Algebraic Coding Theory e-Summer School - ACT21
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Analysis of Low-Density Parity-Check Codes over Finite Integer Rings for the Lee Channel

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joint work with Hannes Bartz, Gianluigi Liva
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Outline

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Linear Block Codes

Let $\mathbb{F}_q$ be a finite field of order q and let n be a positive integer. We will denote by $\mathbb{Z}_q$ the ring of integers modulo q.

Definition [Linear Code]

An $[n, k]_q$-linear code $\mathcal{C} \subset \mathbb{F}_q^n$ is a k-dimensional subspace of $\mathbb{F}_q^n$. The elements of $\mathcal{C}$ are called codewords.

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For any two codewords $x, y \in \mathcal{C}$ we define

- the *Hamming weight* of x, $\mathrm{wt}_H(x) = |\{i \in \{1, \ldots, n\} \mid x_i \neq 0\}|$
- the *Hamming distance* between x and y, $\mathrm{d}_H(x, y) := \mathrm{wt}_H(x - y)$

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An $[n, k]_q$-linear code $\mathcal{C}$ can be represented by an $(n - k) \times n$ matrix H satisfying

$$\mathcal{C} = \ker(H).$$

We call H a *parity-check matrix* of $\mathcal{C}$.

LDPC Codes over Finite Integer Rings

According to Sridhara and Fuja

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An $[n, k]_q$ LDPC code over $\mathbb{Z}_q$ is defined by a sparse parity-check matrix H, whose nonzero entries lie in the set of units $\mathbb{Z}_q^\times$.

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Can be described by a bipartite graph $\mathcal{G}$ consisting of

- variable nodes (VN) $\{v_1, \dots, v_n\}$
- check nodes (CN) $\{c_1, \dots, c_m\}$

VN v_j is connected to CN c_i if and only if $h_{ij} \neq 0$.

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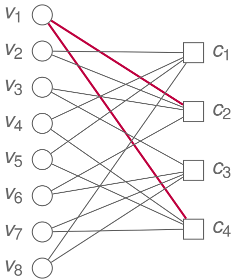
$$H = \begin{bmatrix} 0 & 1 & 0 & 2 & 4 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 3 & 1 \\ 1 & 0 & 0 & 1 & 4 & 0 & 1 & 0 \end{bmatrix} \in \mathbb{Z}_5^{4\times 8}$$



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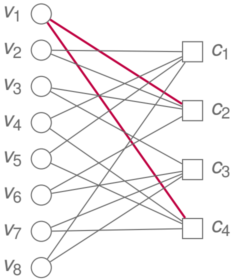

An LDPC code is (k, ℓ)-*regular*, if every VN connects to k CNs and every CN connects to ℓ VNs, for some fixed positive integer k and ℓ.

The Lee Metric

Definition [Lee weight]

For any integer $a \in \mathbb{Z}_q$ we define its *Lee weight* as

$$\mathrm{wt}_L(a) := \min(a,\ q - a). \tag{1}$$

The Lee weight of a vector $x \in \mathbb{Z}_q^n$ is the sum of the Lee weights of its entries, i.e.,

$$\mathrm{wt}_L(x) := \sum_{i=1}^{n} \mathrm{wt}_L(x_i). \tag{2}$$

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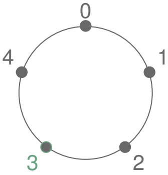

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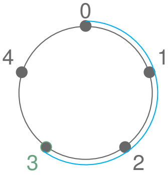

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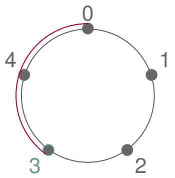

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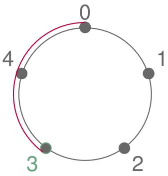

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$\implies \mathrm{wt}_L(3) = 2$

DLR

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Properties

For every $a \in \mathbb{Z}_q$ it holds:

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Definition [Lee Distance]

The Lee distance of two scalars $a, b \in \mathbb{Z}_q$ is $\mathrm{d}_L(a,b) := \mathrm{wt}_L(a-b)$. The Lee distance between two vectors $x, y \in \mathbb{Z}_q^n$ is

$$\mathrm{d}_L(x,y) = \sum_{i=1}^{n} \mathrm{d}_L(x_i, y_i).$$

Channel Coding

Let $\mathcal{X}$ and $\mathcal{Y}$ the input and output alphabet of the channel, respectively.

With a given channel law

$$P_{Y|X}(y \mid x) := \mathbb{P}(Y = y \mid X = x)$$

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Definition [Discrete Memoryless Channel]

A channel is called *discrete memoryless*, if the input and output alphabets are discrete, finite sets and the output $Y = y$ at time t only depends on the input $X = x$ at that time t, i.e.,

$$\mathbb{P}(Y_1 = y_1, \ldots, Y_n = y_n \mid X_1 = x_1, \ldots, X_n = x_n) = \prod_{i=1}^{n} \mathbb{P}(Y_i = y_i \mid X_i = x_i)$$

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Example: q-ary Symmetric Channel

Let $x \in \mathbb{Z}_q^n$ sent and $y \in \mathbb{Z}_q^n$ received.

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Example: q-ary Symmetric Channel

Let $x \in \mathbb{Z}_q^n$ sent and $y \in \mathbb{Z}_q^n$ received. Then $P_{Y_i|X_i}(y_i \mid x_i) := \begin{cases} 1 - \varepsilon & \text{if } y_i = x_i, \\ \frac{\varepsilon}{q-1} & \text{else.} \end{cases} \quad \forall i.$

Outline

The Lee Channel

Define the “Lee Channel” over $\mathbb{Z}_q$ as proposed by Chiang and Wolf:

$$p_i := \mathbb{P}(i \,|\, 0) = \mathbb{P}(-i \,|\, 0), \text{ for } i = 0, \dots, \lfloor q/2 \rfloor. \tag{3}$$

Due to symmetry: $\mathbb{P}(i \,|\, j) = \mathbb{P}(i - j \mod q \,|\, 0)$

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Theorem (Chiang and Wolf)

The channel described in (3) is strictly matched to the Lee metric for maximum likelihood decoding if and only if the following two properties hold.

$$p_0 > p_1 \quad \textit{and} \quad p_i = \frac{p_1^i}{p_0^{i-1}} \quad \textit{for all } i = 2, \dots, \lfloor q/2 \rfloor.$$

The Lee Channel

For $y, x, e \in \mathbb{Z}_q$, consider a discrete memoryless channel (DMC)

$$\underset{\text{channel output}}{y} = \underset{\text{channel input}}{x} + \underset{\text{additive error term}}{e} \tag{4}$$

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The channel law is given by

$$\mathbb{P}(Y = y \,|\, X = x) =: P_{Y|X}(y|x) = \frac{1}{Z} \exp\left(-\lambda \, \mathrm{d}_L(x, y)\right), \tag{5}$$

where $Z := \sum_{e=0}^{q-1} \exp(-\lambda \, \mathrm{wt}_L(e))$ and $\lambda > 0$.

Note:

- The channel defined in (5) is the DMC matched to the Lee metric.
- The conditional distribution (5) arises (in the limit of large n) as the marginal distribution of a channel.

The Constant-Weight Lee Channel

Let $y, x, e \in \mathbb{Z}_q^n$, where $\mathrm{wt}_L(e) = t$ for some fixed positive integer t. Consider again

$$y = x + e.$$

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Note: The error vector e is chosen uniformly at random from the set of all length-n vectors of Lee weight t:

$$\mathcal{S}_t^n := \left\{x \,\middle|\, x \in \mathbb{Z}_q^n, \mathrm{wt}_L(x) = t\right\}.$$

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Question: What would $P_{Y \mid X}(y \mid x)$ look like?

The Constant-weight Lee Channel

Let $\mathbf{p} = (p_0, \ldots, p_{q-1})$, with $p_i := \mathbb{P}(i \mid 0)$ for all $i \in \mathbb{Z}_q$.

Lemma

The constant-weight Lee channel over $\mathbb{Z}_q$ has channel distribution

$$p_i^\star = \kappa \exp\left(-\lambda \operatorname{wt}_L(i)\right), \quad \kappa := \frac{1}{\sum_{j=0}^{q-1} \exp(-\lambda \operatorname{wt}_L(j))},$$

such that it matches under maximum likelihood decoding.

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Sketch of proof
We want that $\mathbf{p} = (p_0, \ldots, p_{q-1})$ maximizes the entropy function

$$H_e(\mathbf{p}) := -\sum_{i=0, p_i \neq 0}^{q-1} p_i \log p_i$$

under the constraint that $\sum_{i=0}^{q-1} \mathrm{wt}_L(i) p_i = t/n =: \delta$.

Outline

Symbol Message Passing

Consider a nonbinary LDPC code $\mathcal{C}$ with VNs $\{v_1, \ldots, v_n\}$ and CNs $\{c_1, \ldots, c_m\}$ and parity-check matrix H. Denote by $\mathcal{N}(v_j)$ and $\mathcal{N}(c_i)$ the set of all connecting elements to VN v_j and CN c_i, respectively.

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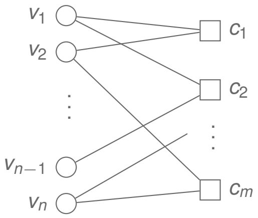

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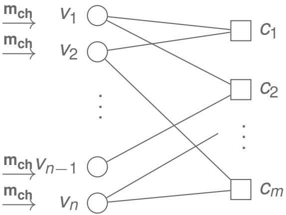
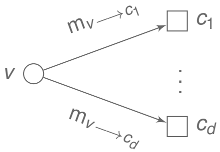
Every VN v receives the channel observation
$\mathbf{m_{ch}} := \left(P_{Y|X}(y \mid 0), \ldots, P_{Y|X}(y \mid q-1)\right)$



Symbol Message Passing

Initialization.
Each VN v sends channel observation to the neighboring CNs $c \in \mathcal{N}(v)$

$$m_{v \longrightarrow c} = \mathbf{m_{ch}}.$$

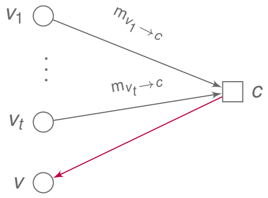


Symbol Message Passing

CN-to-VN step.
Each CN computes for every $v \in \mathcal{N}(c)$

$$m_{c \to v} = h_{c,v}^{-1} \sum_{v' \in \mathcal{N}(c) \setminus \{v\}} h_{c,v'} m_{v' \to c}.$$

Note: $h_{c,v}^{-1}$ exists, since we said the nonzero entries of H are units.



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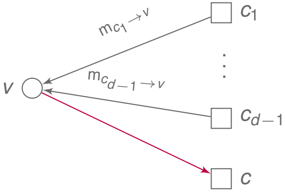
VN-to-CN step.

Define the aggregated extrinsic L-vector

$$E = L(y) + \sum_{\mathrm{c}' \in \mathcal{N}(\mathrm{v}) \setminus \{\mathrm{c}\}} L\left(m_{\mathrm{c}' \to \mathrm{v}}\right),$$

where y is the channel output and
$L(y) = (L_0(y), \ldots, L_{q-1}(y))$ with
$L_x(y) = \log\left(P_{Y|X}(y \mid x)\right)$.
Note: We assume the CN-to-VN messages are modelled as a qSC,

$$P_{M|X}(m|x) = \begin{cases} 1-\xi & \text{if } m = x \\ \xi/(q-1) & \text{otherwise} \end{cases}$$



Then the VN-to-CN messages are

$$m_{\mathrm{v} \to \mathrm{c}} = \arg\max_{x \in \mathbb{Z}_q} E_x.$$

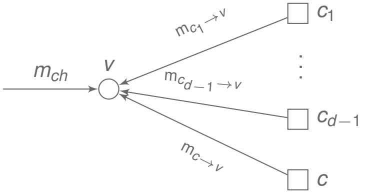
Symbol Message Passing

Final decision.
The final decision at each VN v is

$$\hat{x} = \arg\max_{x \in \mathbb{Z}_q} L_x^{\mathrm{FIN}}$$

where

$$L^{\mathrm{FIN}} = L(m_{\mathrm{ch}}) + \sum_{\mathrm{c} \in \mathcal{N}(\mathrm{v})} L\left(m_{\mathrm{c} \to \mathrm{v}}\right).$$



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  - For a field $\mathbb{Z}_q$ this argument is *independent* of the channel law and hence also valid for the Lee channel.
- If q is not a prime, we treat the messages as qSC anyways, due to the following observations

DLR

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  - Over finite fields, the extrinsic channel transition probabilities, averaged over a uniform distribution of nonzero elements the parity-check matrix, yield (in the limit of a large block length) a qSC.
  - For a field $\mathbb{Z}_q$ this argument is *independent* of the channel law and hence also valid for the Lee channel.
- If q is not a prime, we treat the messages as qSC anyways, due to the following observations
  - The approximation is especially accurate when the fraction of elements of $\mathbb{Z}_q \setminus \{0\}$ that are in $\mathbb{Z}_q^\times$ is large.

The qSC-Assumption for SMP

- Following Lechner, Pedersen, and Kramer, the CN to VN messages are modelled as observations from a q-ary symmetric channel.
- Motivation behind this choice:
  - Over finite fields, the extrinsic channel transition probabilities, averaged over a uniform distribution of nonzero elements the parity-check matrix, yield (in the limit of a large block length) a qSC.
  - For a field $\mathbb{Z}_q$ this argument is *independent* of the channel law and hence also valid for the Lee channel.
- If q is not a prime, we treat the messages as qSC anyways, due to the following observations
  - The approximation is especially accurate when the fraction of elements of $\mathbb{Z}_q \setminus \{0\}$ that are in $\mathbb{Z}_q^{\times}$ is large.
  - The use of the qSC approximation is important from a practical viewpoint, i.e., decoding becomes particularly simple.

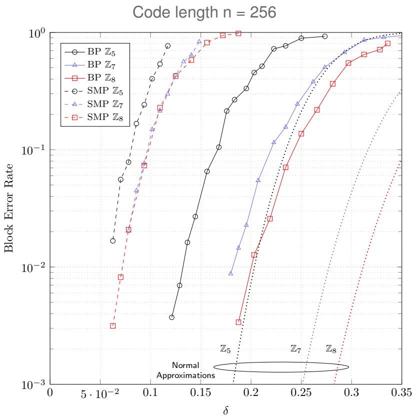
Simulations

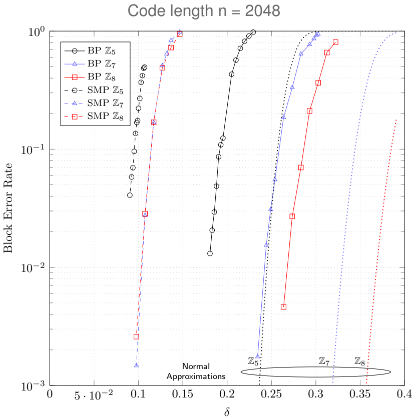
Decoding performance for both BP and SMP over both the Lee channel and the constant-weight Lee channel using

- $(3, 6)$ regular nonbinary LDPC codes of length 256 and 2048,
- For the constant-weight Lee channel, the error vectors are drawn uniformly at random from the set of vectors with a given weight.

Simulations

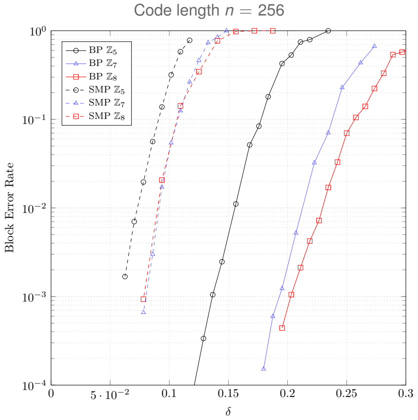
Block error rate vs. average Lee weight δ for regular $(3,6)$ nonbinary LDPC codes in the Lee channel for BP and SMP decoding.





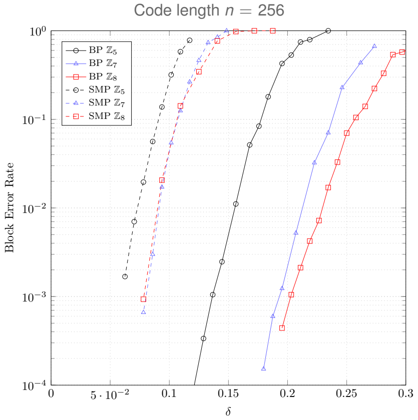
Simulations

Block error rate vs. average Lee weight δ for regular $(3, 6)$ nonbinary LDPC codes in the constant-weight Lee channel for BP and SMP decoding.



Simulations

Block error rate vs. average Lee weight δ for regular $(3, 6)$ nonbinary LDPC codes in the constant-weight Lee channel for BP and SMP decoding.



Thank you very much for your attention!