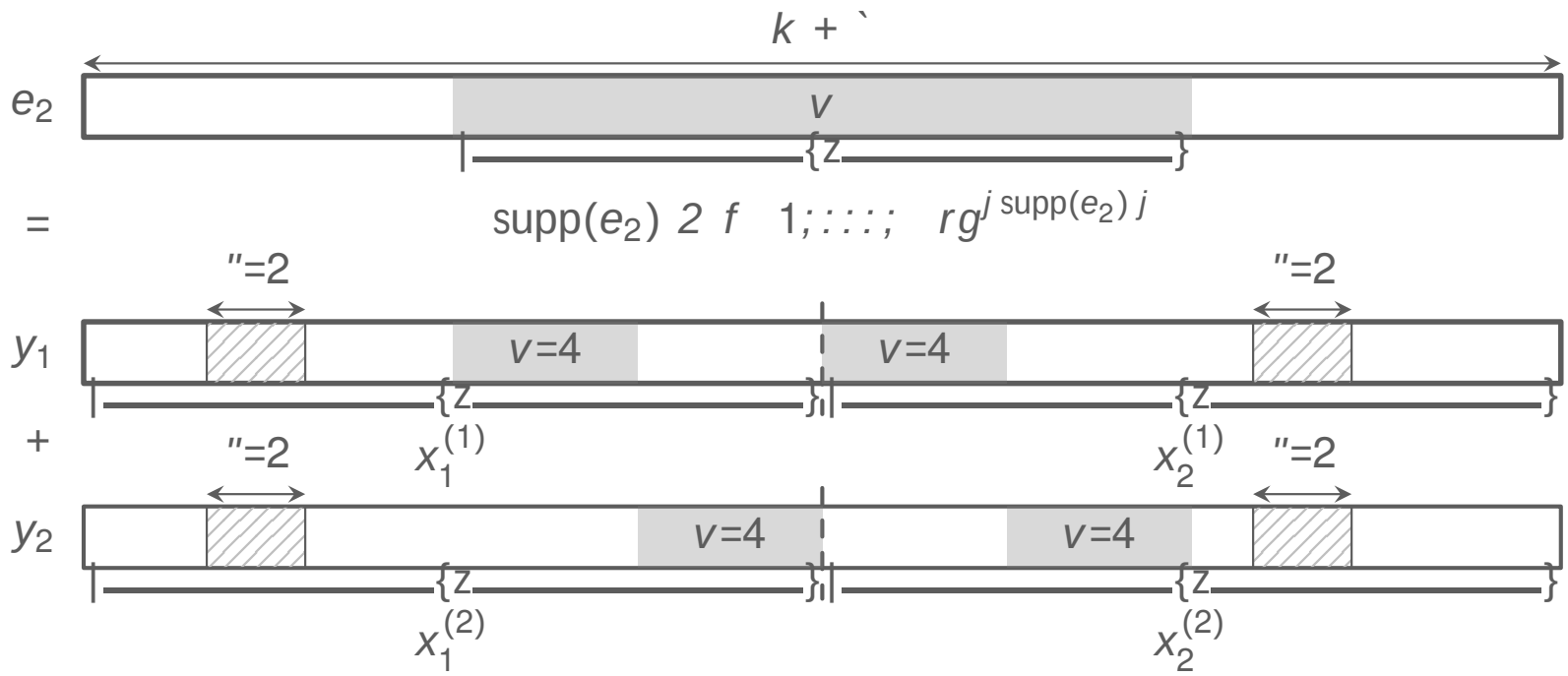

$$B_i = \{(x) \; j \; x_{E_i^c} \; 2 \; f0; : : : ; \; rg^{(k+`\ \ ")=2}; \mathrm{wt_L}(x_{E_i^c}) = v=4; x_{E_i} \; 2 \; \mathbb{Z}=p^s\mathbb{Z}^{\ \ "=2}; \; 2 \; S_{(k+`)=2}\}$$

Minimum Distance Decoding - The BJMM Approach

Recall, $s_2 = e_2 B^{>}$, where $e_2 = y_1 + y_2 = (x_1^{(1)}; x_2^{(1)}) + (x_1^{(2)}; x_2^{(2)})$.

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Recall, $s_2 = e_2 B^\top$, where $e_2 = y_1 + y_2 = (x_1^{(1)}; x_2^{(1)}) + (x_1^{(2)}; x_2^{(2)})$.

1. Splitting $B = (B_1\ B_2)$, for $i = 1; 2$ concatenate all $x_1^{(i)}; x_2^{(i)} \in B_i$ satisfying

$$x_1^{(1)} B_1^\top = u \quad x_2^{(1)} B_2^\top;$$
$$x_1^{(2)} B_1^\top = u\ s_2 \quad x_2^{(2)} B_2^\top:$$

They imply the syndrome equations for y_1 and y_2, respectively.

$$y_1 B^\top = 0 \text{ and } y_2 B^\top = s_2$$

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$$y_1 B^\top = 0 \text{ and } y_2 B^\top = s_2$$

2. Store them in a list L_i.

Minimum Distance Decoding - The BJMM Approach

Recall, $s_2 = e_2 B^\top$, where $e_2 = y_1 + y_2 = (x_1^{(1)}; x_2^{(1)}) + (x_1^{(2)}; x_2^{(2)})$.

1. Splitting $B = (B_1\ B_2)$, for $i = 1; 2$ concatenate all $x_1^{(i)}; x_2^{(i)} \in B_i$ satisfying

$$x_1^{(1)} B_1^\top =_u x_2^{(1)} B_2^\top;$$
$$x_1^{(2)} B_1^\top =_u s_2 - x_2^{(2)} B_2^\top:$$

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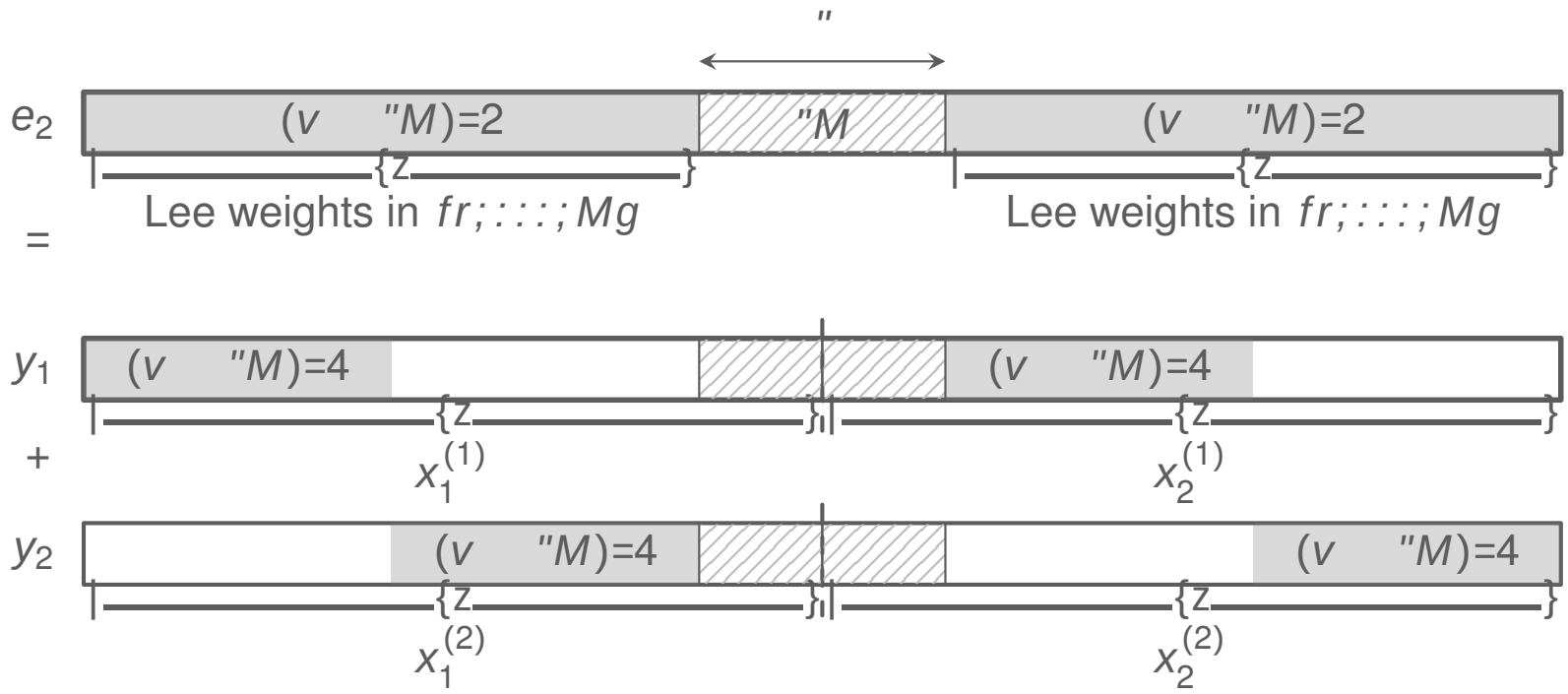
   b) the original LSDP is fulfilled as well

$$\mathrm{wt}_L(s_1 - (y_1 + y_2) A^{>}) = t - v$$

Decoding Beyond the Minimum Distance



Outline

1. Preliminaries
2. Information Set Decoding using Restricted Spheres
   - Up to Minimum Distance Decoding
   - Decoding Beyond the Minimum Distance
3. Comparison

Up to Minimum Distance Decoding - Z=47Z

1

Algorithm	$e(R\ ; p^S)$	R
Lee-BJMM	0.1618	0.451
Restricted Lee-BJMM for $r = 5$	0.1539	0.408
Amortized Lee-BJMM	0.1205	0.396
Amortized Restricted Lee-BJMM	0.1189	0.406
Amortized Lee-Wagner	0.1441	0.445
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[1] André Chailloux, Thomas Debris-Alazard, and Simona Etinski. "Classical and Quantum algorithms for generic Syndrome Decoding problems and applications to the Lee metric". In: *International Conference on Post-Quantum Cryptography*. Springer. 2021, pp. 44–62.

Up to Minimum Distance Decoding - Z=47Z

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Thank you for your attention!

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